Probability 2 : Conditional Probability

Anand Systla

Masters in Financial Engineering Bootcamp
UCLA Anderson

August 9, 2022

Today's Agenda

- Discuss the previous class take home questions
- Independence
- Conditional probability
- Baye's Theorem + Example
- Conditional independence
- Simpson's paradox
- Random Variables

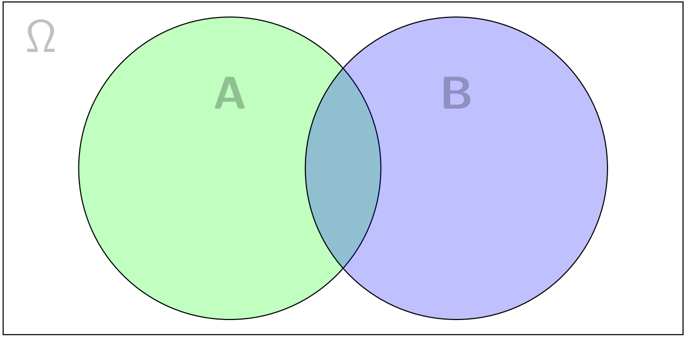
Conditional Probability Motivation

- How do we change our probability of an event occuring given new information?
- Say we want the probability of it snowing tomorrow and we're given
    1. You are located in Antarctica
    2. It is the month of December
    3. Humidity today is 80%
- A really neat way to think about this is using a probability tree diagram

Conditional Probability

- Conditional probability $A|B$ means we condition on a new information set B.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}$$

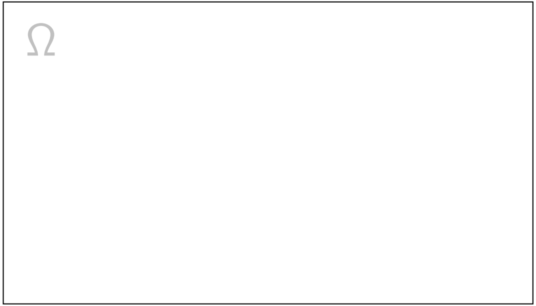


Independence

- Two events A and B are independent iff $P(A \cap B) = P(AB) = P(A) \cdot P(B)$
- We can show independence in two ways
  - Assumption: Two coin tosses, coin toss and the color of the building you're conducting the experiment, etc.
  - Derivation: $P(AB) = P(A) \cdot P(B)$
- **Example:** Say $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2, 3, 4\}, B = \{2, 4, 6\}, C = \{1, 6\}$
  - $P(A) =$ $P(B) =$ $P(C) =$
  - $P(AB) = P(A \cap B = \{2, 4\}) =$
  - $P(A) \cdot P(B) =$
  - $P(AC) = P(A \cap C = \{1\}) =$
  - $P(A) \cdot P(C) =$
  - Alternatively, if B has occurred (we focus on $\{2, 4, 6\}$), then $P(A|B = \{2, 4\}) =$
  - Occurrence of event B has no effect on the likelihood of occurrence of event A (what about C?)
- If A and B are independent, how does $P(A)$ change given B has occurred? i.e. $P(A|B) =$

Independent vs Disjoint Events

- In a single coin toss, is the heads outcome independent of the tails outcome?



- Examples of disjoint events: sides of a dice, positive/negative returns on a day
- Examples of independent events: Two consecutive coin tosses, yearly stock returns (maybe?)

Bayes Theorem

- One of the most important theorems in probability. Rearranging the conditional probability expression, we get

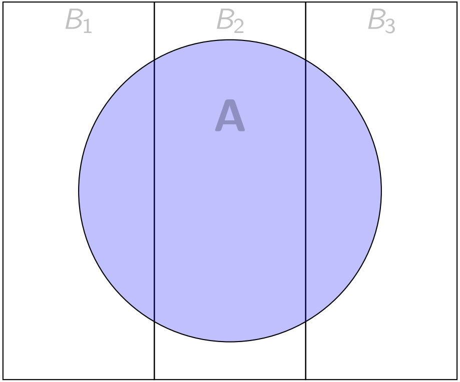
$$\begin{aligned} P(AB) &= P(A|B) \cdot P(B) \\ P(AB) &= P(B|A) \cdot P(A) \\ \implies P(A|B) &= P(B|A) \cdot \frac{P(A)}{P(B)} \end{aligned}$$

- $P(A|B) \neq P(B|A)$
- $P(A|B) \neq 1 - P(A|B^c)$
- $P(A|B) = 1 - P(A^c|B)$
- $P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$ if $A \& B$ are independent
- Intuitively,

$$P(\text{Hypothesis}|\text{Data}) = P(\text{Data}|\text{Hypothesis}) \cdot P(\text{Hypothesis})/P(\text{Data})$$

$$P(\text{Flu}|\text{Symptoms}) = P(\text{Symptoms}|\text{Flu}) \cdot P(\text{Flu})/P(\text{Symptoms})$$

Law of Total Probability



- Generalizing $P(A) = \sum P(AB_i) = \sum P(A|B_i)P(B_i)$. Assumptions?
- Applications: If computing $P(A)$ is a challenge (defective parts sold by an industry)
- How can we write the law if we have two general sets A and B?
- How can we combine the law of total probability and Baye's Theorem?

