Chapter 2

The Classic Petri Net

In this chapter the basic principles of classic Petri net theory are introduced and basic properties are explained. With the latter, the emphasis will not be on a complete listing but on the systematization of qualities as static or dynamic and qualitative or quantitative.

2.1 Definitions

Definition 2.1 (unmarked Petri net) *An unmarked Petri net is a 4-tuple* (P, T, F, V) *such that*

1. P *and* T *are finite sets with* $P \cap T = \emptyset$ *and* $P \cup T \neq \emptyset$.
2. F *is a relation of arity 2 with* $F \subseteq \big(P \times T\big) \cup \big(T \times P\big)$.
3. $V : F \longrightarrow \mathbb{N}^+$.

The elements of P are called *places* and the elements of T are called *transitions*. The elements of F are called *arcs* and F is called the *flow relation* of $\mathcal{N}$. The function V is the *multiplicity (weight)* of the arcs.

This definition covers the static aspects of a Petri net. An unmarked Petri net is therefore a 2-colored, weighted, directed, finite graph. The vertices of one color represent the places and the vertices of the other color the transitions.

L. Popova-Zeugmann, *Time and Petri Nets*, DOI 10.1007/978-3-642-41115-1_2,

In Petri net theory, the places are graphically represented by circles and the transitions by rectangles or bars.

Definition 2.2 (Petri net) *A marked Petri net is a 5-tuple* $\mathcal{N} = (P, T, F, V, m_0)$ *(short: Petri net) such that*

1. (P, T, F, V) *is an unmarked Petri net.*
2. $m_0 : P \longrightarrow \mathbb{N}$ *is the initial marking*[1].

Thus, a Petri net has an initial marking which assigns a natural number to each place. This marking is graphically represented by the corresponding number of tokens (points) on the places, so $m_0(p)$-tokens are drawn in the circle representing the place p. Any distribution of tokens on the places is a marking:

Definition 2.3 (marking) *Let* P *be the set of places of a Petri net* $\mathcal{N}$. *A marking in* $\mathcal{N}$ *is a total function* $m : P \longrightarrow \mathbb{N}$.

Example 2.4

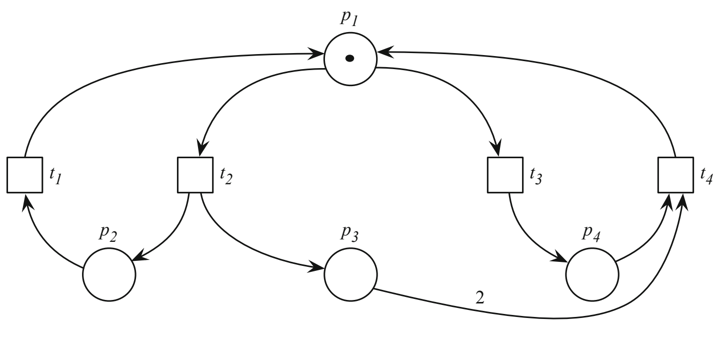

Figure 2.1: $\mathcal{N}_1$ is a Petri net with the initial marking $m_0 = (1, 0, 0, 0)$

[1] The function m_0 is a total one.

The initial marking of a Petri net can generally change into a successor marking according to certain rules and this can itself transform in turn into successor markings. The rules describing the possible changes from one marking to the next one are called *firing rules*, the occurring change itself is called a *firing*. Throughout such firings, the distribution of tokens over the places of a Petri net can change and thereby the whole view of the net changes. In other words: The Petri net also has a *dynamic* aspect which is defined by the firing rules.

Before we define the firing rules for classic Petri nets, some basic static notions need to be explained: Let $\mathcal{N}$ be an arbitrary Petri net with a set of places P, a set of transitions T and a flow relation F. All places which are connected to a transition by an arc form the set of pre-places and post-places of a specific transition. A pre-place of a certain transition is a place which is directly connected with the considered transition through an arc directed from the place to the transition. If the arc points in the opposite direction, that place is a post-place of the transition. The *set of pre-places* of the transition t is denoted by ${}^\bullet t := \{p \mid p \in P \wedge (p,t) \in F\}$. The set $t^\bullet := \{p \mid p \in P \wedge (t,p) \in F\}$ is the *set of post-places* of t. Analogously, for every place p the set ${}^\bullet p := \{t \mid t \in T \wedge (t,p) \in F\}$ denotes the *set of pre-transitions* of p and the set $p^\bullet := \{t \mid t \in T \wedge (p,t) \in F\}$ denotes the *set of post-transitions* of p.

The dynamic aspect of a Petri net is defined by the firing rules. The firing rules reflect causal relations within a permanently changing system: The events of the real system are modeled by transitions of the Petri net. The causes or preconditions of an event are represented by the pre-places of the transition modeling the event. The post-places of the transition describe the post-conditions of the event, which of course in turn can be preconditions of other events. Whenever a pre-place is marked, the respective condition is considered to be fulfilled. In the real system an event can take place provided that all preconditions of the event are fulfilled. In the Petri net the occurrence of the event is represented by firing of the respective transition. After an event has taken place, its preconditions (in general) are not fulfilled any more. The corresponding pre-places are therefore no longer marked. Instead the post-conditions of the event are fulfilled and in the Petri net the post-places of the transition are marked. This atomic process in the real system provides the basic idea for the firing rule in the Petri net. In classic Petri nets this basic idea is carried out with the multiplicity $V \equiv 1$ (these are the so-called *ordinary Petri nets*) and each place holds at most one token, i.e., for each reachable marking m (cf. Def. 2.10) it holds that $m : P \longrightarrow \{0, 1\}$ (these are the so called *1-safe Petri nets*). Ordinary, 1-safe Petri nets are also called *condition/event nets*.

