MSBD 5004 Mathematical Methods for Data Analysis
Homework 2

Due date: 27 March, 11:59pm, Friday

1. Let $(V, \|\cdot\|)$ be a normed vector space.

   (a) Prove that, for all $\boldsymbol{x}, \boldsymbol{y} \in V$,
   $$|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\|.$$

   (b) Let $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$ be a convergent sequence in V with limit $\boldsymbol{x} \in V$. Prove that
   $$\lim_{k\to\infty} \|\boldsymbol{x}_k\| = \|\boldsymbol{x}\|.$$
   (*Hint: Use part (a).*)

   (c) Let $\{\boldsymbol{x}^{(k)}\}_{k\in\mathbb{N}}$ be a sequence in V and $\boldsymbol{x}, \boldsymbol{y} \in V$. Prove that, if
   $$\boldsymbol{x}^{(k)} \to \boldsymbol{x}, \quad \text{and} \quad \boldsymbol{x}^{(k)} \to \boldsymbol{y},$$
   then $\boldsymbol{x} = \boldsymbol{y}$. (In other words, the limit of the same sequence in a normed vector space is unique.)

2. Let V be a vector space, and $\langle \cdot, \cdot \rangle$ be an inner product on V. Use the definition of inner product to prove the following.

   (a) Prove that $\langle \mathbf{0}, \boldsymbol{x} \rangle = \langle \boldsymbol{x}, \mathbf{0} \rangle = 0$ for any $\boldsymbol{x} \in V$. Here $\mathbf{0}$ is the zero vector in V.

   (b) Prove that the second condition
   $$\langle \alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2, \boldsymbol{y} \rangle = \alpha\langle \boldsymbol{x}_1, \boldsymbol{y} \rangle + \beta\langle \boldsymbol{x}_2, \boldsymbol{y} \rangle, \qquad \forall\ \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y} \in V\ , \alpha, \beta \in \mathbb{R}$$
   is equivalent to
   $$\langle \boldsymbol{x}_1 + \boldsymbol{x}_2, \boldsymbol{y} \rangle = \langle \boldsymbol{x}_1, \boldsymbol{y} \rangle + \langle \boldsymbol{x}_2, \boldsymbol{y} \rangle \quad \text{and} \quad \langle \alpha\boldsymbol{x}, \boldsymbol{y} \rangle = \alpha\langle \boldsymbol{x}, \boldsymbol{y} \rangle, \qquad \forall\ \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}, \boldsymbol{y} \in V,\ \alpha \in \mathbb{R}.$$

3. $\mathbb{R}^{m\times n}$ is a vector space over $\mathbb{R}$. Show that $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \text{trace}(\boldsymbol{A}^T\boldsymbol{B})$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}$ is an inner product on $\mathbb{R}^{m\times n}$. Here $\text{trace}(\cdot)$ is the trace of a matrix, i.e., the sum of all diagonal entries.

4. Consider the polynomial kernel $K(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^T\boldsymbol{y} + 1)^2$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. Find an explicit feature map $\phi\ : \mathbb{R}^2 \to \mathbb{R}^6$ satisfying $\langle \phi(\boldsymbol{x}), \phi(\boldsymbol{y}) \rangle = K(\boldsymbol{x}, \boldsymbol{y})$, where the inner product the standard inner product in $\mathbb{R}^6$.

5. (*You don't need to do anything for this question.*) A good Matlab code and demonstration of kernel K-means can be found at
   `http://www.dcs.gla.ac.uk/~srogers/firstcourseml/matlab/chapter6/kernelkmeans.html`
   Read the code. Run the code in Matlab, if possible, to see how kernel K-means works for nonlinear data.