On the Formalization of Martingales

Ata Keskin

October 31, 2023

Abstract

This thesis presents a formalization of martingales in arbitrary Banach spaces using Isabelle/HOL. We begin by examining formalizations in prominent proof repositories and extend the definition of the conditional expectation operator from real numbers to general Banach spaces, drawing inspiration from prior work. We define filtered measure spaces, adapted, progressively measurable, and predictable processes and rigorously formalize martingales, submartingales, and supermartingales. Additionally, our contributions expand the scope of Bochner integration techniques to general Banach spaces and introduce additional lemmas and induction schemes for integrable functions. Our formalization provides a robust framework for future developments within the theory of stochastic processes. Furthermore, we provide an example demonstrating when a coin toss constitutes a martingale, submartingale, or a supermartingale.

Contents

theory *Measure-Space-Supplement*
imports *HOL−Analysis.Measure-Space*
begin

1 Supplementary Lemmas for Measure Spaces

1.1 Sigma Algebra Generated by a Family of Functions

definition *family-vimage-algebra* :: $'a$ *set* $\Rightarrow$ $('a \Rightarrow 'b)$ *set* $\Rightarrow$ $'b$ *measure* $\Rightarrow$ $'a$ *measure* **where**
family-vimage-algebra Ω S M $\equiv$ *sigma* Ω $(\bigcup f \in S.\ \{f -` A \cap \Omega \mid A.\ A \in M\})$

lemma *family-vimage-algebra-singleton*: *family-vimage-algebra* Ω $\{f\}$ M = *vimage-algebra* Ω f M **unfolding** *family-vimage-algebra-def vimage-algebra-def* **by** *simp*

lemma
shows *sets-family-vimage-algebra*: *sets* (*family-vimage-algebra* Ω S M) = *sigma-sets* Ω $(\bigcup f \in S.\ \{f -` A \cap \Omega \mid A.\ A \in M\})$
and *space-family-vimage-algebra*[*simp*]: *space* (*family-vimage-algebra* Ω S M) = Ω
by (*auto simp add*: *family-vimage-algebra-def sets-measure-of-conv space-measure-of-conv*)

lemma *measurable-family-vimage-algebra*:
assumes $f \in S$ $f \in \Omega \to$ *space* M
shows $f \in$ *family-vimage-algebra* Ω S M $\to_M$ M
using *assms* **by** (*intro measurableI*, *auto simp add*: *sets-family-vimage-algebra*)

lemma *measurable-family-vimage-algebra-singleton*:
assumes $f \in \Omega \to$ *space* M
shows $f \in$ *family-vimage-algebra* Ω $\{f\}$ M $\to_M$ M
using *assms measurable-family-vimage-algebra* **by** *blast*

lemma *measurable-family-iff-sets*:
shows $(S \subseteq N \to_M M) \longleftrightarrow S \subseteq$ *space* $N \to$ *space* M $\wedge$ *family-vimage-algebra* (*space* N) S $M \subseteq N$
proof (*standard*, *goal-cases*)
case *1*
hence *subset*: $S \subseteq$ *space* $N \to$ *space* M **using** *measurable-space* **by** *fast*
have $\{f -` A \cap$ *space* N $|A.\ A \in M\} \subseteq N$ **if** $f \in S$ **for** f **using** *measurable-iff-sets*[*unfolded family-vimage-algebra-singleton*[*symmetric*], *of f*] *1 subset that* **by** (*fastforce simp add*: *sets-family-vimage-algebra*)
then show *?case* **unfolding** *sets-family-vimage-algebra* **using** *sets.sigma-algebra-axioms* **by** (*simp add*: *subset*, *intro sigma-algebra.sigma-sets-subset*, *blast+*)
next
case *2*
hence *subset*: $S \subseteq$ *space* $N \to$ *space* M **by** *simp*
show *?case*

proof (*standard*, *goal-cases*)
case (*1 x*)
have *family-vimage-algebra* (*space N*) $\{x\}$ *M* $\subseteq$ *N* **by** (*metis* (*no-types*, *lifting*) *1 2 sets-family-vimage-algebra SUP-le-iff sigma-sets-le-sets-iff singletonD*)
thus *?case* **using** *measurable-iff-sets*[*unfolded family-vimage-algebra-singleton*[*symmetric*]] *subset*[*THEN subsetD*, *OF 1*] **by** *fast*
qed
qed

lemma *family-vimage-algebra-diff*:
shows *family-vimage-algebra* Ω *S M* = *sigma* Ω (*sets* (*family-vimage-algebra* Ω ($S - I$) *M*) $\cup$ *family-vimage-algebra* Ω ($S \cap I$) *M*)
using *sets.space-closed space-measure-of-conv*
unfolding *family-vimage-algebra-def sets-family-vimage-algebra*
by (*intro sigma-eqI*, *blast*, *fastforce*)
(*intro sigma-sets-eqI*, *blast*, *simp add*: *sets-measure-of-conv split*: *if-splits*, *meson Diff-subset Sup-subset-mono in-mono inf-sup-ord*(*1*) *sigma-sets-subseteq subset-image-iff*, *fastforce*+)

end
theory *Elementary-Metric-Spaces-Supplement*
imports *HOL−Analysis.Elementary-Metric-Spaces*
begin

2 Supplementary Lemmas for Elementary Metric Spaces

2.1 Diameter Lemma

lemma *diameter-comp-strict-mono*:
fixes *s* :: *nat* $\Rightarrow$ *'a* :: *metric-space*
assumes *strict-mono r bounded* $\{s\ i\ | i.\ r\ n \leq i\}$
shows *diameter* $\{s\ (r\ i)\ |\ i.\ n \leq i\} \leq$ *diameter* $\{s\ i\ |\ i.\ r\ n \leq i\}$
proof (*rule diameter-subset*)
show $\{s\ (r\ i)\ |\ i.\ n \leq i\} \subseteq \{s\ i\ |\ i.\ r\ n \leq i\}$ **using** *assms*(*1*) *monotoneD strict-mono-mono* **by** *fastforce*
qed (*intro assms*(*2*))

lemma *diameter-bounded-bound'*:
fixes *S* :: *'a* :: *metric-space set*
assumes *S*: *bdd-above* (*case-prod dist* ‘ ($S \times S$)) $x \in S$ $y \in S$
shows *dist x y* $\leq$ *diameter S*
proof −
have $(x,y) \in S \times S$ **using** *S* **by** *auto*
then have *dist x y* $\leq$ (*SUP* $(x,y) \in S \times S$. *dist x y*) **by** (*rule cSUP-upper2*[*OF assms*(*1*)]) *simp*
with ‹$x \in S$› **show** *?thesis* **by** (*auto simp*: *diameter-def*)
qed

lemma *bounded-imp-dist-bounded*:
assumes *bounded* (*range s*)
shows *bounded* $((\lambda(i, j).\ dist\ (s\ i)\ (s\ j))$ ‘ $(\{n..\} \times \{n..\}))$
using *bounded-dist-comp*[*OF bounded-fst bounded-snd*, *OF bounded-Times*(*1*,*1*)[*OF assms*(*1*,*1*)]] **by** (*rule bounded-subset*, *force*)

lemma *cauchy-iff-diameter-tends-to-zero-and-bounded*:
fixes *s* :: *nat* ⇒ *'a* :: *metric-space*
shows *Cauchy s* $\longleftrightarrow$ $((\lambda n.\ diameter\ \{s\ i \mid i.\ i \geq n\}) \longrightarrow 0 \wedge bounded\ (range\ s))$
proof −
have $\{s\ i\ | i.\ N \leq i\} \neq \{\}$ **for** *N* **by** *blast*
hence *diameter-SUP*: $diameter\ \{s\ i\ |i.\ N \leq i\} = (SUP\ (i, j) \in \{N..\} \times \{N..\}.\ dist\ (s\ i)\ (s\ j))$ **for** *N* **unfolding** *diameter-def* **by** (*auto intro*!: *arg-cong*[*of* - - *Sup*])
show *?thesis*
proof (*intro iffI*)
assume *asm*: *Cauchy s*
have $\exists N.\ \forall n \geq N.\ norm\ (diameter\ \{s\ i\ |i.\ n \leq i\}) < e$ **if** *e-pos*: $e > 0$ **for** *e*
proof −
obtain *N* **where** *dist-less*: $dist\ (s\ n)\ (s\ m) < (1/2) * e$ **if** $n \geq N$ $m \geq N$ **for** *n m* **using** *asm e-pos* **by** (*metis Cauchy-def mult-pos-pos zero-less-divide-iff zero-less-numeral zero-less-one*)
{
fix *r* **assume** $r \geq N$
hence $dist\ (s\ n)\ (s\ m) < (1/2) * e$ **if** $n \geq r$ $m \geq r$ **for** *n m* **using** *dist-less that* **by** *simp*
hence $(SUP\ (i, j) \in \{r..\} \times \{r..\}.\ dist\ (s\ i)\ (s\ j)) \leq (1/2) * e$ **by** (*intro cSup-least*) *fastforce*+
also have ... $< e$ **using** *e-pos* **by** *simp*
finally have $diameter\ \{s\ i\ |i.\ r \leq i\} < e$ **using** *diameter-SUP* **by** *presburger*
}
moreover have $diameter\ \{s\ i\ |i.\ r \leq i\} \geq 0$ **for** *r* **unfolding** *diameter-SUP* **using** *bounded-imp-dist-bounded*[*OF cauchy-imp-bounded*, *THEN bounded-imp-bdd-above*, *OF asm*] **by** (*intro cSup-upper2*, *auto*)
ultimately show *?thesis* **by** *auto*
qed
thus $(\lambda n.\ diameter\ \{s\ i\ |i.\ n \leq i\}) \longrightarrow 0 \wedge bounded\ (range\ s)$ **using** *cauchy-imp-bounded*[*OF asm*] **by** (*simp add*: *LIMSEQ-iff*)
next
assume *asm*: $(\lambda n.\ diameter\ \{s\ i\ |i.\ n \leq i\}) \longrightarrow 0 \wedge bounded\ (range\ s)$
have $\exists N.\ \forall n \geq N.\ \forall m \geq N.\ dist\ (s\ n)\ (s\ m) < e$ **if** *e-pos*: $e > 0$ **for** *e*
proof −
obtain *N* **where** *diam-less*: $diameter\ \{s\ i\ |i.\ r \leq i\} < e$ **if** $r \geq N$ **for** *r* **using** *LIMSEQ-D asm*(*1*) *e-pos* **by** *fastforce*
{
fix *n m* **assume** $n \geq N$ $m \geq N$
hence $dist\ (s\ n)\ (s\ m) < e$ **using** *cSUP-lessD*[*OF bounded-imp-dist-bounded*[*THEN bounded-imp-bdd-above*], *OF* - *diam-less*[*unfolded diameter-SUP*]] *asm* **by** *auto*
}

thus *?thesis* **by** *blast*
qed
then show *Cauchy s* **by** (*simp add*: *Cauchy-def*)
qed
qed

end

theory *Bochner-Integration-Supplement*
imports *HOL−Analysis.Bochner-Integration HOL−Analysis.Set-Integral Elementary-Metric-Spaces-Supplement*
begin

3 Supplementary Lemmas for Bochner Integration

3.1 Integrable Simple Functions

lemma *integrable-implies-simple-function-sequence*:
fixes f :: $'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes *integrable M f*
obtains s **where** $\bigwedge i.$ *simple-function M* $(s\ i)$
and $\bigwedge i.$ *emeasure M* $\{y \in$ *space M*. $s\ i\ y \neq 0\} \neq \infty$
and $\bigwedge x.\ x \in$ *space M* $\Longrightarrow (\lambda i.\ s\ i\ x) \longrightarrow f\ x$
and $\bigwedge x\ i.\ x \in$ *space M* $\Longrightarrow$ *norm* $(s\ i\ x) \leq 2 * $ *norm* $(f\ x)$
proof−
have f: $f \in$ *borel-measurable M* $(\int^+ x.$ *norm* $(f\ x)\ \partial M) < \infty$ **using** *assms* **unfolding** *integrable-iff-bounded* **by** *auto*
obtain s **where** s: $\bigwedge i.$ *simple-function M* $(s\ i)$ $\bigwedge x.\ x \in$ *space M* $\Longrightarrow (\lambda i.\ s\ i\ x) \longrightarrow f\ x$ $\bigwedge i\ x.\ x \in$ *space M* $\Longrightarrow$ *norm* $(s\ i\ x) \leq 2 *$ *norm* $(f\ x)$ **using** *borel-measurable-implies-sequence-metric*[*OF* $f(1)$] **unfolding** *norm-conv-dist* **by** *metis*
{
fix i
have $(\int^+ x.$ *norm* $(s\ i\ x)\ \partial M) \leq (\int^+ x.$ *ennreal* $(2 *$ *norm* $(f\ x))\ \partial M)$ **using** s **by** (*intro nn-integral-mono*, *auto*)
also have $\ldots < \infty$ **using** f **by** (*simp add*: *nn-integral-cmult ennreal-mult-less-top ennreal-mult*)
finally have *sbi*: *Bochner-Integration.simple-bochner-integrable M* $(s\ i)$ **using** s **by** (*intro simple-bochner-integrableI-bounded*) *auto*
hence *emeasure M* $\{y \in$ *space M*. $s\ i\ y \neq 0\} \neq \infty$ **by** (*auto intro*: *integrableI-simple-bochner-integrable simple-bochner-integrable.cases*)
}
thus *?thesis* **using** *that s* **by** *blast*
qed

lemma *simple-function-indicator-representation*:
fixes f :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*}
assumes f: *simple-function M f* **and** x: $x \in$ *space M*

shows $f\ x = (\sum y \in f\ `\ space\ M.\ indicator\ (f\ -`\ \{y\} \cap space\ M)\ x *_R\ y)$
(**is** $?l = ?r$)
proof $-$
have $?r = (\sum y \in f\ `\ space\ M.$
$(if\ y = f\ x\ then\ indicator\ (f\ -`\ \{y\} \cap space\ M)\ x *_R\ y\ else\ 0))$ **by** (*auto intro*!: *sum.cong*)
also have ... $=$ $indicator\ (f\ -`\ \{f\ x\} \cap space\ M)\ x *_R\ f\ x$ **using** *assms* **by** (*auto dest*: *simple-functionD*)
also have ... $= f\ x$ **using** *x* **by** (*auto simp*: *indicator-def*)
finally show *?thesis* **by** *auto*
qed

lemma *simple-function-indicator-representation-AE*:
fixes $f :: 'a \Rightarrow 'b :: \{second\text{-}countable\text{-}topology,\ banach\}$
assumes *f*: *simple-function M f*
shows $AE\ x\ in\ M.\ f\ x = (\sum y \in f\ `\ space\ M.\ indicator\ (f\ -`\ \{y\} \cap space\ M)\ x *_R\ y)$
by (*metis* (*mono-tags*, *lifting*) *AE-I2 simple-function-indicator-representation f*)

lemmas *simple-function-scaleR*[*intro*] = *simple-function-compose2*[**where** $h{=}(*_R)$]
lemmas *integrable-simple-function* = *simple-bochner-integrable.intros*[*THEN has-bochner-integral-simple-bochn*
THEN integrable.intros]

Induction rule for simple integrable functions.