If not all multiplicities of arcs of a Petri net are 1, the firing rule is extended consistently. The preconditions of an event are fulfilled if, for each place, the number of tokens it holds is no smaller than the multiplicity of the arc from this place to the respective transition. After an event has taken place, every post-place of the transition obtains the number of tokens equivalent to the multiplicity of the arc from the transition to the respective post-place. A transition t can therefore fire if the Petri net is in a marking m, which assigns at least as many tokens as t needs in each pre-place of t. All preconditions can be regarded as fulfilled. This minimum number of necessary tokens on the places is defined by the marking t^-:

$$t^-(p) := \begin{cases} V(p,t) & , \text{ if } \quad (p,t) \in F \\ 0 & , \text{ if } \quad (p,t) \notin F \end{cases} .$$

Analogously the marking t^+ describes the number of tokens which are added to each place upon firing of t:

$$t^+(p) = \begin{cases} V(t,p) & \text{if} \quad (t,p) \in F \\ 0 & \text{if} \quad (t,p) \notin F \end{cases} .$$

The difference in tokens on the places after firing of transitions t is represented by the marking Δt :

$$\Delta t := t^+ - t^-.$$

Furthermore, let us consider the Petri net $\mathcal{N} = (P, T, F, V, m_0)$ with $T = \{t_1, \cdots, t_n\}$ and $P = \{p_1, \cdots, p_m\}$. The matrix $C_\mathcal{N} = (c_{ij})$ with m rows and n columns and

$$c_{ij} := \Delta t_j(p_i)$$

is called the *incidence matrix* of $\mathcal{N}$.

Example 2.5 *We consider again the Petri net $\mathcal{N}_1$ introduced in Example 2.4. Its incidence matrix $C_{\mathcal{N}_1}$ is:*

$$C_{\mathcal{N}_1} = \begin{array}{cc} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix} & \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \\ \begin{array}{cccc} \Delta t_1 & \Delta t_2 & \Delta t_3 & \Delta t_4 \end{array} & \end{array} .$$

We can now formally introduce the notions *enabled* and *firing*:

Definition 2.6 (enabled) *Let $\mathcal{N} = (P, T, F, V, m_0)$ be a Petri net and let m be a marking in $\mathcal{N}$. A transition $t \in T$ is enabled in m if it holds that: $t^- \leq m$.*

Definition 2.7 (firing) *Let $\mathcal{N} = (P, T, F, V, m_0)$ be a Petri net and let m be a marking in $\mathcal{N}$. A transition $t \in T$ can fire in m (notation: $m \xrightarrow{t}$), if t is enabled in m. After the firing of t the Petri net is in the marking m' (notation: $m \xrightarrow{t} m'$) with*

$$m' := m + \Delta t.$$

This firing rule defines the firing of *a single* transition, i.e., it is a *mono*-firing rule. It is also possible to define a rule firing a set (step) of transitions. This firing rule, also called a *step firing rule*, is usually used in time-dependent Petri nets, see Chapter 4. We will always use the mono-firing rule unless stated otherwise.

We denote the change of a Petri net from marking m into marking m' by firing of transition t by $m \xrightarrow{t} m'$. The relation $\longrightarrow$ which is defined by the firing rule is called the *firing relation*.

A further basic term in Petri Net theory is the notion of a *reachable marking*. In order to introduce it, we first define *firing sequences*:

Definition 2.8 (firing sequence) *Let $\mathcal{N} = (P, T, F, V, m_0)$ be a Petri net, let m be a marking in $\mathcal{N}$ and let $\sigma = t_1 \cdots t_n$ be a sequence of transitions. σ fires from m to m' in $\mathcal{N}$ (short: $m \xrightarrow{\sigma} m'$) if it holds that:*

Basic $\sigma = \varepsilon$

$m' := m$

Step $\sigma = t_1 \cdots t_n t_{n+1}$

There is a marking m'' in $\mathcal{N}$, with

$$m \xrightarrow{t_1 \cdots t_n} m'' \quad \text{and} \quad m'' \xrightarrow{t_{n+1}} m'.$$

σ is called a firing sequence from m in $\mathcal{N}$ if there is a marking m' such that σ fires from m to m'.

A firing sequence σ from m_0 in $\mathcal{N}$ is usually simply called a firing sequence.

Example 2.9 *Let us consider example 2.4 again. The transitions sequence $\sigma_1 = t_2\, t_1 t_2 t_1 t_3 t_4$ is a firing sequence from m_0 in the Petri net $\mathcal{N}_1$. In contrast, the transition sequence $\sigma_2 = t_3 t_4 t_3 t_4$ is not a firing sequence from m_0 in the same net. However, σ_2 is a firing sequence from $m = (1, 0, 4, 0)$ in $\mathcal{N}_1$.*

We write $m \xrightarrow{*} m'$ if there exists a firing sequence σ such that $m \xrightarrow{\sigma} m'$. This means that the relation $\xrightarrow{*}$ is the reflexive-transitive closure of the firing relation $\longrightarrow$.

