Probability 1 : Probability Theory and Counting

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Probability : Week 2

- Class 1
  - Probability theory
  - Counting
  - Gambler's ruin
- Class 2
  - Independence
  - Conditional probability
  - Baye's rule
- Class 3
  - Random variables
  - First order and higher order moments
  - Correlation v causation
  - Common distributions

Today's Agenda

- Discuss the previous class take home questions
- Brief history of probability
- Interpretation
- Probability theory
- Counting
- Gambler's ruin

History and Importance

- Timeline

- Common applications
  - Likelihood of events
  - Optimal sizing using the "Kelly criterion"
  - Stochastic calculus (PDF/CDF), Risk management (VaR, expected shortfall), Econometrics, etc.

Interpretations

- **Classical**: $P(\text{event}) = \frac{\text{Desired Outcomes}}{TotalOutcomes}$
  - Key assumption is that all events are equally likely, i.e principle of indifference
  - Examples: Rolling a dice, drawing a random card, options in a multiple choice exam etc.
  - What is the problem with this interpretation?

- **Frequentist**
  - If we do not know the number of outcomes (unfair dice) or if they are not equally likely, then we can conduct trials to determine the probability

$$P(event \quad A) = \frac{\text{Number of occurrences of A}}{Totalnumberoftrials}$$

  - Frequentists believe that as $n \to \infty$ in the limit, we arrive at the true probability
  - What is the problem with this approach?

- **Bayesian**
  - Probability is subjective and it depends on the degree of belief
  - A Bayesian updates his beliefs conditional on an event occurring
  - For example, what is the probability of life on Mars? What would a Classical and Frequentist say? Or would your house be robbed today?
  - A Bayesian's will update his probability (posterior) condition on new information

Probability Theory

- Experiment: Building block of probability, reproducible, finite outcomes
- Sample Space (Ω): Set of all possible outcomes
- Power Set (U): Collection of all subsets of Ω and empty set $\{\emptyset\}$
- Event: Set of outcomes of an experiment
- Naïve Definition of probability $P =$ Desired Outcomes/Total Outcomes
- Probability can be thought of as a function that maps $P : A \rightarrow [0, 1]$
- Axioms of Probability
  - **Axiom 1:** $P(A) \geq 0$
  - **Axiom 2:** $P(\Omega) = 1$
  - **Axiom 3:** $P\left(\cup_i A_i\right) = \sum_i P(A_i)$

Counting

- Counting lets us figure out the total and number of desirable outcomes

- Permutations
  - n-objects and k-draws
  - Without replacement
  - Ordering matters

$$^{n}P_k = n \cdot (n-1)\dots(n-k+1) \times \frac{(n-k)\cdot(n-k-1)\dots 1}{(n-k)\cdot(n-k-1)\dots 1} = \frac{n!}{(n-k)!}$$

- Combinations
  - n-objects and k-draws
  - Without replacement
  - Ordering does not matter (how many ways can we rearrange k-objects?)
  - Intuitively, we count

$$^{n}C_k = \frac{^{n}P_k}{k!} = \frac{n!}{(n-k)!k!}$$

Sampling Table

With n objects and k draws

	Order matters	Order does not matter
With replacement		
Without replacement		

Take Home Questions (1/2)

1. There are 10 people. In how many ways can you split them into
    1.1 Team of 6 and a team of 4
    1.2 Two teams of 5 each

2. Prove $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. *Hint*: The goal is to prove this mathematically (and not using a Venn diagram). Try writing $A \cup B$ as sum of three disjoint sets and then apply axiom 3 discussed in class.

3. Two people take turns trying to sink a basketball into a net. Person 1 succeeds with probability $1/3$ and person 2 with probability $1/4$. Find the probability that
    3.1 Person 2 succeeds before person 1
    3.2 Person 1 succeeds before person 2

Take Home Questions (2/2)

4. **Birth Month Problem :** This is a variation of the birthday problem. How many people are needed in a room to make it possible that atleast two people have the same birth month with a probability of 50%. There are 12 months in a year and probability of being born in any month is the same. *Hint*: Try expressing the probability of atleast 2 people having the same birth month as a complementary event.
5. **(Optional) Birth Month Problem Advanced Version :** In the previous question, we assumed that the probability of being born in any month is equal. Historically, month of August has seen the largest number of births compared to any other month. If we were to incorporate this new information, say probability of being born in August is p and probability of being born in any other month is q. What would be your approach to solving the above problem with this new information? Hint : The trick here would still be to write the probabilities in terms of its complementary event. Try enumerating possible combinations and compute the probability using the non-uniform distribution.

