SCHOOL OF COMPUTATION,
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TECHNISCHE UNIVERSITÄT MÜNCHEN

Bachelor's Thesis in Informatics

On the Formalization of Martingales

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Eine Formalisierung von Martingalen

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I confirm that this bachelor’s thesis is my own work and I have documented all sources and material used.

Munich, 15 September 2023 Ata Keskin

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Abstract

This thesis presents a formalization of martingales in arbitrary Banach spaces using Isabelle/HOL. The primary focus lies in the formal construction of conditional expectation in Banach spaces, which extends the existing formulation for real-valued functions. We replicate existing lemmas about martingales from the mathematical proof repository mathlib, which is primarily developed in Lean, based on homotopy type theory (HoTT). While mathlib explores formalization in Lean, we choose Isabelle/HOL as the theorem prover due to its powerful locale system that provides a structured and modular framework for representing these dynamic systems. The formalization of martingales and stochastic processes is achieved through Isabelle's locale system. We define the locale `stochastic_process` to formalize stochastic processes over arbitrary Banach spaces. Similarly, we define adapted, progressively measurable and predictable processes via the locales `adapted_process`, `progressive_process` and `predictable_process`. We also show sublocale relations and simple lemmas concerning vector space operations. Filtered measure spaces and σ-finite variants are introduced with the locales `filtered_measure` and `filtered_sigma_finite_measure`. Similarly, the locales `martingale`, `submartingale` and `supermartingale` are introduced to formalize martingales and related constructs in Banach spaces. Our formalization provides a robust mathematical framework for analyzing random processes.

Contents

1 Introduction

Martingales hold a central position in the theory of stochastic processes, making them a fundamental concept for the working mathematician. They provide a powerful way to study and analyze random phenomena, offering a formal framework for understanding the behavior of random variables over time. In various real-world scenarios, we encounter systems that evolve randomly over time. Representing such systems as martingales, we are able to investigate whether these systems remain bounded or converge to certain values in the long run.

In finance and economics, martingales are an invaluable tool for modeling asset prices [Fam65] and option pricing [MR05]. They provide insights into risk assessment, portfolio management, and the efficient market hypothesis, which postulates that asset prices fully reflect all available information [YB89].

Martingales are also closely related to several important probability limit theorems. These theorems, such as the strong law of large numbers and the central limit theorem, formalize the asymptotic behavior of sample means and sums of random variables. They have profound implications in statistics, allowing us to draw conclusions about large datasets and make predictions based on limited information.

In addition to their relevance in mathematics, martingales find applications in various interdisciplinary fields. Their ability to model randomness and analyze dynamic systems makes them useful in physics [Rol+23], biology, and computer science [MU05], among others.

In the scope of this thesis, we present a formalization of martingales in arbitrary Banach spaces using Isabelle/HOL. The background and related work section examines existing formalizations in two prominent formal proof repositories, mathlib (which uses the Lean theorem prover) and the Archive of Formal Proofs (AFP) (which uses Isabelle). Additionally, we conduct a short review of literature on conditional expectation and martingales in Banach spaces, laying a solid foundation for our research.

The current formalization of conditional expectation in the Isabelle library is limited to real-valued functions. To overcome this limitation, we extend the construction of conditional expectation to general Banach spaces, employing an approach similar to the one in [Hyt+16]. We justify our approach, by comparing it to two alternative constructions of the conditional expectation.

Subsequently, we define stochastic processes and introduce the concepts of adapted, progressively measurable and predictable processes using suitable locale definitions. Most importantly, we provide a generalization for the already present locale `filtration` by introducing the locales `filtered_measure` and `filtered_sigma_finite_measure`. These locales serve to formalize the concept of a filtered measure space. The latter also serves to generalize the locale `sigma_finite_subalgebra` which is necessary for the development of the theory of martingales.

Moving forward, we rigorously define martingales, submartingales, and supermartingales, presenting their first consequences and corollaries. Discrete and continuous time martingales are also covered in the formalization, benefiting from the complex and powerful locale system of Isabelle.

Our formalization fully encompasses the introductory mathlib theory on martingales and offers more generalization.

The thesis further contributes by generalizing concepts in Bochner integration, extending their application to general Banach spaces. Induction schemes for simple, integrable, and Borel measurable functions on Banach spaces are introduced, accommodating scenarios with or without a real vector ordering. These amendments expand the applicability of Bochner integration techniques.

The thesis concludes with reflections on the formalization approach, encountered challenges, and suggests future research directions.

2 Background and Related Work

In the following section, we explore existing formalizations of martingales within the mathematical proof repositories mathlib and AFP. Additionally, we will provide a concise introduction to the integration theory in Banach spaces, establishing the mathematical foundation that underpins our formalization efforts.

2.1 Existing Formalizations

2.1.1 Lean Mathematical Library

Our main motivation for formalizing a theory of martingales in Isabelle/HOL comes from the existing in-depth formalization of the same subject in mathlib. As stated on their online platform, "The Lean mathematical library, mathlib, is a community-driven effort to build a unified library of mathematics formalized in the Lean proof assistant." The Lean-formalization of martingales consists of six documents. In the introductory Lean document `basic.lean`, fundamentals of the theory of martingales are formalized. The aim of this bachelor's thesis is to reproduce the results contained within this file in Isabelle/HOL. As will become clear in a moment, this is not a straightforward task, since there are a lot of dependencies missing in the Isabelle/HOL libraries.

The file `basic.lean` contains definitions for martingales, submartingales and supermartingales [DY22b]. Also stated on the official documentation on mathlib, the main results of this document are

→ `measure_theory.martingale` f $\mathcal{F}$ μ:

  f is a martingale with respect to filtration $\mathcal{F}$ and measure μ.

→ `measure_theory.supermartingale` f $\mathcal{F}$ μ:

  f is a supermartingale with respect to filtration $\mathcal{F}$ and measure μ.

→ `measure_theory.submartingale` f $\mathcal{F}$ μ:

  f is a submartingale with respect to filtration $\mathcal{F}$ and measure μ.