lemma *integrable-simple-function-induct*[*consumes 2*, *case-names cong indicator add*, *induct set*: *simple-function*]:
fixes $f :: 'a \Rightarrow 'b :: \{second\text{-}countable\text{-}topology,\ banach\}$
assumes *f*: $simple\text{-}function\ M\ f\ emeasure\ M\ \{y \in space\ M.\ f\ y \neq 0\} \neq \infty$
assumes *cong*: $\bigwedge f\ g.\ simple\text{-}function\ M\ f \Longrightarrow emeasure\ M\ \{y \in space\ M.\ f\ y \neq 0\} \neq \infty$
$\Longrightarrow simple\text{-}function\ M\ g \Longrightarrow emeasure\ M\ \{y \in space\ M.\ g\ y \neq 0\} \neq \infty$
$\Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = g\ x) \Longrightarrow P\ f \Longrightarrow P\ g$
assumes *indicator*: $\bigwedge A\ y.\ A \in sets\ M \Longrightarrow emeasure\ M\ A < \infty \Longrightarrow P\ (\lambda x.\ indicator\ A\ x *_R\ y)$
assumes *add*: $\bigwedge f\ g.\ simple\text{-}function\ M\ f \Longrightarrow emeasure\ M\ \{y \in space\ M.\ f\ y \neq 0\} \neq \infty \Longrightarrow$
$simple\text{-}function\ M\ g \Longrightarrow emeasure\ M\ \{y \in space\ M.\ g\ y \neq 0\} \neq \infty \Longrightarrow$
$(\bigwedge z.\ z \in space\ M \Longrightarrow norm\ (f\ z + g\ z) = norm\ (f\ z) + norm\ (g\ z)) \Longrightarrow$
$P\ f \Longrightarrow P\ g \Longrightarrow P\ (\lambda x.\ f\ x + g\ x)$
shows *P f*
proof$-$
let $?f = \lambda x.\ (\sum y{\in}f\ `\ space\ M.\ indicat\text{-}real\ (f\ -`\ \{y\} \cap space\ M)\ x *_R\ y)$
have *f-ae-eq*: $f\ x = ?f\ x$ **if** $x \in space\ M$ **for** *x* **using** *simple-function-indicator-representation*[*OF f*(*1*) *that*] **.**
moreover have $emeasure\ M\ \{y \in space\ M.\ ?f\ y \neq 0\} \neq \infty$ **by** (*metis* (*no-types*, *lifting*) *Collect-cong calculation f*(*2*))

moreover have P $(\lambda x.\ \sum y{\in}S.\ \textit{indicat-real}\ (f\ -`\ \{y\}\cap \textit{space}\ M)\ x *_R y)$
$\textit{simple-function}\ M\ (\lambda x.\ \sum y{\in}S.\ \textit{indicat-real}\ (f\ -`\ \{y\}\cap \textit{space}\ M)\ x *_R y)$
$\textit{emeasure}\ M\ \{y \in \textit{space}\ M.\ (\sum x{\in}S.\ \textit{indicat-real}\ (f\ -`\ \{x\}\cap \textit{space}\ M)\ y *_R x) \neq \mathit{0}\} \neq \infty$
if $S \subseteq f$ ` *space* M **for** S **using** *simple-functionD*(*1*)[*OF assms*(*1*), *THEN rev-finite-subset*, *OF that*] *that*
proof (*induction rule*: *finite-induct*)
case *empty*
{
case *1*
then show *?case* **using** *indicator*[*of* {}] **by** *force*
next
case *2*
then show *?case* **by** *force*
next
case *3*
then show *?case* **by** *force*
}
next
case (*insert x F*)
have $(f\ -`\ \{x\} \cap \textit{space}\ M) \subseteq \{y \in \textit{space}\ M.\ f\ y \neq \mathit{0}\}$ **if** $x \neq \mathit{0}$ **using** *that* **by** *blast*
moreover have $\{y \in \textit{space}\ M.\ f\ y \neq \mathit{0}\} = \textit{space}\ M - (f\ -`\ \{\mathit{0}\} \cap \textit{space}\ M)$ **by** *blast*
moreover have $\textit{space}\ M - (f\ -`\ \{\mathit{0}\} \cap \textit{space}\ M) \in \textit{sets}\ M$ **using** *simple-functionD*(*2*)[*OF f*(*1*)] **by** *blast*
ultimately have *fin-0*: $\textit{emeasure}\ M\ (f\ -`\ \{x\} \cap \textit{space}\ M) < \infty$ **if** $x \neq \mathit{0}$ **using** *that* **by** (*metis emeasure-mono f*(*2*) *infinity-ennreal-def top.not-eq-extremum top-unique*)
hence *fin-1*: $\textit{emeasure}\ M\ \{y \in \textit{space}\ M.\ \textit{indicat-real}\ (f\ -`\ \{x\} \cap \textit{space}\ M)\ y *_R x \neq \mathit{0}\} \neq \infty$ **if** $x \neq \mathit{0}$ **by** (*metis* (*mono-tags*, *lifting*) *emeasure-mono f*(*1*) *indicator-simps*(*2*) *linorder-not-less mem-Collect-eq scaleR-eq-0-iff simple-functionD*(*2*) *subsetI that*)

have $*$: $(\sum y{\in}\textit{insert}\ x\ F.\ \textit{indicat-real}\ (f\ -`\ \{y\} \cap \textit{space}\ M)\ xa *_R y) = (\sum y{\in}F.\ \textit{indicat-real}\ (f\ -`\ \{y\} \cap \textit{space}\ M)\ xa *_R y) + \textit{indicat-real}\ (f\ -`\ \{x\} \cap \textit{space}\ M)\ xa *_R x$ **for** *xa* **by** (*metis* (*no-types*, *lifting*) *Diff-empty Diff-insert0 add.commute insert.hyps*(*1*) *insert.hyps*(*2*) *sum.insert-remove*)
have $**$: $\{y \in \textit{space}\ M.\ (\sum x{\in}\textit{insert}\ x\ F.\ \textit{indicat-real}\ (f\ -`\ \{x\} \cap \textit{space}\ M)\ y *_R x) \neq \mathit{0}\} \subseteq \{y \in \textit{space}\ M.\ (\sum x{\in}F.\ \textit{indicat-real}\ (f\ -`\ \{x\} \cap \textit{space}\ M)\ y *_R x) \neq \mathit{0}\} \cup \{y \in \textit{space}\ M.\ \textit{indicat-real}\ (f\ -`\ \{x\} \cap \textit{space}\ M)\ y *_R x \neq \mathit{0}\}$ **unfolding** $*$ **by** *fastforce*
{
case *1*
hence *x*: $x \in f$ ` *space* $M$ **and** *F*: $F \subseteq f$ ` *space* M **by** *auto*
show *?case*
proof (*cases* $x = \mathit{0}$)
case *True*

then show *?thesis* **unfolding** $*$ **using** *insert*(*3*)[*OF F*] **by** *simp*
next
case *False*
have *norm-argument*: *norm* $((\sum y{\in}F.\ $ *indicat-real* $(f -` \{y\} \cap$ *space* $M)\ z *_R y) +$ *indicat-real* $(f -` \{x\} \cap$ *space* $M)\ z *_R x) =$ *norm* $(\sum y{\in}F.\ $ *indicat-real* $(f -` \{y\} \cap$ *space* $M)\ z *_R y) +$ *norm* (*indicat-real* $(f -` \{x\} \cap$ *space* $M)\ z *_R x)$ **if** z: $z \in$ *space* M **for** z
proof (*cases* $f\ z = x$)
case *True*
have *indicat-real* $(f -` \{y\} \cap$ *space* $M)\ z *_R y = 0$ **if** $y \in F$ **for** y **using** *True insert*(*2*) *z that 1* **unfolding** *indicator-def* **by** *force*
hence $(\sum y{\in}F.\ $ *indicat-real* $(f -` \{y\} \cap$ *space* $M)\ z *_R y) = 0$ **by** (*meson sum.neutral*)
then show *?thesis* **by** *force*
next
case *False*
then show *?thesis* **by** *force*
qed
show *?thesis* **using** *False simple-functionD*(*2*)[*OF f*(*1*)] *insert*(*3*,*5*)[*OF F*] *simple-function-scaleR fin-0 fin-1* **by** (*subst* $*$, *subst add*, *subst simple-function-sum*) (*blast intro*: *norm-argument indicator*)+
qed
next
case *2*
hence x: $x \in f$ ` *space* $M$ **and** $F$: $F \subseteq f$ ` *space* M **by** *auto*
show *?case*
proof (*cases* $x = 0$)
case *True*
then show *?thesis* **unfolding** $*$ **using** *insert*(*4*)[*OF F*] **by** *simp*
next
case *False*
then show *?thesis* **unfolding** $*$ **using** *insert*(*4*)[*OF F*] *simple-functionD*(*2*)[*OF f*(*1*)] **by** *fast*
qed
next
case *3*
hence x: $x \in f$ ` *space* $M$ **and** $F$: $F \subseteq f$ ` *space* M **by** *auto*
show *?case*
proof (*cases* $x = 0$)
case *True*
then show *?thesis* **unfolding** $*$ **using** *insert*(*5*)[*OF F*] **by** *simp*
next
case *False*
have *emeasure* M $\{y \in$ *space* $M.\ (\sum x{\in}$*insert* $x\ F.\ $ *indicat-real* $(f -` \{x\} \cap$ *space* $M)\ y *_R x) \neq 0\} \leq$ *emeasure* $M$ $(\{y \in$ *space* $M.\ (\sum x{\in}F.\ $ *indicat-real* $(f -` \{x\} \cap$ *space* $M)\ y *_R x) \neq 0\} \cup \{y \in$ *space* $M.$ *indicat-real* $(f -` \{x\} \cap$ *space* $M)\ y *_R x \neq 0\})$
using $**$ *simple-functionD*(*2*)[*OF insert*(*4*)[*OF F*]] *simple-functionD*(*2*)[*OF f*(*1*)] **by** (*intro emeasure-mono*, *force*+)

also have ... $\leq$ *emeasure* M $\{y \in$ *space* M. $(\sum x{\in}F$. *indicat-real* $(f -{`}\ \{x\}$ $\cap$ *space* $M)$ $y *_R x) \neq 0\}$ + *emeasure* $M$ $\{y \in$ *space* $M$. *indicat-real* $(f -{`}\ \{x\} \cap$ *space* $M)$ $y *_R x \neq 0\}$
using *simple-functionD*(2)[*OF insert*(4)[*OF F*]] *simple-functionD*(2)[*OF* $f(1)$] **by** (*intro emeasure-subadditive*, *force*+)
also have ... $< \infty$ **using** *insert*(5)[*OF F*] *fin-1*[*OF False*] **by** (*simp add*: *less-top*)
finally show *?thesis* **by** *simp*
qed
}
qed
moreover have *simple-function* M $(\lambda x. \sum y{\in}f\ {`}$ *space* $M$. *indicat-real* $(f -{`}\ \{y\}$ $\cap$ *space* $M)$ $x *_R y)$ **using** *calculation* **by** *blast*
moreover have P $(\lambda x. \sum y{\in}f\ {`}$ *space* $M$. *indicat-real* $(f -{`}\ \{y\} \cap$ *space* $M)$ x $*_R y)$ **using** *calculation* **by** *blast*
ultimately show *?thesis* **by** (*intro cong*[*OF - - f*(1,2)], *blast*, *presburger*+)
qed

Induction rule for non-negative simple integrable functions

lemma *integrable-simple-function-induct-nn*[*consumes 3*, *case-names cong indicator add*, *induct set*: *simple-function*]:
fixes f :: $'a \Rightarrow\ 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes f: *simple-function* M f *emeasure* M $\{y \in$ *space* M. $f\ y \neq 0\} \neq \infty$ $\bigwedge x$. $x \in$ *space* $M \longrightarrow f\ x \geq 0$
assumes *cong*: $\bigwedge f\ g$. *simple-function* M $f \Longrightarrow$ *emeasure* M $\{y \in$ *space* M. $f\ y$ $\neq 0\} \neq \infty \Longrightarrow (\bigwedge x.\ x \in$ *space* $M \Longrightarrow f\ x \geq 0) \Longrightarrow$ *simple-function* M $g \Longrightarrow$ *emeasure* M $\{y \in$ *space* M. $g\ y \neq 0\} \neq \infty \Longrightarrow (\bigwedge x.\ x \in$ *space* $M \Longrightarrow g\ x \geq 0)$ $\Longrightarrow (\bigwedge x.\ x \in$ *space* $M \Longrightarrow f\ x = g\ x) \Longrightarrow P\ f \Longrightarrow P\ g$
assumes *indicator*: $\bigwedge A\ y.\ y \geq 0 \Longrightarrow A \in$ *sets* $M \Longrightarrow$ *emeasure* M $A < \infty \Longrightarrow$ P $(\lambda x.$ *indicator* A $x *_R y)$
assumes *add*: $\bigwedge f\ g$. $(\bigwedge x.\ x \in$ *space* $M \Longrightarrow f\ x \geq 0) \Longrightarrow$ *simple-function* M f $\Longrightarrow$ *emeasure* M $\{y \in$ *space* M. $f\ y \neq 0\} \neq \infty \Longrightarrow$
$(\bigwedge x.\ x \in$ *space* $M \Longrightarrow g\ x \geq 0) \Longrightarrow$ *simple-function* M $g \Longrightarrow$ *emeasure* M $\{y \in$ *space* M. $g\ y \neq 0\} \neq \infty \Longrightarrow$
$(\bigwedge z.\ z \in$ *space* $M \Longrightarrow$ *norm* $(f\ z + g\ z) =$ *norm* $(f\ z)$ + *norm* $(g\ z)) \Longrightarrow$
$P\ f \Longrightarrow P\ g \Longrightarrow P\ (\lambda x.\ f\ x + g\ x)$
shows $P\ f$
proof−
let $?f = \lambda x.\ (\sum y{\in}f\ {`}$ *space* $M$. *indicat-real* $(f -{`}\ \{y\} \cap$ *space* $M)$ $x *_R y)$
have *f-ae-eq*: $f\ x = ?f\ x$ **if** $x \in$ *space* M **for** x **using** *simple-function-indicator-representation*[*OF* $f(1)$ *that*] **.**
moreover have *emeasure* M $\{y \in$ *space* M. $?f\ y \neq 0\} \neq \infty$ **by** (*metis* (*no-types*, *lifting*) *Collect-cong calculation* $f(2)$)
moreover have P $(\lambda x. \sum y{\in}S$. *indicat-real* $(f -{`}\ \{y\} \cap$ *space* $M)$ $x *_R y)$
simple-function M $(\lambda x. \sum y{\in}S$. *indicat-real* $(f -{`}\ \{y\} \cap$ *space* $M)$ x $*_R y)$
emeasure M $\{y \in$ *space* M. $(\sum x{\in}S$. *indicat-real* $(f -{`}\ \{x\} \cap$ *space*

