1. Find the Fourier series for the following 1-periodic function

$$f(t) = t, \quad -\frac{1}{2} \leq t < \frac{1}{2}.$$

Solution: $f(t) = \sum_{k=-\infty}^{+\infty} C_k e^{2\pi i k t}$

$$C_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} t \, dt = \frac{t^2}{2}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 - \frac{1}{2}\cdot\left(-\frac{1}{2}\right)^2 = 0$$

$$C_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\, e^{-2\pi i k t} dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} t\, e^{-2\pi i k t} dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} t \cdot \left(-\frac{1}{2k\pi i} d e^{-2k\pi i t}\right)$$

$$= -\frac{t}{2k\pi i} e^{-2k\pi i t}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{2k\pi i}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2k\pi i t} dt$$

$$= -\frac{\frac{1}{2}}{2k\pi i} e^{-2k\pi i\cdot\frac{1}{2}} + \frac{\left(-\frac{1}{2}\right)}{2k\pi i} e^{-2k\pi i\left(-\frac{1}{2}\right)}$$

$$+ \frac{1}{2k\pi i}\cdot\left(-\frac{1}{2k\pi i} e^{-2k\pi i t}\right)\Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= -\frac{1}{4k\pi i} e^{-k\pi i} - \frac{1}{4k\pi i} e^{k\pi i}$$

$$- \frac{1}{4k^2\pi^2}\left(e^{-2k\pi i\cdot\frac{1}{2}} - e^{k\pi i}\right)$$

$$= -\frac{1}{4k\pi i}\left(\cos k\pi - i\sin k\pi\right) - \frac{1}{4k\pi i}\left(\cos k\pi + i\sin k\pi\right)$$

$$- \frac{1}{4k^2\pi^2}\left(e^{-k\pi i} - e^{k\pi i}\right)$$

$$= -\frac{1}{2k\pi i}\cos k\pi - \frac{1}{4k^2\pi^2}\left[\cos k\pi - i\sin k\pi - (\cos k\pi + i\sin k\pi)\right]$$

$$= -\frac{1}{2k\pi i}\cos k\pi - \frac{1}{4k^2\pi^2}(-2i\sin k\pi)$$

$$= \frac{i}{2k\pi}\cos k\pi + \frac{\sin k\pi}{2k^2\pi^2} i = \frac{(-1)^k i}{2k\pi}$$

Hence, $f(t) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi i k t} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k i}{2k\pi} e^{2\pi i k t}$

2. Find the sum

$$\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots + \frac{1}{n^4} + \ldots$$

(*Hint: Consider the Fourier series for the function* $f(t) = t^2$ *on* $\left[-\frac{1}{2}, \frac{1}{2}\right)$ *and* $f(t+k) = f(t)$ *for all integer* k.)

Solution: Consider the Fourier series for the function $f(t) = t^2$.

$$f(t) = \sum_{k=-\infty}^{+\infty} C_k e^{2\pi i k t}$$

where $C_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) e^{-2\pi i k t} dt$, $k \neq 0$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} t^2 e^{-2\pi i k t} dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} t^2 \cdot \frac{1}{-2\pi i k} de^{-2\pi i k t}$$

$$= -\frac{1}{2\pi i k} t^2 e^{-2\pi i k t}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{2}{2\pi i k}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i k t} t\, dt$$

$$= -\frac{1}{2\pi i k}\left(\frac{1}{2}\right)^2 e^{-2\pi i k \cdot \frac{1}{2}} + \frac{1}{2\pi i k}\left(-\frac{1}{2}\right)^2 e^{-2\pi i k \cdot (-\frac{1}{2})}$$

$$+ \frac{1}{\pi k i}\int_{-\frac{1}{2}}^{\frac{1}{2}} t \cdot \frac{1}{-2\pi k i}\, de^{-2\pi i k t}$$

$$= -\frac{1}{8\pi k i} e^{-\pi k i} + \frac{1}{8\pi k i} e^{\pi k i}$$

$$+ \frac{1}{2\pi^2 k^2} \left[t e^{-2\pi i k t} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i k t} dt \right]$$