→ `measure_theory.martingale_condexp` f $\mathcal{F}$ μ:

the sequence $(\mu[f|\mathcal{F}_i])_{i\in\mathcal{T}}$ is a martingale with respect to $\mathcal{F}$ and μ, where $\mu[f|\mathcal{F}_i]$ denotes the conditional expectation of f with respect to the subalgebra $\mathcal{F}_i$.

On a first note, we see that this theory relies heavily on the development of a conditional expectation operator. Prior to our development, the only formalization of conditional expectation in Isabelle/HOL was done in the real setting and resides in the theory document `HOL-Probability.Conditional_Expectation`. This formalization was accomplished by Sèbastien Gouëzel, presumably in anticipation of his latter entries [Gou15] and [Gou16]. We will delve further into the existing formalization and how our contribution improves upon it in the upcoming chapter.

Within the mathlib formalization, the majority of lemmata on martingales require the measures in question to be finite. In our formalization of martingales, we will demonstrate that σ-finiteness suffices alone. This approach is also consistent with our generalized formalization of conditional expectation, as it inherits the σ-finiteness requirement from the preexisting formalization in the real setting.

Another short-coming of the mathlib formalization is its treatment of predictable processes. The mathlib formalization contains the definition of adapted processes and progressively measurable processes. No explicit definition of a predictable process is given. Instead predictability is defined only in the discrete case, as a stochastic process which is adapted to the filtration $(\lambda i.\ \mathcal{F}_{i+1})$. In contrast, our formalization defines predictable processes more generally using the concept of a predictable σ-algebra. Similarly, we define adapted and progressively measurable process. One of the major advantages of our formalization is the use of locales and sublocale relations. Concretely, we will show the relationship

$$\texttt{stochastic} \supseteq \texttt{adapted} \supseteq \texttt{progressive} \supseteq \texttt{predictable}$$

Another important point to consider is the restrictions placed on the types in question. In the mathlib formalization, martingales are defined as a family of integrable functions $f : \iota \to \Omega \to E$. The mathlib formalization further requires that

- ι is a preordered set,
- Ω is a measurable space (i.e. a set together with a σ-algebra Σ),
- E is a normed, complete space with an addition operation.

These restrictions are easily replicated in our formalization using type classes and the type $'a$ `measure`. We simply restrict ourselves to functions $f :\ 't \to\ 'a$ `measure` $\to\ 'b$, where the type $'t$ is an instance of the class `linorder_topology` and the type $'b$ is an instance of the class `banach`. Furthermore, we fix an element t_0 of type $'t$ which represents the smallest element for which the function should be integrable. With this approach we can restrict ourselves to the index set $\{i \mid t_0 \leq i\}$. The fact that we only need to consider this index set, which is bounded from below, will prove to be cruicial in certain steps in our development. It will also provide us with the ability to use the reals as an index set. With this specification, our approach mirrors the mathlib formalization, since σ-algebras are encoded as measures in Isabelle/HOL. The additional requirement that $'t$ (equivalently ι in the mathlib case) be linearly ordered is easily justified as well, since in most contexts the index set represents a temporal dimension, which can obviously be linearly ordered. Apart from this, the topology induced must also come from the ordering on $'t$, since otherwise we can't have a useful definition of predictability in the general sense.

The main purpose of the mathlib formalization on martingales is to prove Doob's martingale convergence theorems, which concern discrete time and continuous time martingales (i.e. the naturals or the reals as indices). This justifies their focus on discrete time processes and the formulation of predictability only in the discrete case. More information on the specifics and the development of Doob's martingale convergence theorems is available in [DY22a].

This concludes our review of the mathlib formalization on martingales.

2.1.2 Archive of Formal Proofs

The Archive of Formal Proofs or AFP is a digital repository of formalized proofs and theories developed using the Isabelle theorem prover and proof assistant. The AFP hosts a variety of formalizations and proofs, primarily in the fields of logic, mathematics, and computer science. The repository allows researchers to share their formal proofs, theories, and related materials with the broader community.

The repository offers a search function, which allows us to find if any formalization on martingales has been done previously. A quick search yields the theory file `DiscretePricing.Martingale`. This entry `DiscretePricing`, which is attributed to Mnacho Echenim, focuses on the formalization of the Binomial Options Pricing Model in finance [Ech18]. A development of discrete time real-valued martingales is given in order to introduce the concept of risk-neutral measures. Similar to the development on mathlib, the goal of this entry is not to formalize martingales. A partial formalization of martingales is only given as a byproduct. The actual conference paper detailing the formalization can be found here [EP17].

Apart from this entry, no other development on the theory of martingales is present on AFP.

2.2 Mathematical Foundations and Reference Material

The main focus of our project is to formalize martingales in as general of a setting as possible. In this vein, we will study martingales defined on arbitrary Banach spaces, as opposed to the reals only. The main obstacle we will face is the development of conditional expectation in arbitrary Banach spaces. A great resource on this subject is the book *Analysis in Banach Spaces* by Hytönen et al [Hyt+16]. As a primer for the upcoming chapter, we will quickly cover the basics of integration on Banach spaces. The following information can also be found in the aforementioned book.

Integration on Banach spaces is usually done using the Bochner integral, which is defined similarly to the Lebesgue integral. For (Ω, Σ, μ) a measure space and E a Banach space, we define the Bochner integral as follows

First, we consider simple functions $s : \Omega \to E$. These are functions which can be expressed μ-a.e. as finite sums of the form

$$s = \sum_{i=1}^{n} \mathbf{1}_{A_i} \cdot_{\mathbb{R}} c_i$$

where $\mathbf{1}_A$ is the indicator function of a set $A \in \Sigma$ and $c_i \in E$. Here $\cdot_{\mathbb{R}}$ denotes the scalar multiplication. We call such a function s Bochner integrable if $\mu(A_i) < \infty$ for all $i \in \{1, \ldots, n\}$. In this case, we define the Bochner integral simply as the sum

$$\int s \,\mathrm{d}\mu = \sum_{i=1}^{n} \mu(A_i) \cdot_{\mathbb{R}} c_i$$

If we replace E with $\mathbb{R}$, we can easily see that Bochner integrable simple functions are exactly those functions, which are Lebesgue integrable and simple.

