

Design and Analysis of Algorithms: Homework #4

Due in class on March 27, 2018

Professor Kasturi Varadarajan

Alic Szecsei

Problem 1

Consider the following randomized algorithm for generating biased random bits. The subroutine **FAIRCOIN** returns either 0 or 1 with equal probability; the random bits returned by **FAIRCOIN** are mutually independent.

```

1: function ONEINTHREE
2:   if FAIRCOIN = 0 then
3:     return 0
4:   else
5:     return 1 − ONEINTHREE
6:   end if
7: end function

```

- Prove that **ONEINTHREE** returns 1 with probability $\frac{1}{3}$.
- What is the *exact* expected number of times that this algorithm calls **FAIRCOIN**?
- Now suppose you are *given* a subroutine **ONEINTHREE** that generates a random bit that is equal to 1 with probability $\frac{1}{3}$. Describe a **FAIRCOIN** algorithm that returns either 0 or 1 with equal probability, using **ONEINTHREE** as your only source of randomness.
- What is the *exact* expected number of times that your **FAIRCOIN** algorithm calls **ONEINTHREE**?

Solution

Part A

In order for **ONEINTHREE** to return 1, **FAIRCOIN** must return an odd number of 1s before returning a 0. We can begin to enumerate cases: first, that **FAIRCOIN** returns a 1 and then a 0 is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. The probability that **FAIRCOIN** returns 3 1s and then a 0 is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$. The probability that **FAIRCOIN** returns 5 1s and then a 0 is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{64}$, and so on. Our summation of all cases is then:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots + \frac{1}{2^{2n}} = \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \quad (1)$$

This is a geometric series with common ratio $r = \frac{1}{4}$; we can then use the formula $S = \frac{a_1}{1-r}$ to determine the summation. Using this, we can see that $S = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$. Therefore, **ONEINTHREE** has a $\frac{1}{3}$ probability of returning 1.

Part B

Since **ONEINTHREE** might never terminate, our expected value of $T(n)$ is:

$$E[T(n)] = \sum_{k=1}^{\infty} k \cdot \Pr[T(n) = k] \quad (2)$$

In order for **ONEINTHREE** to terminate with exactly k calls to **FAIRCOIN**, there must have been exactly $k - 1$ times that **FAIRCOIN** returned 1 (and **ONEINTHREE** continued recursing), and 1 time it returned 0 (and **ONEINTHREE** stopped recursing). Thus, the combined probability for **ONEINTHREE** to terminate with exactly k calls to **FAIRCOIN** is $\frac{1}{2^k}$. Plugging this back into our equation for $E[T(n)]$:

$$\begin{aligned}
E[T(n)] &= \sum_{k=1}^{\infty} k \cdot \Pr[T(n) = k] \\
&= \sum_{k=1}^{\infty} k \cdot \frac{1}{2^k} \\
&= 2
\end{aligned} \tag{3}$$

Part C

```

1: function FAIRCOIN
2:    $c_1 \leftarrow \text{ONEINTHREE}$ 
3:    $c_2 \leftarrow \text{ONEINTHREE}$ 
4:   if  $c_1 = 1$  and  $c_2 = 1$  then
5:     return FAIRCOIN
6:   else
7:     if  $c_1 = 1$  or  $c_2 = 1$  then
8:       return 1
9:     else
10:      return 0
11:    end if
12:  end if
13: end function

```

The idea for this algorithm is that we break the probability space into 3 partitions: first, the probability that both calls to `ONEINTHREE` return 1 is $\frac{1}{9}$. The remaining probability is $\frac{8}{9}$; we further divide this into two partitions. The probability that both calls to `ONEINTHREE` returned 0 is $\frac{4}{9}$; thus, the remaining probability (that only one call to `ONEINTHREE` returned 1) is $\frac{4}{9}$.

Now, the probability that any call to `FAIRCOIN` which actually *returns* will return 0 is clearly $\frac{1}{2}$; the same probability applies to any returning call to `FAIRCOIN` returning 1. Thus, `FAIRCOIN` has an equal probability of returning either 0 or 1.

