Algorithms Final Cheat Sheet

Processes and DB

Probability of any success = $p * (1-p)^n - 1$ ... $p = \Theta(1/n)$ ... $t = t$; prob of fail $\leq 1/e$ ... $t = en * cln(n)$; prob of fail $\leq n^{-c}$ ... How long until succeeding at least once: n^{-1}

The Union Bound

Prob [$\bigcup_{i=1}^{\infty} F_i$] <= $\sum_{n=1}^{\infty} Prob[F_i]$

Verifying AB=C

B$\bar{r}->A(B\bar{r})$ and $C\bar{r}$. if $A(B\bar{r})! = C\bar{r}$ then $AB! = C$

Principle of Deferred Decisions

If $AB! = C$ and $\bar{r}$ is chosen uniformly at random with r_n at 0-1 then Prob($AB\bar{r} = C\bar{r}$) $<= 1/2$

Law of Total Probability

Let $E_1 ... E_n$ be mutually disjoint events in the sample space Ω and let $\bigcup_{i=1}^{n} E_i$ then $\sum_{i=1}^{n} Prob[B|E_i]Prob[E_i]$

Repeated trials increase the runtime to $\Theta(kn^2)$ If it returns false, then it this is right, but if it returns true, then it returns so with some probability of mistake.

Median Value

$\sum_{j=1}^{\infty} j * Prob[X = j] = (n+1)/2$ while Prob $[X = j] = 1/n$

To take exactly j steps: Prob $[X = j] = (1-p)^{j-1}p$ $1/p$ for the first success

Median Value

Linearity of Expectation: Given 2 random vars X and Y in the same probability space, $E[X+Y] = E[X] + E[Y]$

Memoryless guessing

Expected correct: 1, independent of n

Memory guessing

$H(n) = \Theta(log(n))$ [harmonic series]

Coupon collection:

$E[X_j] = n/(n-j)$; n= number of; j = collected; $(n-j)/n$ of getting a new one

$E[X] = nH(n) = \Theta(nlog(n))$

Conditional Probability

$E[X|\alpha] = \sum_{j=0}^{\infty} j*$ Prob $[X = j|\alpha]$

Max 3-SAT

There is a randomized algo with polynomial expected run time that is guaranteed to produce a truth assignment satisfying at least a $7/8$ fraction of all clauses. We would need $8k$ trials to get the satisfying assignment.

function SELECT(S, k)
  $a_i \leftarrow randomvarinS$
  for each element a_j of S **do**
    S^-.append(a_j) if $a_j < a_i$
    S^+.append(a_j) if $a_j > a_i$
  end for
  if $S^- = k - 1$ **then**
    return a_i
  else if $S^- \geq k$ **then**
    Select(S^-, k)
  else
    Select($S^+, k - 1 - |S^-|$)
  end if
end function

Quicksort ExpectedComparisons

$2nln(n) + O(n), \Omega(nlogn)$

Uniform

Prob $[h(x) = i] = 1/m$ for all x and all i

Universal

Prob $[h(x) = h(y)] = 1/m$ for all $x! = y$

Near-universal

Prob $[h(x) = h(y)] \leq 2/m$ for all $x! = y$; E[$chainLen$] $\leq 2 * \alpha$; Runtime: $\Theta(1+\alpha)$

k-uniform

Prob $[\wedge_{j=1}^{k} h(x_j) = i_j)] = 1/m^k$ for all distinct $x_1 ... x_k$ and all $i_1 ... i_k$

Load Factor

$\alpha = n/m$

Balanced Binary Tree Search

O(1 + log($chainLen$)) with any hash O(1 + log(α)) for uniform

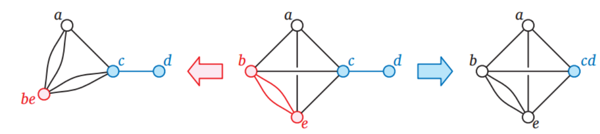
Recursively Hash

$O(log_m n)$

Expected Search w/ binary probe

O(1)

Max Flows and Min Cuts



Blind Guess

function GUESSMINCUT(G)
  for $i \leftarrow n, 2$ **do**
    pick a random edge e in G
    $G \leftarrow G/e$
  end for
  return the only cut in G
end function

