Graph coloring

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Contents

Chapter 1

Preface

These lecture notes come from a master-level course *Graph coloring*, taught by the first author at the Department of Mathematical Sciences of the University of Copenhagen in spring 2016.

Despite the name, this is not what one would call a comprehensive course in graph coloring. Instead, graph coloring problems serve only as a convenient excuse to familiarize the students, who may never have taken a more advanced combinatorics course, with interesting combinatorial techniques. The purpose is rather to give a taste of tools from linear algebra, calculus, combinatorics and geometry and demonstrate a few classical combinatorial theorems and their applications. It also has a bit of an experimental flavour and includes short programming exercises in Sage.

Each chapter ends with a series of exercises, preferably to be solved with the guidance of an instructor. The last chapter contains exam problems with short hints. This division does not imply a gradation of difficulty — some of the chapter exercises are quite challenging!

We are grateful for the contribution by the students who took notes during the course: Giorgia Laura Cassis, Hugrún Fjóla Hafsteinsdóttir, Rolf Jørgensen, Mathis Elmgaard Isaksen, Sokratis Theodoridis, Mortan Janusarson Thomsen, Kristoffer Holm Nielsen. Most of these notes are based on their writeup.

Chapter 2

Temporary

2.1 Notation

A graph is $G = (V, E)$ and it has $n = |V|$, $m = |E|$. If there are more graphs the next one is H. A coloring with c colors is a function $f : V \to \{1, \dots, c\} = C$. The chromatic number is χ. A bipartite graph has parts A, B, or X, Y.

2.2 Aha foo bar

This is just a testing ground for now.

Here is ade finition

Definition 2.1. *We say that a* definition *is a definition is*

$$\int_M d\omega = \int_{\partial M} \omega \tag{2.1}$$

On the other hand, here is a remark:

Remark 2.2. I would like the contents of a remark not to be italicised.

2.3 Section

It would be good to split each chapter into 2-4 sections.

2.4 Including sage code

We include SAGE source code like this:

```
def FunnyGraph(n):
    c = graphs.CompleteGraph(n)
    c.delete_edges(graphs.CycleGraph(n).edges())
    return graphs.MycielskiStep(c).join(graphs.WheelGraph(n+1))

G = FunnyGraph(99)
```

At the very end I will implement it using a pygmentize highlighter for python.

Figures can be drawn in tikz or whatever, or included from another file. In any case, it would be good to have every figure inside a figure environment with a label and caption.

2.5 TODOs

List of todos for MA:

- Remove this file
- Add pygmentize
- Expand and check bibliography

2.6 Notation

(This section is not temporary. In fact, it will become this whole chapter.)

$G, H, \ldots$	graphs
$G = (V, E)$	vertices and edges of a graph
$\overline{G}$	graph complement
$L(G)$	line graph of G
$G \setminus e$	edge removal
G/e	edge contraction
K_n	n-vertex complete graph
$K_{n,m}$	complete bipartite graph
C_n	n-vertex cycle
P_n	n-vertex path
Q_d	the d-dimensional cube graph
$M(G)$	the Mycielski construction on G
$\Delta(G)$	maximal vertex degree in G
$\delta(G)$	minimal vertex degree in G
$\omega(G)$	clique number
$\alpha(G)$	independence number
$\chi(G)$	chromatic number
$\chi_l(G)$	list chromatic number
$\chi'(G)$	edge chromatic number
$P_G(t), P(G, t)$	chromatic polynomial
$G(n, p)$	random graph $G(n, p)$
$G + H$	join of graphs
$G \sqcup H$	disjoint union of graphs
$\mathbb{R}, \mathbb{Z}$	real numbers, integers

Chapter 3

Basic graph theory

Roughly lex1.tex and lec2.tex without defining chromatic number.

Chapter 4

Vertex coloring

Define chromatic number and the lec3.tex, lec4.tex

Chapter 5

Planar graphs

Roughly lec5.tex, lec6.tex

Chapter 6

Chromatic polynomial

Roughly lec7.tex, lec8.tex

Chapter 7

Edge coloring

Roughly lec9.tex, lec10.tex

Chapter 8

Chromatic number of Euclidean spaces

8.1 Chromatic number of Euclidean spaces

In this chapter we are going to work with infinite graphs. Let $d(x,y) = \sqrt{\sum_i (x_i - y_i)^2}$ denote the Euclidean distance.

Definition 8.1. *$\chi(\mathbb{R}^d)$ is the minimal number of colors required to color all points in $\mathbb{R}^d$ so that if $d(x,y) = 1$ then x, y have different colors.*

Definition 8.2. *For $X \subset \mathbb{R}^d$ define a graph U_X (U for "unit") with vertex set X and edges*

$$x_1 x_2 \in E(U_X) \textit{ iff } d(x_1, x_2) = 1.$$

Of course this definition is chosen so that $\chi(\mathbb{R}^d) = \chi(U_{\mathbb{R}^d})$

Example 8.3. $U_{\mathbb{R}}$ is a union of infinitely many (uncountably many) bi-infinite paths. As a consequence $\chi(U_{\mathbb{R}}) = \chi(\mathbb{R}) = 2$.