Furthermore, for $\sigma = t_1 \cdots t_n$ the following is true:

$$m' = m + \sum_{i=1}^{n} \Delta t_i. \tag{1}$$

This equation can equivalently be rewritten in the form:

$$m' = m + \sum_{t \in \sigma} \pi_t \cdot \Delta t, \tag{2}$$

where π_t is the number of appearances of the transition t in the sequence σ. Finally, we obtain the equality

$$m' = m + C_{\mathcal{N}} \cdot \pi_\sigma \tag{3}$$

where $C_{\mathcal{N}}$ is the incidence-matrix of $\mathcal{N}$ and $\pi_\sigma \in \mathbb{N}^T$ the vector with

$$\pi_\sigma(t) = \begin{cases} \pi_t & \text{if } t \in \sigma \\ 0 & \text{otherwise.} \end{cases}$$

The $|T|$-dimensional vector π_σ is called *the Parikh vector* of σ. The equality (3) is called the *state equation of σ in m (in $\mathcal{N}$).*

Definition 2.10 (reachable marking) *A marking m is called reachable from the marking m^* in a Petri net $\mathcal{N}$, if there is a firing sequence σ from m^* to m in $\mathcal{N}$. If $m^* = m_0$ we call m reachable in $\mathcal{N}$.*

Finally, $R_\mathcal{N}(m) := \{m' \mid m \xrightarrow{*} m'\}$ is the notation for the set of all markings m' which are reachable from the marking m in $\mathcal{N}$.

2.2 State Space

For any Petri Net $\mathcal{N}$ the set $R_\mathcal{N}(m_0)$ contains all markings reachable in the net. This set is of particular interest because it gives us information about all the events that can occur in a system modeled by the considered net. The set tells us which of the markings of the net are reachable and thereby also which pre-conditions of events may potentially be fulfilled.

Definition 2.11 (state space) *Let $\mathcal{N} = (P, T, F, V, m_0)$ be a Petri net. The set $R_\mathcal{N} := R_\mathcal{N}(m_0)$ is called the state space of $\mathcal{N}$.*

The state space can be finite or infinite. The fact that the set $R_\mathcal{N}$ is decidable (cf. inter alia [PW03]) is crucial for the analysis of Petri nets as the state space holds information about the reachability/non-reachability of markings in $\mathcal{N}$ and thus about the occurrence/non-occurrence of events.

Definition 2.12 (boundedness) *A Petri net $\mathcal{N}$ is said to be bounded if the set $R_\mathcal{N}$ of all its reachable markings is finite.*

Boundedness can also be introduced using the notion of *bounded place*. A place in a Petri net is bounded if there is a natural number such that the number of tokens on this place never exceeds this number. A net is then bounded if all its places are bounded. These definitions of boundedness are equivalent.

We now consider again the reflexive-transitive closure of the firing relation. This relation generally fulfills none of the common properties of relations like symmetry, asymmetry, anti-symmetry, connexity etc. The graph of the relation is called *the reachability graph* of the Petri net. It is formally defined as follows:

Definition 2.13 (reachability graph) *Let $\mathcal{N} = (P, T, F, V, m_0)$ be a Petri net. The reachability graph of $\mathcal{N}$ is the graph $\mathcal{RG}_{\mathcal{N}}$ with*

1. *the set $R_{\mathcal{N}}$ as set of vertices and*
2. *$(m, m') \in R_{\mathcal{N}} \times R_{\mathcal{N}}$ is an edge in $\mathcal{RG}_{\mathcal{N}}$ if there is a transition $t \in T$ such that $m \xrightarrow{t} m'$.*

Such a reachability graph of a Petri net is a partial deterministic automaton which can be finite or infinite (also compare Fig.2.2). The bounded Petri nets are furthermore exactly those Petri nets whose reachability graphs are finite. In the case of boundedness the Petri net is therefore well analyzable with the help of its reachability graph. Determining whether a certain property holds in an unbounded Petri net however is a lot more challenging.

Example 2.14 *Let us again consider the Petri net $\mathcal{N}_1$ given in Example 2.4. Its reachability graph $\mathcal{RG}_{\mathcal{N}_1}$ is infinite. A part of the reachability graph is presented in Fig.2.2.*

When a Petri net is unbounded its reachability graph is infinite. In this case we can consider the so-called *coverability graph* of the net, which is always finite, to prove the existence or absence of properties. The trade-off for the finiteness of the coverability graph is loss of information.

The vertices of the coverability graph are the so-called *generalized markings* of the Petri net. A generalized marking in a Petri net is a (total) function which assigns to each place a natural number or the value ω. A place has the value ω in a generalized marking when the place is unbounded. We extend the rules of addition, subtraction, and multiplication and the relation $\leq$ for the set $\mathbb{N} \cup \{\omega\}$ as follows:

For each $n \in \mathbb{N}$:

$$\omega \pm n = \pm n + \omega = \omega, \quad n \cdot \omega = \omega \cdot n = \begin{cases} \omega & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}, \quad \omega > n.$$

We say that a (generalized) marking m *covers* another (generalized) marking m' in a Petri net $\mathcal{N}$ (short: $m' \prec m$), if $m' \leq m$ and there is at least one place p in $\mathcal{N}$ with $m(p) \neq m'(p)$. Therefore, $m' \leq m$ implies that either $m' \prec m$ or $m' = m$.