$P(n) = \frac{2}{n(n-1)}$

Repeated Guessing

function KARGERMINCUT(G)
  $mink \leftarrow \infty$
  for $i \leftarrow 1, N$ **do**
    $X \leftarrow$ GUESSMINCUT(G)
    if $|X| < mink$ **then**
      $mink \leftarrow |X|$
      $minX \leftarrow X$
    end if
  end for
  return $minX$
end function

Set $N = c\binom{n}{2} \ln n$ for some constant c. $P(n) \geq 1 - \frac{1}{n^c}$. **KARGERMINCUT** computes the min cut of any n-node graph with high probability in $O(n^4 \log n)$ time.

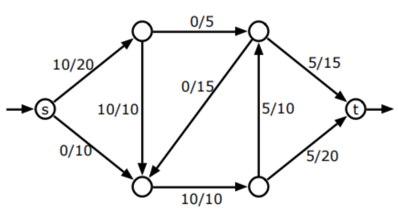
Not-So-Blindly Guessing

function CONTRACT(G, m)
  for $i \leftarrow n, m$ **do**
    pick a random edge e in G
    $G \leftarrow G/e$
  end for
end function
function BETTERGUESS(G)
  if G has more than 8 vertices **then**
    $G_1 \leftarrow$ CONTRACT$(G, n/\sqrt{2}+1)$
    $G_2 \leftarrow$ CONTRACT$(G, n/\sqrt{2}+1)$
    $X_1 \leftarrow$ BETTERGUESS(G_1)
    $X_2 \leftarrow$ BETTERGUESS(G_2)
    return $\min(X_1, X_2)$
  else
    use brute force
  end if
end function

$P(n) \geq 1/\log n$. The running time is $O(n^2 \log n)$.

Flows

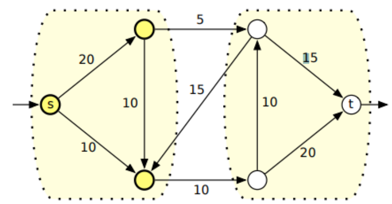
A *flow* is a function f that satisfies the *conservation constraint* at every vertex v: the total flow *into* v is equal to the total flow *out* of v.
A flow f is *feasible* if $f(e) \leq c(e)$ for each edge e. A flow *saturates* edge e if $f(e) = f(c)$, and *avoids* edge e if $f(e) = 0$.



An (s, t)-flow with value 10. Each edge is labeled with its flow/capacity.

Cuts

A *cut* is a partition of the vertices into disjoint subsets S and T - meaning $S \cup T = V$ and $S \cap T = \emptyset$ - where $s \in S$ and $t \in T$.
If we have a capacity function c, the *capacity* of a cut is the sum of the capacities of the edges that start in S and end in T. The definition is asymmetric; edges that start in T and end in S are unimportant. The *min-cut problem* is to compute a cut whose capacity is as large as possible.

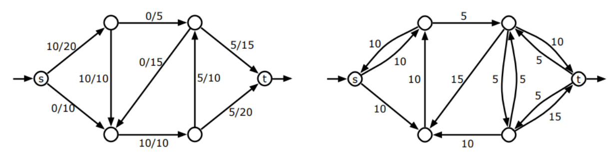


An (s, t)-cut with capacity 15. Each edge is labeled with its capacity.

Theorem 1 (Maxflow Mincut Theorem) *In any flow network, the value of the maximum flow is* equal *to the capacity of the minimum cut.*

Residual Capacity

$$c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(v \to u) & \text{if } v \to u \in E \\ 0 & \text{otherwise} \end{cases}$$


A flow f in a weighted graph G and the corresponding residual graph G_f.

Augmenting Paths

Suppose there is a path $s = v_0 \to v_1 \to \cdots \to v_r$ in the residual graph G_f. This is an *augmenting path*. Let $F = \min_i c_f(v_i \to v_{i+1})$ denote the maximum amount of flow that we can push through the augmenting path in G_f. We can augment the flow into a new flow function f':

$$f'(u \to v) = \begin{cases} f(u \to v) + F & \text{if } u \to v \in s \\ f(u \to v) - F & \text{if } v \to u \in s \\ f(u \to v) & \text{otherwise} \end{cases}$$

Ford-Fulkerson

Starting with the zero flow, repeatedly augment the flow along *any* path from s to t in the residual graph, until there is no such path.

