Design and Analysis of Algorithms: Homework #6

Due in class on April 24, 2018

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Problem 1

For any flow network G and any vertices u and v let $bottleneck_G(u, v)$ denote the maximum, over all paths π in G from u to v, of the minimum-capacity edge along π.

(a) Describe and analyze an algorithm to compute $bottleneck_G(s, t)$ in $O(E \log V)$ time.

Solution

Part A

```
1:  function BOTTLENECK(G, u, v)
2:      queue ← an empty heap
3:      current ← u
4:      for all vertex in G.vertices do
5:          if vertex = u then
6:              vertex.weight ← ∞
7:          else
8:              vertex.weight ← −∞
9:          end if                                  ▷ We'll use these weights in our heap
10:         insert vertex into queue
11:     end for
12:     while ¬v.seen do                            ▷ Find the path with the bottleneck by building a tree
13:         current.seen ← True
14:         for all d in current.neighbors do
15:             if ¬d.seen then
16:                 if edge(current, d).weight > d.weight then
17:                     delete d from queue
18:                     d.weight ← edge(current, d).weight
19:                     d.parent ← current
20:                     insert d into queue
21:                 end if
22:             end if                              ▷ Update the vertex weight of a neighbor of current
23:         end for
24:         current ← dequeue the max vertex in queue
25:     end while
26:     c ← v
27:     min ← ∞
28:     while c ≠ u do                              ▷ Find the minimal edge in the bottlenecking path
29:         if edge_{c,c.parent}.weight < min then
30:             min ← edge_{c,c.parent}.weight
31:         end if
32:         c ← c.parent
33:     end while
34:     return min
35: end function
```

This assumes the function should return the *weight* of the bottlenecking edge; however, it can trivially be modified to return the edge itself.

Problem 1 continued on next page…

This algorithm is essentially a modification of Prim's algorithm, where instead of searching for minimally-weighted edges, we instead search for maximally-weighted edges. To do this, we use a priority queue (using a binary heap).

We can split up our algorithm into three sections. The first is where we add each vertex into the priority queue. This is a total of V insertions into the queue, each of which takes $O(\log V)$ time, for a total runtime of $O(V \log V)$.

Second, we build a spanning tree in G until we have found a route from u to v. To do this, we continually dequeue vertices from our queue. For each unvisited neighboring vertex, we check if we've found a better edge to connect it to our tree. If so, we update the weight of the vertex, and also update its "parent" in the tree. We continue in this fashion until v has been added to the tree. Each time we attempt to update a vertex, it means we have traversed a new edge in the graph; updating the vertex involves a successive deletion and insertion, both of which are $O(\log V)$. Since this can happen at most E times, the total running time for this section is $O(E \log V)$.

Last, we determine the smallest edge in the path from v to u. This is more straightforward, as it is a simple $O(E)$ loop.

Our total running time is thus $O((E + V) \log V)$; since V is $O(E)$ for any connected graph, we can say that it is $O(E \log V)$.

Problem 2

Suppose you have already computed a maximum flow $f*$ in a flow network G with *integer* edge capacities.

(a) Describe and analyze an algorithm to update the maximum flow after the capacity of a single edge is increased by 1.

(b) Describe and analyze an algorithm to update the maximum flow after the capacity of a single edge is decreased by 1.

Both algorithms should be significantly faster than recomputing the maximum flow from scratch.

Solution

Part A

```
1:  function UpdateMaxFlow(G, source, sink)
2:      path ← BFS(G, source, sink)          ▷ Assume this works similarly to original Ford-Fulkerson algorithm
3:      if path exists then                  ▷ Check if there's an augmenting path
4:          r ← the minimum residual capacity of (u, v)∀(u, v) ∈ path
5:          for all edge in path do
6:              if edge is a forward edge then
7:                  G.flow(edge) ← G.flow(edge) + r
8:              else
9:                  G.flow(edge) ← G.flow(edge) − r
10:             end if
11:         end for
12:     end if
13:     return G.flow
14: end function
```

The idea for this algorithm is straightforward. We assume the edge has already been increased before the algorithm is run; we then search for an augmenting path, which runs in $O(V + E)$ time using breadth-first search. If an augmenting path exists, we run through a single iteration of Ford-Fulkerson, updating edges along the path; this runtime is clearly bounded by $O(E)$. If an augmenting path does not exist, no alterations are required. Our algorithm thus runs in $O(V + E)$ time.

We only need to run through the Ford-Fulkerson algorithm at most once, since the maximum flow can only increase by one (the same amount as we increased an edge capacity). The augmenting path, then, must then increase the flow by 1, which satisfies our requirements.

Problem 2 continued on next page...