$$= -\frac{1}{8k\pi i} e^{-\pi k i} + \frac{1}{8\pi k i} e^{\pi k i}$$

$$+ \frac{1}{2\pi^2 k^2} \left[\frac{1}{2} e^{-2\pi i k \cdot \frac{1}{2}} - \left(-\frac{1}{2}\right) e^{-2\pi i k \cdot \left(-\frac{1}{2}\right)} \right.$$

$$\left. + \frac{1}{2\pi i k} e^{-2\pi i k t} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} \right]$$

$$= -\frac{1}{8k\pi i} e^{-\pi k i} + \frac{1}{8k\pi i} e^{\pi k i} + \frac{1}{4\pi^2 k^2} e^{-\pi k i}$$

$$+ \frac{1}{4\pi^2 k^2} e^{\pi k i} + \frac{1}{2\pi^2 k^2} \left[\frac{1}{2k\pi i} e^{-2k\pi i \cdot \frac{1}{2}} - \frac{1}{2k\pi i} e^{\pi k i} \right]$$

$$= \frac{1}{8k\pi i} \left[\cos\pi k + i\sin\pi k - \cos\pi k + i\sin\pi k \right]$$

$$+ \frac{1}{4\pi^2 k^2} \left[\cos k\pi + i\sin k\pi + \cos\pi k - i\sin k\pi \right]$$

$$+ \frac{1}{8\pi^3 k^3 i} \left[\cos k\pi - i\sin k\pi - \cos k\pi - i\sin k\pi \right]$$

$$= \frac{(-1)^k}{2\pi^2 k^2}$$

When $k=0$, then $C_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} t^2 dt = \frac{t^3}{3} \Big|_{-\frac{1}{2}}^{\frac{1}{2}}$

$$= \frac{1}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] = \frac{1}{12}$$

As $\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |C_k|^2$,

Since $C_{-k} = \frac{(-1)^{-k}}{2\pi^2 \cdot (-k)^2} = \frac{1}{2\pi^2 k^2 \cdot (-1)^k} = \frac{(-1)^k}{2\pi^2 k^2}$

$$\sum_{k=-\infty}^{+\infty} |C_k|^2 = |C_0|^2 + 2\sum_{k=1}^{\infty} |C_k|^2$$

$$= \frac{1}{12^2} + 2 \sum_{k=1}^{\infty} \frac{1}{4\pi^4 k^4}$$

and $\int_{-\frac{1}{2}}^{\frac{1}{2}} |f(t)|^2 dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdot t^4 dt = \frac{1}{5} t^5 \Big|_{-\frac{1}{2}}^{\frac{1}{2}}$

$$= \frac{1}{5} \left[\left(\frac{1}{2}\right)^5 - \left(-\frac{1}{2}\right)^5 \right]$$

$$= \frac{1}{80}$$

We have $\sum_{k=-\infty}^{+\infty} |C_k|^2 = \frac{1}{12} + 2\sum_{k=1}^{\infty} \frac{1}{4\pi^4 k^4} = \frac{1}{80}$;

and $\sum_{k=1}^{\infty} \frac{1}{k^4} = 2\pi^4 \left(\frac{1}{80} - \frac{1}{12^2} \right) = \frac{\pi^4}{90}$

Therefore, $\frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{k^4} + \cdots$

$$= \sum_{k=2}^{\infty} \frac{1}{k^4} = \sum_{k=1}^{\infty} \frac{1}{k^4} - 1 = \frac{\pi^4}{90} - 1$$

3. Find a function $g(t)$ such that: for any $f(t)$, the convolution $f * g$ is the ideal low pass filter that retains only the frequencies in the interval $(-1, 1)$.