M) $y *_R x) \neq 0\} \neq \infty$

$\bigwedge x.\ x \in$ *space M* $\Longrightarrow 0 \leq (\sum y{\in}S.$ *indicat-real* (f −' $\{y\} \cap$ *space M*) $x *_R y$)

if $S \subseteq f$ ' *space M* **for** S **using** *simple-functionD*(*1*)[*OF assms*(*1*), *THEN rev-finite-subset*, *OF that*] *that*

proof (*induction rule*: *finite-subset-induct'*)

case *empty*

{

case *1*

then show *?case* **using** *indicator*[*of 0* {}] **by** *force*

next

case *2*

then show *?case* **by** *force*

next

case *3*

then show *?case* **by** *force*

next

case *4*

then show *?case* **by** *force*

}

next

case (*insert x F*)

have (f −' $\{x\} \cap$ *space M*) $\subseteq \{y \in$ *space M*. $f\ y \neq 0\}$ **if** $x \neq 0$ **using** *that* **by** *blast*

moreover have $\{y \in$ *space M*. $f\ y \neq 0\}$ = *space M* − (f −' $\{0\} \cap$ *space M*) **by** *blast*

moreover have *space M* − (f −' $\{0\} \cap$ *space M*) $\in$ *sets M* **using** *simple-functionD*(*2*)[*OF f*(*1*)] **by** *blast*

ultimately have *fin-0*: *emeasure M* (f −' $\{x\} \cap$ *space M*) $< \infty$ **if** $x \neq 0$ **using** *that* **by** (*metis emeasure-mono f*(*2*) *infinity-ennreal-def top.not-eq-extremum top-unique*)

hence *fin-1*: *emeasure M* $\{y \in$ *space M*. *indicat-real* (f −' $\{x\} \cap$ *space M*) $y *_R x \neq 0\} \neq \infty$ **if** $x \neq 0$ **by** (*metis* (*mono-tags*, *lifting*) *emeasure-mono f*(*1*) *indicator-simps*(*2*) *linorder-not-less mem-Collect-eq scaleR-eq-0-iff simple-functionD*(*2*) *subsetI that*)

have *nonneg-x*: $x \geq 0$ **using** *insert f* **by** *blast*

have $*$: $(\sum y{\in}$*insert x F*. *indicat-real* (f −' $\{y\} \cap$ *space M*) $xa *_R y$) = $(\sum y{\in}F.$ *indicat-real* (f −' $\{y\} \cap$ *space M*) $xa *_R y$) + *indicat-real* (f −' $\{x\} \cap$ *space M*) $xa *_R x$ **for** *xa* **by** (*metis* (*no-types*, *lifting*) *add.commute insert.hyps*(*1*) *insert.hyps*(*4*) *sum.insert*)

have $**$: $\{y \in$ *space M*. $(\sum x{\in}$*insert x F*. *indicat-real* (f −' $\{x\} \cap$ *space M*) $y *_R x) \neq 0\} \subseteq \{y \in$ *space M*. $(\sum x{\in}F.$ *indicat-real* (f −' $\{x\} \cap$ *space M*) $y *_R x) \neq 0\} \cup \{y \in$ *space M*. *indicat-real* (f −' $\{x\} \cap$ *space M*) $y *_R x \neq 0\}$ **unfolding** $*$ **by** *fastforce*

{

case *1*

show *?case*

proof (*cases* $x = 0$)

case *True*
then show *?thesis* **unfolding** $*$ **using** *insert* **by** *simp*
next
case *False*
have *norm-argument*: *norm* $((\sum y{\in}F.$ *indicat-real* $(f -` \{y\} \cap$ *space* $M)$ $z *_R y) +$ *indicat-real* $(f -` \{x\} \cap$ *space* $M)$ $z *_R x) =$ *norm* $(\sum y{\in}F.$ *indicat-real* $(f -` \{y\} \cap$ *space* $M)$ $z *_R y) +$ *norm* $($*indicat-real* $(f -` \{x\} \cap$ *space* $M)$ $z *_R x)$ **if** z: $z \in$ *space* M **for** z
proof $($*cases* $f\ z = x)$
case *True*
have *indicat-real* $(f -` \{y\} \cap$ *space* $M)$ $z *_R y = 0$ **if** $y \in F$ **for** y **using** *True insert z that 1* **unfolding** *indicator-def* **by** *force*
hence $(\sum y{\in}F.$ *indicat-real* $(f -` \{y\} \cap$ *space* $M)$ $z *_R y) = 0$ **by** $($*meson sum.neutral*$)$
thus *?thesis* **by** *force*
qed $($*force*$)$
show *?thesis* **using** *False fin-0 fin-1 f norm-argument* **by** $($*subst* $*$, *subst add*, *presburger add*: *insert*, *intro insert*, *intro insert*, *fastforce simp add*: *indicator-def intro*!: *insert*(2) $f(3)$, *auto intro*!: *indicator insert nonneg-x*$)$
qed
next
case *2*
show *?case*
proof $($*cases* $x = 0)$
case *True*
then show *?thesis* **unfolding** $*$ **using** *insert* **by** *simp*
next
case *False*
then show *?thesis* **unfolding** $*$ **using** *insert simple-functionD*$(2)[OF\ f(1)]$ **by** *fast*
qed
next
case *3*
show *?case*
proof $($*cases* $x = 0)$
case *True*
then show *?thesis* **unfolding** $*$ **using** *insert* **by** *simp*
next
case *False*
have *emeasure* M $\{y \in$ *space* $M.$ $(\sum x{\in}$*insert* x $F.$ *indicat-real* $(f -` \{x\} \cap$ *space* $M)$ $y *_R x) \neq 0\} \leq$ *emeasure* $M$ $(\{y \in$ *space* $M.$ $(\sum x{\in}F.$ *indicat-real* $(f -` \{x\} \cap$ *space* $M)$ $y *_R x) \neq 0\} \cup \{y \in$ *space* $M.$ *indicat-real* $(f -` \{x\} \cap$ *space* $M)$ $y *_R x \neq 0\})$
using $**$ *simple-functionD*$(2)[OF$ *insert*$(6)]$ *simple-functionD*$(2)[OF\ f(1)]$ *insert.IH*(2) **by** $($*intro emeasure-mono*, *blast*, *simp*$)$
also have ... $\leq$ *emeasure* M $\{y \in$ *space* $M.$ $(\sum x{\in}F.$ *indicat-real* $(f -` \{x\} \cap$ *space* $M)$ $y *_R x) \neq 0\}$ $+$ *emeasure* $M$ $\{y \in$ *space* $M.$ *indicat-real* $(f -` \{x\} \cap$ *space* $M)$ $y *_R x \neq 0\}$
using *simple-functionD*$(2)[OF$ *insert*$(6)]$ *simple-functionD*$(2)[OF\ f(1)]$

by (*intro emeasure-subadditive*, *force*+)
also have ... $<\infty$ **using** *insert*(*7*) *fin-1*[*OF False*] **by** (*simp add*: *less-top*)
finally show *?thesis* **by** *simp*
qed
next
case (*4 ξ*)
thus *?case* **using** *insert nonneg-x f*(*3*) **by** (*auto simp add*: *scaleR-nonneg-nonneg intro*: *sum-nonneg*)
}
qed
moreover have *simple-function M* ($\lambda x. \sum y \in f$ ' *space M*. *indicat-real* (f −' $\{y\}$ ∩ *space M*) $x *_R y$) **using** *calculation* **by** *blast*
moreover have P ($\lambda x. \sum y \in f$ ' *space M*. *indicat-real* (f −' $\{y\}$ ∩ *space M*) $x *_R y$) **using** *calculation* **by** *blast*
moreover have $\bigwedge x.\ x \in$ *space M* $\Longrightarrow 0 \le f\ x$ **using** *f*(*3*) **by** *simp*
ultimately show *?thesis* **by** (*intro cong*[*OF - - - f*(*1*,*2*)], *blast*, *blast*, *fast*) *presburger*+
qed

lemma *finite-nn-integral-imp-ae-finite*:
fixes f :: $'a \Rightarrow$ *ennreal*
assumes $f \in$ *borel-measurable M* ($\int^+ x.\ f\ x\ \partial M) < \infty$
shows *AE x in M*. $f\ x < \infty$
proof (*rule ccontr*, *goal-cases*)
case *1*
let *?A* = *space M* ∩ $\{x.\ f\ x = \infty\}$
have ∗: *emeasure M ?A* > 0 **using** *1 assms*(*1*) **by** (*metis* (*mono-tags*, *lifting*) *assms*(*2*) *eventually-mono infinity-ennreal-def nn-integral-noteq-infinite top.not-eq-extremum*)
have ($\int^+ x.\ f\ x *$ *indicator ?A x* ∂M) = ($\int^+ x.\ \infty *$ *indicator ?A x* ∂M) **by** (*metis* (*mono-tags*, *lifting*) *indicator-inter-arith indicator-simps*(*2*) *mem-Collect-eq mult-eq-0-iff*)
also have ... $= \infty *$ *emeasure M ?A* **using** *assms*(*1*) **by** (*intro nn-integral-cmult-indicator*, *simp*)
also have ... $= \infty$ **using** ∗ **by** *fastforce*
finally have ($\int^+ x.\ f\ x *$ *indicator ?A x* $\partial M) = \infty$ **.**
moreover have ($\int^+ x.\ f\ x *$ *indicator ?A x* $\partial M) \le (\int^+ x.\ f\ x\ \partial M)$ **by** (*intro nn-integral-mono*, *simp add*: *indicator-def*)
ultimately show *?case* **using** *assms*(*2*) **by** *simp*
qed

Convergence in L1-Norm implies existence of a subsequence which convergences almost everywhere. This lemma is easier to use than the existing one in *HOL−Analysis.Bochner-Integration*

lemma *cauchy-L1-AE-cauchy-subseq*:
fixes s :: *nat* $\Rightarrow 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes [*measurable*]: $\bigwedge n.$ *integrable M* ($s\ n$)
and $\bigwedge e.\ e > 0 \Longrightarrow \exists N.\ \forall i \ge N.\ \forall j \ge N.$ *LINT x*|*M*. *norm* ($s\ i\ x - s\ j\ x) < e$
obtains r **where** *strict-mono r AE x in M*. *Cauchy* ($\lambda i.\ s\ (r\ i)\ x$)
proof−

have $\exists r.\ \forall n.\ (\forall i \geq r\ n.\ \forall j \geq r\ n.\ \mathit{LINT}\ x|M.\ \mathit{norm}\ (s\ i\ x - s\ j\ x) < (1\ /\ 2)\ \hat{}\ n) \wedge (r\ (\mathit{Suc}\ n) > r\ n)$
proof (*intro dependent-nat-choice*, *goal-cases*)
case *1*
then show *?case* **using** *assms*(*2*) **by** *presburger*
next
case ($2\ x\ n$)
obtain N **where** $*$: $\mathit{LINT}\ x|M.\ \mathit{norm}\ (s\ i\ x - s\ j\ x) < (1\ /\ 2)\ \hat{}\ \mathit{Suc}\ n$ **if** $i \geq N\ j \geq N$ **for** $i\ j$ **using** *assms*(*2*)[*of* $(1\ /\ 2)\ \hat{}\ \mathit{Suc}\ n$] **by** *auto*
{
fix $i\ j$ **assume** $i \geq \mathit{max}\ N\ (\mathit{Suc}\ x)\ j \geq \mathit{max}\ N\ (\mathit{Suc}\ x)$
hence $\mathit{integral}^L\ M\ (\lambda x.\ \mathit{norm}\ (s\ i\ x - s\ j\ x)) < (1\ /\ 2)\ \hat{}\ \mathit{Suc}\ n$ **using** $*$ **by** *fastforce*
}
then show *?case* **by** *fastforce*
qed
then obtain r **where** *strict-mono*: *strict-mono* r **and** $\forall i \geq r\ n.\ \forall j \geq r\ n.\ \mathit{LINT}\ x|M.\ \mathit{norm}\ (s\ i\ x - s\ j\ x) < (1\ /\ 2)\ \hat{}\ n$ **for** n **using** *strict-mono-Suc-iff* **by** *blast*
hence *r-is*: $\mathit{LINT}\ x|M.\ \mathit{norm}\ (s\ (r\ (\mathit{Suc}\ n))\ x - s\ (r\ n)\ x) < (1\ /\ 2)\ \hat{}\ n$ **for** n **by** (*simp add*: *strict-mono-leD*)

define g **where** $g = (\lambda n\ x.\ (\sum i \leq n.\ \mathit{ennreal}\ (\mathit{norm}\ (s\ (r\ (\mathit{Suc}\ i))\ x - s\ (r\ i)\ x))))$
define g' **where** $g' = (\lambda x.\ \sum i.\ \mathit{ennreal}\ (\mathit{norm}\ (s\ (r\ (\mathit{Suc}\ i))\ x - s\ (r\ i)\ x)))$

have *integrable-g*: $(\int^+ x.\ g\ n\ x\ \partial M) < 2$ **for** n
proof −
have $(\int^+ x.\ g\ n\ x\ \partial M) = (\int^+ x.\ (\sum i \leq n.\ \mathit{ennreal}\ (\mathit{norm}\ (s\ (r\ (\mathit{Suc}\ i))\ x - s\ (r\ i)\ x)))\ \partial M)$ **using** *g-def* **by** *simp*
also have ... $= (\sum i \leq n.\ (\int^+ x.\ \mathit{ennreal}\ (\mathit{norm}\ (s\ (r\ (\mathit{Suc}\ i))\ x - s\ (r\ i)\ x))\ \partial M))$ **by** (*intro nn-integral-sum*, *simp*)
also have ... $= (\sum i \leq n.\ \mathit{LINT}\ x|M.\ \mathit{norm}\ (s\ (r\ (\mathit{Suc}\ i))\ x - s\ (r\ i)\ x))$ **unfolding** *dist-norm* **using** *assms*(*1*) **by** (*subst nn-integral-eq-integral*[*OF integrable-norm*], *auto*)
also have ... $< \mathit{ennreal}\ (\sum i \leq n.\ (1\ /\ 2)\ \hat{}\ i)$ **by** (*intro ennreal-lessI*[*OF sum-pos sum-strict-mono*[*OF finite-atMost - r-is*]], *auto*)
also have ... $\leq \mathit{ennreal}\ 2$ **unfolding** *sum-gp0*[*of* $1\ /\ 2\ n$] **by** (*intro ennreal-leI*, *auto*)
finally show $(\int^+ x.\ g\ n\ x\ \partial M) < 2$ **by** *simp*
qed

have *integrable-g'*: $(\int^+ x.\ g'\ x\ \partial M) \leq 2$
proof −
have *incseq* $(\lambda n.\ g\ n\ x)$ **for** x **by** (*intro incseq-SucI*, *auto simp add*: *g-def ennreal-leI*)
hence *convergent* $(\lambda n.\ g\ n\ x)$ **for** x **unfolding** *convergent-def* **using** *LIMSEQ-incseq-SUP* **by** *fast*
hence $(\lambda n.\ g\ n\ x) \longrightarrow g'\ x$ **for** x **unfolding** *g-def g'-def* **by** (*intro summable-iff-convergent'*[*THEN iffD2*, *THEN summable-LIMSEQ'*], *blast*)

hence $(\int^+ x.\ g'\ x\ \partial M) = (\int^+ x.\ liminf\ (\lambda n.\ g\ n\ x)\ \partial M)$ **by** (*metis lim-imp-Liminf trivial-limit-sequentially*)
also have ... $\leq$ *liminf* $(\lambda n.\ \int^+ x.\ g\ n\ x\ \partial M)$ **by** (*intro nn-integral-liminf*, *simp add*: *g-def*)
also have ... $\leq$ *liminf* $(\lambda n.\ 2)$ **using** *integrable-g* **by** (*intro Liminf-mono*) (*simp add*: *order-le-less*)
also have ... $= 2$ **using** *sequentially-bot tendsto-iff-Liminf-eq-Limsup* **by** *blast*
finally show *?thesis* **.**
qed
hence *AE x in M*. $g'\ x < \infty$ **by** (*intro finite-nn-integral-imp-ae-finite*) (*auto simp add*: *order-le-less-trans g′-def*)
moreover have *summable* $(\lambda i.\ norm\ (s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x))$ **if** $g'\ x \neq \infty$ **for** x **using** *that* **unfolding** *g′-def* **by** (*intro summable-suminf-not-top*) *fastforce+*

ultimately have *ae-summable*: *AE x in M*. *summable* $(\lambda i.\ s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x)$ **using** *summable-norm-cancel* **unfolding** *dist-norm* **by** *force*

