Graph coloring

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April 11, 2017

Contents

Chapter 1

Preface

These lecture notes come from a master-level course *Graph coloring*, taught by the first author at the University of Copenhagen in spring 2016.

Despite the name it is *not* a comprehensive course in graph coloring. Instead, graph coloring problems serve only as a convenient excuse to familiarize the students, who may never have taken a more advanced combinatorics course, with interesting combinatorial techniques. The purpose is rather to give a taste of tools from linear algebra, calculus, combinatorics and geometry and demonstrate a few classical combinatorial theorems and their applications. It also has a bit of an experimental flavour and includes short programming exercises in Sage.

We are grateful for the contribution by the students who took notes during the course: Giorgia Laura Cassis, Hugrún Fjóla Hafsteinsdóttir, Rolf Jørgensen, Mathis Elmgaard Isaksen, Sokratis Theodoridis, Mortan Janusarson Thomsen, Kristoffer Holm Nielsen.

The website for this course was `http://aszek.net/chromatic`. The LaTeX source of these lecture notes is available under the GNU GPL license from `http://github.com/aszek/chromatic`.

Chapter 2

Temporary

2.1 Notation

A graph is $G = (V, E)$ and it has $n = |V|$, $m = |E|$. If there are more graphs the next one is H. A coloring with c colors is a function $f : V \to \{1, \ldots, c\} = C$. The chromatic number is χ. A bipartite graph has parts A, B, or X, Y.

2.2 Aha foo bar

This is just a testing ground for now.

Here is ade finition

Definition 1. *We say that a* definition *is a definition is*

$$\int_M d\omega = \int_{\partial M} \omega \tag{2.1}$$

On the other hand, here is a remark:

Remark 2. I would like the contents of a remark not to be italicised.

2.3 Section

It would be good to split each chapter into 2-4 sections.

2.4 Including sage code

We include SAGE source code like this:

```
def FunnyGraph(n):
    c = graphs.CompleteGraph(n)
    c.delete_edges(graphs.CycleGraph(n).edges())
    return graphs.MycielskiStep(c).join(graphs.WheelGraph(n+1))

G = FunnyGraph(99)
```

At the very end I will implement it using a pygmentize highlighter for python.

Figures can be drawn in tikz or whatever, or included from another file. In any case, it would be good to have every figure inside a figure environment with a label and caption.

2.5 TODOs

List of todos for MA:

- Remove this file
- Add pygmentize
- Expand and check bibliography

2.6 Notation

(This section is not temporary. In fact, it will become this whole chapter.)

$G, H, \ldots$	graphs
$G = (V, E)$	vertices and edges of a graph
$\overline{G}$	graph complement
$L(G)$	line graph of G
$G \setminus e$	edge removal
G/e	edge contraction
K_n	n-vertex complete graph
$K_{n,m}$	complete bipartite graph
C_n	n-vertex cycle
P_n	n-vertex path
Q_d	the d-dimensional cube graph
$\omega(G)$	clique number
$\alpha(G)$	independence number
$\chi(G)$	chromatic number
$\chi_l(G)$	list chromatic number
$\chi'(G)$	edge chromatic number
$P_G(t), P(G, t)$	chromatic polynomial
$G(n, p)$	random graph $G(n, p)$
$G + H$	join of graphs
$G \sqcup H$	disjoint union of graphs

Chapter 3

Basic graph theory

Roughly lex1.tex and lec2.tex without defining chromatic number.

Chapter 4

Vertex coloring

Define chromatic number and the lec3.tex, lec4.tex

Chapter 5

Planar graphs

Roughly lec5.tex, lec6.tex

Chapter 6

Chromatic polynomial

Roughly lec7.tex, lec8.tex

Chapter 7

Edge coloring

Roughly lec9.tex, lec10.tex

Chapter 8

Chromatic number of Euclidean spaces

Roughly lec11.tex, lec12.tex, lec13.tex

Chapter 9

Coloring and topology

To be written up, MA

Chapter 10

Exam problems

10.1 Problems

The number of stars indicates the difficulty level.