Part B

```
1:  function UpdateMaxFlow(G, source, sink)
2:      e ← Null                                  ▷ We need to make our flow satisfy the new constraint
3:      for all edge in G do
4:          if G.flow(edge) − G.capacity(edge) > 0 then
5:              e ← edge                          ▷ Find an edge with negative residual flow
6:          end if
7:      end for
8:      if e = Null then
9:          return G.flow                         ▷ If the edge has non-negative residual flow, we didn't affect a bottleneck
10:     end if
11:     path ← a path from source to sink which includes e        ▷ DFS would likely be best here
12:     for all edge in path do                   ▷ Decrease the flow down the path from source to sink
13:         G.flow(edge) ← G.flow(edge) − 1        ▷ Assume that the path does not contain any backwards edges
14:     end for
15:     path ← BFS(G, source, sink)               ▷ Again, do a single iteration of Ford-Fulkerson
16:     if path exists then                       ▷ Check if there's an augmenting path
17:         r ← the minimum residual capacity of edge∀edge ∈ path
18:         for all edge in path do
19:             if edge is a forward edge then
20:                 G.flow(edge) ← G.flow(edge) + r
21:             else
22:                 G.flow(edge) ← G.flow(edge) − r
23:             end if
24:         end for
25:     end if
26:     return G.flow
27: end function
```

For this algorithm, we must first resolve the new constraints. Specifically, if an edge has had its capacity reduced such that its residual flow has become negative, we must decrease the flow through that edge so that its residual flow becomes zero. Finding an edge from $source$ to $sink$ which includes that negative edge is relatively easy; we can perform a depth-first search, attempting to follow the flow from this overloaded edge through the graph. In the worst case, this has a running time of $O(V + E)$.

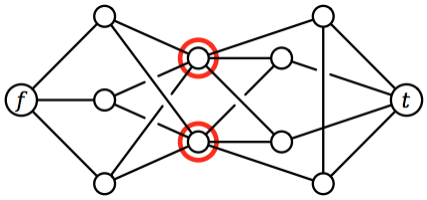
Now, all of our constraints have been satisfied, but the resultant flow may be sub-optimal. However, the flow can only be at most 1 less than the maximum flow, as we have decreased edges by 1; therefore, similar to part (a), we can simply try to find an augmenting path and run a single iteration of Ford-Fulkerson. As in part (a), this has a run-time of $O(V + E)$, and so our final algorithm also has a run-time of $O(V + E)$.

Problem 3

The Island of Sodor is home to a large number of towns and villages, connected by an extensive rail network. Recently, several cases of a deadly contagious disease (either swine flu or zombies; reports are unclear) have been reported in the village of Ffarquhar. The controller of the Sodor railway plans to close down certain railway stations to prevent the disease from spreading to Tidmouth, his home town. No trains can pass through a closed station. To minimize expense (and public notice), he wants to close down as few station as possible. However, he cannot close the Ffarquhar station, because that would expose him to the disease, and he cannot close the Tidmouth station, because then he couldn't visit his favorite pub.

Describe and analyze an algorithm to find the minimum number of stations that must be closed to block all rail travel from Ffarquhar to Tidmouth. The Sodor rail network is represented by an undirected graph, with a vertex for each station and an edge for each rail connection between two stations. Two special vertices f and t represent the stations in Ffarquhar and Tidmouth.

For example, given the following input graph, your algorithm should return the number 2.



Solution

```
1:  function MinStationClosures(G, f, t)
2:      N ← a new graph
3:      for all v in G.vertices do
4:          if v = f ∨ v = t then
5:              add v to N
6:          else
7:              add v_i and v_o to N
8:              connect v_i → v_o with weight 1
9:          end if
10:     end for
11:     for all v in G.vertices do
12:         for all w in v.neighbors do
13:             if ¬(v = t) then
14:                 connect v_o → w_i with weight ∞
15:             end if
16:             if ¬(v = f) then
17:                 connect w_o → v_i with weight ∞
18:             end if
19:         end for
20:     end for
21:     return the minimum cut of N
22: end function
```

The minimum-cut algorithm is here assumed to determine the lowest-weighted edges to cut; an inverse assignment can be used if the min-cut algorithm instead prioritizes the highest-capacity edges.

This algorithm is designed to duplicate all vertices in G which are not s or t. For all $v \in G$ (again, aside from s or t), we construct two vertices, v_i and v_o. We then construct edges in a directed graph, such that for all nodes w connected to v, there exists a connection from w_o to v_i and a connection from v_o to w_i. Without loss of generality, we assume s as a source and t as destination; thus, for all neighbors n of s, we construct an edge from s to n_i; conversely, for all neighbors n of t, we construct an edge from n_o to t.

Having split each vertex into two, we can construct a connection from v_i to v_o, and thus all paths from s to t are preserved. Now, we can perform a standard min-cut algorithm; our edge weightings will ensure that the algorithm only selects our newly-created "vertex" edges (those from v_i to v_o); cutting this edge has the equivalent effect to removing v from the graph, as now all paths from s to t passing through v are broken.