We call a function $f : \Omega \to E$ strongly measurable, if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions converging to f μ-almost everywhere. A strongly measurable function f is called Bochner integrable with respect to μ, if there exists a sequence of Bochner integrable simple functions $f_n : \Omega \to E$ such that

$$\lim_{n\to\infty} \int_{\Omega} \|f - f_n\| \,\mathrm{d}\mu = 0$$

The integral used in this definition is the ordinary Lebesgue integral. This definition makes sense, since $w \mapsto \|f(w) - f_n(w)\|$ is μ-measurable and non-negative.

It can be shown via the triangle inequality that the integrals $\int f_n \,\mathrm{d}\mu$ form a Cauchy sequence. By completeness, this sequence converges to some element $\lim_{n\to\infty} \int f_n \,\mathrm{d}\mu \in E$. This limit is called the Bochner integral of f with respect to the measure μ

$$\int f \, \mathrm{d}\mu = \lim_{n \to \infty} \int f_n \, \mathrm{d}\mu$$

A formalization of the Bochner integral is available in Isabelle/HOL in the theory file `HOL-Analysis.Bochner_Integration` [HH11]. This formalization, which is due to Johannes Hölzl, has the additional assumption that the space E be second-countable. In the context of a metric space, this is the same as requiring separability.

Remark. One can show that a function f is strongly measurable if and only if it is essentially separably valued and for all $A \in \mathcal{B}(E)$ we have $f^{-1}(A) \in \Sigma$. Here $\mathcal{B}(E)$ denotes the Borel σ-algebra on E. A function is called essentially separably valued if there exists a μ-null set $N \subseteq \Omega$, such that $f(\Omega \setminus N)$ is separable as a subspace of E. Therefore, if E is already a separable Banach space, a function $f : \Omega \to E$ is strongly measurable if and only if it is Σ-measurable.

Consequently, we don't need to concern ourselves with definining strong measurability when working within separable (or equivalently second-countable) Banach spaces.

The book also contains an in depth section on the construction of the conditional expectation operator on Banach spaces. For our purposes, we only need to focus on the case where $f : \Omega \to E$ is a Bochner integrable function. In this case, the conditional expectation can be thought of as a linear operator $\mathbb{E}(\cdot|\mathcal{F}) : L^1(E) \to L^1(E)$ with respect to a sub-σ-algebra. The book contains theorems for the existence and uniqueness of conditional expectations (up to μ-null sets) for functions not only in $L^1(E)$, but also $L^2(E)$ and $L^0(E)$, which is the space of strongly measurable functions with codomain E. Unsuprisingly, the definition of conditional expectation in the last case is a bit more complicated, since it has to take into account the case where f is not integrable.

Another extensive reference regarding martingales in Banach spaces is the book *Martingales in Banach Spaces* by Gilles Pisier [Pis16]. This resource provides an in-depth exploration of the theory of martingales in Banach spaces at a graduate level. Given the limited scope of this thesis, the book serves as a supplementary resource, as only a select few of its results are applicable to our elementary objectives.

3 Conditional Expectation in Banach Spaces

Conditional expectation extends the concept of expected value to situations where we have additional information about the outcomes. In a discrete setting, i.e. when the range of the random variables in question is countable, the setup is quite simple.

Without loss of generality, let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let E be a complete normed vector space, i.e. a Banach space, and $S \subseteq E$ be some countable subset. Given a random variable $X : \Omega \to S$ and an event $A \in \mathcal{F}$, the conditional expectation of X given A, denoted as $\mathbb{E}(X|A)$, represents the expected value of X given that A occurs. In this simple case, we can directly define the conditional expectation as:

$$\mathbb{E}(X|A) = \sum_{w \in S} w \cdot \frac{\mu(\{X = w\} \cap A)}{\mu(A)}$$

Of course, this definition only makes sense if the value on the right hand side is finite and $\mu(A) \neq 0$. Defined this way, the conditional expectation satisfies the following equality

$$\begin{aligned} \int_A X \, \mathrm{d}\mu &= \sum_{w \in S} w \cdot \mu(\{\mathbf{1}_A \cdot X = w\}) \\ &= \mu(A) \cdot \mathbb{E}(X|A) \\ &= \int_A \mathbb{E}(X|A) \, \mathrm{d}\mu \end{aligned}$$

This observation motivates us to generalize the definition of conditional expectation to take into account not just a single event, but a collection of events. Fix $X : \Omega \to E$. Given a sub-σ-algebra $\mathcal{H} \subseteq \mathcal{F}$, we call an $\mathcal{H}$-measurable function $g : \Omega \to E$ a conditional expectation of X with respect to the sub-σ-algebra $\mathcal{H}$, denoted as $\mathbb{E}(X|\mathcal{H})$, if the following equality holds for all $A \in \mathcal{H}$

$$\int_A X \, \mathrm{d}\mu = \int_A g \, \mathrm{d}\mu$$

In the case that $E = \mathbb{R}$, it is straightforward to show that such a function g always exists (via Radon-Nikodym), and is unique up to a μ-null set. Notice that $\mathbb{E}(X|\mathcal{H})$ is a function $\Omega \to E$, as opposed to some value in E.

The suitable setting for defining the conditional expectation is when the sub-σ-algebra $\mathcal{H}$ gives rise to a σ-finite measure space. This is the case when $\mu|_{\mathcal{H}}$, the restriction of μ to $\mathcal{H}$ is a σ-finite measure. To see what goes wrong, consider the trivial sub-σ-algebra $\{\varnothing, \Omega\}$. A function which is measurable with respect to this σ-algebra is necessarily constant. Therefore, if $\mu(\Omega) = \infty$, no conditional expectation can exist, since it would have to be equal to 0 μ-almost everywhere in order to be integrable.