1. ($\star$) Find the chromatic number of the graph G defined below in Sage:

```
def FunnyGraph(n):
    c = graphs.CompleteGraph(n)
    c.delete_edges(graphs.CycleGraph(n).edges())
    return graphs.MycielskiStep(c).join(graphs.WheelGraph(n+1))

G = FunnyGraph(99)
```

2. ($\star\star$) Consider the following algorithm for vertex coloring. Find the largest independent set of vertices, and color them with color 1. Remove those vertices, find the largest independent set in the remaining graph and color it with color 2, and so on until there are no more vertices left to color. Prove that there are infinitely many graphs G for which this algorithm will use more than $\chi(G)$ colors.

3. ($\star$) Prove that $\max\{\chi(G), \chi(\overline{G})\} \geq \sqrt{|V(G)|}$ for any graph G.

4. ($\star\star$) Let G, H be two graphs. The *substitution of H into G*, denoted $G[H]$, is the graph obtained by replacing every vertex of G with a copy of H, and replacing every original edge of G with a complete bipartite graph between the corresponding copies of H. Formally $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ iff either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. Sage calls this operation `G.lexicographic_product(H)`. Prove that

$$\omega(G)\chi(H) \leq \chi(G[H]) \leq \chi(G)\chi(H).$$

   Find an example with $\chi(G[H]) < \chi(G)\chi(H)$.

5. ($\star\star$) Let $\gcd(a, b)$ denote the greatest common divisor of a and b. Let $n = 20162016$. Define G as the graph with vertex set $\{1, \ldots, n\}$ where two numbers $1 \leq a < b \leq n$ are adjacent if and only if $\gcd(a, b) = 1$. Find the exact value of $\chi(G)$.

6. ($\star$)

   a) Prove or disprove: if the only complex roots of $P_G(t)$ are 0 and 1 then G is a forest.

   b) How many non-isomorphic graphs have chromatic polynomial $t^2(t-1)^8$?

   c) Find all non-isomorphic graphs with chromatic polynomial $t(t-1)^3(t-2)$.

7. ($\star$) A vertex coloring of G will be called *brilliant* if (1) every two adjacent vertices have different colors and (2) every two vertices which have a common neighbour also have different colors. Let $\chi_b(G)$ be the minimal number of colors required for a brilliant coloring of a simple graph G, and let $P_b(G, t)$ be the number of brilliant colorings of G with colors $\{1, \ldots, t\}$.

   Find all graphs G with $\chi_b(G) \leq 2$ and show that $P_b(G, t)$ is a polynomial in t for every graph G.

8. ($\star\star\star$) Prove that $\chi_l(G) + \chi_l(\overline{G}) \leq |V(G)| + 1$ for any graph G, where χ_l is the list chromatic number.

9. ($\star\star$) (This is an experimental problem; formal proofs are not expected.) Let $g(n)$ be the expected number of colors used by the greedy algorithm to color a random graph from $G(n, \frac{1}{2})$.

   - Compute and plot an experimental approximation of $g(n)$ for a sequence of reasonably large values of n, for example $n = 100, 200, \ldots, 2000$.
   - Speculate about the asymptotic behaviour of $g(n)$ as $n \to \infty$. In particular, what do you think about $\lim_{n\to\infty} \frac{g(n)}{n/\log_2 n}$?
   - Find information about the expected value of $\chi(G)$ for $G \in G(n, \frac{1}{2})$. How well does the greedy algorithm perform?

10. ($\star\star\star$) Let G be a nonempty graph. Simplify the expression

$$\sum_I P(G - I, -1)$$

   where the sum runs over all independent sets I in G (including the empty one) and, as always, $G - X$ denotes the subgraph of G induced by the vertex set $V(G) - X$.

11. ($\star\star\star$) A vertex v of a directed graph is called a *source* if all the edges incident to v are pointing out of v. Suppose G is a nonempty graph with n vertices. Prove that the number of acyclic orientations of G having exactly one source equals $n \cdot (-1)^{n-1} \cdot [t]P_G(t)$.

12. ($\star$) Find the edge-chromatic number χ' of `FunnyGraph(99)` from Problem 1.

