Design and Analysis of Algorithms: Homework #6

Due in class on April 24, 2018

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Problem 1

For any flow network G and any vertices u and v let $bottleneck_G(u, v)$ denote the maximum, over all paths π in G from u to v, of the minimum-capacity edge along π.

(a) Describe and analyze an algorithm to compute $bottleneck_G(s, t)$ in $O(E \log V)$ time.

Solution

Part A

```
1:  function BOTTLENECK(G, u, v)
2:      queue ← an empty heap
3:      current ← u
4:      for all vertex in G.vertices do
5:          if vertex = u then
6:              vertex.weight ← ∞
7:          else
8:              vertex.weight ← −∞
9:          end if                                  ▷ We'll use these weights in our heap
10:         insert vertex into queue
11:     end for
12:     while ¬v.seen do                            ▷ Find the path with the bottleneck by building a tree
13:         current.seen ← True
14:         for all d in current.neighbors do
15:             if ¬d.seen then
16:                 if edge(current, d).weight > d.weight then
17:                     delete d from queue
18:                     d.weight ← edge(current, d).weight
19:                     d.parent ← current
20:                     insert d into queue
21:                 end if
22:             end if                              ▷ Update the vertex weight of a neighbor of current
23:         end for
24:         current ← dequeue the max vertex in queue
25:     end while
26:     c ← v
27:     min ← ∞
28:     while c ≠ u do                              ▷ Find the minimal edge in the bottlenecking path
29:         if edge_{c,c.parent}.weight < min then
30:             min ← edge_{c,c.parent}.weight
31:         end if
32:         c ← c.parent
33:     end while
34:     return min
35: end function
```

This assumes the function should return the *weight* of the bottlenecking edge; however, it can trivially be modified to return the edge itself.

Problem 1 continued on next page…

This algorithm is essentially a modification of Prim's algorithm, where instead of searching for minimally-weighted edges, we instead search for maximally-weighted edges. To do this, we use a priority queue (using a binary heap).

We can split up our algorithm into three sections. The first is where we add each vertex into the priority queue. This is a total of V insertions into the queue, each of which takes $O(\log V)$ time, for a total runtime of $O(V \log V)$.

Second, we build a spanning tree in G until we have found a route from u to v. To do this, we continually dequeue vertices from our queue. For each unvisited neighboring vertex, we check if we've found a better edge to connect it to our tree. If so, we update the weight of the vertex, and also update its "parent" in the tree. We continue in this fashion until v has been added to the tree. Each time we attempt to update a vertex, it means we have traversed a new edge in the graph; updating the vertex involves a successive deletion and insertion, both of which are $O(\log V)$. Since this can happen at most E times, the total running time for this section is $O(E \log V)$.

Last, we determine the smallest edge in the path from v to u. This is more straightforward, as it is a simple $O(E)$ loop.

Our total running time is thus $O((E + V) \log V)$; since V is $O(E)$ for any connected graph, we can say that it is $O(E \log V)$.

Problem 2

Suppose you have already computed a maximum flow $f*$ in a flow network G with *integer* edge capacities.

(a) Describe and analyze an algorithm to update the maximum flow after the capacity of a single edge is increased by 1.

(b) Describe and analyze an algorithm to update the maximum flow after the capacity of a single edge is decreased by 1.

Both algorithms should be significantly faster than recomputing the maximum flow from scratch.

Solution

Part A

Solution

Part B

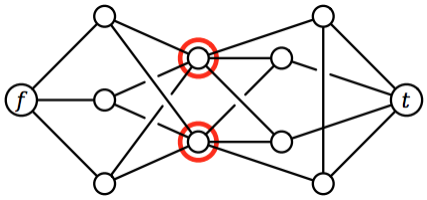
Solution

Problem 3

The Island of Sodor is home to a large number of towns and villages, connected by an extensive rail network. Recently, several cases of a deadly contagious disease (either swine flu or zombies; reports are unclear) have been reported in the village of Ffarquhar. The controller of the Sodor railway plans to close down certain railway stations to prevent the disease from spreading to Tidmouth, his home town. No trains can pass through a closed station. To minimize expense (and public notice), he wants to close down as few station as possible. However, he cannot close the Ffarquhar station, because that would expose him to the disease, and he cannot close the Tidmouth station, because then he couldn?t visit his favorite pub.

Describe and analyze an algorithm to find the minimum number of stations that must be closed to block all rail travel from Ffarquhar to Tidmouth. The Sodor rail network is represented by an undirected graph, with a vertex for each station and an edge for each rail connection between two stations. Two special vertices f and t represent the stations in Ffarquhar and Tidmouth.

For example, given the following input graph, your algorithm should return the number 2.



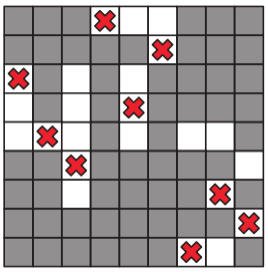
Solution

Solution

Problem 4

Suppose we are given an $n \times n$ square grid, some of whose squares are colored black and the rest white. Describe and analyze an algorithm to determine whether tokens can be placed on the grid so that

1. every token is on a white square;
2. every row of the grid contains exactly one token; and
3. every column of the grid contains exactly one token



Your input is a two dimensional array $IsWhite[1..n, 1..n]$ of booleans, indicated which squares are white. Your output is a single boolean. For example, given the grid above as input, your algorithm should return **True**.

Solution

Solution

Problem 5

Ad-hoc networks are made up of low-powered wireless devices. In principle, these networks can be used on battlefields, in regions that have recently suffered from natural disasters, and in other hard-to-reach areas. The idea is that a large collection of cheap, simple devices could be distributed through the area of interest (for example, by dropping them from an airplane); the devices would then automatically configure themselves into a functioning wireless network.

These devices can communicate only within a limited range. We assume all the devices are identical; there is a distance D such that two devices can communicate if and only if the distance between them is at most D.

We would like our ad-hoc network to be reliable, but because the devices are cheap and low-powered, they frequently fail. If a device detects that it is likely to fail, it should transmit its information to some other $backup$ device within its communication range. We require each device x to have k potential backup devices, all within distance D of x; we call these k devices the ***backup set*** of x. Also, we do not want any device to be in the backup set of too many other devices; otherwise, a single failure might affect a large fraction of the network.

So suppose we are given the communication radius D, parameters b and k, and an array $d[1 \,..\, n, 1 \,..\, n]$ of distances, where $d[i, j]$ is the distance between device i and device j. Describe an algorithm that either computes a backup set of size k for each of the n devices, such that no device appears in more than b backup sets, or reports (correctly) that no good collection of backup sets exists.

Solution

Solution