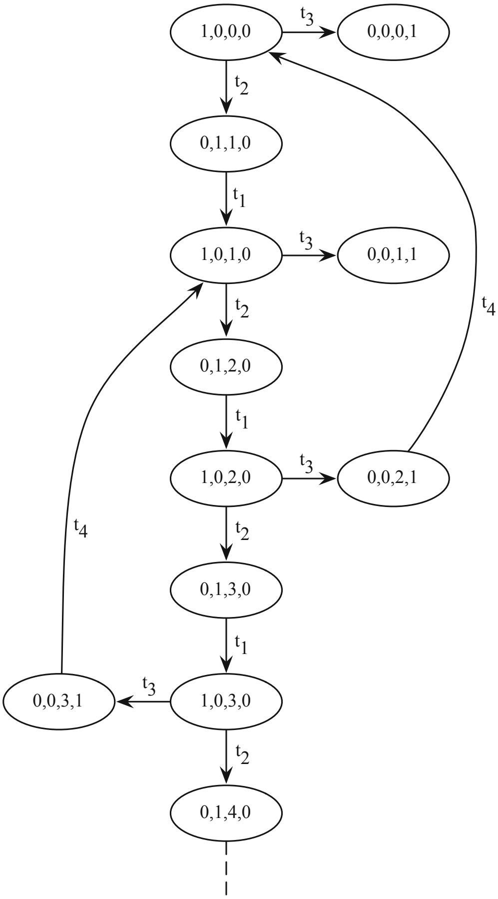

Figure 2.2: Part of the reachability graph $\mathcal{RG}_{\mathcal{N}_1}$

Definition 2.15 (coverability graph) *Let* $\mathcal{N} = (P, T, F, V, m_0)$ *be a Petri net. The edge-labeled digraph* $\mathcal{CG}_{\mathcal{N}} := (W, E, T)$ *is said to be a coverability* *graph of $\mathcal{N}$ if the set of vertices W, the set of edges E and the set of labels T are defined using the following algorithm* [2]*:*

begin $R := \{m_0\}$; $ancestor\_marking(m_0) := *$; $W := \emptyset$; $E := \emptyset$;

while $R \neq \emptyset$ **do**

`choose` m `from` R; $R := R - \{m\}$; $W := W \cup \{m\}$;

enabled_set$:=\{t \mid t^- \leq m\}$;

for $t \in$ *enabled_set* **do**

$m' := m + \Delta t$;

$m^* := m$;

while $(m^* \neq *)$ `and` $(m^* \not\leq m')$ **do**

$m^* := ancestor\_marking(m^*)$;

end;

if $m^* \neq *$ **then**

$m' := m' + (m' - m^*) \cdot \omega$;

end;

$E := E \cup \{(m, t, m')\}$;

if $m' \notin W \cup R$ **then**

$R := R \cup \{m'\}$; $ancestor\_marking(m') := m$;

end;

end;

end;

end.

The coverability graph is not unique for a given Petri net. The notion "coverability graph" was first defined by Karp and Miller (see [KM69]). Later, in [Fin93], Finkel introduced an algorithm which computes a minimal coverability graph of a Petri net. This graph is unique and has the minimum number of vertices.

[2] Adapted from [Sta90]

Example 2.16 *Using Definition 2.15 we obtain for the Petri net $\mathcal{N}_1$ considered in Example 2.4 the coverability graph $\mathcal{CG}_{\mathcal{N}_1}$, represented in Fig. 2.3:*

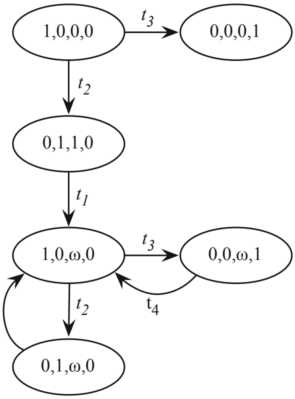

Figure 2.3: A coverability graph $\mathcal{CG}_{\mathcal{N}_1}$ of the Petri net $\mathcal{N}_1$

2.3 PN-Computability

The expressive power of Petri nets is less than that of Turing machines (TM). This means that there are algorithms which cannot be described by any classic Petri net: Not every Turing-computable function can also be computed by a Petri net.

A number-theoretical n-ary function $f : \mathbb{N}^n \longrightarrow \mathbb{N}$ is called *Turing-computable* if there is a Turing machine M_f which given an n-tuple $(x_1, \cdots, x_n)$ as input stops if and only if that n-tuple belongs to the domain of the function and in this case also returns the value $f(x)$. For more on Turing-computability compare [HMU02].

We now need to clarify what a *PN-computable function* is. Regardless of how we define that notion, we first have to ensure that there is a unique presentation for every natural number in a Petri net. The infinity of the set of all natural numbers prevents it from being modeled by places, transitions or arcs. But the set of reachable markings in a Petri net can in general be infinite. Moreover, transitions might fire infinitely often and places can change their number of tokens infinitely often, even if the places themselves are bounded.

In order for a place to change its number of tokens infinitely often, at least one transition in the net needs to fire infinitely often. The reverse is also true and therefore the two properties are equivalent.

It can furthermore be shown that the number of firings in an arbitrary Petri net is equal to the number of reachable markings in another Petri net derived from the first one as follows:

Let $\mathcal{N} = (P, T, F, V, m_0)$ be an arbitrary Petri net. Using $\mathcal{N}$ we construct the Petri net $\mathcal{N}' = (P', T', F', V', m_0')$ where:

$$P' := P \cup \{p^*\} \text{ with } p^* \notin P, \quad T' := T, \quad F' := F \cup \{(t, p^*) \mid t \in T\} \quad \text{and}$$

$$V'(u) := \begin{cases} V(u) & \text{if } u \neq (t, p^*) \\ 1 & \text{if } u = (t, p^*) \end{cases}, \quad m_o'(p) := \begin{cases} m_0(p) & \text{if } p \neq p^* \\ 0 & \text{if } p = p^* \end{cases}$$

for all $p \in P'$ and for all $t \in T'$.