Take Home Questions : Suggested Solutions (1/2)

1. 1.1 Selecting a team of 6 and a team of 4 people can be done ${}^{10}C_6 ={}^{10} C_4$ ways
   1.2 By selecting one team of 5, you automatically choose another team of 5, so this can be done ${}^{10}C_5/2$
2. $A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B) \implies P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$

$$P(A \cap B^c) = P(A) - P(A \cap B) \text{ and } P(A^c \cap B) = P(B) - P(A \cap B)$$

$$\implies P(A \cup B) = P(A) + P(B) - 2 \cdot P(A \cap B) + P(A \cap B) = P(A) + P(B) - P(A \cap B)$$

3. If P1 goes first, defining
   - $A_1 =$P1 succeeds in try 1 $\implies P(A_1) = 1/3$
   - $A_2 =$P1 succeeds in try 2 given P1 & P2 miss in try 1 $\implies P(A_2) = (2/3 \cdot 3/4) \cdot 1/3 = (1/2) \cdot 1/3$
   - Similarly $P(A_3) = (1/2)^2 \cdot 1/3$
   - Finally,
     $p = P(\text{P1 succeeds before P2}) = P(A_1) + P(A_2) + \ldots = \sum_{i=1}^{\infty}(1/2)^{i-1} \cdot 1/3 = 1/3 \cdot \frac{1}{1-1/2} = 2/3$

   If P2 plays after P1, then $P(\text{P2 succeeds before P1}) = 1 - p = 1 - 2/3 = 1/3$

   If P2 goes first, then $P(\text{P2 succeeds before P1}) = \sum(1/2)^{i-1} \cdot 1/4 = 1/4 \cdot \frac{1}{1-1/2} = 1/2$

Take Home Questions : Suggested Solutions (2/2)

4. If we have 13 people, atleast 2 of them will have the same birth month (uninteresting case). Say we have *n*-people in our sample ($1 < n < 12$). Since each person is unique
   - If n = 2,
   $$P(\text{Nobody Has Same Birthmonth}) = \frac{12}{12} \cdot \frac{11}{12}$$
   $$P(\text{Atleast 2 Have Same Birthmonth}) = 1 - \frac{12 \cdot 11}{12^2} = 0.083$$
   - If n = 3,
   $$P(\text{Nobody Has Same Birthmonth}) = \frac{12}{12} \cdot \frac{11}{12} \cdot \frac{10}{12}$$
   $$P(\text{Atleast 2 Have Same Birthmonth}) = 1 - \frac{12 \cdot 11 \cdot 10}{12^3} = 0.236$$
   - If n = 4,
   $$P(\text{Atleast 2 Have Same Birthmonth}) = 1 - \frac{12 \cdot 11 \cdot 10 \cdot 9}{12^4} = 0.427$$
   - So we need atleast $n = 5$ people to have a probability of 61.8%.

Take Home Questions : Suggested Solutions (2/2)

5. Intuitively, nothing changes with our approach. We still want to think of expressing the probability of atleast 2 people having the same birthmonth as a complementary event. This question is motivation for thinking of independence and conditional probability.

$$P(\text{Atleast 2 have same birthmonth}) = 1 - \underbrace{P(\text{All have different birthmonths})}_{=p}$$

$$p = P(\text{All have different birthmonths}|\text{Nobody in August}) \cdot P(\text{Born in non-Aug months}) +$$
$$P(\text{All have different birthmonths}|\text{One in August}) \cdot P(\text{One in Aug, Rest in other months})$$

$$P(\text{All have different birthmonths}|\text{Nobody in August}) = \frac{{}^{11}P_k}{11^k}$$
$$P(\text{All have different birthmonths}|\text{One in August}) = \frac{{}^{11}P_{k-1}}{11^{k-1}} \quad {}^{k}C_1$$
$$P(\text{Born in non-Aug months}) = q^k$$
$$P(\text{One in Aug, Rest in other months}) = p \cdot q^{k-1}$$