Part D

Since `FAIRCOIN` might never terminate, our expected value of $T(n)$ is:

$$E[T(n)] = \sum_{k=1}^{\infty} k \cdot \Pr[T(n) = k] \tag{4}$$

In order for `FAIRCOIN` to terminate with exactly $2k$ calls to `ONEINTHREE`, there must have been exactly $k - 1$ times that both calls of `ONEINTHREE` returned 1 (and `FAIRCOIN` continued recursing), and 1 time at least one call returned 0 (and `FAIRCOIN` stopped recursing). Thus, the combined probability for `FAIRCOIN` to terminate with exactly $2k$ calls to `FAIRCOIN` is $\frac{1}{9^{k-1}} \cdot \frac{8}{9} = \frac{8}{9^k}$. Plugging this back into our equation for $E[T(n)]$:

$$\begin{aligned}
E[T(n)] &= \sum_{k=1}^{\infty} 2k \cdot \Pr[T(n) = k] \\
&= \sum_{k=1}^{\infty} 2k \cdot \frac{8}{9^k} \\
&= \frac{9}{4}
\end{aligned} \tag{5}$$

Problem 2

Consider the following algorithm for finding the smallest element in an unsorted array:

```

1: function RANDOMMIN( $A[1..n]$ )
2:    $min \leftarrow \infty$ 
3:   for  $i \leftarrow 1$  to  $n$  in random order do
4:     if  $A[i] < min$  then
5:        $min \leftarrow A[i]$   $\triangleright$  (★)
6:     end if
7:   end for
8:   return  $min$ 
9: end function

```

- (a) In the worst case, how many times does **RANDOMMIN** execute line (★)?
- (b) What is the probability that line (★) is executed during the i th iteration of the for loop?
- (c) What is the *exact* expected number of executions of line (★)?

Solution

Part A

In the worst case, **RANDOMMIN** retrieves the elements in descending order, so it must re-assign min (in line (★)) a total of n times.

Part B

If line (★) is executed during the i th iteration of the for loop, this means that the randomly selected element is smaller than the previous $i - 1$ randomly selected elements. Essentially, we have a set of i elements, and we want to know the probability that a specific element (the most recently added element) is the smallest element in that set. We have a single minimal element, and a $\frac{1}{i}$ chance of choosing that minimal element; therefore, the probability that the most recently added element is the minimal element of the set is $\frac{1}{i}$. We can thus conclude that the probability that line (★) is executed during the i th iteration of the for loop is also $\frac{1}{i}$.

Part C

We know that line (★) has a $\frac{1}{i}$ probability of being executed during the i th iteration of the for loop, so the expected value at each iteration of the for loop is:

$$E[T(n)]_{\text{iter}} = 0 \cdot \frac{i-1}{i} + 1 \cdot \frac{1}{i} = \frac{1}{i} \quad (6)$$

We can sum all of these expected values together to determine the expected number of executions of line (★) during the entire algorithm:

$$E[T(n)] = \sum_{k=1}^n \frac{1}{k} = H_n \quad (7)$$

Problem 3

Suppose we have a circular linked list of numbers, implemented as a pair of arrays, one storing the actual numbers and the other storing successor pointers. Specifically, let $X[1..n]$ be an array of n distinct real numbers, and let $N[1..n]$ be an array of indices with the following property: If $X[i]$ is the largest element of X , then $X[N[i]]$ is the smallest element of X ; otherwise, $X[N[i]]$ is the smallest among the set of elements in X larger than $X[i]$. For example:

i	1	2	3	4	5	6	7	8	9
$X[i]$	83	54	16	31	45	99	78	62	27
$N[i]$	6	8	9	5	2	3	1	7	4

Describe and analyze a randomized algorithm that determines whether a given number x appears in the array X in $O(\sqrt{n})$ expected time. **Your algorithm may not modify the arrays X and N .**

Solution

Our algorithm selects \sqrt{n} random pivots in the linked list, and searches them to find the greatest lower bound on our target value. If our target value is less than all of the pivots, it selects the largest pivot. We then walk through the linked list until we either find our target or exceed its value (with suitable edge cases handled if we need to wrap around).

First, we would like to show that the expected size S of any of the \sqrt{n} partitions our algorithm generates is, itself, \sqrt{n} . As an example, we will have 3 partitions of an array of length 9. We randomly select an element of the array r and two pivots, p_1 and p_2 ; without loss of generality, assume $p_1 < p_2$. The probability that r is between the two pivots is $\frac{1}{3}$, as the other two equiprobable alternatives are that $r < p_1$ and $r > p_2$. The probability that any value is in one of the partitions is $\frac{S}{L}$, where L is the length of the array and S is the partition size; thus, the expected length of each partition must be $\frac{1}{3} = \frac{S}{L} \Rightarrow \frac{1}{3} = \frac{S}{9} \Rightarrow S = 3$.