Remark 8.4. All invariants $\omega, \chi, \alpha, \Delta, \ldots$ we defined in previous lectures make sense for infinite graphs, except that they might be equal to ∞. Inequalities such as $\omega(G) \leq \chi(G)$ and $H \subset G \Rightarrow \chi(H) \leq \chi(G)$ etc. still hold.

8.2 Compactness

Here is a question we should probably address before seriously considering the chromatic number of infinite graphs: how does the chromatic number of G relate to those of its finite subgraphs?

Theorem 8.5. *Suppose G is a graph (which may be infinite). If every* finite *subgraph of G can be colored with k colors, then G can be colored with k colors.*

Proof. Let $G = (V, E)$ be a graph and let X be the set of all functions $f \colon V \to \{1, \ldots, k\}$, i.e. $X = \prod_{v \in V} \{1, \ldots, k\} = \{1, \ldots, k\}^V$. View $\{1, \ldots, k\}$ as a discrete topological space and equip X with the product topology. $\{1, \ldots, k\}$ is finite, so it is compact. By Tychonoff's theorem X is compact. For any $F \subset E$ let $X_F \subset X$ be defined as those $f \colon V \to \{1, \ldots, k\}$ which are proper colorings of (V, F).

- $X_{\{e\}}$ is closed in X since

  $$X_{\{e\}} = \bigcup_{i \neq j} \{f \in X : f(u) = i, f(v) = j, e = uv\}$$

  is a finite union of closed sets.

- $X_{F_1} \cap X_{F_2} = X_{F_1 \cup F_2}$.

- For any $F \subset E$, X_F is closed since $X_F = \bigcap_{e \in F} X_{\{e\}}$, is an intersection of closed sets, hence closed.

Now: Take the family $\mathcal{F} = \{X_F\}_{\substack{F \subset E \\ F \text{ finite}}}$. All sets in $\mathcal{F}$ are closed, and all intersections of finitely many from $\mathcal{F}$ are non-empty (second claim: $X_{F_1} \cap \cdots \cap X_{F_n} = X_{F_1 \cup \cdots \cup F_n} \neq \emptyset$ because $(V, F_1 \cup \cdots \cup F_n)$ is finite, hence k-colorable) Then the intersection of all sets in $\mathcal{F}$ is non-empty (by compactness of X). $f \in \bigcap_{\substack{F \subset E \\ |F| < \infty}} X_F$ is a proper coloring on *every* edge of G. ☐

8.3 Chromatic number of $\mathbb{R}^2$

Let us now discuss colorings of the unit graph $U_{\mathbb{R}^2}$ of the plane. Here is an easy upper bound.

Lemma 8.6. $\chi(\mathbb{R}^2) \leq 9$.

Proof. Take the 3×3-square where the length of the diagonal in each of the 9 parts is 0.99. Color every such square with 9 colors (choose any color on the common edges). Use this square to tile the plane. Now take two points x, y of the same color. Then

- either x, y are in the same small square and so $d(x, y) \leq 0.99$, or
- x, y are in two different big squares and $d(x, y) \geq 2 \cdot 0.99 \cdot 1/\sqrt{2} > 1$

so $d(x, y) \neq 1$. ☐

Next we prove the state-of-the-art bounds $4 \leq \chi(\mathbb{R}^2) \leq 7$.

Proposition 8.7. $\chi(\mathbb{R}^2) \leq 7$.

Proof. Consider a covering of $\mathbb{R}^2$ with squares of diagonal 1 (that is, of side $1/\sqrt{2}$):
[FIGURE]
Use the colors as indicated, remembering to color the interior of the square, its top-right corner, the top edge without the top-left corner and the right edge without the bottom-right corner. ☐

Remark 8.8. Another coloring uses a covering by hexagons. Each small hexagon should have diameter 0.99:
[FIGURE]

Proposition 8.9. $\chi(\mathbb{R}^2) \geq 4$.

Proof. Suppose, on the contrary, that $c : \mathbb{R}^2 \to \{1, 2, 3\}$ is a coloring of $U_{\mathbb{R}^2}$ with 3 colors. I claim that

$$d(x, y) = \sqrt{3} \implies c(x) = c(y).$$

Indeed, if $d(x, y) = \sqrt{3}$ then there are points z, t with $d(x, z) = d(y, z) = d(x, t) = d(y, t) = d(z, t) = 1$, because x, y are the "opposite" vertices of two equilateral triangles joined at one base. Since $c(x), c(z), c(t)$ are all different and so are $c(y), c(z), c(t)$, we conclude $c(x) = c(y)$.

It means that for any $x \in \mathbb{R}^2$, all points on the circle centered at x of radius $\sqrt{3}$ have the same color. But on that circle we can find two points in distance 1, contradiction. ☐

Remark 8.10. The proof above can be turned into a construction of a finite subgraph H of $U_{\mathbb{R}^2}$ with $\chi(H) = 4$. It is called the Moser graph, left. Another graph with this property is the Golomb graph, right.

