

# Derivation for Inflation of Heritability by Parentally-Imbalanced VCT

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Feb 24, 2026

## Background

Define the phenotype as

$$P = A + C + \epsilon$$

where  $A$  is the additive genetic value,  $C$  is the vertically culturally transmitted trait value, and  $\epsilon$  is i.i.d. Gaussian noise.  $\text{Var}(P)$  is normalized to 1, so  $\text{Var}(A) = h^2$ ,  $\text{Var}(C) = b^2$ , and  $\text{AVar}(\epsilon) = 1 - h^2 - b^2$ .

In the original model, we assumed that  $C$  for any given offspring would simply be the mean of the parental values:

$$C_i = \frac{C_p + C_m}{2}$$

However, this can be generalized to a weighted sum over both parents, where the father contributes variance

$$b_p^2 = w b^2$$

and the mother

$$b_m^2 = (1 - w) b^2$$

Crucially, by the properties of ratios, this means that if  $w = 0.75$ , then the ratio of contributions will be

$$\frac{b_p^2}{b_m^2} = \frac{w}{1 - w} = (0.75)(0.25) = 3$$

In the parental mean case where  $w = 0.5 = 1 - w$ :

$$\frac{b_p^2}{b_m^2} = \frac{w}{1 - w} = 1$$

(John, I believe this is the point you were making at our meeting, but forgive me if I misinterpreted your words).

The interesting empirical result is that the average phenotypic covariance  $\text{Cov}(P_i, P_j)$  for a given  $(h^2, b^2)$  parameter set appears to be higher as  $w$  moves away from 0.5 and toward 0 or 1.

## Theory

Here I present a small derivation attempting to explain this result analytically. The VCT value for one individual:

$$c_i = \sum_j^2 w_j c_j = w c_p + (1 - w) c_m$$

For siblings sharing parents:

$$\begin{aligned}
\text{Cov}(c_i, c_j) &= \text{Cov}(wc_p + (1-w)c_m + \epsilon_i, wc_p + (1-w)c_m + \epsilon_j) \\
&= \text{Cov}(wc_p + (1-w)c_m, wc_p + (1-w)c_m) \\
&= \text{Var}(wc_p + (1-w)c_m) \\
&= w^2\text{Var}(c_p) + (1-w)^2\text{Var}(c_m) \\
&= b^2(w^2 + (1-w)^2)
\end{aligned}$$

This is a polynomial  $f(w)$  in  $w$ .

The total phenotype correlation is

$$\begin{aligned}
\text{Cov}(P_i, P_j) &= \text{Cov}(a_i, a_j) + \text{Cov}(c_i, c_j) \\
\text{Cov}(a_i, a_j) &= 2k_{ij}h^2
\end{aligned}$$

where  $k_{ij}$  is the kinship coefficient.

Therefore:

$$\text{Cov}(P_i, P_j) = 2k_{ij}h^2 + b^2(w^2 + (1-w)^2)$$

$\text{Cov}(P_i, P_j)$  is therefore a second-degree polynomial  $w$  with a minimum at  $w = 0.5$ , and with a minimum value and intercept determined by  $k_{ij}$  and  $h^2$ .

Initially, it would seem like this is also a full expression for phenotype covariance for arbitrary relatives, but I believe that the earlier assumption that  $\text{Cov}(c_i, c_j) = \text{Var}(c_i)$  only holds for the case of sampling from the same parents, i.e. full siblings. In the more general case, we'd need an expression for the expected parental  $c$  values given the distance between parent sets.

```

phenotype_cov <- function(w, b2, k = 0.5, h2 = 0.2) {
  return(2 * k * h2 + b2 * ((w^2) + (1 - w)^2))
}

w_vals <- seq(0, 1, length.out = 100)
b2_vals <- seq(0, 0.8, by = 0.2)

p_covariances_matrix <- sapply(
  X = b2_vals,
  FUN = phenotype_cov,
  w = w_vals,
  h2 = 0.2,
  k = 0.5
)

colors <- c("black", "red", "blue", "purple", "lightblue")
matplot(
  x = w_vals,
  y = p_covariances_matrix,
  xlab = bquote(w),
  ylab = bquote(Cov(P[i], P[j])),
  col = colors,
  type = "l",
  lty = "solid",
  lwd = 2
)
legend(

```

```

"top",
legend = paste("w =", b2_vals),
fill = colors
)

```