We can generalize this for any array length L and number of partitions P ; the expected partition size must be $\frac{L}{P}$. Since we have \sqrt{n} partitions and an array of size n , our expected partition size is $\frac{n}{\sqrt{n}} = \sqrt{n}$.

Our algorithm uses $O(\sqrt{n})$ runtime to generate our pivot list and determine which partition contains x ; it then linearly traverses the relevant partition. Since our expected partition size is *also* \sqrt{n} , we can determine our final expected runtime to be $O(\sqrt{n})$.

```

1: function RANDOMCONTAINS( $X[1..n], N[1..n], x$ )
2:    $T \leftarrow$  an array of  $\sqrt{n}$  random numbers between 1 and  $n$ 
3:    $hasLowerBound \leftarrow False$   $\triangleright$  We need to be able to handle the case where we do not select any elements
   less than  $x$ 
4:   for  $i \leftarrow 1, \sqrt{n}$  do
5:     if  $X[T[i]] < x$  then
6:        $hasLowerBound \leftarrow True$  break
7:     end if
8:   end for
9:   if  $hasLowerBound$  then
10:     $minDist \leftarrow \infty$ 
11:    for  $i \leftarrow 1, \sqrt{n}$  do
12:      if  $X[T[i]] < x$  and  $x - X[T[i]] < minDist$  then
13:         $minDist \leftarrow x - X[T[i]]$ 
14:         $l \leftarrow T[i]$ 
15:      end if
16:    end for
17:  else
18:     $l \leftarrow T[1]$ 
19:    for  $i \leftarrow 2, \sqrt{n}$  do
20:      if  $X[T[i]] > X[l]$  then
21:         $l \leftarrow T[i]$ 
22:      end if
23:    end for
24:  end if  $\triangleright$  We now have the “closest” element before  $x$  (if it exists) on our linked list
25:  while  $True$  do
26:    if  $X[l] = x$  then
27:      return  $True$ 
28:    else
29:       $prev \leftarrow X[l]$ 
30:       $l \leftarrow N[l]$ 
31:      if  $X[l] < prev$  then
32:        if  $hasLowerBound$  then
33:          return  $False$   $\triangleright$  We wrapped around before we found  $x$ 
34:        else
35:           $hasLowerBound \leftarrow True$   $\triangleright$  We wrapped around, so we should have a lower bound
36:        end if
37:      end if
38:      if  $X[l] > x$  and  $hasLowerBound$  then
39:        return  $False$   $\triangleright$  We had a lower bound, but now we’ve passed where  $x$  should have been
40:      end if
41:    end if
42:  end while
43: end function

```

Problem 4

A *majority tree* is a complete binary tree with depth n , where every leaf is labeled either 0 or 1. The *value* of a leaf is its label; the *value* of any internal node is the majority of the values of its three children. Consider the problem of computing the value of the root of a majority tree, given the sequence of 3^n leaf labels as input. For example, if $n = 2$ and the leaves are labeled 1, 0, 0, 0, 1, 0, 1, 1, 1, the root has value 0.

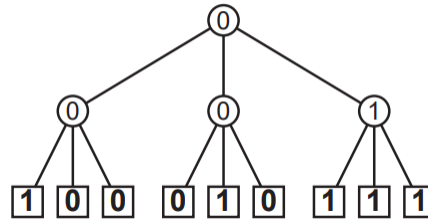


Figure 1: A majority tree with depth $n = 2$.

- (a) Prove that *any* deterministic algorithm that computes the value of the root of a majority tree *must* examine every leaf. [Hint: Consider the special case $n = 1$. Recurse.]
- (b) Describe and analyze a randomized algorithm that computes the value of the root in worst-case expected time $O(c^n)$ for some constant $c < 3$. [Hint: Consider the special case $n = 1$. Recurse.]

Solution

Problem 5

Suppose you are given a graph G with weighted edges, and your goal is to find a cut whose total weight (not just number of edges) is smallest.

- (a) Describe an algorithm to select a random edge of G , where the probability of choosing edge e is proportional to the weight of e .
- (b) Prove that if you use the algorithm from part (a), instead of choosing edges uniformly at random, the probability that `GUESSMINCUT` returns a minimum-weight is still $\Omega(1/n^2)$.
- (c) What is the running time of your modified `GUESSMINCUT` algorithm?

Solution