[FIGURE], [FIGURE]

Remark 8.11. Surprising as it may sound, the bounds $4 \leq \chi(\mathbb{R}^2) \leq 7$ are all that we know in general about $\chi(\mathbb{R}^2)$.

We can now move on to higher dimensions, where the gaps in our knowledge are even bigger. For example, it is only known that

$$6 \leq \chi(\mathbb{R}^3) \leq 15.$$

However, the rate of growth of $\chi(\mathbb{R}^d)$ as $d \to \infty$ is generally understood to be exponential.

Theorem 8.12. *There are constants $1 < c_1 < c_2$ such that for all d:*

$$c_1^d \leq \chi(\mathbb{R}^d) \leq c_2^d.$$

The current best are $c_1 \approx 1.23$ and $c_2 = 3 + \varepsilon$. We are not going to prove the theorem with the optimal constants, but we are going to show some weaker exponential upper and lower bounds. There will be some especially nice mathematics involved in the lower bounds, in particular!

8.4 Upper bound on $\chi(\mathbb{R}^d)$

We start with an upper bound.

Theorem 8.13. *For sufficiently large d we have $\chi(\mathbb{R}^d) \leq 14^d$.*

Proof. We will tile $\mathbb{R}^d$ with small cubes, and color each cube with one color, similarly to the 3×3 strategy used to show $\chi(\mathbb{R}^2) \leq 9$. As you can prove in one of the exercises, repetitive coloring may not work for $d \geq 4$. Instead, we will color the small cubes greedily.

For simplicity, suppose first that $d = 2k$. We need two prerequisites: the formula for the volume of the d-dimensional ball $B_d(x, r)$ with center x and radius r for $d = 2k$ is

$$\text{vol}(B_d(x, r)) = \frac{\pi^k}{k!} r^{2k}.$$

We will also need the inequality $k! \geq (k/e)^k$, or equivalently $k^k/k! \leq e^k$.

Divide $\mathbb{R}^d$ into "small cubes" of size

$$0.99\frac{1}{\sqrt{d}} \times 0.99\frac{1}{\sqrt{d}} \times \cdots \times 0.99\frac{1}{\sqrt{d}}.$$

Each cube has diameter (main diagonal) $0.99 < 1$ and volume $0.99^d d^{-d/2} = 0.99^{2k}(2k)^{-k}$. Denote any such small cube by C.

Now we ask: how many small cubes are completely contained in any ball $B_d(x, 3)$ of radius 3? Comparing volumes gives that this number is *at most*

$$\frac{\text{vol}(B_d(x,3))}{\text{vol}(C)} = \frac{3^{2k}\pi^k(2k)^k}{k!0.99^{2k}} = \frac{k^k}{k!} \cdot (\frac{18\pi}{0.99^2})^k \leq (\frac{18\pi e}{0.99^2})^k < 163^k < 13^d.$$

Now order the (countably many) small cubes into a sequence $C_1, C_2, \ldots$ and let x_i be the center of C_i. We color each C_i according to the greedy rule: choose any color that is not used for cubes C_j, $j < i$ such that $C_j \subseteq B_d(x_i, 3)$. By the previous observation there are at most $13^d - 1$ cubes C_j we have to consider, and there always is a spare color so that the greedy algorithm will do with at most 13^d colors. (It does not matter which color we use on points common to more than one cube, for example we can color closed cubes and repaint anything that is already colored).

Now, two points inside one small cube are in distance at most $0.99 < 1$. Consider two points x, y with $d(x, y) = 1$ and let $x \in C_i$, $y \in C_j$, with $i > j$. By the triangle inequality we have that for any point $z \in C_j$

$$d(x_i, z) \leq d(x_i, x) + d(x, y) + d(y, z) \leq 0.99 + 1 + 0.99 < 3$$

so $C_j \subseteq B_d(x_i, 3)$. It means that the greedy algorithm used different colors for $x \in C_i$ and $y \in C_j$, and so we showed $\chi(\mathbb{R}^d) \leq 13^d$.

If d is odd and large enough then $\chi(\mathbb{R}^d) \leq \chi(\mathbb{R}^{d+1}) \leq 13^{d+1} < 14^d$ by what we already showed. $\square$

8.5 Cube-line unit distance graphs in $\mathbb{R}^d$

We would like to move on to lower bounds. Clearly, we have $\chi(\mathbb{R}^d) \geq \omega(U_{\mathbb{R}^d}) = d + 1$, but that is far from exponential.

Definition 8.14. *We say a finite graph G is a* unit distance graph in $\mathbb{R}^d$ *if there is a set $X \subseteq \mathbb{R}^d$ with $|X| = |V(G)|$ for which G is isomorphic to U_X.*

Lemma 8.15. *If G is a unit distance graph in $\mathbb{R}^d$ then $\chi(\mathbb{R}^d) \geq \chi(G)$.*

Proof. For X as in the definition, we have $U_X \subseteq U_{\mathbb{R}^d}$, so $\chi(\mathbb{R}^d) = \chi(U_{\mathbb{R}^d}) \geq \chi(U_X) = \chi(G)$. $\square$

Example 8.16.

