MSBD 5004
Spring 2020
Final Exam Sample Questions

The questions should be answered with reasoning, otherwise you may loose points.

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1. Let X be an inner product space, and $\boldsymbol{x}, \boldsymbol{y}$ be two non-zero vectors in X. Prove that
   (a) $\boldsymbol{x}, \boldsymbol{y}$ are orthogonal if and only if $\|\boldsymbol{x} + \alpha\boldsymbol{y}\| = \|\boldsymbol{x} - \alpha\boldsymbol{y}\|$ for all $\alpha \in \mathbb{R}$.
   (b) $\boldsymbol{x}, \boldsymbol{y}$ are orthogonal if and only if $\|\boldsymbol{x} + \alpha\boldsymbol{y}\| \geq \|\boldsymbol{x}\|$ for all $\alpha \in \mathbb{R}$.

2. Let $S_i, i = 1, \ldots, k$ be k hyperplanes in $\mathbb{R}^n$ defined by

$$S_i = \{\boldsymbol{x} \in V \mid \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle = b_i\}, \quad i = 1, \ldots, k.$$

   Let $\boldsymbol{y} \in \mathbb{R}^n$ be given. We consider the projection of $\boldsymbol{y}$ onto $\cap_{i=1}^k S_i$, i.e., the solution of

$$\min_{\boldsymbol{x} \in \cap_{i=1}^k S_i} \|\boldsymbol{x} - \boldsymbol{y}\|_2. \tag{1}$$

   Prove that $\boldsymbol{z}$ is a solution of (1) if and only if $\boldsymbol{z} \in \cap_{i=1}^k S_i$ and

$$\langle \boldsymbol{z} - \boldsymbol{y}, \boldsymbol{z} - \boldsymbol{x} \rangle = 0, \quad \forall \boldsymbol{x} \in \cap_{i=1}^k S_i. \tag{2}$$

3. Let $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^N$ are given, $\boldsymbol{x}_i \in \mathbb{R}^n$ and $y_i \in \mathbb{R}$. Assume $N < n$, $\boldsymbol{x}_i$ are linearly independent. Consider the following optimization problem:

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} F(\boldsymbol{x}), \quad \text{with} \quad F(\boldsymbol{x}) = \sum_{i=1}^N f(\boldsymbol{a}_i^\top \boldsymbol{x} - \boldsymbol{b}_i) + \lambda \|\boldsymbol{x}\|^2 \tag{3}$$

   with $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function.
   (a) Compute $\nabla F(\boldsymbol{x})$.
   (b) Compute $\nabla^2 F(\boldsymbol{x})$.
   (c) Prove the minimizer of F must be in the form of $\boldsymbol{x}^* = \sum_{i=1}^N c_i \boldsymbol{a}_i$, with $\boldsymbol{c} = [c_1, c_2, ..., c_N]^\top$.
   (d) Re-express the minimization problem (3) in terms of $\boldsymbol{c}$ with fewer unknowns.

4. Compute the gradient and Hessian of the following functions.
   (a) $f(\boldsymbol{x}) = \sum_{i=1}^n \sqrt{x_{2i-1}^2 + x_{2i}^2}$, where $\boldsymbol{x} \in \mathbb{R}^{2n}$.
   (b) $f(\boldsymbol{X}) = \log\det(\boldsymbol{X})$, where $\boldsymbol{X} \in \mathbb{R}^{n \times n}$ is symmetric positive definite.

5. (a) Compute the Fourier transform of the function

$$f(x) = \begin{cases} 2|t| - 1, & -1 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Compute the discrete Fourier transform of

$$[1\ 3\ 4\ 2]^T.$$