{
fix x **assume** *summable* $(\lambda i.\ s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x)$
hence $(\lambda n.\ \sum i{<}n.\ s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x) \longrightarrow (\sum i.\ s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x)$ **using** *summable-LIMSEQ* **by** *blast*
moreover have $(\lambda n.\ (\sum i{<}n.\ s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x)) = (\lambda n.\ s\ (r\ n)\ x - s\ (r\ 0)\ x)$ **using** *sum-lessThan-telescope* **by** *fastforce*
ultimately have $(\lambda n.\ s\ (r\ n)\ x - s\ (r\ 0)\ x) \longrightarrow (\sum i.\ s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x)$ **by** *argo*
hence $(\lambda n.\ s\ (r\ n)\ x - s\ (r\ 0)\ x + s\ (r\ 0)\ x) \longrightarrow (\sum i.\ s\ (r\ (Suc\ i))\ x - s\ (r\ i)\ x) + s\ (r\ 0)\ x$ **by** (*intro isCont-tendsto-compose*[*of* - $\lambda z.\ z + s\ (r\ 0)\ x$], *auto*)
hence *Cauchy* $(\lambda n.\ s\ (r\ n)\ x)$ **by** (*simp add*: *LIMSEQ-imp-Cauchy*)
}
hence *AE x in M*. *Cauchy* $(\lambda i.\ s\ (r\ i)\ x)$ **using** *ae-summable* **by** *fast*
thus *?thesis* **by** (*rule that*[*OF strict-mono*(*1*)])
qed

3.2 Totally Ordered Banach Spaces

lemma *integrable-max*[*simp*, *intro*]:
fixes $f :: {}'a \Rightarrow {}'b ::$ {*second-countable-topology*, *banach*, *linorder-topology*}
assumes *fg*[*measurable*]: *integrable M f integrable M g*
shows *integrable M* $(\lambda x.\ max\ (f\ x)\ (g\ x))$
proof (*rule Bochner-Integration.integrable-bound*)
{
fix $x\ y :: {}'b$
have $norm\ (max\ x\ y) \leq max\ (norm\ x)\ (norm\ y)$ **by** *linarith*
also have ... $\leq norm\ x + norm\ y$ **by** *simp*
finally have $norm\ (max\ x\ y) \leq norm\ (norm\ x + norm\ y)$ **by** *simp*
}
thus *AE x in M*. $norm\ (max\ (f\ x)\ (g\ x)) \leq norm\ (norm\ (f\ x) + norm\ (g\ x))$ **by** *simp*
qed (*auto intro*: *Bochner-Integration.integrable-add*[*OF integrable-norm*[*OF fg*(*1*)]]

integrable-norm[*OF fg*(*2*)]])

lemma *integrable-min*[*simp*, *intro*]:
fixes f :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*}
assumes [*measurable*]: *integrable M f integrable M g*
shows *integrable M* (λx. *min* (*f x*) (*g x*))
proof −
have *norm* (*min* (*f x*) (*g x*)) $\leq$ *norm* (*f x*) $\vee$ *norm* (*min* (*f x*) (*g x*)) $\leq$ *norm* (*g x*) **for** *x* **by** *linarith*
thus *?thesis* **by** (*intro integrable-bound*[*OF integrable-max*[*OF integrable-norm*(*1*,*1*), *OF assms*]], *auto*)
qed

lemma *integral-nonneg-AE-banach*:
fixes f :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes [*measurable*]: $f \in$ *borel-measurable M* **and** *nonneg*: *AE x in M*. *0* $\leq$ *f x*
shows *0* $\leq$ *integral*^{L} *M f*
proof *cases*
assume *integrable*: *integrable M f*
hence *max*: (λx. *max 0* (*f x*)) $\in$ *borel-measurable M* $\bigwedge x$. *0* $\leq$ *max 0* (*f x*) *integrable M* (λx. *max 0* (*f x*)) **by** *auto*
hence *0* $\leq$ *integral*^{L} *M* (λx. *max 0* (*f x*))
proof −
obtain *s* **where** $*$: $\bigwedge i$. *simple-function M* (*s i*)
$\bigwedge i$. *emeasure M* {$y \in$ *space M*. *s i y* $\neq$ *0*} $\neq \infty$
$\bigwedge x$. $x \in$ *space M* $\Longrightarrow$ (λi. *s i x*) $\longrightarrow$ *f x*
$\bigwedge x$ *i*. $x \in$ *space M* $\Longrightarrow$ *norm* (*s i x*) $\leq$ *2* $*$ *norm* (*f x*) **using** *integrable-implies-simple-function-sequence*[*OF integrable*] **by** *blast*
have *simple*: $\bigwedge i$. *simple-function M* (λx. *max 0* (*s i x*)) **using** $*$ **by** *fast*
have $\bigwedge i$. {$y \in$ *space M*. *max 0* (*s i y*) $\neq$ *0*} $\subseteq$ {$y \in$ *space M*. *s i y* $\neq$ *0*} **unfolding** *max-def* **by** *force*
moreover have $\bigwedge i$. {$y \in$ *space M*. *s i y* $\neq$ *0*} $\in$ *sets M* **using** $*$ **by** *measurable*
ultimately have $\bigwedge i$. *emeasure M* {$y \in$ *space M*. *max 0* (*s i y*) $\neq$ *0*} $\leq$ *emeasure M* {$y \in$ *space M*. *s i y* $\neq$ *0*} **by** (*rule emeasure-mono*)
hence $**$:$\bigwedge i$. *emeasure M* {$y \in$ *space M*. *max 0* (*s i y*) $\neq$ *0*} $\neq \infty$ **using** $*$(*2*) **by** (*auto intro*: *order.strict-trans1 simp add*: *top.not-eq-extremum*)
have $\bigwedge x$. $x \in$ *space M* $\Longrightarrow$ (λi. *max 0* (*s i x*)) $\longrightarrow$ *max 0* (*f x*) **using** $*$(*3*) *tendsto-max* **by** *blast*
moreover have $\bigwedge x$ *i*. $x \in$ *space M* $\Longrightarrow$ *norm* (*max 0* (*s i x*)) $\leq$ *norm* (*2* $*_R$ *f x*) **using** $*$(*4*) **unfolding** *max-def* **by** *auto*
ultimately have *tendsto*: (λi. *integral*^{L} *M* (λx. *max 0* (*s i x*))) $\longrightarrow$ *integral*^{L} *M* (λx. *max 0* (*f x*))
using *borel-measurable-simple-function simple integrable* **by** (*intro integral-dominated-convergence*[*OF max*(*1*) - *integrable-norm*[*OF integrable-scaleR-right*], *of - 2 f*], *blast+*)
{
fix *h* :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}

assume *simple-function M h emeasure M* $\{y \in$ *space M*. *h y* $\neq 0\} \neq \infty$ $\bigwedge x$. $x \in$ *space M* $\longrightarrow$ *h x* ≥ 0
hence $*$: *integral*^{L} *M h* ≥ 0
proof (*induct rule*: *integrable-simple-function-induct-nn*)
case (*cong f g*)
then show *?case* **using** *Bochner-Integration.integral-cong* **by** *force*
next
case (*indicator A y*)
hence $A \neq \{\} \Longrightarrow y \geq 0$ **using** *sets.sets-into-space* **by** *fastforce*
then show *?case* **using** *indicator* **by** (*cases* $A = \{\}$, *auto simp add*: *scaleR-nonneg-nonneg*)
next
case (*add f g*)
then show *?case* **by** (*simp add*: *integrable-simple-function*)
qed
}
thus *?thesis* **using** *LIMSEQ-le-const*[*OF tendsto*, *of 0*] $**$ *simple* **by** *fastforce*
qed
also have ... = *integral*^{L} *M f* **using** *nonneg* **by** (*auto intro*: *integral-cong-AE*)
finally show *?thesis* **.**
qed (*simp add*: *not-integrable-integral-eq*)

lemma *integral-mono-AE-banach*:
fixes *f g* :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes *integrable M f integrable M g AE x in M*. *f x* $\leq$ *g x*
shows *integral*^{L} *M f* $\leq$ *integral*^{L} *M g*
using *integral-nonneg-AE-banach*[*of* λx. *g x* $-$ *f x*] *assms Bochner-Integration.integral-diff*[*OF assms*(*1*,*2*)] **by** *force*

lemma *integral-mono-banach*:
fixes *f g* :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes *integrable M f integrable M g* $\bigwedge x$. $x \in$ *space M* $\Longrightarrow$ *f x* $\leq$ *g x*
shows *integral*^{L} *M f* $\leq$ *integral*^{L} *M g*
using *integral-mono-AE-banach assms* **by** *blast*

3.3 Integrability and Measurability of the Diameter

context
fixes *s* :: *nat* $\Rightarrow 'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*} **and** *M*
assumes *bounded*: $\bigwedge x$. $x \in$ *space M* $\Longrightarrow$ *bounded* (*range* (λi. *s i x*))
begin

lemma *borel-measurable-diameter*:
assumes [*measurable*]: $\bigwedge i$. (*s i*) $\in$ *borel-measurable M*
shows (λx. *diameter* {*s i x* |*i*. $n \leq i$}) $\in$ *borel-measurable M*
proof −
have {*s i x* |*i*. $N \leq i$} $\neq \{\}$ **for** *x N* **by** *blast*

hence *diameter-SUP*: *diameter* $\{s\ i\ x\ |i.\ N \leq i\} = (SUP\ (i, j) \in \{N..\} \times \{N..\}.$ *dist* $(s\ i\ x)\ (s\ j\ x))$ **for** $x\ N$ **unfolding** *diameter-def* **by** (*auto intro*!: *arg-cong*[*of* - - *Sup*])

have *case-prod dist* ' $(\{s\ i\ x\ |i.\ n \leq i\} \times \{s\ i\ x\ |i.\ n \leq i\}) = ((\lambda(i, j).$ *dist* $(s\ i\ x)\ (s\ j\ x))$ ' $(\{n..\} \times \{n..\}))$ **for** x **by** *fast*
hence $*$: $(\lambda x.$ *diameter* $\{s\ i\ x\ |i.\ n \leq i\}) = (\lambda x.$ *Sup* $((\lambda(i, j).$ *dist* $(s\ i\ x)\ (s\ j\ x))$ ' $(\{n..\} \times \{n..\})))$ **using** *diameter-SUP* **by** (*simp add*: *case-prod-beta'*)

have *bounded* $((\lambda(i, j).$ *dist* $(s\ i\ x)\ (s\ j\ x))$ ' $(\{n..\} \times \{n..\}))$ **if** $x \in$ *space* M **for** x **by** (*rule bounded-imp-dist-bounded*[*OF bounded*, *OF that*])
hence *bdd*: *bdd-above* $((\lambda(i, j).$ *dist* $(s\ i\ x)\ (s\ j\ x))$ ' $(\{n..\} \times \{n..\}))$ **if** $x \in$ *space* M **for** x **using** *that bounded-imp-bdd-above* **by** *presburger*
have *fst* $p \in$ *borel-measurable* M *snd* $p \in$ *borel-measurable* M **if** $p \in s$ ' $\{n..\} \times s$ ' $\{n..\}$ **for** p **using** *that* **by** *fastforce+*
hence $(\lambda x.$ *fst* $p\ x -$ *snd* $p\ x) \in$ *borel-measurable* M **if** $p \in s$ ' $\{n..\} \times s$ ' $\{n..\}$ **for** p **using** *that borel-measurable-diff* **by** *simp*
hence $(\lambda x.$ *case* p *of* $(f, g) \Rightarrow$ *dist* $(f\ x)\ (g\ x)) \in$ *borel-measurable* M **if** $p \in s$ ' $\{n..\} \times s$ ' $\{n..\}$ **for** p **unfolding** *dist-norm* **using** *that* **by** *measurable*
moreover have *countable* $(s$ ' $\{n..\} \times s$ ' $\{n..\})$ **by** (*intro countable-SIGMA countable-image*, *auto*)
ultimately show *?thesis* **unfolding** $*$ **by** (*auto intro*!: *borel-measurable-cSUP bdd*)
qed

lemma *integrable-bound-diameter*:
fixes f :: *'a* $\Rightarrow$ *real*
assumes *integrable M f*
and [*measurable*]: $\bigwedge i.\ (s\ i) \in$ *borel-measurable* M
and $\bigwedge x\ i.\ x \in$ *space* $M \Longrightarrow$ *norm* $(s\ i\ x) \leq f\ x$
shows *integrable* M $(\lambda x.$ *diameter* $\{s\ i\ x\ |i.\ n \leq i\})$
proof −
have $\{s\ i\ x\ |i.\ N \leq i\} \neq \{\}$ **for** $x\ N$ **by** *blast*
hence *diameter-SUP*: *diameter* $\{s\ i\ x\ |i.\ N \leq i\} = (SUP\ (i, j) \in \{N..\} \times \{N..\}.$ *dist* $(s\ i\ x)\ (s\ j\ x))$ **for** $x\ N$ **unfolding** *diameter-def* **by** (*auto intro*!: *arg-cong*[*of* - - *Sup*])
{
fix x **assume** x: $x \in$ *space* M
let $?S = (\lambda(i, j).$ *dist* $(s\ i\ x)\ (s\ j\ x))$ ' $(\{n..\} \times \{n..\})$
have *case-prod dist* ' $(\{s\ i\ x\ |i.\ n \leq i\} \times \{s\ i\ x\ |i.\ n \leq i\}) = (\lambda(i, j).$ *dist* $(s\ i\ x)\ (s\ j\ x))$ ' $(\{n..\} \times \{n..\})$ **by** *fast*
hence $*$: *diameter* $\{s\ i\ x\ |i.\ n \leq i\} =$ *Sup ?S* **using** *diameter-SUP* **by** (*simp add*: *case-prod-beta'*)

have *bounded ?S* **by** (*rule bounded-imp-dist-bounded*[*OF bounded*[*OF x*]])
hence *Sup-S-nonneg*: $0 \leq$ *Sup ?S* **by** (*auto intro*!: *cSup-upper2 x bounded-imp-bdd-above*)

have *dist* $(s\ i\ x)\ (s\ j\ x) \leq 2 * f\ x$ **for** $i\ j$ **by** (*intro dist-triangle2*[*THEN order-trans*, *of* - *0*]) (*metis norm-conv-dist assms*(*3*) *x add-mono mult-2*)

hence $\forall c \in ?S.\ c \leq 2 * f\ x$ **by** *force*
hence *Sup* $?S \leq 2 * f\ x$ **by** (*intro cSup-least*, *auto*)
hence *norm* (*Sup* $?S$) $\leq 2 *$ *norm* ($f\ x$) **using** *Sup-S-nonneg* **by** *auto*
also have ... = *norm* ($2 *_R f\ x$) **by** *simp*
finally have *norm* (*diameter* $\{s\ i\ x\ |i.\ n \leq i\}$) $\leq$ *norm* ($2 *_R f\ x$) **unfolding** $*$ **.**
}
hence *AE x in M*. *norm* (*diameter* $\{s\ i\ x\ |i.\ n \leq i\}$) $\leq$ *norm* ($2 *_R f\ x$) **by** *blast*
thus *integrable M* ($\lambda x.$ *diameter* $\{s\ i\ x\ |i.\ n \leq i\}$) **using** *borel-measurable-diameter* **by** (*intro Bochner-Integration.integrable-bound*[*OF assms*(*1*)[*THEN integrable-scaleR-right*[*of 2*]]], *measurable*)
qed
end