Solution: $\widehat{f * g} = \hat{f} \cdot \hat{g}$, $\hat{g} = 1_{s \in [-1,1]}$

$$g = \mathcal{F}^{-1}(\hat{g}) = \int_{-\infty}^{+\infty} 1_{s \in [-1,1]} \, e^{2\pi i t s} ds$$

$$= \int_{-1}^{1} e^{2\pi i t s}\, ds$$

when $t=0$, $\check{g} = \int_{-1}^{1} e^{0} ds = t\Big|_{-1}^{1} = 2$

when $t \neq 0$, $\check{g}(t) = \int_{-1}^{1} e^{2\pi i s t} ds$

$$= \frac{e^{2\pi i s t}}{2\pi i t}\Big|_{-1}^{1}$$

$$= \frac{e^{2\pi i t} - e^{-2\pi i t}}{2\pi i t}$$

$$= \frac{2 i \sin 2\pi t}{2\pi i t}$$

$$= \frac{\sin 2\pi t}{\pi t}$$

$$= 2 \operatorname{sinc}(2t)$$

4. Find the Fourier transform of the function

$$f(t) = \begin{cases} 1 - |t|, & -1 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Solution: $\hat{f}(s) = \int_{-\infty}^{+\infty} f(t)\, e^{-2\pi i s t} dt$

when $s=0$, $\hat{f} = \int_{-\infty}^{+\infty} (1-|t|)\, dt$

$$= \int_{-1}^{0} (1+t) dt + \int_{0}^{1} (1-t) dt$$

$$= t + \frac{t^2}{2}\Big|_{-1}^{0} + \left(t - \frac{t^2}{2}\right)\Big|_{0}^{1}$$

$$= -\left(-1 + \frac{1}{2}\right) + 1 - \frac{1}{2} = 1$$

when $s \neq 0$, $\hat{f} = \int_{-\infty}^{+\infty} (1-|t|)\, e^{-2\pi i s t}\, dt$

$$= \int_{-1}^{0} (1+t) e^{-2\pi i s t} dt + \int_{0}^{1} (1-t) e^{-2\pi i s t} dt$$

$$= \int_{-1}^{0} (1+t) \cdot \frac{1}{-2\pi i s} d e^{-2\pi i s t}$$

$$+ \int_{0}^{1} (1-t) \cdot \frac{1}{-2\pi i s} d e^{-2\pi i s t}$$

$$= (1+t) \cdot \frac{1}{-2\pi i s} e^{-2\pi i s t} \Big|_{-1}^{0} + \frac{1}{2\pi i s} \int_{-1}^{0} e^{-2\pi i s t} dt$$

$$+ \left(-\frac{1}{2\pi i s}\right)(1-t) e^{-2\pi i s t} \Big|_{0}^{1}$$

$$+ \frac{1}{2\pi i s} \int_{0}^{1} e^{-2\pi i s t} (-1) dt$$

$$= -\frac{1}{2\pi i s} + \frac{1}{2\pi i s} \cdot \frac{1}{-2\pi i s} e^{-2\pi i s t} \Big|_{-1}^{0}$$

$$+ \frac{1}{2\pi i s} - \frac{1}{2\pi i s} \cdot \left(-\frac{1}{2\pi i s} e^{-2\pi i s t}\right)\Big|_{0}^{1}$$

$$= -\frac{1}{2\pi i s} - \frac{1}{(2\pi i s)^2} + \frac{1}{(2\pi i s)^2} e^{2\pi i s} + \frac{1}{2\pi i s}$$

$$+ \frac{1}{(2\pi i s)^2} \left(e^{-2\pi i s} - 1\right)$$

$$= \frac{1}{4\pi^2 s^2} - \frac{1}{4\pi^2 s^2} e^{2\pi i s} - \frac{1}{4\pi^2 s^2} e^{-2\pi i s} + \frac{1}{4\pi^2 s^2}$$

$$= \frac{1}{2\pi^2 s^2} - \frac{1}{4\pi^2 s^2} \left(e^{2\pi i s} + e^{-2\pi i s}\right)$$