Example. Let $\mathcal{H} \subseteq \mathcal{F}$ be a sub-σ-algebra such that $\mu|_{\mathcal{H}}$ is a σ-finite measure. Given an integrable function $X : \Omega \to \mathbb{R}$, we can define a measure ν on $(\Omega, \mathcal{F})$ via

$$\nu(A) := \int_A X \, \mathrm{d}\mu$$

It is easy to verify that $\mu|_{\mathcal{H}}(A) = 0$ implies $\nu|_{\mathcal{H}}(A) = 0$, i.e. $\nu|_{\mathcal{H}}$ is absolutely continuous with respect to $\mu|_{\mathcal{H}}$. Using the Radon-Nikodym Theorem, we obtain an $\mathcal{H}$-measurable function $g : \Omega \to \mathbb{R}$ such that

$$\nu|_{\mathcal{H}}(A) = \int_A g \, \mathrm{d}\mu|_{\mathcal{H}}$$

Thus for any $A \in \mathcal{H}$, we have

$$\int_A X \, \mathrm{d}\mu = \int_A g \, \mathrm{d}\mu|_{\mathcal{H}} = \int_A g \, \mathrm{d}\mu$$

In the second equality, we use the fact that g is $\mathcal{H}$-measurable. Radon-Nikodym also guarentees that this function g is unique up to a $\mu|_{\mathcal{H}}$-null set. Since all $\mu|_{\mathcal{H}}$-null sets are also μ-null sets, the function g satisfies the requirements of a conditional expectation.

Technicalities aside, this shows that the conditional expectation always exists and is unique up to μ-null set for all $X \in \mathcal{L}^1(\mathbb{R})$. Our job now will be to construct a similar operator on arbitrary Banach spaces using methods from functional analysis and measure theory.

3.1 Preliminaries

In anticipation of our construction, we need to lift some results from the real setting to our more general setting. Our fundamental tool in this regard will be the **averaging theorem**. The proof of this theorem is due to Serge Lang [Lan93]. The theorem allows us to make statements about a function's value almost everywhere, depending on the value it's integral takes on various sets of the measure space.

3.1.1 Averaging Theorem

Before we introduce and prove the averaging theorem, we will first show the following lemma which is crucial for our proof. While not stated exactly in this manner, our proof makes use of the characterization of second-countable topological spaces given in the book General Topology by Ryszard Engelking (Theorem 4.1.15) [Eng89].

Lemma 3.1.1. *Let E be a separable metric space. Then there exists a countable set $D \subseteq E$, such that the set of open balls*

$$\mathcal{B} = \{B_\varepsilon(x) \mid x \in D,\ \varepsilon \in \mathbb{Q} \cap (0, \infty)\}$$

generates the topology on E. Here $B_\varepsilon(x)$ is the open ball of radius ε with centre x.

Proof. In the context of metric spaces, second-countability is equivalent to separability. Consequently, there exists some non-empty countable subset $D \subseteq E$, which is dense in E. We want to show that this D fulfills the statement above. For this end we will use the following equivalence which is valid for any $\mathcal{A} \subseteq \mathcal{P}(E)$

$$\mathcal{A} \text{ is topological basis} \iff \forall \text{open } U.\ \forall x \in U.\ \exists A \in \mathcal{A}.\ x \in A \wedge A \subseteq U$$

Let $U \subseteq E$ be open. Fix $x \in U$. Since U is open and we are working with the metric topology, there is some $\varepsilon > 0$, such that $B_\varepsilon(x) \subseteq U$. Furthermore, we know that a set D is dense if and only if for any non-empty open subset $O \subseteq E$, $D \cap O$ is also non-empty. Therefore, there exists some $y \in D \cap B_{\varepsilon/3}(x)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists some $r \in \mathbb{Q}$ with $e/3 < r < e/2$. It is easy to check that $x \in B_r(y)$ and $B_r(y) \subseteq U$ with $y \in D$ and $r \in \mathbb{Q} \cap (0, \infty)$. This concludes the proof. □

Now we are ready to state and subsequently prove the averaging theorem

Theorem 3.1.2. *(Averaging Theorem) Let $(\Omega, \mathcal{F}, \mu)$ be some σ-finite measure space. Let $f \in L^1(E)$. Let S be a closed subset of E and assume that for all measurable sets $A \in \mathcal{F}$ with finite and non-zero measure the following holds*

$$\frac{1}{\mu(A)} \int_A f \, d\mu \in S$$

Then $f(x) \in S$ for μ-almost all x.

Proof. Without loss of generality we will show the statement assuming $\mu(\Omega) < \infty$. Let $v \in E$ and $v \notin S$.

We show by contradiction that if $B_r(v) \cap S = \varnothing$, then $A := f^{-1}(B_r(v))$, the set of all $x \in \Omega$ such that $f(x) \in B_r(v)$, is a μ-null set. Assume $\mu(A) > 0$. We have

$$\begin{aligned}\left\|\frac{1}{\mu(A)}\int_A f \,\mathrm{d}\mu - v\right\| &= \left\|\frac{1}{\mu(A)}\int_A f - v \,\mathrm{d}\mu\right\| \\ &\leq \frac{1}{\mu(A)}\int_A \|f - v\| \,\mathrm{d}\mu \\ &< r\end{aligned}$$

The last inequality follows from the fact that $f(x) \in B_r(v)$ for $x \in A$. This contradicts our first assumption. Therefore $\mu(A) = 0$.

Similar to the notation in Isabelle, we will use $-S$ to denote the complement of S. $-S$ is an open subset of E. By the previous lemma, there exist open balls $B_{r_i}(w_i)$ with $r_i \in \mathbb{Q}_{\geq 0}$, $w_i \in D$ for $i \in \mathbb{N}$ such that $\bigcup_i B_{r_i}(w_i) = -S$. Obviously, $w_i \in E \setminus S$ and $B_{r_i}(w_i) \cap S = \varnothing$ for $i \in \mathbb{N}$. It follows

$$\begin{aligned}\mu(f^{-1}(-S)) &= \mu\left(\bigcup_i f^{-1}(B_{r_i}(w_i))\right) \\ &\leq \sum_i \mu(f^{-1}(B_{r_i}(w_i))) \\ &= 0\end{aligned}$$

Thus $\{f \notin S\}$ is a μ-null set, which completes the proof.