3.4 Auxiliary Lemmas for Set Integrals

lemma *set-integral-scaleR-left*:
assumes $A \in$ *sets M* $c \neq 0 \Longrightarrow$ *integrable M f*
shows *LINT t*:*A*|*M*. $f\ t *_R c$ = (*LINT t*:*A*|*M*. $f\ t$) $*_R c$
unfolding *set-lebesgue-integral-def*
using *integrable-mult-indicator*[*OF assms*]
by (*subst integral-scaleR-left*[*symmetric*], *auto*)

lemma *nn-set-integral-eq-set-integral*:
assumes [*measurable*]:*integrable M f*
and *AE* $x \in A$ *in M*. $0 \leq f\ x$ $A \in$ *sets M*
shows ($\int^+ x{\in}A.\ f\ x\ \partial M$) = ($\int\ x \in A.\ f\ x\ \partial M$)
proof−
have ($\int^+ x.$ *indicator* $A\ x *_R f\ x\ \partial M$) = ($\int\ x \in A.\ f\ x\ \partial M$)
unfolding *set-lebesgue-integral-def* **using** *assms*(*2*) **by** (*intro nn-integral-eq-integral*[*of - λx. indicat-real A x* $*_R$ *f x*], *blast intro*: *assms integrable-mult-indicator*, *fastforce*)
moreover have ($\int^+ x.$ *indicator* $A\ x *_R f\ x\ \partial M$) = ($\int^+ x{\in}A.\ f\ x\ \partial M$) **by** (*metis ennreal-0 indicator-simps*(*1*) *indicator-simps*(*2*) *mult.commute mult-1 mult-zero-left real-scaleR-def*)
ultimately show *?thesis* **by** *argo*
qed

lemma *set-integral-restrict-space*:
fixes f :: $'a \Rightarrow\ 'b$::{*banach*, *second-countable-topology*}
assumes $\Omega \cap$ *space M* $\in$ *sets M*
shows *set-lebesgue-integral* (*restrict-space M* Ω) *A f* = *set-lebesgue-integral M A* ($\lambda x.$ *indicator* $\Omega\ x *_R f\ x$)
unfolding *set-lebesgue-integral-def*
by (*subst integral-restrict-space*, *auto intro*!: *integrable-mult-indicator assms simp*: *mult.commute*)

lemma *set-integral-const*:
fixes c :: $'b$::{*banach*, *second-countable-topology*}
assumes $A \in$ *sets M* *emeasure M A* $\neq \infty$

shows *set-lebesgue-integral M A* (λ-. *c*) = *measure M A* $*_R$ *c*
unfolding *set-lebesgue-integral-def*
using *assms* **by** (*metis has-bochner-integral-indicator has-bochner-integral-integral-eq infinity-ennreal-def less-top*)

lemma *set-integral-mono-banach*:
fixes *f g* :: $'a \Rightarrow\, 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes *set-integrable M A f set-integrable M A g*
$\bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x$
shows (*LINT x*:*A*|*M*. *f x*) $\leq$ (*LINT x*:*A*|*M*. *g x*)
using *assms* **unfolding** *set-integrable-def set-lebesgue-integral-def*
by (*auto intro*: *integral-mono-banach split*: *split-indicator*)

lemma *set-integral-mono-AE-banach*:
fixes *f g* :: $'a \Rightarrow\, 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes *set-integrable M A f set-integrable M A g AE x*$\in$*A in M*. *f x* $\leq$ *g x*
shows *set-lebesgue-integral M A f* $\leq$ *set-lebesgue-integral M A g* **using** *assms* **unfolding** *set-lebesgue-integral-def* **by** (*auto simp add*: *set-integrable-def intro*!: *integral-mono-AE-banach*[*of M* λx. *indicator A x* $*_R$ *f x* λx. *indicator A x* $*_R$ *g x*], *simp add*: *indicator-def*)

3.5 Averaging Theorem

lemma *balls-countable-basis*:
obtains *D* :: $'a$ :: {*metric-space*, *second-countable-topology*} *set*
where *topological-basis* (*case-prod ball* ‘ (*D* $\times$ ($\mathbb{Q} \cap$ {*0*<..})))
and *countable D*
and *D* $\neq$ {}
proof –
obtain *D* :: $'a$ *set* **where** *dense-subset*: *countable D D* $\neq$ {} ⟦*open U*; *U* $\neq$ {}⟧ $\Longrightarrow \exists\, y \in D.\ y \in U$ **for** *U* **using** *countable-dense-exists* **by** *blast*
have *topological-basis* (*case-prod ball* ‘ (*D* $\times$ ($\mathbb{Q} \cap$ {*0*<..})))
proof (*intro topological-basis-iff*[*THEN iffD2*], *fast*, *clarify*)
fix *U* **and** *x* :: $'a$ **assume** *asm*: *open U x* $\in$ *U*
obtain *e* **where** *e*: *e* > *0 ball x e* $\subseteq$ *U* **using** *asm openE* **by** *blast*
obtain *y* **where** *y*: *y* $\in$ *D y* $\in$ *ball x* (*e* / *3*) **using** *dense-subset*(*3*)[*OF open-ball*, *of x e* / *3*] *centre-in-ball*[*THEN iffD2*, *OF divide-pos-pos*[*OF e*(*1*), *of 3*]] **by** *force*
obtain *r* **where** *r*: *r* $\in \mathbb{Q} \cap$ {*e*/*3*<..<*e*/*2*} **unfolding** *Rats-def* **using** *of-rat-dense*[*OF divide-strict-left-mono*[*OF* - *e*(*1*)], *of 2 3*] **by** *auto*

have $*$: *x* $\in$ *ball y r* **using** *r y* **by** (*simp add*: *dist-commute*)
hence *ball y r* $\subseteq$ *U* **using** *r* **by** (*intro order-trans*[*OF* - *e*(*2*)], *simp*, *metric*)
moreover have *ball y r* $\in$ (*case-prod ball* ‘ (*D* $\times$ ($\mathbb{Q} \cap$ {*0*<..}))) **using** *y*(*1*) *r* **by** *force*
ultimately show $\exists\, B' \in$(*case-prod ball* ‘ (*D* $\times$ ($\mathbb{Q} \cap$ {*0*<..}))). *x* $\in B' \wedge B' \subseteq$ *U* **using** $*$ **by** *meson*
qed

thus *?thesis* **using** *that dense-subset* **by** *blast*
qed

context *sigma-finite-measure*
begin

lemma *sigma-finite-measure-induct*[*case-names finite-measure*, *consumes 0*]:
assumes $\bigwedge(N :: 'a\ measure)\ \Omega.\ \mathit{finite\text{-}measure}\ N$
$\Longrightarrow N = \mathit{restrict\text{-}space}\ M\ \Omega$
$\Longrightarrow \Omega \in \mathit{sets}\ M$
$\Longrightarrow \mathit{emeasure}\ N\ \Omega \neq \infty$
$\Longrightarrow \mathit{emeasure}\ N\ \Omega \neq 0$
$\Longrightarrow \mathit{almost\text{-}everywhere}\ N\ Q$
and [*measurable*]: *Measurable.pred M Q*
shows *almost-everywhere M Q*
proof −
have ∗: *almost-everywhere N Q* **if** *finite-measure N* $N = \mathit{restrict\text{-}space}\ M\ \Omega\ \Omega \in \mathit{sets}\ M\ \mathit{emeasure}\ N\ \Omega \neq \infty$ **for** $N\ \Omega$ **using** *that* **by** (*cases* $\mathit{emeasure}\ N\ \Omega = 0$, *auto intro*: *emeasure-0-AE assms*(*1*))

obtain $A :: \mathit{nat} \Rightarrow 'a\ \mathit{set}$ **where** *A*: $\mathit{range}\ A \subseteq \mathit{sets}\ M\ (\bigcup i.\ A\ i) = \mathit{space}\ M$ **and** *emeasure-finite*: $\mathit{emeasure}\ M\ (A\ i) \neq \infty$ **for** *i* **using** *sigma-finite* **by** *metis*
note *A*(*1*)[*measurable*]
have *space-restr*: $\mathit{space}\ (\mathit{restrict\text{-}space}\ M\ (A\ i)) = A\ i$ **for** *i* **unfolding** *space-restrict-space* **by** *simp*
{
fix *i*
have ∗: $\{x \in A\ i \cap \mathit{space}\ M.\ Q\ x\} = \{x \in \mathit{space}\ M.\ Q\ x\} \cap (A\ i)$ **by** *fast*
have *Measurable.pred* (*restrict-space M* (*A i*)) *Q* **using** *A* **by** (*intro measurableI*, *auto simp add*: *space-restr intro*!: *sets-restrict-space-iff*[*THEN iffD2*], *measurable*, *auto*)
}
note *this*[*measurable*]
{
fix *i*
have *finite-measure* (*restrict-space M* (*A i*)) **using** *emeasure-finite* **by** (*intro finite-measureI*, *subst space-restr*, *subst emeasure-restrict-space*, *auto*)
hence $\mathit{emeasure}\ (\mathit{restrict\text{-}space}\ M\ (A\ i))\ \{x \in A\ i.\ \neg Q\ x\} = 0$ **using** *emeasure-finite* **by** (*intro AE-iff-measurable*[*THEN iffD1*, *OF - -* ∗], *measurable*, *subst space-restr*[*symmetric*], *intro sets.top*, *auto simp add*: *emeasure-restrict-space*)
hence $\mathit{emeasure}\ M\ \{x \in A\ i.\ \neg\ Q\ x\} = 0$ **by** (*subst emeasure-restrict-space*[*symmetric*], *auto*)
}
hence $\mathit{emeasure}\ M\ (\bigcup i.\ \{x \in A\ i.\ \neg\ Q\ x\}) = 0$ **by** (*intro emeasure-UN-eq-0*, *auto*)
moreover have $(\bigcup i.\ \{x \in A\ i.\ \neg\ Q\ x\}) = \{x \in \mathit{space}\ M.\ \neg\ Q\ x\}$ **using** *A* **by** *auto*
ultimately show *?thesis* **by** (*intro AE-iff-measurable*[*THEN iffD2*], *auto*)
qed

The following lemma allows us to make statements about the behavious of a function almost everywhere, depending on the value it takes on average.

lemma *averaging-theorem*:
fixes f::- $\Rightarrow$ $'b$::{*second-countable-topology*, *banach*}
assumes [*measurable*]:*integrable M f*
and *closed*: *closed S*
and $\bigwedge A$. $A \in$ *sets M* $\Longrightarrow$ *measure M A* $> 0 \Longrightarrow$ (1 / *measure M A*) $*_R$ *set-lebesgue-integral M A f* $\in S$
shows *AE x in M*. *f x* $\in S$
proof (*induct rule*: *sigma-finite-measure-induct*)
case (*finite-measure N* Ω)

interpret *finite-measure N* **by** (*rule finite-measure*)

have *integrable*[*measurable*]: *integrable N f* **using** *assms finite-measure* **by** (*auto simp*: *integrable-restrict-space integrable-mult-indicator*)
have *average*: (1 / *Sigma-Algebra.measure N A*) $*_R$ *set-lebesgue-integral N A f* $\in S$ **if** $A \in$ *sets N measure N A* > 0 **for** A
proof −
have $*$: $A \in$ *sets M* **using** *that* **by** (*simp add*: *sets-restrict-space-iff finite-measure*)
have $A = A \cap \Omega$ **by** (*metis finite-measure*(2,3) *inf.orderE sets.sets-into-space space-restrict-space that*(1))
hence *set-lebesgue-integral N A f* $=$ *set-lebesgue-integral M A f* **unfolding** *finite-measure* **by** (*subst set-integral-restrict-space*, *auto simp add*: *finite-measure set-lebesgue-integral-def indicator-inter-arith*[*symmetric*])
moreover have *measure N A* $=$ *measure M A* **using** *that* **by** (*auto intro*!: *measure-restrict-space simp add*: *finite-measure sets-restrict-space-iff*)
ultimately show *?thesis* **using** *that* $*$ *assms*(3) **by** *presburger*
qed

obtain D :: $'b$ *set* **where** *balls-basis*: *topological-basis* (*case-prod ball* ‘ ($D \times$ ($\mathbb{Q}$ $\cap$ {0<..}))) **and** *countable-D*: *countable D* **using** *balls-countable-basis* **by** *blast*
have *countable-balls*: *countable* (*case-prod ball* ‘ ($D \times$ ($\mathbb{Q}$ $\cap$ {0<..}))) **using** *countable-rat countable-D* **by** *blast*

obtain B **where** *B-balls*: $B \subseteq$ *case-prod ball* ‘ ($D \times$ ($\mathbb{Q}$ $\cap$ {0<..})) $\bigcup B = -S$ **using** *topological-basis*[*THEN iffD1*, *OF balls-basis*] *open-Compl*[*OF assms*(2)] **by** *meson*
hence *countable-B*: *countable B* **using** *countable-balls countable-subset* **by** *fast*

define b **where** $b =$ *from-nat-into* ($B \cup$ {{}})
have $B \cup$ {{}} $\neq$ {} **by** *simp*
have *range-b*: *range b* $= B \cup$ {{}} **using** *countable-B* **by** (*auto simp add*: *b-def intro*!: *range-from-nat-into*)
have *open-b*: *open* (b i) **for** i **unfolding** *b-def* **using** *B-balls open-ball from-nat-into*[*of* $B \cup$ {{}} i] **by** *force*
have *Union-range-b*: $\bigcup$(*range b*) $= -S$ **using** *B-balls range-b* **by** *simp*

{

fix *v r* **assume** *ball-in-Compl*: *ball v r* $\subseteq -S$
define *A* **where** $A = f$ −' *ball v r* $\cap$ *space N*
have *dist-less*: *dist* (*f x*) *v* $<$ *r* **if** $x \in A$ **for** *x* **using** *that* **unfolding** *A-def* *vimage-def* **by** (*simp add*: *dist-commute*)
hence *AE-less*: *AE* $x \in A$ *in N*. *norm* ($f\ x - v$) $<$ *r* **by** (*auto simp add*: *dist-norm*)
have $*$: $A \in$ *sets N* **unfolding** *A-def* **by** *simp*
have *emeasure N A* = *0*
proof −
{
assume *asm*: *emeasure N A* $>$ *0*
hence *measure-pos*: *measure N A* $>$ *0* **unfolding** *emeasure-eq-measure* **by** *simp*
hence (*1* / *measure N A*) $*_R$ *set-lebesgue-integral N A f* − *v* = (*1* / *measure N A*) $*_R$ *set-lebesgue-integral N A* (λx. $f\ x - v$) **using** *integrable integrable-const* $*$ **by** (*subst set-integral-diff*(*2*), *auto simp add*: *set-integrable-def set-integral-const*[*OF* $*$] *algebra-simps intro*!: *integrable-mult-indicator*)
moreover have *norm* ($\int x{\in}A.\ (f\ x - v)\partial N$) $\leq$ ($\int x{\in}A.$ *norm* ($f\ x - v$)∂N) **using** $*$ **by** (*auto intro*!: *integral-norm-bound*[*of N* λx. *indicator A x* $*_R$ ($f\ x - v$), *THEN order-trans*] *integrable-mult-indicator integrable simp add*: *set-lebesgue-integral-def*)
ultimately have *norm* ((*1* / *measure N A*) $*_R$ *set-lebesgue-integral N A f* − *v*) $\leq$ *set-lebesgue-integral N A* (λx. *norm* ($f\ x - v$)) / *measure N A* **using** *asm* **by** (*auto intro*: *divide-right-mono*)
also have ... $<$ *set-lebesgue-integral N A* (λx. *r*) / *measure N A*
unfolding *set-lebesgue-integral-def*
using *asm* $*$ *integrable integrable-const AE-less measure-pos*
by (*intro divide-strict-right-mono integral-less-AE*[*of* - - *A*] *integrable-mult-indicator*)
(*fastforce simp add*: *dist-less dist-norm indicator-def*)+
also have ... = *r* **using** $*$ *measure-pos* **by** (*simp add*: *set-integral-const*)
finally have *dist* ((*1* / *measure N A*) $*_R$ *set-lebesgue-integral N A f*) *v* $<$ *r* **by** (*subst dist-norm*)
hence *False* **using** *average*[*OF* $*$ *measure-pos*] **by** (*metis ComplD dist-commute in-mono mem-ball ball-in-Compl*)
}
thus *?thesis* **by** *fastforce*
qed
}
note $*$ = *this*
{
fix *b'* **assume** $b' \in B$
hence *ball-subset-Compl*: $b' \subseteq -S$ **and** *ball-radius-pos*: $\exists\, v \in D.\ \exists\, r{>}0.\ b' =$ *ball v r* **using** *B-balls* **by** (*blast*, *fast*)
}
note $**$ = *this*
hence *emeasure N* (*f* −' *b i* $\cap$ *space N*) = *0* **for** *i* **by** (*cases b i* = {}, *simp*) (*metis UnE singletonD* $*$ *range-b*[*THEN eq-refl*, *THEN range-subsetD*])
hence *emeasure N* ($\bigcup i$. *f* −' *b i* $\cap$ *space N*) = *0* **using** *open-b* **by** (*intro emeasure-UN-eq-0*) *fastforce*+

moreover have $(\bigcup i.\ f -` b\ i \cap space\ N) = f -` (\bigcup (range\ b)) \cap space\ N$ **by** *blast*
ultimately have $emeasure\ N\ (f -` (-S) \cap space\ N) = 0$ **using** *Union-range-b* **by** *argo*
hence $AE\ x\ in\ N.\ f\ x \notin -S$ **using** *open-Compl*[*OF assms*(*2*)] **by** (*intro AE-iff-measurable*[*THEN iffD2*], *auto*)
thus *?case* **by** *force*
qed (*simp add*: *pred-sets2*[*OF borel-closed*] *assms*(*2*))

lemma *density-zero*:
fixes $f::'a \Rightarrow 'b::\{second\text{-}countable\text{-}topology,\ banach\}$
assumes *integrable M f*
and *density-0*: $\bigwedge A.\ A \in sets\ M \Longrightarrow set\text{-}lebesgue\text{-}integral\ M\ A\ f = 0$
shows $AE\ x\ in\ M.\ f\ x = 0$
using *averaging-theorem*[*OF assms*(*1*), *of* $\{0\}$] *assms*(*2*)
by (*simp add*: *scaleR-nonneg-nonneg*)

