MSBD 5004 **Name:** ______________________
Spring 2020
Final Exam Paper
27-May-2020
Time Limit: 7:30-10:30PM **Student ID:** ______________________

Total of points is 0.

Please answer all the questions with reasoning, otherwise you may loose points.

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1. (20 points) Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be two norms on a vector space V.
   1. Show that $\|\cdot\|_M$ defined by $\|\boldsymbol{x}\|_M = \max\{\|\boldsymbol{x}\|_A, \|\boldsymbol{x}\|_B\}$ is also a norm on V.
   2. Give an example that $\|\boldsymbol{x}\|_m = \min\{\|\boldsymbol{x}\|_A, \|\boldsymbol{x}\|_B\}$ does not define a norm on V.

2. (10 points) Let V be a Hilbert space. Let S_1 and S_2 be two hyperplanes in V defined by
$$S_1 = \{\boldsymbol{x} \in V \mid \langle \boldsymbol{a}_1, \boldsymbol{x}\rangle = b_1\}, \quad S_2 = \{\boldsymbol{x} \in V \mid \langle \boldsymbol{a}_2, \boldsymbol{x}\rangle = b_2\}.$$
Let $\boldsymbol{y} \in V$ be given. We consider the projection of $\boldsymbol{y}$ onto $S_1 \cap S_2$, i.e., the solution of
$$\min_{\boldsymbol{x} \in S_1 \cap S_2} \|\boldsymbol{x} - \boldsymbol{y}\|. \tag{1}$$
Prove that $\boldsymbol{z}$ is a solution of (1) if and only if $\boldsymbol{z} \in S_1 \cap S_2$ and
$$\langle \boldsymbol{z} - \boldsymbol{y}, \boldsymbol{z} - \boldsymbol{x}\rangle = 0, \quad \forall \boldsymbol{x} \in S_1 \cap S_2. \tag{2}$$

3. (20 points) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be symmetric. Let $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} - \frac{\lambda}{2}\|\boldsymbol{x}\|_2^2 + \boldsymbol{b}^\top \boldsymbol{x}$. Consider the optimization problem:
$$\min_{\boldsymbol{x} \in \mathbf{R}^n} \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} - \frac{\lambda}{2}\|\boldsymbol{x}\|_2^2 + \boldsymbol{b}^\top \boldsymbol{x}$$
   1) Calculate $\nabla f(\boldsymbol{x})$
   2) Calculate $\nabla^2 f(\boldsymbol{x})$.
   3) Assume $\boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x} > \lambda\|\boldsymbol{x}\|^2$ for all $\boldsymbol{x} \neq 0$. Show that $\boldsymbol{x}_0$ is the unique minimizer of $f(\boldsymbol{x})$ (i.e., $f(\boldsymbol{x}) > f(\boldsymbol{x}_0)$ for all $\boldsymbol{x} \neq \boldsymbol{x}_0$) if and only $\nabla f(\boldsymbol{x}_0) = \mathbf{0}$.

4. (30 points) Compute the gradient and Hessian of the following functions.
   1) $f(\boldsymbol{X}) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left((x_{i,j} - x_{i+1,j})^2 + (x_{i,j} - x_{i,j+1})^2\right)^{1/2}$, where $\boldsymbol{X} \in \mathbb{R}^{n\times n}$ with entries $x_{i,j}$. Here we assume $x_{i,j} - x_{i+1,j} \neq 0$ and $x_{i,j} - x_{i,j+1} \neq 0$ for all i, j. This function is the total variation of $\boldsymbol{X}$, which is used in image processing.

2) $f(\boldsymbol{X}) = \operatorname{trace}(\boldsymbol{X}^T \boldsymbol{A} \boldsymbol{X})$, where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is given, and the variable $\boldsymbol{X} \in \mathbb{R}^{n\times r}$.

5. (20 points) 1) Compute the Fourier transform of the function

$$f(x) = \begin{cases} 0, & -1 \le t \le 0, \\ 1, & 0 < t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

2) Compute the discrete Fourier transform of

$$[3\ 3\ 4\ 4]^T.$$