□

At the beginning of our proof, we assumed $\mu(\Omega) < \infty$ without loss of generality. This is only possible, since we assumed the measure space in question to be σ-finite. To simplify the formalization of proofs employing this argument, we have introduced the following induction scheme

Lemma 3.1.3

```
lemma sigma_finite_measure_induct:
  assumes "⋀N Ω. finite_measure N
          ⟹ N = restrict_space M Ω
          ⟹ Ω ∈ sets M
          ⟹ emeasure N Ω ≠ ∞
          ⟹ emeasure N Ω ≠ 0
          ⟹ almost_everywhere N Q"
  and "Measurable.pred M Q"
  shows "almost_everywhere M Q"
```

This induction scheme allows us prove results about a σ-finite measure space M, assuming that we can show the property on arbitrary subspaces of M with finite measure. For ease of use we include additional assumptions such as `emeasure` $N\ \Omega \neq 0$ which let us to avoid unnecessary trivial cases. The proof of this induction scheme is straightforward.

Proof. Let $M = (\Omega, \Sigma, \mu)$ be a σ-finite measure space. There exists a family of sets with finite measure $(\Omega_i)_{i\in\mathbb{N}}$ such that $\bigcup_{i\in\mathbb{N}} \Omega_i = \Omega$. By assumption, the property Q holds μ-almost everywhere on all Ω_i. Therefore the sets $\Omega_i \cap \{\neg Q\} \in \Sigma|_{\Omega_i} \subseteq \Sigma$ are all μ-null sets. This means that $\bigcup_{i\in\mathbb{N}}(\Omega_i \cap \{\neg Q\}) = \{\neg Q\}$ is also μ-null set, which completes the proof. $\square$

Now that we have the averaging theorem at our disposal, we can lift the following results from the real case, to our more general setting.

Corollary 3.1.4. *Let $f \in L^1(E)$ and $\int_A f\, d\mu = 0$ for all measurable sets $A \subseteq \Omega$. Then $f = 0$ μ-almost everywhere.*

Proof. Apply the averaging theorem with $S = \{0\}$. $\square$

Corollary 3.1.5. *(Uniqueness of Densities) Let $f, g \in L^1(E)$ and $\int_A f\, d\mu = \int_A g\, d\mu$ for all measurable sets $A \subseteq \Omega$. Then $f = g$ μ-almost everywhere.*

Proof. Follows directly from the previous corollary. $\square$

Corollary 3.1.6. *Let E be linearly orderable. Let $f \in L^1(E)$ and $\int_A f\, d\mu \geq 0$ for all measurable sets $A \subseteq \Omega$. Then f is non-negative μ-almost everywhere.*

Proof. Our first assumption guarantees that $\{y \in E \mid y \geq 0\}$ is a closed subset of E. Applying the averaging theorem on this set, yields the desired result. $\square$

The corollary on the uniqueness of densities is crucial in showing that the conditional expectation is unique as an element of $L^1(E)$.

3.1.2 Diameter Lemma

The goal of this subsection is to prove the diameter lemma, which provides a characterization of Cauchy sequences in metric spaces.

Definition 3.1.7. Let E be a metric space with metric $d : E \times E \to \mathbb{R}$. The diameter of a set A is defined as

$$\operatorname{diam}(A) = \sup_{x,y\in A} d(x, y)$$

Intuitively the diameter of a set A measures how spread out or "large" the set A is with respect to the distance defined by the metric.

Lemma 3.1.8. *(Diameter Lemma) Let E be a metric space with metric $d : E \times E \to \mathbb{R}$ and $(s_i)_{i\in\mathbb{N}} \subseteq E$ a sequence. Define $S_n = \{s_i \mid i \geq n\}$. The sequence $(s_i)_{i\in\mathbb{N}}$ is Cauchy, if and only if the S_0 is bounded and*

$$\lim_{n\to\infty} \operatorname{diam}(S_n) = 0$$

Proof.
$\Longrightarrow$: Assume $(s_i)_{i\in\mathbb{N}}$ is Cauchy.

Recall that a set A is bounded if there exists some $x \in E$ and $\varepsilon \in \mathbb{R}$ such that $d(x,y) \leq \varepsilon$ for all $y \in A$. Since $(s_i)_{i\in\mathbb{N}}$ Cauchy, there exists some $N \in \mathbb{N}$ such that $d(s_n, s_m) < 1$ for all $n, m \geq N$. The set $\{s_i \mid i \in \{0, \ldots, N\}\}$ is bounded since it is finite. Thus there exists some $a \in \mathbb{R}$ such that $d(s_N, s_i) < a$ for all $i \in \{0, \ldots, N\}$. Therefore $d(s_N, s_i) < \max(a, 1)$ for all $i \in \mathbb{N}$, which shows that S_0 is bounded.

We know $S_n \subseteq S_m$ for $n \geq m$. Therefore $\operatorname{diam}(S_n) < \infty$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. Then there exists some $N \in \mathbb{N}$ such that $d(s_n, s_m) < \frac{1}{2}\varepsilon$ for all $n, m \geq N$. Hence

$$\operatorname{diam}(S_N) = \sup_{x,y\in S_N} d(x,y) \leq \frac{1}{2}\varepsilon < \varepsilon$$

Furthermore, we have $\operatorname{diam}(S_n) \leq \operatorname{diam}(S_N)$ for $n \geq N$ because of the subset relation stated above. Thus $\lim_{n\to\infty} \operatorname{diam}(S_n) = 0$.
$\Longleftarrow$: Assume $\lim_{n\to\infty} \operatorname{diam}(S_n) = 0$ and that S_0 is bounded.

Hence $\operatorname{diam}(S_n) < \infty$ for all $n \in \mathbb{N}$ with the same argument as above.