This lemma shows that densities are unique in Banach spaces.

lemma *density-unique-banach*:
fixes $f\ f'::'a \Rightarrow 'b::\{second\text{-}countable\text{-}topology,\ banach\}$
assumes *integrable M f integrable M f'*
and *density-eq*: $\bigwedge A.\ A \in sets\ M \Longrightarrow set\text{-}lebesgue\text{-}integral\ M\ A\ f = set\text{-}lebesgue\text{-}integral\ M\ A\ f'$
shows $AE\ x\ in\ M.\ f\ x = f'\ x$
proof−
{
fix *A* **assume** *asm*: $A \in sets\ M$
hence $LINT\ x|M.\ indicat\text{-}real\ A\ x *_R (f\ x - f'\ x) = 0$ **using** *density-eq assms*(*1*,*2*) **by** (*simp add*: *set-lebesgue-integral-def algebra-simps Bochner-Integration.integral-diff*[*OF integrable-mult-indicator*(*1*,*1*)])
}
thus *?thesis* **using** *density-zero*[*OF Bochner-Integration.integrable-diff*[*OF assms*(*1*,*2*)]] **by** (*simp add*: *set-lebesgue-integral-def*)
qed

lemma *density-nonneg*:
fixes $f::\text{-} \Rightarrow 'b::\{second\text{-}countable\text{-}topology,\ banach,\ linorder\text{-}topology,\ ordered\text{-}real\text{-}vector\}$
assumes *integrable M f*
and $\bigwedge A.\ A \in sets\ M \Longrightarrow set\text{-}lebesgue\text{-}integral\ M\ A\ f \geq 0$
shows $AE\ x\ in\ M.\ f\ x \geq 0$
using *averaging-theorem*[*OF assms*(*1*), *of* $\{0..\}$, *OF closed-atLeast*] *assms*(*2*)
by (*simp add*: *scaleR-nonneg-nonneg*)

corollary *integral-nonneg-AE-eq-0-iff-AE*:
fixes $f :: 'a \Rightarrow 'b :: \{second\text{-}countable\text{-}topology,\ banach,\ linorder\text{-}topology,\ ordered\text{-}real\text{-}vector\}$
assumes *f*[*measurable*]: *integrable M f* **and** *nonneg*: $AE\ x\ in\ M.\ 0 \leq f\ x$
shows $integral^L\ M\ f = 0 \longleftrightarrow (AE\ x\ in\ M.\ f\ x = 0)$
proof

assume $*$: *integral*^{L} *M f* = *0*
{
fix *A* **assume** *asm*: *A* ∈ *sets M*
have *0* ≤ *integral*^{L} *M* (λ*x*. *indicator A x* $*_R$ *f x*) **using** *nonneg* **by** (*subst integral-zero*[*of M*, *symmetric*], *intro integral-mono-AE-banach integrable-mult-indicator asm f integrable-zero*, *auto simp add*: *indicator-def*)
moreover have ... ≤ *integral*^{L} *M f* **using** *nonneg* **by** (*intro integral-mono-AE-banach integrable-mult-indicator asm f*, *auto simp add*: *indicator-def*)
ultimately have *set-lebesgue-integral M A f* = *0* **unfolding** *set-lebesgue-integral-def* **using** $*$ **by** *force*
}
thus *AE x in M*. *f x* = *0* **by** (*intro density-zero f*, *blast*)
qed (*auto simp add*: *integral-eq-zero-AE*)

corollary *integral-eq-mono-AE-eq-AE*:
fixes *f g* :: *'a* ⇒ *'b* :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes *integrable M f integrable M g integral*^{L} *M f* = *integral*^{L} *M g AE x in M*. *f x* ≤ *g x*
shows *AE x in M*. *f x* = *g x*
proof −
define *h* **where** *h* = (λ*x*. *g x* − *f x*)
have *AE x in M*. *h x* = *0* **unfolding** *h-def* **using** *assms* **by** (*subst integral-nonneg-AE-eq-0-iff-AE*[*symmetric*]) *auto*
then show *?thesis* **unfolding** *h-def* **by** *auto*
qed

end

end

theory *Conditional-Expectation-Banach*
imports *HOL−Probability.Conditional-Expectation HOL−Probability.Independent-Family Bochner-Integration-Supplement*
begin

4 Conditional Expectation in Banach Spaces

definition *has-cond-exp* :: *'a measure* ⇒ *'a measure* ⇒ (*'a* ⇒ *'b*) ⇒ (*'a* ⇒ *'b*::{*real-normed-vector*, *second-countable-topology*}) ⇒ *bool* **where**
has-cond-exp M F f g = ((∀ *A* ∈ *sets F*. (∫ *x* ∈ *A*. *f x* ∂*M*) = (∫ *x* ∈ *A*. *g x* ∂*M*))
∧ *integrable M f*
∧ *integrable M g*
∧ *g* ∈ *borel-measurable F*)

lemma *has-cond-expI'*:
assumes ⋀*A*. *A* ∈ *sets F* ⟹ (∫ *x* ∈ *A*. *f x* ∂*M*) = (∫ *x* ∈ *A*. *g x* ∂*M*)

integrable M f
integrable M g
g ∈ *borel-measurable F*
shows *has-cond-exp M F f g*
using *assms* **unfolding** *has-cond-exp-def* **by** *simp*

lemma *has-cond-expD*:
assumes *has-cond-exp M F f g*
shows $\bigwedge A.\ A \in$ *sets F* $\Longrightarrow (\int x \in A.\ f\ x\ \partial M) = (\int x \in A.\ g\ x\ \partial M)$
integrable M f
integrable M g
g ∈ *borel-measurable F*
using *assms* **unfolding** *has-cond-exp-def* **by** *simp+*

definition *cond-exp* :: *'a measure* ⇒ *'a measure* ⇒ (*'a* ⇒ *'b*) ⇒ (*'a* ⇒ *'b*::{*banach*, *second-countable-topology*}) **where**
cond-exp M F f = (*if* ∃ *g*. *has-cond-exp M F f g then* (*SOME g*. *has-cond-exp M F f g*) *else* (λ-. *0*))

lemma *borel-measurable-cond-exp*[*measurable*]: *cond-exp M F f* ∈ *borel-measurable F*
by (*metis cond-exp-def someI has-cond-exp-def borel-measurable-const*)

lemma *integrable-cond-exp*[*intro*]: *integrable M* (*cond-exp M F f*)
by (*metis cond-exp-def has-cond-expD*(*3*) *integrable-zero someI*)

lemma *set-integrable-cond-exp*[*intro*]:
assumes *A* ∈ *sets M*
shows *set-integrable M A* (*cond-exp M F f*) **using** *integrable-mult-indicator*[*OF assms integrable-cond-exp*, *of F f*] **by** (*auto simp add*: *set-integrable-def intro*!: *integrable-mult-indicator*[*OF assms integrable-cond-exp*])

lemma *has-cond-exp-self*:
assumes *integrable M f*
shows *has-cond-exp M* (*vimage-algebra* (*space M*) *f borel*) *f f*
using *assms* **by** (*auto intro*!: *has-cond-expI' measurable-vimage-algebra1*)

lemma *has-cond-exp-sets-cong*:
assumes *sets F* = *sets G*
shows *has-cond-exp M F* = *has-cond-exp M G*
using *assms* **unfolding** *has-cond-exp-def* **by** *force*

lemma *cond-exp-sets-cong*:
assumes *sets F* = *sets G*
shows *AE x in M*. *cond-exp M F f x* = *cond-exp M G f x*
by (*intro AE-I2*, *simp add*: *cond-exp-def has-cond-exp-sets-cong*[*OF assms*, *of M*])

context *sigma-finite-subalgebra*
begin

lemma *borel-measurable-cond-exp*′[*measurable*]: *cond-exp M F f* $\in$ *borel-measurable M*
by (*metis cond-exp-def someI has-cond-exp-def borel-measurable-const subalg measurable-from-subalg*)

lemma *cond-exp-null*:
assumes $\nexists$ *g*. *has-cond-exp M F f g*
shows *cond-exp M F f* = (λ-. *0*)
unfolding *cond-exp-def* **using** *assms* **by** *argo*

We state the tower property of the conditional expectation in terms of the predicate *has-cond-exp*.

lemma *has-cond-exp-nested-subalg*:
fixes *f* :: ′*a* $\Rightarrow$ ′*b*::{*second-countable-topology*, *banach*}
assumes *subalgebra G F has-cond-exp M F f h has-cond-exp M G f h*′
shows *has-cond-exp M F h*′ *h*
by (*intro has-cond-expI*′) (*metis assms has-cond-expD in-mono subalgebra-def*)+

The following lemma shows that the conditional expectation is unique as an element of L1, given that it exists.

lemma *has-cond-exp-charact*:
fixes *f* :: ′*a* $\Rightarrow$ ′*b*::{*second-countable-topology*, *banach*}
assumes *has-cond-exp M F f g*
shows *has-cond-exp M F f* (*cond-exp M F f*)
AE x in M. *cond-exp M F f x* = *g x*
proof −
show *cond-exp*: *has-cond-exp M F f* (*cond-exp M F f*) **using** *assms someI cond-exp-def* **by** *metis*
let *?MF* = *restr-to-subalg M F*
interpret *sigma-finite-measure ?MF* **by** (*rule sigma-fin-subalg*)
{
fix *A* **assume** *A* $\in$ *sets ?MF*
then have [*measurable*]: *A* $\in$ *sets F* **using** *sets-restr-to-subalg*[*OF subalg*] **by** *simp*
have ($\int$ *x* $\in$ *A*. *g x* ∂*?MF*) = ($\int$ *x* $\in$ *A*. *g x* ∂*M*) **using** *assms subalg* **by** (*auto simp add*: *integral-subalgebra2 set-lebesgue-integral-def dest*!: *has-cond-expD*)
also have ... = ($\int$ *x* $\in$ *A*. *cond-exp M F f x* ∂*M*) **using** *assms cond-exp* **by** (*simp add*: *has-cond-exp-def*)
also have ... = ($\int$ *x* $\in$ *A*. *cond-exp M F f x* ∂*?MF*) **using** *subalg* **by** (*auto simp add*: *integral-subalgebra2 set-lebesgue-integral-def*)
finally have ($\int$ *x* $\in$ *A*. *g x* ∂*?MF*) = ($\int$ *x* $\in$ *A*. *cond-exp M F f x* ∂*?MF*) **by** *simp*
}
hence *AE x in ?MF*. *cond-exp M F f x* = *g x* **using** *cond-exp assms subalg* **by** (*intro density-unique-banach*, *auto dest*: *has-cond-expD intro*!: *integrable-in-subalg*)

then show *AE x in M. cond-exp M F f x = g x* **using** *AE-restr-to-subalg*[*OF subalg*] **by** *simp*
qed

corollary *cond-exp-charact*:
fixes f :: $'a \Rightarrow 'b$::{*second-countable-topology*, *banach*}
assumes $\bigwedge A.\ A \in$ *sets F* $\Longrightarrow (\int x \in A.\ f\ x\ \partial M) = (\int x \in A.\ g\ x\ \partial M)$
integrable M f
integrable M g
$g \in$ *borel-measurable F*
shows *AE x in M. cond-exp M F f x = g x*
by (*intro has-cond-exp-charact has-cond-expI′ assms*) *auto*

corollary *cond-exp-F-meas*[*intro*, *simp*]:
fixes f :: $'a \Rightarrow 'b$::{*second-countable-topology*, *banach*}
assumes *integrable M f*
$f \in$ *borel-measurable F*
shows *AE x in M. cond-exp M F f x = f x*
by (*rule cond-exp-charact*, *auto intro*: *assms*)