13. ($\star\star$) In the lectures we defined a family of graphs $Q_d(u, s)$, and we used the fact that they are unit distance graphs in $\mathbb{R}^d$.

    a) Show that each $Q_d(u, s)$ is in fact a unit distance graph in $\mathbb{R}^{d-1}$.
    b) Use the graphs $Q_{10}(u, s)$ to prove $\chi(\mathbb{R}^9) \geq C$ for a constant C as large as you can.

14. ($\star\star$) The *supremum metric* (or ℓ_∞ metric) in $\mathbb{R}^d$, $d \geq 1$ is given by

$$d_\infty((x_1, \ldots, x_d), (y_1, \ldots, y_d)) = \max\{|x_1 - y_1|, \ldots, |x_d - y_d|\}.$$

   Find the smallest number of colors required to color $\mathbb{R}^d$ so that any two points whose distance in the supremum metric equals 1 have different colors.

15. ($\star\star$) Let $P_{2\times n} = P_2 \Box P_n$ be the $2 \times n$ grid graph. Find the number of edge-colorings of $P_{2\times n}$ with 3 colors. ($P_{2\times n}$ is called `graphs.Grid2dGraph(2,n)` in Sage).

10.2 Hints

1. The graph is $M(K_n \setminus C_n) + C_n + K_1$ and so its chromatic number is $(1 + \lceil \frac{n}{2} \rceil) + (2 + (n \mod 2)) + 1$ for $n > 3$.
2. Take two n-vertex stars and connect their central vertices with an edge.
3. $\chi(G)\chi(\overline{G}) \geq \chi(G)\alpha(G) \geq |V(G)|$.

4. For the lower bound G contains an $\omega(G)$-fold join of copies of H. For the upper bound take a cartesian product of colorings of G an H. For the example take $G = C_5$, $H = K_2$.

5. Let $\pi(n)$ be the number of prime numbers up to n. The graph contains a clique of size $1 + \pi(n)$. There is also a coloring with $1 + \pi(n)$ colors: map each number to the smallest prime in its factorization.

6.
   - True, because each connected component has n_i vertices and $n_i - 1$ edges.
   - Count all forests with 10 vertices, 8 edges and 2 connected components.
   - Look among connected graphs with 5 vertices and 5 edges.

7. If $\chi_b(G) \leq 2$ then every connected component of G has at most two vertices. Let G^2 denote the graph where all pairs originally at distance 2 are given an edge. Then $P_b(G,t) = P(G^2,t)$.

8. Induction.

9. The expected value of the chromatic number of $G(n, \frac{1}{2})$ is $(1+o(1))\frac{n}{2\log_2 n}$, while the expected number of colors used by the greedy algorithm is $(1+o(1))\frac{n}{\log_2 n}$. It is hard to observe the asymptotics within the bounds of this problem, though.

10. Prove combinatorially the identity
$$P(G, t+1) = \sum_I P(G - I, t).$$

11. Prove a deletion-contraction rule for $a(G, v_0)$ defined as the number of acyclic orientations in which the fixed vertex v_0 is the unique source.

12. Find the degree sequence. There is only one vertex of maximal degree.

13. All vertices lie in the same $(d-1)$-hyperplane in $\mathbb{R}^d$. We have $\alpha(Q_{10}(4,5)) = 12$, so $\chi(\mathbb{R}^9) \geq \chi(Q_{10}(4,5)) \geq \binom{10}{5}/12 = 21$. This proof comes from `http://arxiv.org/abs/1409.1278`.

14. A coloring $c : \mathbb{R}^d \to \{0,1\}^d$ given by
$$c(x_1, \ldots, x_d) = (\lfloor x_1 \rfloor \bmod 2, \ldots, \lfloor x_d \rfloor \bmod 2)$$
uses 2^d colors and is optimal.

15. There are many ways to observe inductively that $c_n = 2c_{n-1} - 6$ for $n \geq 3$, where $c_n = P(L(P_{2\times n}), 3)$.

Bibliography

[D] R. Diestel, Graph Theory, Graduate Texts in Mathematics 173, Springer–Verlag