Let $\varepsilon > 0$. There exists some $N \in \mathbb{N}$ such that $\sup_{x,y\in S_n} d(x,y) < \varepsilon$ for all $n \geq N$. Hence $d(x,y) < \varepsilon$ for all $x, y \in S_n$ for $n \geq N$. This implies $d(s_i, s_j) < \varepsilon$ for all $i, j \geq n \geq N$, which shows that $(s_i)_{i\in\mathbb{N}}$ is Cauchy. □

In our construction of the conditional expectation, we will use the diameter lemma to show that the limit of a sequence of simple functions admits a conditional expectation. In anticipation of this, we present the following lemmas concerning measurability and integrability.

Lemma 3.1.9

```
lemma borel_measurable_diameter:
  assumes "⋀x. x ∈ space M ⟹ bounded (range (λi. s i x))"
          "⋀i. (s i) ∈ borel_measurable M"
  shows "(λx. diameter {s i x | i. n ≤ i}) ∈ borel_measurable M"
```

Lemma 3.1.10

```
lemma integrable_bound_diameter:
  assumes "integrable M f"
  and "⋀i. (s i) ∈ borel_measurable M"
  and "⋀x i. x ∈ space M ⟹ norm (s i x) ≤ f x"
  shows "integrable M (λx. diameter {s i x | i. n ≤ i})"
```

3.1.3 Induction Schemes for Simple Integrable Functions

In the upcoming sections of our work, we will frequently need to prove statements about simple integrable functions. For simple functions $s : \Omega \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, the Isabelle theory `HOL_Analysis.Nonnegative_Lebesgue_Integration` already provides an induction scheme `simple_function_induct`. For our purposes we extend this scheme to cover simple integrable functions $s : \Omega \to E$. Notice that a simple function s is integrable if and only if $\mu(\{s \neq 0\}) < \infty$. The new induction scheme is as follows

Lemma 3.1.11

```
lemma simple_integrable_function_induct[case_names cong indicator add]:
  assumes "simple_function M f" "emeasure M {y ∈ space M. f y ≠ 0} ≠ ∞"
  assumes cong: "⋀f g. simple_function M f ⟹ emeasure M {y ∈ space M. f y ≠ 0} ≠ ∞
                  ⟹ simple_function M g ⟹ emeasure M {y ∈ space M. g y ≠ 0} ≠ ∞
                  ⟹ (⋀x. x ∈ space M ⟹ f x = g x) ⟹ P f ⟹ P g"
  assumes indicator: "⋀A y. A ∈ sets M ⟹ emeasure M A < ∞
                  ⟹ P (λx. indicator A x ·ℝ y)"
  assumes add: "⋀f g. simple_function M f ⟹ emeasure M {y ∈ space M. f y ≠ 0} ≠ ∞
                  ⟹ simple_function M g ⟹ emeasure M {y ∈ space M. g y ≠ 0} ≠ ∞
                  ⟹ (⋀z. z ∈ space M ⟹ norm (f z + g z) = norm (f z) + norm (g z))
                  ⟹ P f ⟹ P g ⟹ P (λx. f x + g x)"
  shows "P f"
```

The idea of the induction scheme is simple. We know f can be represented μ-a.e. as a finite sum $\sum_{i=1}^{n} \mathbf{1}_{A_i} \cdot_{\mathbb{R}} c_i$ for some collection of measurable sets $(A_i)_{i=1,\ldots,n}$ and elements $c_i \in E$. We do induction on n. In this sense, "indicator" corresponds to the induction basis, while "add" corresponds to the induction step. Since f is representable as a finite sum μ-a.e. we need the additional assumption "cong" to make sure P is a well defined predicate on the space $L^1(E)$.

Remark. To make proving certain properties easier, we have the additional assumption $\|f(x) + g(x)\| = \|f(x)\| + \|g(x)\|$ in the induction step "add". It is easy to see why we can assume this without loss of generality. If we have some simple function $s = \sum_{i=1}^{n} \mathbf{1}_{A_i} \cdot_{\mathbb{R}} c_i$, we can assume the sets A_i to be pairwise disjoint. Thus, if $x \in A_j$ for some $j \leq n$ we have $\|s(x)\| = \|\mathbf{1}_{A_j}(x) \cdot c_j\| = \sum_{i=1}^{n} \mathbf{1}_{A_i} \cdot \|c_i\|$.

When working with an ordering on E, we may need to concern ourselves with non-negative simple functions. For this goal, we have the following induction scheme.

Lemma 3.1.12

```
lemma simple_integrable_function_induct_nn[case_names cong indicator add]:
  assumes "simple_function M f" "emeasure M {y ∈ space M. f y ≠ 0} ≠ ∞"
          "⋀ x. x ∈ space M ⟶ f x ≥ 0"
  assumes cong: "⋀ f g. simple_function M f ⟹ emeasure M {y ∈ space M. f y ≠ 0} ≠ ∞
                    ⟹ (⋀ x. x ∈ space M ⟹ f x ≥ 0)
                    ⟹ simple_function M g ⟹ emeasure M {y ∈ space M. g y ≠ 0} ≠ ∞
                    ⟹ (⋀ x. x ∈ space M ⟹ g x ≥ 0)
                    ⟹ (⋀ x. x ∈ space M ⟹ f x = g x) ⟹ P f ⟹ P g"
  assumes indicator: "⋀ A y. y ≥ 0 ⟹ A ∈ sets M ⟹ emeasure M A < ∞
                         ⟹ P (λx. indicator A x ·ℝ y)"
  assumes add: "⋀ f g. simple_function M f ⟹ emeasure M {y ∈ space M. f y ≠ 0} ≠ ∞
                  ⟹ (⋀ x. x ∈ space M ⟹ f x ≥ 0)
                  ⟹ simple_function M g ⟹ emeasure M {y ∈ space M. g y ≠ 0} ≠ ∞
                  ⟹ (⋀ x. x ∈ space M ⟹ g x ≥ 0)
                  ⟹ (⋀ z. z ∈ space M ⟹ norm (f z + g z) = norm (f z) + norm (g z))
                  ⟹ P f ⟹ P g ⟹ P (λx. f x + g x)"
  shows "P f"
```

The induction scheme looks complicated and cumbersome, but in essence it is the same induction scheme as the previous one with the added assumption of non-negativity everywhere. The proof is also largely the same. We just need to show that the partial sums stay non-negative all the way through.