- The Moser and Golomb graphs are unit distance graphs in $\mathbb{R}^2$.
- The d-cube graph Q_d is a unit distance graph in $\mathbb{R}^d$. However, $\chi(Q_d) = 2$, so it does not provide useful lower bounds.
- Take the vertices of the 3-cube Q_3, but this time instead of the edges of the cube, take a graph formed by all the diagonals of the faces of the cube. There are 12 of them, each of length $\sqrt{2}$, so they form a unit distance graph in $\mathbb{R}^3$ (after rescaling by $1/\sqrt{2}$). We see that this graph is isomorphic to $K_4 \sqcup K_4$, so it yields $\chi(\mathbb{R}^3) \geq 4$. We knew that already, but it is better than with the standard cube.

Definition 8.17. *For $1 \leq u \leq d$ let $Q_d(u)$ be the graph whose vertices are all binary sequences of length d:*

$$V(Q_d(u)) = \{(x_1, \ldots, x_d) \ : \ x_i \in \{0, 1\}\}$$

and two sequences are adjacent in $Q_d(u)$ if and only if they differ in exactly u positions.

Example 8.18.

- $Q_d(1) = Q_d$.
- $Q_3(2) = K_4 \sqcup K_4$ is the graph from the previous example.
- $Q_d(d)$ is a disjoint union of 2^{d-1} copies of K_2.

Lemma 8.19. *Each $Q_d(u)$ is a unit distance graph in $\mathbb{R}^d$.*

Proof. Every vertex of $Q_d(u)$ can be treated as a point in $\mathbb{R}^d$ with the same coordinates. If $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ are sequences of 0s and 1s which differ in exactly u positions, then their Euclidean distance is $\sqrt{u}$. $\square$

The graphs $Q_d(u)$ give pretty good lower bounds on $\chi(\mathbb{R}^d)$ already for small d. Here are results which can be verified using Sage.

[CODE]

- $\chi(Q_5(2)) = 8$. Consequently, $\chi(\mathbb{R}^5) \geq 8$. The best known lower bound is 9.
- $\alpha(Q_{10}(4)) = 40$ (this will take about 20min in Sage). Consequently

  $$\chi(\mathbb{R}^{10}) \geq \chi(Q_{10}(4)) \geq \frac{|V(Q_{10}(4))|}{\alpha(Q_{10}(4))} = \frac{2^{10}}{40} = 25.6,$$

  that is $\chi(\mathbb{R}^d) \geq 26$. This is the best known bound!

In order to prove some lower bounds valid for all d we need to add a further complication to $Q_d(u)$.

Definition 8.20. *The graph $Q_d(u, s) \subseteq Q_d(u)$ is the subgraph of $Q_d(u)$ induced by the vertices with exactly s coordinates equal to 1. Precisely:*

$$V(Q_d(u, s)) = \{\overline{x} = (x_1, \ldots, x_d) \ : \ x_i \in \{0, 1\}, \ \sum_{i=1}^{d} x_i = s\}$$

and $\overline{x}$ and $\overline{y}$ are adjacent in $Q_d(u, s)$ iff they differ in exactly u positions.

Example 8.21. $Q_3(2, 1)$ has vertex set $\{001, 010, 100\}$ and it is isomorphic to K_3.

8.6 Lower bound on $\chi(\mathbb{R}^d)$

As in the computational examples above, it is usually easier to say something about the independence number α than directly about the chromatic number χ. Our main theorem, which we will prove in the next part of the lecture, is the following.

Theorem 8.22. *If p is a prime then*

$$\alpha(Q_d(2p, 2p-1)) \leq \binom{d}{0} + \binom{d}{1} + \cdots + \binom{d}{p-1}.$$

We will prove this theorem in a moment. Let us just note that the condition "p is a prime" suggests that this fact is somewhat algebraic in nature. For now, let us see what this theorem buys us when it comes to chromatic numbers.

Theorem 8.23. *We have $\chi(\mathbb{R}^d) \geq 1.05^d$ for sufficiently large d.*

Proof. For any prime $p \leq d/2$ we have

$$\chi(\mathbb{R}^d) \geq \chi(Q_d(2p)) \geq \chi(Q_d(2p, 2p-1)) \geq \frac{|V(Q_d(2p, 2p-1))|}{\alpha(Q_d(2p, 2p-1))} \geq \frac{\binom{d}{2p-1}}{p\binom{d}{p-1}}$$

where in the last step we used the inequality of Theorem 8.22 and the observation $|V(Q_d(u, s))| = \binom{d}{s}$.