The net $\mathcal{N}'$ is a copy of the net $\mathcal{N}$ with one additional place p^* such that every time a transition in the copy of $\mathcal{N}$ fires, the number of tokens in p^* increases by 1. Thus the place p^* in $\mathcal{N}'$ is unbounded if and only if there is at least one transition t in the net $\mathcal{N}$ which fires infinitely often.

It is thereby clearly possible to represent natural numbers – and therefore also n-tuples of natural numbers – by markings in a Petri net.

Next we define the notion of *PN-computability*:

Definition 2.17 (PN-computable) *An n-ary function $f : \mathbb{N}^n \longrightarrow \mathbb{N}$ is called Petri-net-computable (PN-computable) if there is an initial Petri net*[3] *$\mathcal{N}_f = (P_f, T_f, F_f, V_f, m_0^f)$ such that for each n-tuple $x = (x_1, \cdots, x_n) \in \mathbb{N}^n$*

[3] An initial Petri net is an arbitrary Petri net which is fixed for a function f. The extension of the initial marking m_0^f of $\mathcal{N}_f$ to initial markings $m_0^{f,x}$ modeling the arguments $x = (x_1, \cdots, x_n)$ generates the Petri nets $\mathcal{N}_f^x$. The marking m_0^f is better understood after reading the similar definition for Time Petri nets, Definition 3.15, as well as Examples 3.17 and 3.18.

and for the Petri net $\mathcal{N}_f^x = (P_f, T_f, F_f, V_f, m_0^{f,x})$*, where the marking* $m_0^{f,x}$ *models the* n*-tuple* $(x_1, \cdots, x_n)$*, it holds that:*

Case 1 *If the tuple* $(x_1, \cdots, x_n)$ *belongs to the domain of* f *then the Petri net* $\mathcal{N}_f^x$ *stops (cannot fire anymore) and the last-reached marking* $m^{f(x_1,\cdots,x_n)}$ *uniquely represents the number* $f(x_1, \cdots, x_n)$*.*

Case 2 *If the tuple* $(x_1, \cdots, x_n)$ *does not belong to the domain of* f *then the Petri net* $\mathcal{N}_f^x$ *never stops, i.e., for each marking* m *with* $m_0^{f,x} \xrightarrow{*} m$ *there exists at least one transition* t *which is enabled in* m*.*

Evidently, every PN-computable function is also Turing-computable.

We *assume* now that every Turing-computable function is also PN-computable. Let f be an arbitrary, n-ary, Turing-computable function. According to the assumption f is PN-computable, too. Let M_f be a Turing machine which computes f and $\mathcal{N}_f$ a Petri net which computes f. Furthermore, let $(x_1, \cdots, x_n)$ be an n-tuple of natural numbers. Then it holds that:

$$\begin{aligned} &M_f \text{ started on } (x_1, \cdots, x_n) \text{ stops} \\ &\qquad\qquad\qquad\qquad \text{if and only if} \\ &\mathcal{N}_f \text{ started in } (x_1, \cdots, x_n) \text{ stops.} \end{aligned} \tag{4}$$

As detailed before, we can define the Petri net $\mathcal{N}'$ for every Petri net $\mathcal{N}$. The construction of the respective Petri net $\mathcal{N}_f'$ for $\mathcal{N}_f$ ensures that for any $\mathcal{N}_f$ the following holds:

$$\begin{aligned} &\mathcal{N}_f \text{ started in } (x_1, \cdots, x_n) \text{ stops} \\ &\qquad\qquad\qquad\qquad \text{if and only if} \\ &\text{the place } p^* \text{ in } \mathcal{N}_f' \text{ is bounded.} \end{aligned} \tag{5}$$

We can determine whether the place p^* is bounded in $\mathcal{N}_f'$ by means of the coverability graph of $\mathcal{N}_f'$. Taking into account (4) and (5) it is consequently decidable for an arbitrary Turing-computable function f whether the Turing machine M_f started with an arbitrary n-tuple $(x_1, \cdots, x_n)$ stops or not. This contradicts, however, the undecidability of the halting problem for Turing machines (cf. among others [HMU02]). Therefore the assumption that every Turing-computable function is also PN-computable is disproved.

Thus we have proved the property:

Theorem 2.18 *The class of all marked Petri nets (as defined in Definition 2.2 and using the mono-firing rule) is not Turing-complete.*

2.4 Basic Properties

The potential of Petri nets for modeling results from their graphical representability and makes them applicable in a wide variety of areas. Many kinds of systems can be represented easily but nevertheless clearly and formally using Petri nets. Furthermore, the nature of Petri nets allows modeling of a system at each stage of abstraction and refinement of the model where necessary. Therefore modeling with Petri nets is a hierarchical procedure as well as a modular one. Without analyzing methods for Petri nets however their use as a modeling instrument is very limited. We want on the one hand to ascertain the presence of certain properties in a Petri net in order to evaluate its "quality" as a model of a real system, on the other hand an analysis of the model might detect properties of the real system previously overlooked.

Some properties of Petri nets can be determined very easily, these in general being properties which result from the structure of the Petri net itself. We call such properties *static*. In contrast to this, we describe properties which result from firing and which characterize permanent changes in the net as *dynamic* properties. These dynamic properties are usually more difficult to identify and sometimes impossible to analyze.

Among the static properties there are such features as the existence of static conflicts, deadlocks, traps, etc., or whether a net is a marked graph, a state machine, a free-choice net, an extended-free-choice net, an asymmetric-choice net, a homogeneous one, etc. All of these properties are decidable independently of the state space of the considered Petri nets. Some of them are relevant for our studies and will be defined below, although they are not the main subject of our analysis.

