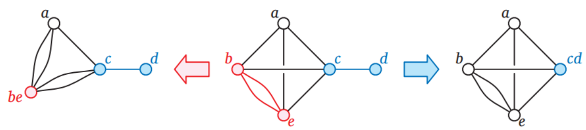
Algorithms Final Cheat Sheet

RandomizedAlgorithms

Max Flows and Min Cuts



Blind Guess

function GuessMinCut(G)
  for $i \leftarrow n, 2$ **do**
    pick a random edge e in G
    $G \leftarrow G/e$
  end for
  return the only cut in G
end function

$P(n) = \frac{2}{n(n-1)}$

Repeated Guessing

function KargerMinCut(G)
  $mink \leftarrow \infty$
  for $i \leftarrow 1, N$ **do**
    $X \leftarrow$ GuessMinCut(G)
    if $|X| < mink$ **then**
      $mink \leftarrow |X|$
      $minX \leftarrow X$
    end if
  end for
  return $minX$
end function

Set $N = c\binom{n}{2} \ln n$ for some constant c. $P(n) \geq 1 - \frac{1}{n^c}$. KargerMinCut computes the min cut of any n-node graph with high probability in $O(n^4 \log n)$ time.

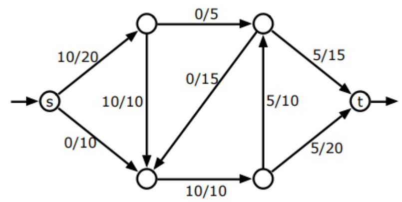
Not-So-Blindly Guessing

function Contract(G, m)
  for $i \leftarrow n, m$ **do**
    pick a random edge e in G
    $G \leftarrow G/e$
  end for
end function
function BetterGuess(G)
  if G has more than 8 vertices **then**
    $G_1 \leftarrow$ Contract($G, n/\sqrt{2}+1$)
    $G_2 \leftarrow$ Contract($G, n/\sqrt{2}+1$)
    $X_1 \leftarrow$ BetterGuess(G_1)
    $X_2 \leftarrow$ BetterGuess(G_2)
    return $\min(X_1, X_2)$
  else
    use brute force
  end if
end function

$P(n) \geq 1/\log n$. The running time is $O(n^2 \log n)$.

Flows

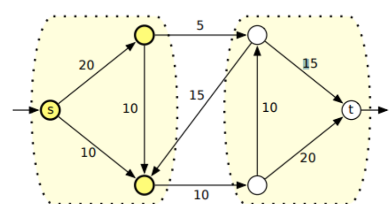
A *flow* is a function f that satisfies the *conservation constraint* at every vertex v: the total flow *into* v is equal to the total flow *out* of v.
A flow f is *feasible* if $f(e) \leq c(e)$ for each edge e. A flow *saturates* edge e if $f(e) = f(c)$, and *avoids* edge e if $f(e) = 0$.



An (s,t)-flow with value 10. Each edge is labeled with its flow/capacity.

Cuts

A *cut* is a partition of the vertices into disjoint subsets S and T - meaning $S \cup T = V$ and $S \cap T = \emptyset$ - where $s \in S$ and $t \in T$.
If we have a capacity function c, the *capacity* of a cut is the sum of the capacities of the edges that start in S and end in T. The definition is asymmetric; edges that start in T and end in S are unimportant. The *min-cut problem* is to compute a cut whose capacity is as large as possible.

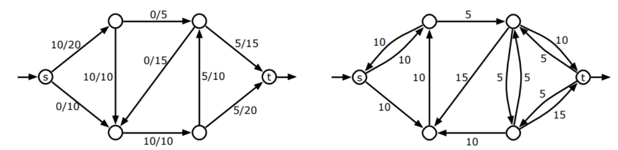


An (s,t)-cut with capacity 15. Each edge is labeled with its capacity.

Theorem 1 (Maxflow Mincut Theorem) *In any flow network, the value of the maximum flow is* equal *to the capacity of the minimum cut.*

Residual Capacity

$$c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(v \to u) & \text{if } v \to u \in E \\ 0 & \text{otherwise} \end{cases}$$



A flow f in a weighted graph G and the corresponding residual graph G_f.

Augmenting Paths

Suppose there is a path $s = v_0 \to v_1 \to \cdots \to v_r$ in the residual graph G_f. This is an *augmenting path*. Let $F = \min_i c_f(v_i \to v_{i+1})$ denote the maximum amount of flow that we can push through the augmenting path in G_f. We can augment the flow into a new flow function f':

$$f'(u \to v) = \begin{cases} f(u \to v) + F & \text{if } u \to v \in s \\ f(u \to v) - F & \text{if } v \to u \in s \\ f(u \to v) & \text{otherwise} \end{cases}$$

Ford-Fulkerson

Starting with the zero flow, repeatedly augment the flow along *any* path from s to t in the residual graph, until there is no such path.