Intuitively, the last fraction will be maximized if the binomial coefficient $\binom{d}{2p-1}$ is close to the middle of the d-th row of the Pascal triangle, that is when $p \approx d/4$. Since we can only use p primes, we resort to a classical number-theoretic result of Czebyschev: every interval $[n, 2n]$ contains a prime. That allows us to choose a prime p such that $\frac{d}{8} \leq p \leq \frac{d}{4}$. By carefully cancelling common factors in the binomial coefficients we obtain:

$$\chi(\mathbb{R}^d) \geq \frac{1}{p} \cdot \frac{d-p+1}{2p-1} \cdot \frac{d-p}{2p-2} \cdots \frac{d-2p+2}{p}.$$

Under the condition $d \geq 4p$ each of the last p factors is $\geq \frac{3}{2}$, so:

$$\chi(\mathbb{R}^d) \geq \frac{1}{p}\left(\frac{3}{2}\right)^p \geq \frac{4}{d}\left(\left(\frac{3}{2}\right)^{\frac{1}{8}}\right)^d \geq \frac{4}{d} \cdot 1.051^d \geq 1.05^d$$

where the last inequality holds for sufficiently large d. □

8.7 Linear algebra in action

Let us now review two combinatorial methods of proving inequalities like $A \leq B$, where A, B are some combinatorially defined quantities.

Method 1 — set comparison. If a set of size B contains a subset of size A then $A \leq B$.

Example 8.24. We will show that $\binom{n}{k} \leq 2^n$. The family of all subsets of $\{1, \ldots, n\}$ has size 2^n, and it contains the family of all k-element subsets, the latter of size $\binom{n}{k}$. Our inequality follows.

That was an easy and completely standard argument. Our next method is also based on an elementary observation in linear algebra.

Method 2 — vector space comparison. If a vector space of dimension B contains A linearly independent vectors then $A \leq B$.

This may seem like an overkill, but it is actually a useful strategy in many otherwise complicated situations (like our Theorem 3). Here is an example of how the method works: the (rather classical) problem known as Odd–Town.

Example 8.25. n people participate in m clubs. Every club has an odd number of members, and every two clubs have an even number of common members. Prove that $m \leq n$.

First let's note that we may have $m = n$, for example when every person forms its own one-element club.

To solve the problem, encode the clubs $C_1, \ldots, C_m$ via "membership vectors" $\overline{c_1}, \ldots, \overline{c_m}$ of length n, where

$$(\overline{c_i})_j = \begin{cases} 1 & \text{if person } j \text{ belongs to club } i, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n$. If we write $\langle \overline{x}, \overline{y} \rangle = \sum_i x_i y_i$ for the standard inner product, then

$$\begin{aligned} \langle \overline{c_i}, \overline{c_k} \rangle &= \text{number of common members of } C_i \text{ and } C_k, \\ \langle \overline{c_i}, \overline{c_i} \rangle &= \text{number of members of } \ C_i. \end{aligned}$$

We will show that $\overline{c_1}, \ldots, \overline{c_m}$ are linearly independent. Suppose, for a contradiction, that it is not true. Then we have a linear relation

$$\sum_i a_i \overline{c_i} = 0$$

where not all a_i are zero. Since the coordinates of $\overline{c_i}$ are integers, we can assume that all $a_i \in \mathbb{Z}$ and moreover $\gcd(a_1, \ldots, a_m) = 1$. In particular, a_k is odd for some k. Now:

$$0 = \langle \sum_i a_i \overline{c_i}, \overline{c_k} \rangle = a_k \langle \overline{c_k}, \overline{c_k} \rangle + \sum_{i \neq k} a_i \langle \overline{c_i}, \overline{c_k} \rangle$$

which is a contradiction, because $a_k \langle \overline{c_k}, \overline{c_k} \rangle$ is odd, while all the other terms are even.

We showed that $\overline{c_1}, \ldots, \overline{c_m}$ are linearly independent vectors in $\mathbb{R}^n$. It follows that $m \leq n$.

Very similar arguments will now appear in the proof Theorem 8.22.

Proof of Theorem 8.22. As always, we write $\langle \overline{x}, \overline{y} \rangle = \sum_{i=1}^d x_i y_i$. Let $\overline{x}$ and $\overline{y}$ be two different vertices of $Q_d(2p, 2p-1)$. Using the fact that both $\overline{x}$ and $\overline{y}$ have exactly $2p-1$ coordinates equal to 1, we easily get

$$|\{j \ : \ x_j \neq y_j\}| = 2(2p - 1 - |\{j \ : \ x_j = y_j = 1\}|) = 2(2p - 1 - \langle \overline{x}, \overline{y} \rangle),$$

hence

$$\langle \overline{x}, \overline{y} \rangle = 2p - 1 - \frac{1}{2}|\{j \ : \ x_j \neq y_j\}|.$$

Now if $\overline{x}$ and $\overline{y}$ are adjacent in $Q_d(2p, 2p-1)$ then they differ in exactly $2p$ places, and we get $\langle \overline{x}, \overline{y} \rangle = 2p - 1 - p = p - 1$. Otherwise we get some other inner product between 0 and $2p-2$ (because $\overline{x} \neq \overline{y}$). The upshot is that

$$\langle \overline{x}, \overline{y} \rangle \begin{cases} = p - 1 & \text{if } \overline{xy} \in E(Q_d(2p, 2p-1)), \\ \not\equiv p - 1 \pmod p & \text{if } \overline{xy} \notin E(Q_d(2p, 2p-1)). \end{cases}$$

Moreover $\langle \overline{x}, \overline{x} \rangle = 2p - 1$ for all $\overline{x}$.