The notions *free-choice net, extended-free-choice net, asymmetric-choice net* and *marked graph* were originally defined for ordinary Petri nets and are still mainly used in that context. For arbitrary multiplicity these notions were first introduced in [Sta90]. In this book, we use more recent and more general definitions. Therefore the classes defined here, like free-choice nets, extended-free-choice nets, asymmetric-choice nets and marked graphs for homogeneous Petri nets, are supersets of the corresponding classes of ordinary nets.

Definition 2.19 *Let $\mathcal{N} = (P, T, F, V, m_0)$ be a Petri net.*

1. *Two transitions t and t' from T are in a* **static conflict**, *if they have at least one common pre-place, i.e., ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$.*

2. *Two transitions t and t' from T are in a* **dynamic conflict** *in the marking m if they are in a static conflict and by firing one of the transitions in marking m the other one may become disabled, i.e., ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$ and $t^- \leq m$ and $t'^- \leq m$ but $t^- + t'^- \not\leq m$.*

3. *$\mathcal{N}$ is a* **marked graph** *if $\mathcal{N}$ is an ordinary Petri net and if each place p has exactly one pre-transition and one post-transition, i.e., $V(f) = 1$ for each $f \in F$ and $|\,{}^\bullet p\,| = |\,p^\bullet\,| = 1$.*

4. *$\mathcal{N}$ is a* **free-choice net** *(FC net) if each shared place is the only pre-place of its post-transitions, i.e., if $t, t' \in p^\bullet$ then ${}^\bullet t = \{p\} = {}^\bullet t'$.*

5. *$\mathcal{N}$ is an* **extended-free-choice net** *(EFC net) if the post-transitions of each shared place have the same pre-places, i.e., if $t, t' \in p^\bullet$ then ${}^\bullet t = {}^\bullet t'$.*

6. *$\mathcal{N}$ is an* **asymmetric-choice net** *(AC net) if it holds that if two places have at least one common post-transition then the set of all post-transitions of one of the places is a subset of the set of all post-transitions of the other one. Formally: For each two places p and p' it holds: If $p^\bullet \cap p'^\bullet \neq \emptyset$ then $p^\bullet \subseteq p'^\bullet$ or $p'^\bullet \subseteq p^\bullet$.*

7. *$\mathcal{N}$ is* **homogeneous** *if all output arcs of a place have the same multiplicity, i.e., for each place $p \in P$ it is true that if $t, t' \in p^\bullet$, then $V(p, t) = V(p, t')$.*

Example 2.20 *Let us consider the Petri net $\mathcal{N}_1$ from Example 2.4 . The transitions t_2 and t_3 are in a static conflict. $\mathcal{N}_1$ is a FC net and therefore also an EFC net and an ES net. It is homogeneous.*

We refrain from defining further static properties and refer to the common literature on classic Petri nets, such as [Mur89] or [Sta90].

Liveness, boundedness, reachability, the existence of invariants, etc. are the basic properties of a Petri net and are dynamic properties. They provide information about the behavior of the examined Petri net.

The notion of *liveness* was established by Petri, Genrich and Lautenbach at the GMD, Bonn at the beginning of the 1960s and developed further in cooperation with Commoner, Even, Holt, Pnueli and Hack at MIT. It was initially examined in special classes of Petri nets such as marked graphs (cf. [Gen68], [CHEP71]) and AC-nets (cf. [Hac72], [Com73]). The *four levels* of liveness introduced by Lautenbach in his PhD thesis [Lau73] in 1973 constituted an important advance and form the basis of liveness studies up to the present day.

Definition 2.21 *Let $\mathcal{N} = (P, T, F, V, m_0)$ be a Petri net, m be an arbitrary marking in $\mathcal{N}$ and t a transition in T.*

1. *t is called* **live in** *m in $\mathcal{N}$ if for each marking $m' \in R_{\mathcal{N}}(m)$ there exists a marking $m'' \in R_{\mathcal{N}}(m')$ such that $m'' \xrightarrow{t}$.*
2. *t is called* **dead in** *m in $\mathcal{N}$ if for each marking $m' \in R_{\mathcal{N}}(m)$ it holds that $t^- \not\leq m'$.*
3. *m is called* **live in** *$\mathcal{N}$ if each transition $t \in T$ is live in m.*
4. *m is called* **dead in** *$\mathcal{N}$ if each transition $t \in T$ is dead in m.*
5. *t is called* **live / dead in** *$\mathcal{N}$ if t is live / dead in m_0.*
6. *$\mathcal{N}$ is called* **live / dead** *if m_0 is live / dead in $\mathcal{N}$.*
7. *$\mathcal{N}$ is called* **blocking-free**[4] *if at least one transition is enabled in each reachable marking $m \in R_{\mathcal{N}}(m_0)$.*

As is obvious from the definition, a transition t is live in the marking m if it is not dead in any successor marking of m. If a marking is dead, no transition can fire from that marking. The marking therefore constitutes a leaf in the

[4] Some authors call this property *deadlock-free*.

reachability graph of the net. Furthermore we observe that a Petri net which is not dead does not have to be blocking-free (deadlock-free). The reason for this being that a Petri net which fires only finitely often and then stops is not dead. A blocking-free Petri net obviously is not dead. Every live Petri net is blocking-free, the reverse however does not hold. A Petri net which is not dead need not be live and a Petri net which is not live is not necessarily dead.