Conditional Independence

- A and B are conditionally independent given event C iff

$$P(AB|C) = P(A|C) \cdot P(B|C)$$

- What is the relationship between unconditional and conditional independence?
- Three situations we will study
  - Unconditionally independent and conditionally independent
  - Unconditionally independent, but conditionally dependent
  - Unconditionally dependent, but conditionally independent

Unconditionally Independent, Conditionally Dependent

- $\Omega = \{1, 2, 3, 4, 5, 6\}, A = \{1, 2\}, B = \{2, 4, 6\}, C = \{1, 2, 3, 4, 5\}$
- $P(A) =$ $\qquad$ $P(B) =$ $\qquad$ $P(C) =$ $\qquad$ $P(AB) =$
- Are A and B unconditionally independent?
- $P(A|C) =$ $\qquad$ $P(B|C) =$ $\qquad$ $P(AB|C) =$
- Conditional on event C, are A and B conditionally independent?
- What would a real life example look like?

Unconditionally Dependent, Conditionally Independent

- $\Omega = \{1, 2, 3, 4, 5, 6\}, A = \{1, 2, 5\}, B = \{2, 4, 6\}, C = \{1, 2, 3, 4\}$
- $P(A) =$ $P(B) =$ $P(C) =$ $P(AB) =$
- Are A and B unconditionally independent?
- $P(A|C) =$ $P(B|C) =$ $P(AB|C) =$
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Unconditionally Independent, Conditionally Independent

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Simpson's Paradox

- Classic example of how conditioning and aggregation affect things. Prof. Joe Blitzstein calls conditioning the "soul of statistics" and that all probabilities are conditional.

- Whenever we have data, we want to think about how it was sampled. We need to be careful what we are conditioning on.

Bayes Theorem Example

- Probability of a car theft in a neighbourhood is 1%. All cars are now equipped with state-of-the-art alarms that have a success rate of 90% (will go off 90% of the times if there is a theft). However, the alarms sometimes go off randomly (due to a bird, ball etc.) around 20% of the times. What is the probability that there is a theft happening conditional on the alarm going off?

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$$P(\text{Theft}) = 0.01$$

$$P(\text{Alarm}|\text{Theft}) = 0.9$$

$$P(\text{Alarm}|\text{No Theft}) = 0.2$$

$$P(\text{Alarm}) = P(\text{Alarm}|\text{Theft}) \cdot P(\text{Theft}) + P(\text{Alarm}|\text{No Theft}) \cdot P(\text{No Theft})$$

$$P(\text{Alarm}) = 0.9 \times 0.01 + 0.2 \times 0.99 = 0.009 + 0.198 = 0.207$$

$$\implies P(\text{Theft}|\text{Alarm}) = P(\text{Alarm}|\text{Theft}) \cdot \frac{P(\text{Theft})}{P(\text{Alarm})} = 0.9 \cdot \frac{0.01}{0.207} = 0.043$$

Random Variables

- Formally, a random variable is neither nor a variable, it is a function that maps events to the real line. A Random Variable (henceforth RV) X is defined as

$$X : \omega \to \mathbb{R}$$

- The randomness comes from the underlying experiment involved. Variable comes from storing events generated by the experiment in a variable X
- Example1: $\Omega = \{H, T\}$. RV can be defined as $X = 1$ if heads and $X = 0$ if Tails
- Example2: Say we toss a coin 3 times. $X = \{\text{No. of H}\}$. Sketching our RV

Take Home Questions

1. **Quant Interview:** A quant driven hedge fund wants to interview all the UCLA MFE students for an internship. Say 50% of all students who received their first interview received a second interview. 95% of your friends that got a second interview said they had a good first interview. 75% of your friends that did not get a second interview said they had a good first interview. If you felt you had a good first interview, what is the probability that you will receive a second interview? Alternatively, if you felt you had a bad first interview, what is the probability that you will receive a second interview?
2. There are two biased coins A and B in a bag. Probability of heads for coin A is 0.75 and the probability of heads for coin B is 0.3. You pick a coin randomly and perform 10 tosses (without knowing which coin you picked). Hint: To solve the two problems below, compute the posterior probability $P(Hypothesis|Data)$ and argue that one of the coin has a higher posterior probability. You will have to test and compare the two hypothesis - picking coin A given data and picking coin B given data. Since we are choosing a coin randomly, P(picking coin A) = P(picking coin B) = 1/2. Key takeaway is how the data changes your beliefs about which coin you picked.
   - 2.1 You observe that you get 8 heads and 2 tails from your coin tosses. What is the probability that you picked coin A from the bag given the data. Compare this with the posterior probability of picking the other coin.
   - 2.2 Now, say you observed 8 tails and 2 heads from your coin tosses. What is the probability that you picked coin B given the data. Compare this with the posterior probability of picking the other coin.