## MSBD 5004 Mathematical Methods for Data Analysis Homework 4

Due date: 1 May, 11:59pm, Friday

- 1. Let V be a Hilbert space. Let  $a \in V$  be a given vector. The function  $Ax := \langle a, x \rangle$  can be viewed as a linear transformation from V to  $\mathbb{R}$ . Find the operator norm ||A||.
- 2. Let  $f_1, f_2, \ldots, f_n$  are differentiable functions from  $V \mapsto \mathbb{R}$  with V a Hilbert space. Define  $F: V \mapsto \mathbb{R}^n$  by

$$F(oldsymbol{x}) = egin{bmatrix} f_1(oldsymbol{x}) \ f_2(oldsymbol{x}) \ dots \ f_n(oldsymbol{x}) \end{bmatrix}, & orall oldsymbol{x} \in V.$$

Prove that

$$DF(oldsymbol{x})(oldsymbol{y}) = egin{bmatrix} \langle 
abla f_1(oldsymbol{x}), oldsymbol{y} \\ \langle 
abla f_2(oldsymbol{x}), oldsymbol{y} \\ dots \\ \langle 
abla f_n(oldsymbol{x}), oldsymbol{y} 
angle \end{bmatrix}, \quad orall oldsymbol{x} \in V.$$

- 3. Find  $\nabla f(\boldsymbol{x})$  and  $\nabla^2 f(\boldsymbol{x})$ .
  - (a)  $f(\boldsymbol{x}) = \frac{1}{2} \|\boldsymbol{A}\boldsymbol{x} \boldsymbol{b}\|_2^2 + \lambda \|\boldsymbol{x}\|_2^2$ , where  $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ ,  $\boldsymbol{b} \in \mathbb{R}^m$ , and  $\lambda > 0$  are given.
  - (b)  $f(\mathbf{X}) = \mathbf{b}^T \mathbf{X} \mathbf{c}$ , where  $\mathbf{X} \in \mathbb{R}^{n \times n}$  and  $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ .
  - (c)  $f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$ , where  $\mathbf{x} \in \mathbb{R}^n$ , and  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is non-symmetric.
  - (d)  $f(X) = \mathbf{b}^T X^T X \mathbf{c}$ , where  $X \in \mathbb{R}^{n \times n}$  and  $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ .
  - (e) f(X) = trace(XAXB), where  $X, A, B \in \mathbb{R}^{n \times n}$ .