MSBD 5004 Mathematical Methods for Data Analysis
Homework 4

Due date: 1 May, 11:59pm, Friday

1. Let V be a Hilbert space. Let $\boldsymbol{a} \in V$ be a given vector. The function $A\boldsymbol{x} := \langle \boldsymbol{a}, \boldsymbol{x} \rangle$ can be viewed as a linear transformation from V to $\mathbb{R}$. Find the operator norm $\|A\|$.

2. Let $f_1, f_2, \ldots, f_n$ are differentiable functions from $V \mapsto \mathbb{R}$ with V a Hilbert space. Define $F : V \mapsto \mathbb{R}^n$ by
$$F(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{bmatrix}, \quad \forall\, \boldsymbol{x} \in V.$$
Prove that
$$DF(\boldsymbol{x})(\boldsymbol{y}) = \begin{bmatrix} \langle \nabla f_1(\boldsymbol{x}), \boldsymbol{y} \rangle \\ \langle \nabla f_2(\boldsymbol{x}), \boldsymbol{y} \rangle \\ \vdots \\ \langle \nabla f_n(\boldsymbol{x}), \boldsymbol{y} \rangle \end{bmatrix}, \quad \forall\, \boldsymbol{x} \in V.$$

3. Find $\nabla f(\boldsymbol{x})$ and $\nabla^2 f(\boldsymbol{x})$.
   (a) $f(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \lambda\|\boldsymbol{x}\|_2^2$, where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\lambda > 0$ are given.
   (b) $f(\boldsymbol{X}) = \boldsymbol{b}^T \boldsymbol{X} \boldsymbol{c}$, where $\boldsymbol{X} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$.
   (c) $f(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}$, where $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is non-symmetric.
   (d) $f(\boldsymbol{X}) = \boldsymbol{b}^T \boldsymbol{X}^T \boldsymbol{X} \boldsymbol{c}$, where $\boldsymbol{X} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$.
   (e) $f(\boldsymbol{X}) = \mathrm{trace}(\boldsymbol{X}\boldsymbol{A}\boldsymbol{X}\boldsymbol{B})$, where $\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n \times n}$.