Take any independent set I in $Q_d(2p, 2p-1)$. For any $\overline{x} \in I$ consider the function $f_{\overline{x}} : \{0,1\}^d \to \mathbb{R}$ defined for $\overline{t} = (t_1, \ldots, t_d)$ by the formula

$$f_{\overline{x}}(\overline{t}) = \langle \overline{x}, \overline{t} \rangle^{\underline{p-1}}$$

(recall that $z^{\underline{p-1}} = z(z-1)\cdots(z-(p-2))$ is the falling factorial). The functions $f_{\overline{x}}$ are naturally elements of the $\mathbb{R}$-vector space of all functions $\{0,1\}^d \to \mathbb{R}$. Let us check that the set $\{f_{\overline{x}}\}_{\overline{x} \in I}$ is linearly independent in that space. If not, then we would have a linear relation

$$\sum_{\overline{x} \in I} a_{\overline{x}} f_{\overline{x}} = 0$$

for $a_{\overline{x}}$ not all zero. As in the example before, we can assume that $a_{\overline{x}} \in \mathbb{Z}$ and $\gcd(a_{\overline{x}}) = 1$. In particular, some $a_{\overline{x_0}}$ is not divisible by p. We have

$$0 = \sum_{\overline{x} \in I} a_{\overline{x}} f_{\overline{x}}(\overline{x_0}) = a_{\overline{x_0}} \langle \overline{x_0}, \overline{x_0} \rangle^{\underline{p-1}} + \sum_{I \ni \overline{x} \neq \overline{x_0}} a_{\overline{x}} \langle \overline{x}, \overline{x_0} \rangle^{\underline{p-1}}.$$

We have $\langle \overline{x_0}, \overline{x_0} \rangle^{\underline{p-1}} = (2p-1)(2p-2)\cdots(p+1) \neq 0 \pmod p$. Here we use that p is a prime! Since I is an independent set, each $\langle \overline{x}, \overline{x_0} \rangle$ is different from $p-1 \pmod p$, hence one of the factors in the falling factorial formula for $\langle \overline{x}, \overline{x_0} \rangle^{\underline{p-1}}$ is divisible by p. That is a contradiction, since all the terms in the formula above are now divisible by p except for the first one.

We would now like to know $\dim(\mathrm{span}\{f_{\overline{x}}\}_{\overline{x} \in I})$. A more explicit representation of $f_{\overline{x}}$

$$f_{\overline{x}}(t_1, \ldots, t_d) = \Big(\sum x_i t_i\Big)\Big(\sum x_i t_i - 1\Big)\cdots\Big(\sum x_i t_i - (p-2)\Big)$$

reveals, after opening the brackets, that $f_{\overline{x}}$ is a linear combination of monomials of degree at most $p-1$ in the d variables $t_1, \ldots, t_d$. Since $t_i \in \{0,1\}$, we have $t_i^2 = t_i$, so $f_{\overline{x}}$ is in fact equal to a linear combination of square-free monomials of degree at most $p-1$ in d variables. The dimension of the vector space of such functions is $\binom{d}{0} + \cdots + \binom{d}{p-1}$, where $\binom{d}{i}$ is the number of square-free monomials of degree i (that is, products of i out of d variables).

To conclude, $\{f_{\overline{x}}\}_{\overline{x} \in I}$ is a set of linearly independent vectors in a vector space of dimension $\binom{d}{0} + \cdots + \binom{d}{p-1}$, which means that $|I| \leq \binom{d}{0} + \cdots + \binom{d}{p-1}$, as we wanted to prove. $\square$

Remark 8.26. The proof above is based on [Mat, Chapter 17].

8.8 Exercises

1. What is $\omega(U_{\mathbb{R}^2})$?

2. What is the length of the side in a d-dimensional cube whose main diagonal has length 1?

3. Recall our naive coloring of $U_{\mathbb{R}^2}$ by translates of a 9-colored 3×3 square, where each small square has diameter almost 1. Does the same strategy work in $\mathbb{R}^3$ and produce a coloring of $U_{\mathbb{R}^3}$ with 27 colors by translates of a 27-colored $3 \times 3 \times 3$ cube? What about $\mathbb{R}^d$?

4. We say that two points A and B of the integer lattice $\mathbb{Z}^2$ *see each other* if the line segment AB contains no other point of $\mathbb{Z}^2$. Find a 4-coloring of $\mathbb{Z}^2$ so that any two points which see each other have different colors. Hint: use colors $00, 01, 10, 11$.

5. Show that the following are equivalent definitions of $Q_d(u)$:

    - The vertices are all vertices of Q_d. Two vertices are adjacent if their distance in Q_d is exactly u.
    - The vertices are all subsets of $\{1, \ldots, d\}$. Two subsets A, B are adjacent if $|A \triangle B| = u$, where $\triangle$ is the symmetric difference $A \triangle B = (A \cup B) \setminus (A \cap B)$
    - The vertices are all integers in $\{0, \ldots, 2^d - 1\}$. Two numbers x, y are adjacent if $x \oplus y$ has exactly u nonzero digits in base 2, where $\oplus$ is bitwise XOR.
    - The vertices are the vertices of the cube $[0,1]^d \subseteq \mathbb{R}^d$. Two of them are adjacent if their Euclidean distance is $\sqrt{u}$.