3.1.4 Bochner Integration on Linearly Ordered Banach Spaces

In the context of real numbers, the following statement is easy to show.

$$\text{Let } f, g : \Omega \to \mathbb{R} \text{ be integrable and } f \geq g \ \mu\text{-a.e., then } \int f \, d\mu \geq \int g \, d\mu.$$

In this subsection, we aim to provide similar results for functions $f, g : \Omega \to E$ with E a linearly ordered Banach space. For the remainder of our discourse a topological space E is linearly ordered, if there exists a total ordering on E such that the topology on E and the order topology induced by the ordering coincide.

We start with the following lemma

Lemma 3.1.13. *Let $f \in L^1(E)$ and $f \geq 0$ μ-a.e. Then $\int f \, d\mu \geq 0$.*

Proof. Since $f \in L^1(E)$, there exists a sequence of integrable simple functions $(s_n)_{n \in \mathbb{N}}$, such that $\lim_{n \to \infty} s_n(x) = f(x)$ μ-a.e. and $\lim_{n \to \infty} \int s_n \, d\mu = \int f \, d\mu$. At first, we have no further information about s_n. However, since we know that $f \geq 0$ μ-a.e, it follows that $f = \max(0, f)$ μ-a.e. Using dominated convergence and the fact that the function $\max(0, \cdot)$ is continuous w.r.t to the order topology on E, we can show

$$\lim_{n\to\infty} \max(0, s_n(x)) = \max(0, f(x)) \ \mu\text{-a.e.}$$

and

$$\lim_{n\to\infty} \int \max(0, s_n) \, \mathrm{d}\mu = \int \max(0, f) \, \mathrm{d}\mu$$

The function $\max(0, s_n)$ is still a simple and integrable function, which has the additional property of being always non-negative.

We will now show that if h is a non-negative simple function, then $\int h \, \mathrm{d}\mu \geq 0$. For this purpose we will use the induction scheme for non-negative simple integrable functions that we proved in the previous subsection.

Case "cong": Let $h = g$ μ-a.e. and $\int g \, \mathrm{d}\mu \geq 0$. It follows directly

$$\int h \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu \geq 0$$

Case "indicator": Let $h = \mathbf{1}_A \cdot_{\mathbb{R}} y$ for some measurable set A with finite measure and $y \in E$ with $y \geq 0$. It follows directly

$$\int h \, \mathrm{d}\mu = \mu(A) \cdot_{\mathbb{R}} y \geq 0$$

Case "add": Let $h = h_1 + h_2$ for some simple integrable functions h_1 and h_2. By the induction hypothesis, we have $\int h_i \, \mathrm{d}\mu \geq 0$ for $i = 1, 2$. Therefore

$$\int h \, \mathrm{d}\mu = \int h_1 \, \mathrm{d}\mu + \int h_2 \, \mathrm{d}\mu \geq 0$$

Hence, we know $\int \max(0, s_n) \, \mathrm{d}\mu \geq 0$ for all $n \in \mathbb{N}$. Therefore, the same must hold for the limit $\lim_{n\to\infty} \int \max(0, s_n) \, \mathrm{d}\mu = \int \max(0, f) \, \mathrm{d}\mu$. Since $f = \max(0, f)$ μ-a.e., we have $\int f \, \mathrm{d}\mu = \int \max(0, f) \, \mathrm{d}\mu$ and the statement follows.

$\square$

Remark. For the proof of this statement, we need the topology on E to coincide with the order topology. Otherwise, we can't guarantee statements such as $(\forall i.\ x_i \geq 0) \implies \lim_{i\to\infty} x_i \geq 0$ or the continuity of the max function.

This lemma entails the following corollary.

Corollary 3.1.14. *Let $f, g \in L^1(E)$ and $f \geq g$ μ-a.e. Then $\int f \, d\mu \geq \int g \, d\mu$.*

In Isabelle, we can replace the assumption $f \in L^1(E)$ with Borel measurability, since a non-integrable function has the value of its integral set to 0 by default. The lemma above can be stated as

```
lemma integral_nonneg_AE_banach:
  assumes "f ∈ borel_measurable M" and "AE x in M. 0 ≤ f x"
  shows "0 ≤ integral^L M f"
proof (cases "integrable M f")
...
qed
```

3.2 Constructing the Conditional Expectation

Before we can talk about *the* conditional expectation, we must define what it means for a function to have *a* conditional expectation. For this purpose we define the following predicate

Definition 3.2.1

```
definition has_cond_exp :: "'a measure ⇒ 'a measure ⇒ ('a ⇒ 'b) ⇒ ('a ⇒ 'b) ⇒ bool" where
  "has_cond_exp M F f g = (∀A ∈ sets F. ∫_A f ∂M = ∫_A g ∂M)
                  ∧ integrable M f
                  ∧ integrable M g
                  ∧ g ∈ borel_measurable F"
```

This predicate precisely characterizes what it means for a function f to have a conditional expectation g w.r.t the measure M and the sub-σ-algebra F. Now we can use Hilbert's ϵ-operator, `SOME` in Isabelle [NPW02], to define *the* conditional expectation, if it exists.

Definition 3.2.2

```
definition cond_exp :: "'a measure ⇒ 'a measure ⇒ ('a ⇒ 'b) ⇒ ('a ⇒ 'b) ⇒ bool" where
  "cond_exp M F f = (if ∃g. has_cond_exp M F f g then (SOME g. has_cond_exp M F f g) else (λ_.0))"
```

A major advantage of defining the conditional expectation this way is that it allows us to make statements about its measurability and integrability, without needing to show existence or uniqueness. We have

Lemma 3.2.3

```
lemma borel_measurable_cond_exp: "cond_exp M F f ∈ borel_measurable F"
  by (metis cond_exp_def someI has_cond_exp_def borel_measurable_const)
```

Lemma 3.2.4

```
lemma integrable_cond_exp: "integrable M (cond_exp M F f)"
  by (metis cond_exp_def has_cond_expD(3) integrable_zero someI)
```

3.2.1 Uniqueness

3.2.2 Existence

3.2.3 Properties of the Conditional Expectation

Tower Property

Contractivity

Pulling Out What's Known

3.3 Conditional Expectation on Linearly Ordered Banach Spaces

4 Stochastic Processes

4.1 Filtered Measure Spaces

4.2 Adapted Processes

4.3 Progressively Measurable Processes

4.4 Predictable Processes

4.5 Discrete Time Processes

5 Martingales

5.1 Definitions

5.2 Necessary and Sufficient Conditions

5.3 Discrete-Time Martingales

6 Discussion

6.1 Formalization Approach

6.2 Comparison with Existing Formalizations

The following tables provide a list of the entries in the mathlib formalization of martingales, all of which have counterparts in our formalization.