Further Work

The fastest known maximum flow algorithm, announced by James Orlin in 2012, runs in $O(VE)$ time.

Flow/Cut Applications

Edge-Disjoint Paths

A set of paths in G is *edge-disjoint* if each edge in G appears in at most one of the paths; several edge-disjoint paths may pass through the same vertex, however.
Assign each edge capacity 1. The number of edge-disjoint paths is exactly equal to the value of the flow. Using Orlin's algorithm is overkill; the the maximum flow has value at most $V - 1$, so Ford-Fulkerson's original augmenting path algorithm also runs in $O(|f^*|E) = O(VE)$ time.

Vertex Capacities and Vertex-Disjoint Paths

If we require the total flow into (and out of) any vertex v other than s and t is at most some value $c(v)$, we transform the input into a new graph. We replace each vertex v with two vertices v_{in} and v_{out}, connected by an edge $v_{in} \to v_{out}$ with capacity $c(v)$, and then replace every directed edge $u \to v$ with the edge $u_{out} \to v_{in}$ (keeping the same capacity).
Computing the maximum number of *vertex-disjoint* paths from s to t in any directed graph simply involves giving every vertex capacity 1, and computing a maximum flow.

SAT and CNF-SAT

Formula Satisfiability/SAT

Given a boolean formula like $\overline{(a \vee b \vee c \vee \overline{d})} \Leftrightarrow ((b \wedge \overline{c}) \vee \overline{(\overline{a} \Rightarrow d)} \vee (c \neq a \wedge b))$, is it possible to assign boolean values to $a, b, c, \ldots$ so that the formula evaluates to True?
We can transform any boolean circuit to a formula in linear time (using depth-first search), and the size of the resulting formula is only a constant factor larger than the size of the circuit. Thus, we have a polynomial-time reduction from circuit satisfiability to SAT. Thus, SAT is NP-hard.
To prove that a boolean formula is satisfiable, we only have to specify an assignment to the variables that makes the formula True. We can check the proof in linear time just by reading the formula from left to right, evaluating as we go. So SAT is also in NP, and thus is NP-complete.

CNF

A boolean form is in *conjunctive normal form* if it is a conjunction (AND) of several clauses, each of which is the disjunction (OR) of several *literals*, each of which is either a variable or its negation. For example, $(a \vee b \vee c \vee d) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b})$.

3CNF

A 3CNF formula is a CNF formula with exactly 3 literals per clause; the previous example is not a 3CNF formula, since its first clause has four literals and its last clause has only two. 3SAT is just SAT restricted to 3CNF formulas: Given a 3CNF formula, is there an assignment to the variables that makes the formula evaluate to True?

1. Make sure every AND and OR gate has only two inputs. If any gate has $k > 2$ inputs, replace it with a binary tree of $k - 1$ two-input gates.
2. Write down the circuit as a formula, with one clause per gate.
3. Change every gate clause into a CNF formula
4. Make sure every clause has exactly three literals

Every binary gate in the original circuit will be transformed into at most 5 clauses.

NP-Hardness

Definition 2 P *is the set of decision problems that can be solved in polynomial time. Intuitively, P is the set of problems that can be solved quickly.*

Definition 3 NP *is the set of decision problems with the following property: if the answer is Yes, then there is a* proof *of this fact that can be checked in polynomial time. Intuitively, NP is the set of decision problems where we can verify a Yes answer quickly if we have the solution in front of us.*

Definition 4 co-NP *is essentially the opposite of NP. If the answer to a problem in co-NP is No, then there is a proof of this fact that can be checked in polynomial time.*

Every decision problem in P is also in NP and also in co-NP.

Definition 5 *A problem* Π *is* NP-hard *if a polynomial-time algorithm for* Π *would imply a polynomial-time algorithm for every problem in NP.*

Definition 6 *A problem* Π *is* NP-complete *if it is both NP-hard and an element of NP.*

Theorem 7 (Cook-Levin Theorem) *Circuit satisfiability is NP-complete.*

To prove that problem A is NP-hard, reduce a known NP-hard problem to A.