There are two stages to the algorithm to analyze for run-time: graph construction and then the min-cut algorithm.

Graph construction is straightforward; we re-create the original graph G, doubling vertices and edges as we go; since V is $O(E)$, our total run-time here is simply $O(E)$.

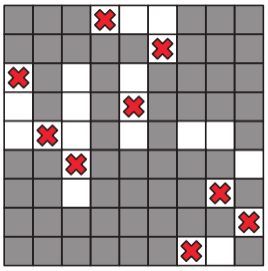
For the min-cut algorithm, the fastest algorithm for computing maximum flows is $O(VE)$, as discovered by James Orlin. Since the min-cut is equivalent to the maximum flow, we can use this algorithm to determine the min-cut of our new graph.

Thus, the total runtime of our algorithm is $O(VE)$.

Problem 4

Suppose we are given an $n \times n$ square grid, some of whose squares are colored black and the rest white. Describe and analyze an algorithm to determine whether tokens can be placed on the grid so that

1. every token is on a white square;
2. every row of the grid contains exactly one token; and
3. every column of the grid contains exactly one token



Your input is a two dimensional array $IsWhite[1..n, 1..n]$ of booleans, indicated which squares are white. Your output is a single boolean. For example, given the grid above as input, your algorithm should return **True**.

Solution

```
1:  function CanPlaceTokens(IsWhite[1..n, 1..n])
2:      G ← an empty graph
3:      add s and t to G
4:      for i ← 1, n do
5:          add row_i to G
6:          add an edge from s to row_i with capacity 1
7:      end for                    ▷ This corresponds to the requirement of having exactly one token per row
8:      for i ← 1, n do
9:          add col_i to G
10:         add an edge from col_i to t with capacity 1
11:     end for                    ▷ This corresponds to the requirement of having exactly one token per column
12:     for i ← 1, n do
13:         for j ← 1, n do
14:             if IsWhite[i, j] then
15:                 add white_{i,j} to G
16:                 add an edge from row_i to white_{i,j} with capacity 1
17:                 add an edge from white_{i,j} to col_j with capacity 1
18:             end if
19:         end for
20:     end for                    ▷ Add possible nodes (in this case, white nodes)
21:     return MaxFlow(G) = n
22: end function
```

For this algorithm, we must transform our problem into a graph. We first require our source node s to output flow (of capacity 1) to nodes representing each of the n rows. We also require that each of the n nodes representing

individual rows can output flow (again, of capacity 1) to our target node t.

Now, we can construct nodes representing possible positions. Each can *receive* flow from their respective row, and can *send* flow to their respective column; as the edges from each row and column have capacity 1, this essentially locks the number of interior nodes that can be "chosen" to 1.

We can then determine maximum flow using Orlin's method; if our maximum flow is equal to n, it means there is a unique node connecting each row and column, and thus there exists a solution to the given board.

V is, at worst $2 + 2n + n^2$ - a source node, a target node, two sets of n vertices for each row and column, and then at most n^2 interior vertices for each possible position. Thus, V is $O(n^2)$. Similarly, E is at worst $2n + 2n^2$: one set of n edges to connect the source node to each row node, another set of n edges to connect each column node to the target node, and finally at most $2n^2$ edges to connect each interior vertex to its corresponding row node and column node. Therefore, E is *also* $O(n^2)$.

The cost to build this graph is simply $O(V + E) = O(n^2)$; Orlin's algorithm provides the bulk of our computation at $O(VE) = O(n^4)$, which is thus the runtime of our algorithm.

Problem 5

Ad-hoc networks are made up of low-powered wireless devices. In principle, these networks can be used on battlefields, in regions that have recently suffered from natural disasters, and in other hard-to-reach areas. The idea is that a large collection of cheap, simple devices could be distributed through the area of interest (for example, by dropping them from an airplane); the devices would then automatically configure themselves into a functioning wireless network.

These devices can communicate only within a limited range. We assume all the devices are identical; there is a distance D such that two devices can communicate if and only if the distance between them is at most D.

We would like our ad-hoc network to be reliable, but because the devices are cheap and low-powered, they frequently fail. If a device detects that it is likely to fail, it should transmit its information to some other $backup$ device within its communication range. We require each device x to have k potential backup devices, all within distance D of x; we call these k devices the ***backup set*** of x. Also, we do not want any device to be in the backup set of too many other devices; otherwise, a single failure might affect a large fraction of the network.

So suppose we are given the communication radius D, parameters b and k, and an array $d[1 \,..\, n, 1 \,..\, n]$ of distances, where $d[i, j]$ is the distance between device i and device j. Describe an algorithm that either computes a backup set of size k for each of the n devices, such that no device appears in more than b backup sets, or reports (correctly) that no good collection of backup sets exists.

Solution

Solution