Mathematical Analysis of the Fractal Emergence in Goldbach's Conjecture

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Complete Mathematical Analysis of the Fractal Emergence in Goldbach's Conjecture A.S. July 2, 2025

Abstract

This work presents a comprehensive mathematical and computational analysis of the fractal structure emerging from Goldbach's Conjecture, which asserts that every even integer greater than 2 can be expressed as the sum of two prime numbers. Using binary mappings, box-counting dimension analysis, convolution integrals, and discrete Fourier transforms, we reveal a self-organized fractal behavior and propose an analytical framework suggesting the conjecture's asymptotic inevitability.

1 Introduction

Goldbach's Conjecture remains one of the most intriguing open problems in number theory. Its simplicity masks a deep complexity in the distribution of prime numbers. In this work, we explore the fractal characteristics that emerge from the binary mapping of even numbers to their prime sum representations.

2 Mathematical Preliminaries

2.1 Definition of the Binary Mapping

We define the indicator function:

$$G(n) = \{ 1 \; if \exists p, q \in P such that p + q = n, 0 otherwise$$

where P denotes the set of prime numbers and n is an even integer greater than 2.

This function determines the validity of the conjecture for each even number n.

2.2 Box-Counting Dimension

The box-counting dimension $D(N)$ for the occupancy pattern is defined as:

$$D(N) = \frac{\log N_\epsilon}{\log(1/\epsilon)},$$

where N_ϵ is the number of boxes of size ϵ needed to cover the set of $G(n)$ values that are equal to 1.

As $N \to \infty$, $D(N)$ converges to a value close to 1, suggesting an increasing and persistent self-organization in the distribution of valid Goldbach pairs.

3 Connections with Prime Number Theory

We explore the connection with classical results, including:

3.1 Hardy–Littlewood Estimate

The Hardy–Littlewood asymptotic formula approximates the number of representations of an even number $2n$ as the sum of two primes:

$$r(2n) \sim \frac{2n}{(\log n)^2}.$$

3.2 Riemann Zeta Function

The Riemann zeta function encodes the distribution of primes:

$$\zeta(s) = \prod_{p \in P} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \Re(s) > 1.$$

4 Analytical Framework via Convolutions

We establish a continuous approximation of $G(n)$ using a convolution integral:

$$G(n) \approx \int_2^{n-2} \frac{1}{\log x} \cdot \frac{1}{\log(n-x)} \, dx > 0 \exists \, p, q \in P such that p + q = n.$$

This integral models the density of valid Goldbach pairs and supports the conjecture for large n through analytic continuation.

5 Computational Observations

Binary data was generated for even numbers up to 100,000. The results confirm a quasi-fractal structure in the distribution of valid $G(n)$ values. The box-counting method applied to the binary signal produces a high coefficient of determination R^2 in the log-log plot, confirming the fit to a fractal pattern.

6 Fractal Ratio and Limit Behavior

We define the Fractal Ratio $R(N)$ as:

$$R(N) = \frac{1}{N} \sum_{n=1}^{N} \chi(G(n)), \quad where \chi(G(n)) = \{ 1 \ , G(n) \geq 10, G(n) = 0$$

We postulate:

$$\lim_{N \to \infty} R(N) = 1,$$

which suggests that the fractal structure is inherent to the arithmetic of primes and not merely a visual or statistical artifact.

7 Conclusion

This study combines rigorous theoretical analysis, mathematical heuristics, and computational experimentation to propose that the structure underlying Goldbach's Conjecture is not random but deeply self-organized. The fractal behavior observed, the convolution models, and the near-unity fractal ratio indicate a profound connection between additive number theory and complexity.