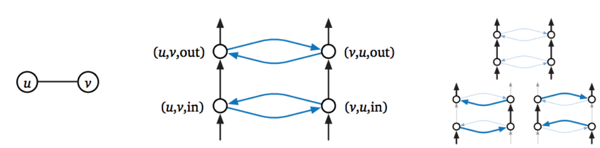
Definition 8 *A* many-one *reduction from one language* $L' \subseteq \Sigma^*$ *is a function* $f : \Sigma^* \to \Sigma^*$ *such that* $x \in L'$ *iff* $f(x) \in L$. *A* language L *is NP-hard iff, for any language* $L' \in NP$, *there is a* many-one *reduction from* L' *to* L *that can be computed in polynomial time.*

NP-Hard Problems

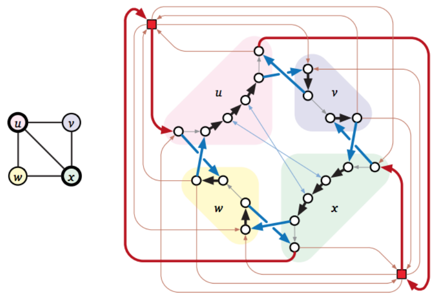
- SAT
- 3SAT
- Maximum Independent Set: find the size of the largest subset of the vertices of a graph with no edges between them
- Clique: Compute the number of nodes in its largest complete subgraph
- Vertex Cover: Smallest set of vertices that touch every edge in the graph
- Graph Coloring: Find the smallest possible number of colors in a legal coloring such that every edge has two different colors at its endpoints
- Hamiltonian Cycle: find a cycle that visits each vertex in a graph exactly once
- Subset Sum: Given a set X of positive integers and an integer t, determine whether X has a subset whose elements sum to t
- Planar Circuit SAT: Given a boolean circuit that can be embedded in the plane so that no two wires cross, is there an input that makes the circuit output True
- Not All Equal 3SAT: Given a 3CNF formula, is there an assignment of values to the variables so that every clause contains at least one True literal *and* at least one False literal?
- Exact 3-Dimensional Matching: Given a set S and a collection of three-element subsets of S, called *triples*, is there a sub-collection of disjoint triples that exactly cover S?
- Partition: Given a set S of n integers, are there subsets A and B such that $A \cup B = S$, $A \cap B = \emptyset$, and $\sum_{a \in A} a = \sum_{b \in B} b$?
- 3Partition: Given a set S of $3n$ integers, can it be partitioned into n disjoint three-element subsets, such that every subset has exactly the same sum?
- Set Cover: Given a collection of sets $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, find the smallest sub-collection of S_i's that contains all the elements of $\bigcup_i S_i$
- Hitting Set: Given a collection of sets $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, find the minimum number of elements of $\bigcup_i S_i$ that hit every set in $\mathcal{S}$
- Hamiltonian Path: Given a graph G, is there a path in G that visits every vertex exactly once?
- Longest Path: Given a non-negatively weighted graph G and two vertices u and v, what is the longest simple path from u to v in the graph? A path is *simple* if it visits each vertex at most once.
- Steiner Tree: Given a weighted, undirected graph G with some of the vertices marked, what is the minimum-weight subtree of G that contains every marked vertex?

Hamiltonian Cycle [From Vertex Cover]

For each vertex u, all the edge gadgets are connected in H into a single directed path, a vertex chain. H has $d - 1$ additional edges for each i. H also contains k cover vertices, $1 - k$, with a directed edge to the first vertex in each vertex chain and a directed edge from the last vertex in each vertex chain.
Start at cover vertex 1 and traverse vertex chain for vu_2, then visit cover vertex 2 and so on and so forth before returning to 1. If v is a part of the vertex cover, follow the edge from (u_i, v, in) to (u_i, v, out), else, detour from (u_i, v, in) > (v, u_i, in) > (v, u_i, out) > (u_i, v, out).
G contains a vertex cover of size K if and only if H contains a Hamiltonian cycle



An edge gadget for uv and its only possible intersections with a Hamiltonian cycle.