We can summarize as follows:

$$\mathcal{N}\text{– dead} \quad \underset{\not\longleftarrow}{\longrightarrow} \quad \mathcal{N}\text{– not blocking-free} \quad \underset{\not\longleftarrow}{\longrightarrow} \quad \mathcal{N}\text{– not live}$$

and

$$\mathcal{N}\text{– live} \quad \underset{\not\longleftarrow}{\longrightarrow} \quad \mathcal{N}\text{– blocking-free} \quad \underset{\not\longleftarrow}{\longrightarrow} \quad \mathcal{N}\text{– not dead.}$$

Whether a Petri net is live, dead or blocking-free can be decided by means of its reachability graph. The coverability graph of a Petri net however is of no use in deciding liveness and blocking-freedom but does still contain the information of whether the net is dead (cf. [PW03]). There are a number of algorithms which decide these properties more effectively on restricted classes of nets.

Finally we remark that the notion of t *being live in a marking* m introduced in Definition 2.21 is equivalent to the *4-liveness* defined by Lautenbach in [Lau73]. The notion of t *being dead in a marking* m is equivalent to the *0-liveness* also defined in [Lau73]. In other words, t is dead in m according to Definition 2.21 if and only if t is not 1-live in m. This analogously applies to the liveness of a marking and to the liveness of a Petri net. Finally, at least one transition in a Petri net is *3-live* as defined by Lautenbach if the Petri net is blocking-free. The converse does not hold.

We already introduced the notions of *boundedness* and *reachable markings* with Definitions 2.12 and 2.10.

Example 2.22 *Let us consider the Petri net $\mathcal{N}_1$ from Example 2.4 again.*

- *The place P_3 is unbounded.*
- *$\mathcal{N}_1$ is unbounded.*

- *$\mathcal{N}_1$ is not live.*
- *$\mathcal{N}_1$ is not dead.*
- *$\mathcal{N}_1$ is not blocking-free.*

With the last two definitions in this chapter we will now introduce two (dual) notions of invariants in Petri nets:

Definition 2.23 (T-invariant) *Let $C_\mathcal{N}$ be the incidence matrix of the Petri net $\mathcal{N}$.*

1. *Each non-trivial solution $x \in \mathbb{N}^{|T|}$ of the homogeneous equality*

$$C_\mathcal{N} \cdot x = 0$$

   is called a transitions-invariant (short: T-invariant) of $\mathcal{N}$.

2. *A Parikh vector of a firing sequence which is also a T-invariant is called a feasible T-invariant.*

It is not difficult to see that if a Parikh vector π_σ is a T-invariant the marking is not changed by the firing of the sequence σ. This follows from the state equality for σ:

$$\begin{gathered} m \xrightarrow{\sigma} m' \\ \text{iff} \\ m' = m + C_\mathcal{N} \cdot \pi_\sigma = m \\ \text{i.e.,} \\ m' = m. \end{gathered}$$

The path representing the sequence σ in the reachability graph is obviously a cycle.

Example 2.24 *Let us consider the Petri net $\mathcal{N}_1$ shown in Example 2.4. The integer solutions of the homogeneous equality system*

$$\mathcal{C}_{\mathcal{N}_1} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

$$\textit{are} \quad x_1 = x_2 = 2 \cdot k, \quad x_3 = x_4 = k \ \textit{ for } \ k \in \mathbb{N}.$$

Furthermore, we note that the transition sequence $\sigma = t_2t_1t_2t_1t_3t_4$ *is a firing sequence starting in* m_0 *and its Parikh vector* π_σ *is a solution of the equality (6).*

We can see in Example 2.14 that the path σ *is a cycle in the reachability graph* $\mathcal{RG}_{\mathcal{N}_1}$*.*

Definition 2.25 (P-invariant) *Let* $C_{\mathcal{N}}$ *be the incidence matrix of the Petri net* $\mathcal{N}$*. Each non-trivial solution* $y \in \mathbb{N}^{|P|}$ *of the homogeneous equality*

$$y^T \cdot C_{\mathcal{N}} = 0$$

is called a place-invariant (short: P-invariant) of $\mathcal{N}$*.*

We can view a P-invariant as an equation of weights for the places that always holds:

Let $y = (y_1, \cdots, y_{|P|})$ be a P-invariant in a Petri net $\mathcal{N}$. It is true for each reachable marking m in $\mathcal{N}$ that :

$$m = m_0 + C_{\mathcal{N}} \cdot \pi_\sigma$$

and therefore

$$y^T \cdot m = y^T \cdot m_0 + y^T \cdot C_{\mathcal{N}} \cdot \pi_\sigma$$

and thus subsequently

$$\sum_{i=1}^{|P|} y_i \cdot m(p_i) = const. = \sum_{i=1}^{|P|} y_i \cdot m_0(p_i).$$

Example 2.26 *The homogeneous system of equations*

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}^T \cdot \mathcal{C}_{\mathcal{N}_1} = 0 \ \textit{in} \ \mathbb{N}$$

has the solutions

$$y_1 = y_2 = y_4 = s \ \ \textit{for} \ \ s \in \mathbb{N} \ \textit{and} \ y_3 = 0.$$

Therefore, the following equality holds for each marking $m \in \mathcal{R}_{\mathcal{N}_1}$*:*

$$s \cdot m(P_1) + s \cdot m(P_2) + 0 \cdot m(P_3) + s \cdot m(P_4) = \\ s \cdot m_0(P_1) + s \cdot m_0(P_2) + 0 \cdot m_0(P_3) + s \cdot m_0(P_4)$$

i.e.,

$$m(P_1) + m(P_2) + m(P_4) = 1.$$

All the properties introduced here are qualitative properties, even though some of them might also be considered as quantitative properties. The question of boundedness of a place for instance is equivalent to the question of the maximum number of tokens on the place, which deals with quantities. We will consider quantitative properties that do not have direct qualitative equivalents for the time-dependent Petri nets presented in the following chapters.