6. If u is odd then show that $Q_d(u)$ is bipartite (hence $\chi(Q_d(u)) = 2$).

7. If u is even then show that $Q_d(u)$ has at least two connected components.

8. If $Q_d(u)$ has two components, we denote any of them by $Q'_d(u)$. Implement your favourite definition of $Q_d(u)$ in Sage and compute $\chi(Q'_5(2))$ and $\alpha(Q'_{10}(4))$ to prove the results mentioned in this chapter.

Chapter 9

Coloring and topology

9.1 Exercises

1. Can the directed graph formed in the proof of oriented Sperner's lemma really have cycles, or is it just a union of directed paths?

2. Prove that Sperner's lemma and Brouwer's fixed point theorem are equivalent in the following sense: Find a proof of the "classical" version of Sperner's lemma from Brouwer's theorem.

   Hint: Construct a continuous map $f : \Delta \to \Delta$ by defining it on the vertices of the triangulation and extending linearly to the interiors of the edges and triangles of the triangulation. Arrange it so that any fixed point of f must lie inside a 3-colored triangle.

3. Use Sperner's lemma to prove the following classical fact about rectangle dissections:

   Suppose a rectangle is partitioned into smaller rectangles, and that each small rectangle has at least one side of integer length. Prove that the big rectangle also has at least one side of integer length.

   Hint: Subdivide each small rectangle with one diagonal to get a triangulation of the big rectangle. Place everything in a coordinate system with one corner at $(0, 0)$. Color each vertex $P = (x, y)$ with

   - color 1 if $x \in \mathbb{Z}$,
   - color 2 if $x \notin \mathbb{Z}$ and $y \in \mathbb{Z}$,
   - color 3 if $x, y \notin \mathbb{Z}$.

   Prove that there is no triangle with colors $1, 2, 3$. Finally show that assuming both sides of the big rectangle are non-integers contradicts Sperner's lemma.

Chapter 10

Exam problems

10.1 Problems

The number of stars indicates the difficulty level.

1. ($\star$) Find the chromatic number of the graph G defined below in Sage:

```
def FunnyGraph(n):
    c = graphs.CompleteGraph(n)
    c.delete_edges(graphs.CycleGraph(n).edges())
    return graphs.MycielskiStep(c).join(graphs.WheelGraph(n+1))

G = FunnyGraph(99)
```

2. ($\star\star$) Consider the following algorithm for vertex coloring. Find the largest independent set of vertices, and color them with color 1. Remove those vertices, find the largest independent set in the remaining graph and color it with color 2, and so on until there are no more vertices left to color. Prove that there are infinitely many graphs G for which this algorithm will use more than $\chi(G)$ colors.

3. ($\star$) Prove that $\max\{\chi(G), \chi(\overline{G})\} \geq \sqrt{|V(G)|}$ for any graph G.

4. ($\star\star$) Let G, H be two graphs. The *substitution of H into G*, denoted $G[H]$, is the graph obtained by replacing every vertex of G with a copy of H, and replacing every original edge of G with a complete bipartite graph between the corresponding copies of H. Formally $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ iff either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. Sage calls this operation `G.lexicographic_product(H)`. Prove that

$$\omega(G)\chi(H) \leq \chi(G[H]) \leq \chi(G)\chi(H).$$

Find an example with $\chi(G[H]) < \chi(G)\chi(H)$.

5. ($\star\star$) Let $\gcd(a, b)$ denote the greatest common divisor of a and b. Let $n = 20162016$. Define G as the graph with vertex set $\{1, \ldots, n\}$ where two numbers $1 \leq a < b \leq n$ are adjacent if and only if $\gcd(a, b) = 1$. Find the exact value of $\chi(G)$.

6. ($\star$)

   a) Prove or disprove: if the only complex roots of $P_G(t)$ are 0 and 1 then G is a forest.

   b) How many non-isomorphic graphs have chromatic polynomial $t^2(t-1)^8$?

   c) Find all non-isomorphic graphs with chromatic polynomial $t(t-1)^3(t-2)$.

7. ($\star$) A vertex coloring of G will be called *brilliant* if (1) every two adjacent vertices have different colors and (2) every two vertices which have a common neighbour also have different colors. Let $\chi_b(G)$ be the minimal number of colors required for a brilliant coloring of a simple graph G, and let $P_b(G,t)$ be the number of brilliant colorings of G with colors $\{1,\dots,t\}$.

   Find all graphs G with $\chi_b(G) \le 2$ and show that $P_b(G,t)$ is a polynomial in t for every graph G.

8. ($\star\star\star$) Prove that $\chi_l(G) + \chi_l(\overline{G}) \le |V(G)| + 1$ for any graph G, where χ_l is the list chromatic number.

9. ($\star\star$) (This is an experimental problem; formal proofs are not expected.) Let $g(n)$ be the expected number of colors used by the greedy algorithm to color a random graph from $G(n, \frac{1}{2})$.