Congruence

lemma *has-cond-exp-cong*:
assumes *integrable M f* $\bigwedge x.\ x \in$ *space M* $\Longrightarrow f\ x = g\ x$ *has-cond-exp M F g h*
shows *has-cond-exp M F f h*
proof (*intro has-cond-expI′*[*OF - assms*(*1*)], *goal-cases*)
case (*1 A*)
hence *set-lebesgue-integral M A f = set-lebesgue-integral M A g* **by** (*intro set-lebesgue-integral-cong*) (*meson assms*(*2*) *subalg in-mono subalgebra-def sets.sets-into-space subalgebra-def subsetD*)+
then show *?case* **using** *1 assms*(*3*) **by** (*simp add*: *has-cond-exp-def*)
qed (*auto simp add*: *has-cond-expD*[*OF assms*(*3*)])

lemma *cond-exp-cong*:
fixes f :: $'a \Rightarrow 'b$::{*second-countable-topology*,*banach*}
assumes *integrable M f integrable M g* $\bigwedge x.\ x \in$ *space M* $\Longrightarrow f\ x = g\ x$
shows *AE x in M. cond-exp M F f x = cond-exp M F g x*
proof (*cases* $\exists h$. *has-cond-exp M F f h*)
case *True*
then obtain *h* **where** *h*: *has-cond-exp M F f h has-cond-exp M F g h* **using** *has-cond-exp-cong assms* **by** *metis*
show *?thesis* **using** *h*[*THEN has-cond-exp-charact*(*2*)] **by** *fastforce*
next
case *False*
moreover have $\nexists h$. *has-cond-exp M F g h* **using** *False has-cond-exp-cong assms* **by** *auto*
ultimately show *?thesis* **unfolding** *cond-exp-def* **by** *auto*
qed

lemma *has-cond-exp-cong-AE*:

assumes *integrable M f AE x in M. f x = g x has-cond-exp M F g h*
shows *has-cond-exp M F f h*
using *assms(1,2) subalg subalgebra-def subset-iff*
by (*intro has-cond-expI′*, *subst set-lebesgue-integral-cong-AE*[*OF - assms*(*1*)[*THEN borel-measurable-integrable*] *borel-measurable-integrable*(*1*)[*OF has-cond-expD*(*2*)[*OF assms*(*3*)]]])
(*fast intro*: *has-cond-expD*[*OF assms*(*3*)] *integrable-cong-AE-imp*[*OF - - AE-symmetric*])+

lemma *has-cond-exp-cong-AE′*:
assumes *h ∈ borel-measurable F AE x in M. h x = h′ x has-cond-exp M F f h′*
shows *has-cond-exp M F f h*
using *assms(1, 2) subalg subalgebra-def subset-iff*
using *AE-restr-to-subalg2*[*OF subalg assms*(*2*)] *measurable-from-subalg*
by (*intro has-cond-expI′* , *subst set-lebesgue-integral-cong-AE*[*OF - measurable-from-subalg*(*1*,*1*)[*OF subalg*], *OF - assms*(*1*) *has-cond-expD*(*4*)[*OF assms*(*3*)]])
(*fast intro*: *has-cond-expD*[*OF assms*(*3*)] *integrable-cong-AE-imp*[*OF - - AE-symmetric*])+

lemma *cond-exp-cong-AE*:
fixes *f* :: *′a ⇒ ′b*::{*second-countable-topology*,*banach*}
assumes *integrable M f integrable M g AE x in M. f x = g x*
shows *AE x in M. cond-exp M F f x = cond-exp M F g x*
proof (*cases ∃ h. has-cond-exp M F f h*)
case *True*
then obtain *h* **where** *h*: *has-cond-exp M F f h has-cond-exp M F g h* **using** *has-cond-exp-cong-AE assms* **by** (*metis* (*mono-tags*, *lifting*) *eventually-mono*)
show *?thesis* **using** *h*[*THEN has-cond-exp-charact*(*2*)] **by** *fastforce*
next
case *False*
moreover have *∄ h. has-cond-exp M F g h* **using** *False has-cond-exp-cong-AE assms* **by** *auto*
ultimately show *?thesis* **unfolding** *cond-exp-def* **by** *auto*
qed

lemma *has-cond-exp-real*:
fixes *f* :: *′a ⇒ real*
assumes *integrable M f*
shows *has-cond-exp M F f* (*real-cond-exp M F f*)
by (*intro has-cond-expI′*, *auto intro*!: *real-cond-exp-intA assms*)

lemma *cond-exp-real*[*intro*]:
fixes *f* :: *′a ⇒ real*
assumes *integrable M f*
shows *AE x in M. cond-exp M F f x = real-cond-exp M F f x*
using *has-cond-exp-charact has-cond-exp-real assms* **by** *blast*

lemma *cond-exp-cmult*:
fixes *f* :: *′a ⇒ real*
assumes *integrable M f*
shows *AE x in M. cond-exp M F (λx. c ∗ f x) x = c ∗ cond-exp M F f x*

using *real-cond-exp-cmult*[*OF assms*(*1*), *of c*] *assms*(*1*)[*THEN cond-exp-real*] *assms*(*1*)[*THEN integrable-mult-right*, *THEN cond-exp-real*, *of c*] **by** *fastforce*

4.1 Existence

Indicator functions

lemma *has-cond-exp-indicator*:
assumes $A \in$ *sets M emeasure M A* $< \infty$
shows *has-cond-exp M F* (λx. *indicat-real A x* $*_R$ *y*) (λx. *real-cond-exp M F* (*indicator A*) *x* $*_R$ *y*)
proof (*intro has-cond-expI′*, *goal-cases*)
case (*1 B*)
have $\int$ *x*$\in$*B*. (*indicat-real A x* $*_R$ *y*) ∂M = ($\int$ *x*$\in$*B*. *indicat-real A x* ∂M) $*_R$ *y* **using** *assms* **by** (*intro set-integral-scaleR-left*, *meson 1 in-mono subalg subalgebra-def*, *blast*)
also have ... = ($\int$ *x*$\in$*B*. *real-cond-exp M F* (*indicator A*) *x* ∂M) $*_R$ *y* **using** *1 assms* **by** (*subst real-cond-exp-intA*, *auto*)
also have ... = $\int$ *x*$\in$*B*. (*real-cond-exp M F* (*indicator A*) *x* $*_R$ *y*) ∂M **using** *assms* **by** (*intro set-integral-scaleR-left*[*symmetric*], *meson 1 in-mono subalg subalgebra-def*, *blast*)
finally show *?case* **.**
next
case *2*
then show *?case* **using** *integrable-scaleR-left integrable-real-indicator assms* **by** *blast*
next
case *3*
show *?case* **using** *assms* **by** (*intro integrable-scaleR-left*, *intro real-cond-exp-int*, *blast+*)
next
case *4*
then show *?case* **by** (*intro borel-measurable-scaleR*, *intro Conditional-Expectation.borel-measurable-cond-exp*, *simp*)
qed

lemma *cond-exp-indicator*[*intro*]:
fixes *y* :: *′b*::{*second-countable-topology*,*banach*}
assumes [*measurable*]: $A \in$ *sets M emeasure M A* $< \infty$
shows *AE x in M*. *cond-exp M F* (λx. *indicat-real A x* $*_R$ *y*) *x* = *cond-exp M F* (*indicator A*) *x* $*_R$ *y*
proof −
have *AE x in M*. *cond-exp M F* (λx. *indicat-real A x* $*_R$ *y*) *x* = *real-cond-exp M F* (*indicator A*) *x* $*_R$ *y* **using** *has-cond-exp-indicator*[*OF assms*] *has-cond-exp-charact* **by** *blast*
thus *?thesis* **using** *cond-exp-real*[*OF integrable-real-indicator*, *OF assms*] **by** *fastforce*
qed

Addition

lemma *has-cond-exp-add*:
  fixes $f\ g$ :: $'a \Rightarrow 'b$::{*second-countable-topology*,*banach*}
  assumes *has-cond-exp M F f f′ has-cond-exp M F g g′*
  shows *has-cond-exp M F* $(\lambda x.\ f\ x + g\ x)$ $(\lambda x.\ f'\ x + g'\ x)$
proof (*intro has-cond-expI′*, *goal-cases*)
  case (*1 A*)
   have $\int x{\in}A.\ (f\ x + g\ x)\partial M = (\int x{\in}A.\ f\ x\ \partial M) + (\int x{\in}A.\ g\ x\ \partial M)$ **using** *assms*[*THEN has-cond-expD*(*2*)] *subalg 1* **by** (*intro set-integral-add*(*2*), *auto simp add*: *subalgebra-def set-integrable-def intro*: *integrable-mult-indicator*)
   also have ... $= (\int x{\in}A.\ f'\ x\ \partial M) + (\int x{\in}A.\ g'\ x\ \partial M)$ **using** *assms*[*THEN has-cond-expD*(*1*)[*OF - 1*]] **by** *argo*
  also have ... $= \int x{\in}A.\ (f'\ x + g'\ x)\partial M$ **using** *assms*[*THEN has-cond-expD*(*3*)] *subalg 1* **by** (*intro set-integral-add*(*2*)[*symmetric*], *auto simp add*: *subalgebra-def set-integrable-def intro*: *integrable-mult-indicator*)
  finally show *?case* **.**
next
  case *2*
  then show *?case* **by** (*metis Bochner-Integration.integrable-add assms has-cond-expD*(*2*))
next
  case *3*
  then show *?case* **by** (*metis Bochner-Integration.integrable-add assms has-cond-expD*(*3*))
next
  case *4*
  then show *?case* **using** *assms borel-measurable-add has-cond-expD*(*4*) **by** *blast*
qed

lemma *has-cond-exp-scaleR-right*:
  fixes f :: $'a \Rightarrow 'b$::{*second-countable-topology*,*banach*}
  assumes *has-cond-exp M F f f′*
  shows *has-cond-exp M F* $(\lambda x.\ c *_R f\ x)$ $(\lambda x.\ c *_R f'\ x)$
  using *has-cond-expD*[*OF assms*] **by** (*intro has-cond-expI′*, *auto*)

lemma *cond-exp-scaleR-right*:
  fixes f :: $'a \Rightarrow 'b$::{*second-countable-topology*,*banach*}
  assumes *integrable M f*
  shows *AE x in M. cond-exp M F* $(\lambda x.\ c *_R f\ x)\ x = c *_R$ *cond-exp M F f x*
proof (*cases* $\exists f'.$ *has-cond-exp M F f f′*)
  case *True*
  then show *?thesis* **using** *assms has-cond-exp-charact has-cond-exp-scaleR-right* **by** *metis*
next
  case *False*
  show *?thesis*
  proof (*cases* $c = 0$)
    case *True*
    then show *?thesis* **by** *simp*
  next
    case *c-nonzero*: *False*
    have $\nexists f'.$ *has-cond-exp M F* $(\lambda x.\ c *_R f\ x)\ f'$

proof (*standard*, *goal-cases*)
case *1*
then obtain f' **where** f': *has-cond-exp M F* ($\lambda x.$ *c* $*_R$ *f x*) f' **by** *blast*
have *has-cond-exp M F f* ($\lambda x.$ *inverse c* $*_R$ f' *x*) **using** *has-cond-expD*[*OF* f'] *divideR-right*[*OF c-nonzero*] *assms* **by** (*intro has-cond-expI*$'$, *auto*)
then show *?case* **using** *False* **by** *blast*
qed
then show *?thesis* **using** *cond-exp-null*[*OF False*] *cond-exp-null* **by** *force*
qed
qed

lemma *cond-exp-uminus*:
fixes f :: $'a \Rightarrow$ $'b$::{*second-countable-topology*,*banach*}
assumes *integrable M f*
shows *AE x in M*. *cond-exp M F* ($\lambda x.$ $-$ *f x*) x = $-$ *cond-exp M F f x*
using *cond-exp-scaleR-right*[*OF assms*, *of* -1] **by** *force*

corollary *has-cond-exp-simple*:
fixes f :: $'a \Rightarrow$ $'b$::{*second-countable-topology*,*banach*}
assumes *simple-function M f emeasure M* $\{y \in$ *space M*. *f y* $\neq 0\} \neq \infty$
shows *has-cond-exp M F f* (*cond-exp M F f*)
using *assms*
proof (*induction rule*: *integrable-simple-function-induct*)
case (*cong f g*)
then show *?case* **using** *has-cond-exp-cong* **by** (*metis* (*no-types*, *opaque-lifting*) *Bochner-Integration.integrable-cong has-cond-expD*(*2*) *has-cond-exp-charact*(*1*))
next
case (*indicator A y*)
then show *?case* **using** *has-cond-exp-charact*[*OF has-cond-exp-indicator*] **by** *fast*
next
case (*add u v*)
then show *?case* **using** *has-cond-exp-add has-cond-exp-charact*(*1*) **by** *blast*
qed

lemma *cond-exp-contraction-real*:
fixes f :: $'a \Rightarrow$ *real*
assumes *integrable*[*measurable*]: *integrable M f*
shows *AE x in M*. *norm* (*cond-exp M F f x*) $\leq$ *cond-exp M F* ($\lambda x.$ *norm* (*f x*)) *x*
proof$-$
have *int*: *integrable M* ($\lambda x.$ *norm* (*f x*)) **using** *assms* **by** *blast*
have $*$: *AE x in M*. $0 \leq$ *cond-exp M F* ($\lambda x.$ *norm* (*f x*)) *x* **using** *cond-exp-real*[*THEN AE-symmetric*, *OF integrable-norm*[*OF integrable*]] *real-cond-exp-ge-c*[*OF integrable-norm*[*OF integrable*], *of 0*] *norm-ge-zero* **by** *fastforce*
have $**$: $A \in$ *sets F* $\Longrightarrow$ $\int x{\in}A.\ |f\ x|\ \partial M = \int x{\in}A.$ *real-cond-exp M F* ($\lambda x.$ *norm* (*f x*)) *x* ∂M **for** *A* **unfolding** *real-norm-def* **using** *assms integrable-abs real-cond-exp-intA* **by** *blast*

have *norm-int*: $A \in$ *sets F* $\Longrightarrow$ $(\int x{\in}A.\ |f\ x|\ \partial M) = (\int^{+} x{\in}A.\ |f\ x|\ \partial M)$ **for** *A* **using** *assms* **by** (*intro nn-set-integral-eq-set-integral*[*symmetric*], *blast*, *fastforce*)

(*meson subalg subalgebra-def subsetD*)

have *AE x in M. real-cond-exp M F* (λx. *norm* (*f x*)) $x \geq 0$ **using** *int real-cond-exp-ge-c* **by** *force*
hence *cond-exp-norm-int*: $A \in$ *sets F* $\Longrightarrow$ ($\int x{\in}A$. *real-cond-exp M F* (λx. *norm* (*f x*)) $x\ \partial M$) = ($\int^+ x{\in}A$. *real-cond-exp M F* (λx. *norm* (*f x*)) $x\ \partial M$) **for** *A* **using** *assms* **by** (*intro nn-set-integral-eq-set-integral*[*symmetric*], *blast*, *fastforce*) (*meson subalg subalgebra-def subsetD*)

have $A \in$ *sets F* $\Longrightarrow$ $\int^+ x{\in}A$. $|f\ x| \partial M$ = $\int^+ x{\in}A$. *real-cond-exp M F* (λx. *norm* (*f x*)) $x\ \partial M$ **for** *A* **using** ** *norm-int cond-exp-norm-int* **by** (*auto simp add*: *nn-integral-set-ennreal*)
moreover have (λx. *ennreal* $|f\ x|$) $\in$ *borel-measurable M* **by** *measurable*
moreover have (λx. *ennreal* (*real-cond-exp M F* (λx. *norm* (*f x*)) *x*)) $\in$ *borel-measurable F* **by** *measurable*
ultimately have *AE x in M. nn-cond-exp M F* (λx. *ennreal* $|f\ x|$) *x* = *real-cond-exp M F* (λx. *norm* (*f x*)) *x* **by** (*intro nn-cond-exp-charact*[*THEN AE-symmetric*], *auto*)
hence *AE x in M. nn-cond-exp M F* (λx. *ennreal* $|f\ x|$) $x \leq$ *cond-exp M F* (λx. *norm* (*f x*)) *x* **using** *cond-exp-real*[*OF int*] **by** *force*
moreover have *AE x in M.* |*real-cond-exp M F f x*| = *norm* (*cond-exp M F f x*) **unfolding** *real-norm-def* **using** *cond-exp-real*[*OF assms*] * **by** *force*
ultimately have *AE x in M. ennreal* (*norm* (*cond-exp M F f x*)) $\leq$ *cond-exp M F* (λx. *norm* (*f x*)) *x* **using** *real-cond-exp-abs*[*OF assms*[*THEN borel-measurable-integrable*]] **by** *fastforce*
hence *AE x in M. enn2real* (*ennreal* (*norm* (*cond-exp M F f x*))) $\leq$ *enn2real* (*cond-exp M F* (λx. *norm* (*f x*)) *x*) **using** *ennreal-le-iff2* **by** *force*
thus *?thesis* **using** * **by** *fastforce*
qed

lemma *cond-exp-contraction-simple*:
fixes *f* :: $'a \Rightarrow\ 'b$::{*second-countable-topology*, *banach*}
assumes *simple-function M f emeasure M* {$y \in$ *space M*. *f y* $\neq 0$} $\neq \infty$
shows *AE x in M. norm* (*cond-exp M F f x*) $\leq$ *cond-exp M F* (λx. *norm* (*f x*)) *x*
using *assms*
proof (*induction rule*: *integrable-simple-function-induct*)
case (*cong f g*)
hence *ae*: *AE x in M. f x = g x* **by** *blast*
hence *AE x in M. cond-exp M F f x = cond-exp M F g x* **using** *cong has-cond-exp-simple* **by** (*subst cond-exp-cong-AE*) (*auto intro*!: *has-cond-expD*(*2*))
hence *AE x in M. norm* (*cond-exp M F f x*) = *norm* (*cond-exp M F g x*) **by** *force*
moreover have *AE x in M. cond-exp M F* (λx. *norm* (*f x*)) *x* = *cond-exp M F* (λx. *norm* (*g x*)) *x* **using** *ae cong has-cond-exp-simple* **by** (*subst cond-exp-cong-AE*) (*auto dest*: *has-cond-expD*)
ultimately show *?case* **using** *cong*(*6*) **by** *fastforce*
next
case (*indicator A y*)
hence *AE x in M. cond-exp M F* (λa. *indicator A a* $*_R$ *y*) *x* = *cond-exp M F*