2.5 Bibliographical Notes

With his thesis "Kommunikation mit Automaten"[5] [Pet62] Petri established a theory of communication in arbitrarily large, non-globally defined systems. The starting point of his considerations is the opinion that the notion *state* in the sense of one global state is unsuitable for the description of causal relationships. The reason for this is that the use of a notion of global state

[5] "Communication with Automata". Published English translation in: Technical Report RADC-TR-65-377, Vol.1, January 1966, Suppl. 1, Griffiss Air Force Base, New York.

presupposes an explicit or implicit time scale implying simultaneity of independent events about whose independence we do not necessarily want to make any presumptions. Petri uses the notions *condition* and *event* for the description of causal relations and thereby defines action nets (or A-nets). Further works by him, as well as Genrich [Gen68], Lautenbach [Lau73], Holt et al. [CHEP71], Commoner [Com73], Hack [Hac72], etc. form the foundation of net theory. This net theory is today known as the theory of Petri nets. The graphic representation of these nets as well as their name "Petri nets" goes back to A. Holt. Innumerable introductory articles and books on the theory of Petri nets have been published since then. Among the first ones and still standard references in the field are the articles by Peterson [Pet77] and Murata [Mur89] as well as the books by Starke [Sta80] and Peterson [Pet81].

The latest book publications include a book by Reisig [Rei13] in which the author presents a thorough introduction to the essentials of Petri nets.

Mayr set another milestone for Petri net theory with his Ph.D. Thesis (cf. [May80]). Among other things he proved the decidability of the reachability of an arbitrary marking in a Petri net.

Algorithms deciding properties for different restricted classes of Petri nets can be found in [Esp98]. Priese and Wimmel give an almost complete introduction to the theory of Petri nets with their book [PW03].

2.6 Exercises

Exercise 2.1

Let $\mathcal{N}_1 = (P, T, F, V, m_0)$ be the following Petri net:

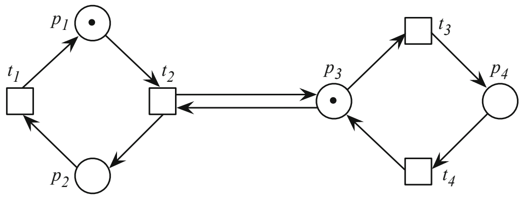

Figure 2.4: The Petri net $\mathcal{N}_1$

(a) Give the incidence matrix $C_{\mathcal{N}_1}$.

(b) Give the Parikh vectors of the two transition sequences $\sigma_1 = t_3t_4t_3t_4t_2t_3$ and $\sigma_2 = t_1t_3t_4t_3^3t_2t_1$. Compute the markings m_i which are reached after the firing of the transition sequences $\sigma_i, i = 1, 2$, in the Petri net $\mathcal{N}_1$ starting at m_0 by means of its state equations. What can you say about the reachability of m_1 and m_2 in $\mathcal{N}_1$?

(c) Compute the P- and T-invariants for the Petri net $\mathcal{N}_1$ if it has any. Give a feasible T-invariant if there is one.

(d) Compute the reachability graph of $\mathcal{N}_1$. Is $\mathcal{N}_1$ live?

Exercise 2.2

Let $\mathcal{N}_2 = (P, T, F, V, m_0)$ be the following Petri net:

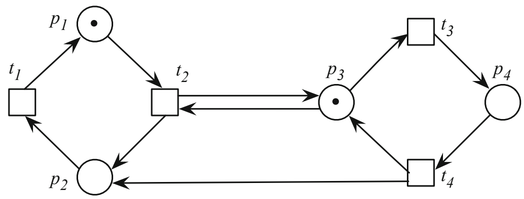

Figure 2.5: The Petri net $\mathcal{N}_2$

(a) Show by means of the state equation that the empty marking $(0, 0, 0, 0)$ is not reachable in $\mathcal{N}_2$.

(b) Compute the coverability graph of $\mathcal{N}_2$. Is $\mathcal{N}_2$ bounded?

Exercise 2.3

(a) Is it always possible to show the non-reachability of an arbitrary marking in a Petri net using the state equation? Give a proof of your answer.

(b) Is the liveness of an unbounded Petri net decidable from its coverability graph?

Exercise 2.4

Let $\mathcal{PN}$ be the set of all unmarked Petri nets and let $\mathcal{M}$ be the set of all (finite) matrices over the natural numbers.

(a) Is the following mapping Φ bijective:

$$\Phi : \mathcal{PN} \longrightarrow \mathcal{M}, \text{ where } \Phi(\mathcal{N}) := C_{\mathcal{N}}?$$

Justify your answer.

(b) If Φ is not bijective in general then give, if possible, a subset of $\mathcal{PN}$ such that the mapping Φ is bijective on this subset. Justify your answer.

Exercise 2.5

An arbitrary Petri net property α is called *monotone* if it holds that: For any Petri net $\mathcal{N} = (P, T, F, V, m_0)$ with the property α, α holds for all Petri nets $\mathcal{N}_k := (P, T, F, V, m_0^k)$ with $m_0^k(p) := k \cdot m_0(p)$ for all $p \in P$.

Is liveness a monotone property? Give a proof.

http://www.springer.com/978-3-642-41114-4

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