   - Compute and plot an experimental approximation of $g(n)$ for a sequence of reasonably large values of n, for example $n = 100, 200, \dots, 2000$.
   - Speculate about the asymptotic behaviour of $g(n)$ as $n \to \infty$. In particular, what do you think about $\lim_{n\to\infty} \frac{g(n)}{n/\log_2 n}$?
   - Find information about the expected value of $\chi(G)$ for $G \in G(n, \frac{1}{2})$. How well does the greedy algorithm perform?

10. ($\star\star\star$) Let G be a nonempty graph. Simplify the expression

$$\sum_I P(G-I,-1)$$

   where the sum runs over all independent sets I in G (including the empty one) and, as always, $G - X$ denotes the subgraph of G induced by the vertex set $V(G) - X$.

11. ($\star\star\star$) A vertex v of a directed graph is called a *source* if all the edges incident to v are pointing out of v. Suppose G is a nonempty graph with n vertices. Prove that the number of acyclic orientations of G having exactly one source equals $n \cdot (-1)^{n-1} \cdot [t]P_G(t)$.

12. ($\star$) Find the edge-chromatic number χ' of `FunnyGraph(99)` from Problem 1.

13. ($\star\star$) In the lectures we defined a family of graphs $Q_d(u,s)$, and we used the fact that they are unit distance graphs in $\mathbb{R}^d$.

    a) Show that each $Q_d(u,s)$ is in fact a unit distance graph in $\mathbb{R}^{d-1}$.

    b) Use the graphs $Q_{10}(u,s)$ to prove $\chi(\mathbb{R}^9) \ge C$ for a constant C as large as you can.

14. ($\star\star$) The *supremum metric* (or ℓ_∞ metric) in $\mathbb{R}^d$, $d \ge 1$ is given by

$$d_\infty((x_1,\dots,x_d),(y_1,\dots,y_d)) = \max\{|x_1-y_1|,\dots,|x_d-y_d|\}.$$

   Find the smallest number of colors required to color $\mathbb{R}^d$ so that any two points whose distance in the supremum metric equals 1 have different colors.

15. ($\star\star$) Let $P_{2\times n} = P_2 \square P_n$ be the $2\times n$ grid graph. Find the number of edge-colorings of $P_{2\times n}$ with 3 colors. ($P_{2\times n}$ is called `graphs.Grid2dGraph(2,n)` in Sage).

10.2 Hints

1. The graph is $M(K_n \setminus C_n) + C_n + K_1$ and so its chromatic number is $(1+\lceil \frac{n}{2} \rceil) + (2 + (n \mod 2)) + 1$ for $n > 3$.

2. Take two n-vertex stars and connect their central vertices with an edge.

3. $\chi(G)\chi(\overline{G}) \ge \chi(G)\alpha(G) \ge |V(G)|$.

4. For the lower bound G contains an $\omega(G)$-fold join of copies of H. For the upper bound take a cartesian product of colorings of G an H. For the example take $G = C_5$, $H = K_2$.

5. Let $\pi(n)$ be the number of prime numbers up to n. The graph contains a clique of size $1 + \pi(n)$. There is also a coloring with $1 + \pi(n)$ colors: map each number to the smallest prime in its factorization.

6.
   - True, because each connected component has n_i vertices and $n_i - 1$ edges.
   - Count all forests with 10 vertices, 8 edges and 2 connected components.
   - Look among connected graphs with 5 vertices and 5 edges.

7. If $\chi_b(G) \leq 2$ then every connected component of G has at most two vertices. Let G^2 denote the graph where all pairs originally at distance 2 are given an edge. Then $P_b(G,t) = P(G^2,t)$.

8. Induction.

9. The expected value of the chromatic number of $G(n, \frac{1}{2})$ is $(1 + o(1))\frac{n}{2\log_2 n}$, while the expected number of colors used by the greedy algorithm is $(1+o(1))\frac{n}{\log_2 n}$. See [Kri02] for a nice introduction. It is hard to observe the asymptotics within the bounds of this problem, though.

10. Prove combinatorially the identity
$$P(G, t+1) = \sum_I P(G - I, t).$$

11. Prove a deletion-contraction rule for $a(G, v_0)$ defined as the number of acyclic orientations in which the fixed vertex v_0 is the unique source.

12. Find the degree sequence. There is only one vertex of maximal degree.

13. All vertices lie in the same $(d-1)$-hyperplane in $\mathbb{R}^d$. We have $\alpha(Q_{10}(4,5)) = 12$, so $\chi(\mathbb{R}^9) \geq \chi(Q_{10}(4,5)) \geq \binom{10}{5}/12 = 21$. This proof comes from [KT15].

14. A coloring $c : \mathbb{R}^d \to \{0,1\}^d$ given by
$$c(x_1, \ldots, x_d) = (\lfloor x_1 \rfloor \bmod 2, \ldots, \lfloor x_d \rfloor \bmod 2)$$
uses 2^d colors and is optimal.

15. There are many ways to observe inductively that $c_n = 2c_{n-1} - 6$ for $n \geq 3$, where $c_n = P(L(P_{2\times n}), 3)$.

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