Further Work

The fastest known maximum flow algorithm, announced by James Orlin in 2012, runs in $O(VE)$ time.

Flow/Cut Applications

Edge-Disjoint Paths

A set of paths in G is *edge-disjoint* if each edge in G appears in at most one of the paths; several edge-disjoint paths may pass through the same vertex, however.
Assign each edge capacity 1. The number of edge-disjoint paths is exactly equal to the value of the flow. Using Orlin's algorithm is overkill; the the maximum flow has value at most $V - 1$, so Ford-Fulkerson's original augmenting path algorithm also runs in $O(|f^*|E) = O(VE)$ time.

Vertex Capacities and Vertex-Disjoint Paths

If we require the total flow into (and out of) any vertex v other than s and t is at most some value $c(v)$, we transform the input into a new graph. We replace each vertex v with two vertices v_{in} and v_{out}, connected by an edge $v_{in} \to v_{out}$ with capacity $c(v)$, and then replace every directed edge $u \to v$ with the edge $u_{out} \to v_{in}$ (keeping the same capacity).
Computing the maximum number of *vertex-disjoint* paths from s to t in any directed graph simply involves giving every vertex capacity 1, and computing a maximum flow.
SATandCNFSAT

NP-Hardness

Definition 2 P *is the set of decision problems that can be solved in polynomial time. Intuitively, P is the set of problems that can be solved quickly.*

Definition 3 NP *is the set of decision problems with the following property: if the answer is Yes, then there is a* proof *of this fact that can be checked in polynomial time. Intuitively, NP is the set of decision problems where we can verify a Yes answer quickly if we have the solution in front of us.*

Definition 4 co-NP *is essentially the opposite of NP. If the answer to a problem in co-NP is No, then there is a proof of this fact that can be checked in polynomial time.*

Every decision problem in P is also in NP and also in co-NP.

Definition 5 *A problem* Π *is* NP-hard *if a polynomial-time algorithm for* Π *would imply a polynomial-time algorithm for every problem in NP.*

Definition 6 *A problem* Π *is* NP-complete *if it is both NP-hard and an element of NP.*

Theorem 7 (Cook-Levin Theorem) *Circuit satisfiability is NP-complete.*

To prove that problem A is NP-hard, reduce a known NP-hard problem to A.

Definition 8 *A* many-one *reduction from one language* $L' \subseteq \Sigma^*$ *is a function* $f : \Sigma^* \to \Sigma^*$ *such that* $x \in L'$ *iff* $f(x) \in L$. *A* language L *is NP-hard iff, for any language* $L' \in NP$, *there is a* many-one *reduction from* L' *to* L *that can be computed in polynomial time.*

NP-Hard Problems

- SAT
- 3SAT
- Maximum Independent Set: find the size of the largest subset of the vertices of a graph with no edges between them
- Clique: Compute the number of nodes in its largest complete subgraph
- Vertex Cover: Smallest set of vertices that touch every edge in the graph
- Graph Coloring: Find the smallest possible number of colors in a legal coloring such that every edge has two different colors at its endpoints
- Hamiltonian Cycle: find a cycle that visits each vertex in a graph exactly once
- Subset Sum: Given a set X of positive integers and an integer t, determine whether X has a subset whose elements sum to t
- Planar Circuit SAT: Given a boolean circuit that can be embedded in the plane so that no two wires cross, is there an input that makes the circuit output True
- Not All Equal 3SAT: Given a 3CNF formula, is there an assignment of values to the variables so that every clause contains at least one True literal *and* at least one False literal?
- Exact 3-Dimensional Matching: Given a set S and a collection of three-element subsets of S, called *triples*, is there a sub-collection of disjoint triples that exactly cover S?
- Partition: Given a set S of n integers, are there subsets A and B such that $A \cup B = S$, $A \cap B = \emptyset$, and $\sum_{a \in A} a = \sum_{b \in B} b$?
- 3Partition: Given a set S of $3n$ integers, can it be partitioned into n disjoint three-element subsets, such that every subset has exactly the same sum?
- Set Cover: Given a collection of sets $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, find the smallest sub-collection of S_i's that contains all the elements of $\bigcup_i S_i$
- Hitting Set: Given a collection of sets $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, find the minimum number of elements of $\bigcup_i S_i$ that hit every set in $\mathcal{S}$
- Hamiltonian Path: Given a graph G, is there a path in G that visits every vertex exactly once?
- Longest Path: Given a non-negatively weighted graph G and two vertices u and v, what is the longest simple path from u to v in the graph? A path is *simple* if it visits each vertex at most once.
- Steiner Tree: Given a weighted, undirected graph G with some of the vertices marked, what is the minimum-weight subtree of G that contains every marked vertex?