Lean	Isabelle
`martingale`	`martingale (locale)`
`martingale.adapted`	`adapted_process.adapted`
`martingale.add`	`martingale.add`
`martingale.condexp_ae_eq`	`martingale.martingale_property`
`martingale.eq_zero_of_predictable`	`martingale.predictable_eq_zero`
`martingale.integrable`	`martingale.integrable`
`martingale.neg`	`martingale.uminus`
`martingale.set_integral_eq`	`martingale.set_integral_eq`
`martingale.smul`	`martingale.scaleR`
`martingale.strongly_measurable`	`stochastic_process.random_variable`
`martingale.sub`	`martingale.diff`
`martingale.submartingale`	via sublocale relation
`martingale.supermartingale`	via sublocale relation
`martingale_condexp`	`filtered_sigma_finite_measure.martingale_cond_exp`
`martingale_const`	`filtered_sigma_finite_measure.martingale_const`
`martingale_const_fun`	`filtered_sigma_finite_measure.martingale_const`
`martingale_iff`	`martingale_iff`
`martingale_nat`	`nat_sigma_finite_adapted_process.martingale_nat`
`martingale_of_condexp_sub_eq_zero_nat`	`nat_sigma_finite_adapted_process.martingale_of _cond_exp_diff_Suc_eq_zero`
`martingale_of_set_integral_eq_succ`	`nat_sigma_finite_adapted_process.martingale_of _set_integral_eq_Suc`
`martingale_zero`	`filtered_sigma_finite_measure.martingale_zero`

Table 6.1: Lookup table for martingale lemmas and definitions

Lean	Isabelle
submartingale	submartingale (locale)
submartingale.adapted	adapted_process.adapted
submartingale.add	submartingale.add
submartingale.add_martingale	submartingale.add
submartingale.ae_le_condexp	submartingale_property
submartingale.condexp_sub_nonneg	submartingale.cond_exp_diff_nonneg
submartingale.integrable	submartingale.integrable
submartingale.neg	submartingale.uminus
submartingale.pos	submartingale.max_0
submartingale.set_integral_le	submartingale.set_integral_le
submartingale.smul_nonneg	submartingale.scaleR_nonneg
submartingale.smul_nonpos	submartingale.scaleR_nonpos
submartingale.strongly_measurable	stochastic_process.random_variable
submartingale.sub_martingale	submartingale.diff
submartingale.sub_supermartingale	submartingale.diff
submartingale.sum_mul_sub	nat_submartingale.partial_sum_scaleR
submartingale.sum_mul_sub'	nat_submartingale.partial_sum_scaleR'
submartingale.sup	submartingale.max
submartingale.zero_le_of_predictable	nat_submartingale.predictable_ge_bot
submartingale_nat	nat_sigma_finite_adapted_process.submartingale_nat
submartingale_of_condexp_sub_nonneg	sigma_finite_adapted_process.submartingale_of _cond_exp_diff_nonneg
submartingale_of_condexp_sub_nonneg_nat	nat_sigma_finite_adapted_process.submartingale_of _cond_exp_diff_Suc_nonneg
submartingale_of_set_integral_le	sigma_finite_adapted_process.submartingale_of _set_integral_le
submartingale_of_set_integral_le_succ	nat_sigma_finite_adapted_process.submartingale_of _set_integral_le_Suc

Table 6.2: Lookup table for submartingale lemmas and definitions

Lean	Isabelle
supermartingale	supermartingale (locale)
supermartingale.adapted	adapted_process.adapted
supermartingale.add	supermartingale.add
supermartingale.add_martingale	supermartingale.add
supermartingale.condexp_ae_le	supermartingale_property
supermartingale.integrable	supermartingale.integrable
supermartingale.le_zero_of_predictable	supermartingale.predictable_le_zero
supermartingale.neg	supermartingale.uminus
supermartingale.set_integral_le	supermartingale.set_integral_ge

`supermartingale.smul_nonneg`	`supermartingale.scaleR_nonneg`
`supermartingale.smul_nonpos`	`supermartingale.scaleR_nonpos`
`supermartingale.strongly_measurable`	`stochastic_process.random_variable`
`supermartingale.sub_martingale`	`supermartingale.diff`
`supermartingale.sub_submartingale`	`supermartingale.diff`
`supermartingale_nat`	`nat_sigma_finite_adapted_process.supermartingale_nat`
`supermartingale_of_condexp_sub_nonneg_nat`	`nat_sigma_finite_adapted_process.supermartingale_of _cond_exp_diff_Suc_nonneg`
`supermartingale_of_set_integral_succ_le`	`nat_sigma_finite_adapted_process.supermartingale_of _set_integral_le_Suc`

Table 6.3: Lookup table for supermartingale lemmas and definitions

6.3 Challenges and Limitations

6.3.1

6.4 Future Research

Semimartingales

Doob's Martingale Convergence

Fundamental Theorem of Arbitrage The fundamental theorem of asset pricing relates the concept of a fair market price for a financial asset to the notion of a risk-neutral measure.

It provides the necessary and sufficient conditions for a market to be arbitrage-free.

In this framework, the prices of financial assets can be treated as martingales, ensuring that there is no arbitrage opportunity.

7 Conclusion

Concluded.

Abbreviations

AFP Archive of Formal Proofs

List of Figures

List of Tables

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