A vertex cover $\{u, x\}$ in G and the corresponding Hamiltonian cycle in H.

Subset Sum

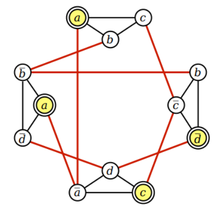
Given a graph G and an integer k, first number edges from 0 to $m-1$; set X contains the integer $b_i = 4^i$ for each edge i, and the integer $a_v = 4^m + \sum_{i \in \delta(v)} 4^i$ where $\delta(v)$ is the set of edges that have v as an endpoint. Finally, we set the target sum: $t = k * 4^m + \sum_{i=0}^{m-1} 2 * 4^i$

Longest Increasing Subsequence [from DAG]

Turn every number in the sequence into a vertex in a graph. Construct a special vertex d. For each vertex v_1, construct an edge to another vertex v_2 if (1) v_2 comes after v_1 in the sequence, and (2) $v_2 > v_1$. Also construct an edge to d.
Every path on this graph is a valid increasing subsequence. The problem of finding the LIS is now the problem of finding the longest path on this graph. Apply DAG.

Maximum Independent Set [from 3SAT]

Construct a graph G which has one vertex for each instance of each literal in the 3SAT formula. Two vertices are connected by an edge if (1) they correspond to literals in the same clause, or (2) they correspond to a variable and its inverse. For example, the formula $(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$ is transformed into:

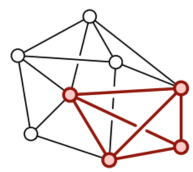


A graph derived from a 3CNF formula, and an independent set of size 4.
Black edges join literals from the same clause; red (heavier) edges join contradictory literals.

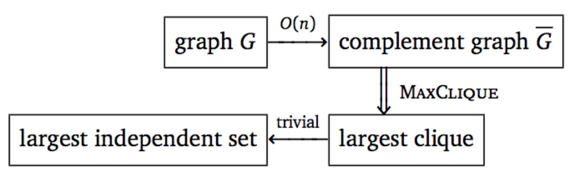
Suppose the original formula had k clauses. Then the formula is satisfiable iff the graph has an independent set of size k.

Clique [from Independent Set]

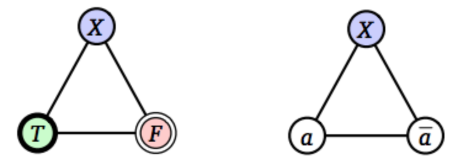
Any graph G has an *edge-complement* $\overline{G}$ with the same vertices, but with exactly the opposite set of edges - (u, v) is an edge in $\overline{G}$ if and only if it is *not* an edge in G. A set of vertices is independent in G if and only if the same vertices define a clique in $\overline{G}$. Thus, we can compute the largest independent set in a graph by computing the largest clique in the complement of the graph.



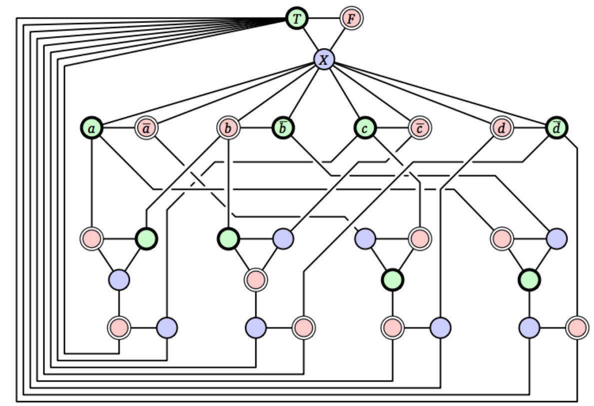
A graph with maximum clique size 4.



3Color [from 3SAT]



The truth gadget and a variable gadget for a.



A 3-colorable graph derived from the satisfiable 3CNF formula
$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$

Truth gadget: T, F, and X for true/false/other, variable gadget for variable a connecting and $\bar{a}$ which must be opposite bools. Clause gadget joining three literal nodes to node T in the truth gadget using give new unlabeled nodes and ten edges.