$$= \frac{1}{2\pi^2 s^2} - \frac{1}{4\pi^2 s^2} \left(\cos 2\pi s + i \sin 2\pi s + \cos 2\pi s - i \sin 2\pi s\right)$$

$$= \frac{1}{2\pi^2 s^2} - \frac{1}{4\pi^2 s^2} \cdot 2\cos 2\pi s$$

$$= \frac{1}{2\pi^2 s^2} - \frac{\cos 2\pi s}{2\pi^2 s^2}$$

$$= \frac{2\sin^2 \pi s}{2\pi^2 s^2}$$

$$= \sin^2 c(\pi s)$$

5. Compute the Discrete Fourier Transform of $[1\ 1\ 2\ 2]^T$.

Solution: $N=4$, $W_4 = e^{\frac{2\pi i}{4}} = e^{\frac{\pi}{2} i} = i$

$$W_4^{-n} = (i)^{-n} = (-i)^n = \begin{cases} -i, & n=4k+1 \\ -1, & n=4k+2 \\ i, & n=4k+3 \\ 1, & n=4(k+1) \end{cases}$$

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ 1 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ 1 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$F = A_4 \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+1+2+2 \\ 1-i-2+2i \\ 1-1+2-2 \\ 1+i-2-2i \end{pmatrix} = \begin{pmatrix} 6 \\ i-1 \\ 0 \\ -1-i \end{pmatrix}$$

6. Prove the discrete convolution theorem:

$$A_N(f * g) = (A_N f) \cdot (A_N g),$$

where $f, g \in \mathbb{C}^N$ are vectors, $A_N \in \mathbb{C}^{N\times N}$ is the discrete Fourier transform matrix, $\cdot$ is the entrywise multiplication, and $*$ is the discrete convolution defined by $(f * g)(m) = \sum_{n=0}^{N-1} f(n)g(m - n \mod N)$ for $m = 0, 1, \ldots, N-1$.

Solution: $(A_N(f*g))_i = \sum_{k=0}^{N-1} W_N^{-ik} \left(\sum_{n=0}^{N-1} f(n) g(k-n)\right)$

$$= \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} W_N^{-ik} f(n)\, g(k-n)$$

$$= \sum_{n=0}^{N-1} f(n) \sum_{k=0}^{N-1} W_N^{-ik} g(k-n)$$

$$= \sum_{n=0}^{N-1} W_N^{-in} f(n) \sum_{k=0}^{N-1} W_N^{-i(k-n)} g(k-n)$$

$$= \sum_{n=0}^{N-1} W_N^{-in} f(n) \sum_{m=0}^{N-1} W_N^{-im} g(m)$$

$$= (A_N f)_i \cdot (A_N g)_i$$

Therefore, $\vec{F} = (A_N f) \cdot (A_N g)$.

7. Let f be a vector and let $\tau(f)$ be the cyclic shift by 1 position to the right. What is $F(\tau(f))$ in relation to $F(f)$? Here $F(f)$ is the discrete Fourier transform of f.

Solution: $f = (f_0, f_1, \cdots, f_{N-1})^T$

$\tau(f) = (f_{N-1}, f_0, \cdots, f_{N-2})^T$

$f_{N-1} = f_{-1}$

$\vec{F} = A_N f$, $\quad \vec{F}_j = \sum_{n=0}^{N-1} W_N^{-jn} f(n)$

$(F(\tau(f)))_j = \sum_{n=0}^{N-1} W_N^{-jn} f(n-1)$

$$= W_N^{-j} \sum_{n=0}^{N-1} W_N^{-j(n-1)} f(n-1)$$

$$= W_N^{-j} \vec{F}_j$$

Hence $F(\tau(f)) = \left(W_N^0 \vec{F}_0 , W_N^{-1} \vec{F}_1 , \cdots , W_N^{-(N-1)} \vec{F}_{N-1} \right)^T$

$$= \begin{pmatrix} W_N^0 \\ W_N^{-1} \\ \vdots \\ W_N^{-(N-1)} \end{pmatrix} \cdot F(f)$$