(*indicator A*) *x* $*_R$ *y* **by** *blast*
hence ∗: *AE x in M. norm* (*cond-exp M F* (λ*a. indicat-real A a* $*_R$ *y*) *x*) $\leq$ *norm y* ∗ *cond-exp M F* (λ*x. norm* (*indicat-real A x*)) *x* **using** *cond-exp-contraction-real*[*OF integrable-real-indicator*, *OF indicator*] **by** *fastforce*

have *AE x in M. norm y* ∗ *cond-exp M F* (λ*x. norm* (*indicat-real A x*)) *x* = *norm y* ∗ *real-cond-exp M F* (λ*x. norm* (*indicat-real A x*)) *x* **using** *cond-exp-real*[*OF integrable-real-indicator*, *OF indicator*] **by** *fastforce*
moreover have *AE x in M. cond-exp M F* (λ*x. norm y* ∗ *norm* (*indicat-real A x*)) *x* = *real-cond-exp M F* (λ*x. norm y* ∗ *norm* (*indicat-real A x*)) *x* **using** *indicator* **by** (*intro cond-exp-real*, *auto*)
ultimately have *AE x in M. norm y* ∗ *cond-exp M F* (λ*x. norm* (*indicat-real A x*)) *x* = *cond-exp M F* (λ*x. norm y* ∗ *norm* (*indicat-real A x*)) *x* **using** *real-cond-exp-cmult*[*of* λ*x. norm* (*indicat-real A x*) *norm y*] *indicator* **by** *fastforce*
moreover have (λ*x. norm y* ∗ *norm* (*indicat-real A x*)) = (λ*x. norm* (*indicat-real A x* $*_R$ *y*)) **by** *force*
ultimately show *?case* **using** ∗ **by** *force*
next
case (*add u v*)
have *AE x in M. norm* (*cond-exp M F* (λ*a. u a* + *v a*) *x*) = *norm* (*cond-exp M F u x* + *cond-exp M F v x*) **using** *has-cond-exp-charact*(*2*)[*OF has-cond-exp-add*, *OF has-cond-exp-simple*(*1*,*1*), *OF add*(*1*,*2*,*3*,*4*)] **by** *fastforce*
moreover have *AE x in M. norm* (*cond-exp M F u x* + *cond-exp M F v x*) $\leq$ *norm* (*cond-exp M F u x*) + *norm* (*cond-exp M F v x*) **using** *norm-triangle-ineq* **by** *blast*
moreover have *AE x in M. norm* (*cond-exp M F u x*) + *norm* (*cond-exp M F v x*) $\leq$ *cond-exp M F* (λ*x. norm* (*u x*)) *x* + *cond-exp M F* (λ*x. norm* (*v x*)) *x* **using** *add*(*6*,*7*) **by** *fastforce*
moreover have *AE x in M. cond-exp M F* (λ*x. norm* (*u x*)) *x* + *cond-exp M F* (λ*x. norm* (*v x*)) *x* = *cond-exp M F* (λ*x. norm* (*u x*) + *norm* (*v x*)) *x* **using** *integrable-simple-function*[*OF add*(*1*,*2*)] *integrable-simple-function*[*OF add*(*3*,*4*)] **by** (*intro has-cond-exp-charact*(*2*)[*OF has-cond-exp-add*[*OF has-cond-exp-charact*(*1*,*1*)], *THEN AE-symmetric*], *auto intro*: *has-cond-exp-real*)
moreover have *AE x in M. cond-exp M F* (λ*x. norm* (*u x*) + *norm* (*v x*)) *x* = *cond-exp M F* (λ*x. norm* (*u x* + *v x*)) *x* **using** *add*(*5*) *integrable-simple-function*[*OF add*(*1*,*2*)] *integrable-simple-function*[*OF add*(*3*,*4*)] **by** (*intro cond-exp-cong*, *auto*)
ultimately show *?case* **by** *force*
qed

lemma *has-cond-exp-simple-lim*:
fixes *f* :: *'a* $\Rightarrow$ *'b*::{*second-countable-topology*, *banach*}
assumes *integrable*[*measurable*]: *integrable M f*
and $\bigwedge$*i. simple-function M* (*s i*)
and $\bigwedge$*i. emeasure M* {*y* $\in$ *space M. s i y* $\neq$ *0*} $\neq \infty$
and $\bigwedge$*x. x* $\in$ *space M* $\Longrightarrow$ (λ*i. s i x*) $\longrightarrow$ *f x*
and $\bigwedge$*x i. x* $\in$ *space M* $\Longrightarrow$ *norm* (*s i x*) $\leq$ *2* ∗ *norm* (*f x*)
obtains *r*
where *strict-mono r has-cond-exp M F f* (λ*x. lim* (λ*i. cond-exp M F* (*s* (*r i*)) *x*))

AE x in M. convergent (λ*i. cond-exp M F* (*s* (*r i*)) *x*)
proof −
have [*measurable*]: (*s i*) $\in$ *borel-measurable M* **for** *i* **using** *assms*(*2*) **by** (*simp add*: *borel-measurable-simple-function*)
have *integrable-s*: *integrable M* (λ*x. s i x*) **for** *i* **using** *assms integrable-simple-function* **by** *blast*
have *integrable-4f*: *integrable M* (λ*x. 4* $*$ *norm* (*f x*)) **using** *assms*(*1*) **by** *simp*
have *integrable-2f*: *integrable M* (λ*x. 2* $*$ *norm* (*f x*)) **using** *assms*(*1*) **by** *simp*
have *integrable-2-cond-exp-norm-f*: *integrable M* (λ*x. 2* $*$ *cond-exp M F* (λ*x. norm* (*f x*)) *x*) **by** *fast*

have *emeasure M* {*y* $\in$ *space M. s i y* − *s j y* $\neq$ *0*} $\leq$ *emeasure M* {*y* $\in$ *space M. s i y* $\neq$ *0*} + *emeasure M* {*y* $\in$ *space M. s j y* $\neq$ *0*} **for** *i j* **using** *simple-functionD*(*2*)[*OF assms*(*2*)] **by** (*intro order-trans*[*OF emeasure-mono emeasure-subadditive*], *auto*)
hence *fin-sup*: *emeasure M* {*y* $\in$ *space M. s i y* − *s j y* $\neq$ *0*} $\neq$ ∞ **for** *i j* **using** *assms*(*3*) **by** (*metis* (*mono-tags*) *ennreal-add-eq-top linorder-not-less top.not-eq-extremum infinity-ennreal-def*)

have *emeasure M* {*y* $\in$ *space M. norm* (*s i y* − *s j y*) $\neq$ *0*} $\leq$ *emeasure M* {*y* $\in$ *space M. s i y* $\neq$ *0*} + *emeasure M* {*y* $\in$ *space M. s j y* $\neq$ *0*} **for** *i j* **using** *simple-functionD*(*2*)[*OF assms*(*2*)] **by** (*intro order-trans*[*OF emeasure-mono emeasure-subadditive*], *auto*)
hence *fin-sup-norm*: *emeasure M* {*y* $\in$ *space M. norm* (*s i y* − *s j y*) $\neq$ *0*} $\neq$ ∞ **for** *i j* **using** *assms*(*3*) **by** (*metis* (*mono-tags*) *ennreal-add-eq-top linorder-not-less top.not-eq-extremum infinity-ennreal-def*)

have *Cauchy*: *Cauchy* (λ*n. s n x*) **if** *x* $\in$ *space M* **for** *x* **using** *assms*(*4*) *LIMSEQ-imp-Cauchy that* **by** *blast*
hence *bounded-range-s*: *bounded* (*range* (λ*n. s n x*)) **if** *x* $\in$ *space M* **for** *x* **using** *that cauchy-imp-bounded* **by** *fast*

have *AE x in M.* (λ*n. diameter* {*s i x* | *i. n* $\leq$ *i*}) $\longrightarrow$ *0* **using** *Cauchy cauchy-iff-diameter-tends-to-zero-and-bounded* **by** *fast*
moreover have (λ*x. diameter* {*s i x* |*i. n* $\leq$ *i*}) $\in$ *borel-measurable M* **for** *n* **using** *bounded-range-s borel-measurable-diameter* **by** *measurable*
moreover have *AE x in M. norm* (*diameter* {*s i x* |*i. n* $\leq$ *i*}) $\leq$ *4* $*$ *norm* (*f x*) **for** *n*
proof −
{
fix *x* **assume** *x*: *x* $\in$ *space M*
have *diameter* {*s i x* |*i. n* $\leq$ *i*} $\leq$ *2* $*$ *norm* (*f x*) + *2* $*$ *norm* (*f x*) **by** (*intro diameter-le*, *blast*, *subst dist-norm*[*symmetric*], *intro dist-triangle3*[*THEN order-trans*, *of 0*], *intro add-mono*) (*auto intro*: *assms*(*5*)[*OF x*])
hence *norm* (*diameter* {*s i x* |*i. n* $\leq$ *i*}) $\leq$ *4* $*$ *norm* (*f x*) **using** *diameter-ge-0*[*OF bounded-subset*[*OF bounded-range-s*], *OF x*, *of* {*s i x* |*i. n* $\leq$ *i*}] **by** *force*
}
thus *?thesis* **by** *fast*

qed
ultimately have *diameter-tendsto-zero*: $(\lambda n.\ LINT\ x|M.\ diameter\ \{s\ i\ x\ |\ i.\ n \leq i\}) \longrightarrow 0$ **by** (*intro integral-dominated-convergence*[*OF borel-measurable-const*[*of 0*] - *integrable-4f*, *simplified*]) (*fast+*)

have *diameter-integrable*: *integrable M* $(\lambda x.\ diameter\ \{s\ i\ x\ |\ i.\ n \leq i\})$ **for** *n* **using** *assms*(*1*,*5*)
by (*intro integrable-bound-diameter*[*OF bounded-range-s integrable-2f*], *auto*)

have *dist-integrable*: *integrable M* $(\lambda x.\ dist\ (s\ i\ x)\ (s\ j\ x))$ **for** *i j* **using** *assms*(*5*) *dist-triangle3*[*of s i - - 0*, *THEN order-trans*, *OF add-mono*, *of - 2 * norm* (*f* -)]
by (*intro Bochner-Integration.integrable-bound*[*OF integrable-4f*]) *fastforce+*

have $\exists N.\ \forall i \geq N.\ \forall j \geq N.\ LINT\ x|M.\ norm\ (cond\text{-}exp\ M\ F\ (s\ i)\ x - cond\text{-}exp\ M\ F\ (s\ j)\ x) < e$ **if** *e-pos*: $e > 0$ **for** *e*
proof –
obtain *N* **where** $*$: $LINT\ x|M.\ diameter\ \{s\ i\ x\ |\ i.\ n \leq i\} < e$ **if** $n \geq N$ **for** *n* **using** *that order-tendsto-iff*[*THEN iffD1*, *OF diameter-tendsto-zero*, *unfolded eventually-sequentially*] *e-pos* **by** *presburger*
{
fix *i j x* **assume** *asm*: $i \geq N\ j \geq N\ x \in space\ M$
have *case-prod dist* ‘ $(\{s\ i\ x\ |i.\ N \leq i\} \times \{s\ i\ x\ |i.\ N \leq i\}) = case\text{-}prod\ (\lambda i\ j.\ dist\ (s\ i\ x)\ (s\ j\ x))$ ‘ $(\{N..\} \times \{N..\})$ **by** *fast*
hence $diameter\ \{s\ i\ x\ |\ i.\ N \leq i\} = (SUP\ (i, j) \in \{N..\} \times \{N..\}.\ dist\ (s\ i\ x)\ (s\ j\ x))$ **unfolding** *diameter-def* **by** *auto*
moreover have $(SUP\ (i, j) \in \{N..\} \times \{N..\}.\ dist\ (s\ i\ x)\ (s\ j\ x)) \geq dist\ (s\ i\ x)\ (s\ j\ x)$ **using** *asm bounded-imp-bdd-above*[*OF bounded-imp-dist-bounded*, *OF bounded-range-s*] **by** (*intro cSup-upper*, *auto*)
ultimately have $diameter\ \{s\ i\ x\ |\ i.\ N \leq i\} \geq dist\ (s\ i\ x)\ (s\ j\ x)$ **by** *presburger*
}
hence $LINT\ x|M.\ dist\ (s\ i\ x)\ (s\ j\ x) < e$ **if** $i \geq N\ j \geq N$ **for** *i j* **using** *that* $*$ **by** (*intro integral-mono*[*OF dist-integrable diameter-integrable*, *THEN order.strict-trans1*], *blast+*)
moreover have $LINT\ x|M.\ norm\ (cond\text{-}exp\ M\ F\ (s\ i)\ x - cond\text{-}exp\ M\ F\ (s\ j)\ x) \leq LINT\ x|M.\ dist\ (s\ i\ x)\ (s\ j\ x)$ **for** *i j*
proof –
have $LINT\ x|M.\ norm\ (cond\text{-}exp\ M\ F\ (s\ i)\ x - cond\text{-}exp\ M\ F\ (s\ j)\ x) = LINT\ x|M.\ norm\ (cond\text{-}exp\ M\ F\ (s\ i)\ x + - 1 *_R cond\text{-}exp\ M\ F\ (s\ j)\ x)$ **unfolding** *dist-norm* **by** *simp*
also have ... $= LINT\ x|M.\ norm\ (cond\text{-}exp\ M\ F\ (\lambda x.\ s\ i\ x - s\ j\ x)\ x)$ **using** *has-cond-exp-charact*(*2*)[*OF has-cond-exp-add*[*OF - has-cond-exp-scaleR-right*, *OF has-cond-exp-charact*(*1*,*1*), *OF has-cond-exp-simple*(*1*,*1*)[*OF assms*(*2*,*3*)]], *THEN AE-symmetric*, *of i* -1 *j*] **by** (*intro integral-cong-AE*) *force+*
also have ... $\leq LINT\ x|M.\ cond\text{-}exp\ M\ F\ (\lambda x.\ norm\ (s\ i\ x - s\ j\ x))\ x$ **using** *cond-exp-contraction-simple*[*OF - fin-sup*, *of i j*] *integrable-cond-exp assms*(*2*) **by** (*intro integral-mono-AE*, *fast+*)
also have ... $= LINT\ x|M.\ norm\ (s\ i\ x - s\ j\ x)$ **unfolding** *set-integral-space*(*1*)[*OF integrable-cond-exp*, *symmetric*] *set-integral-space*[*OF dist-integrable*[*unfolded dist-norm*],

symmetric] **by** (*intro has-cond-expD*(*1*)[*OF has-cond-exp-simple*[*OF - fin-sup-norm*], *symmetric*]) (*metis assms*(*2*) *simple-function-compose1 simple-function-diff*, *metis sets.top subalg subalgebra-def*)
finally show *?thesis* **unfolding** *dist-norm* **.**
qed
ultimately show *?thesis* **using** *order.strict-trans1* **by** *meson*
qed
then obtain *r* **where** *strict-mono-r*: *strict-mono r* **and** *AE-Cauchy*: *AE x in M*. *Cauchy* (λi. *cond-exp M F* (*s* (*r i*)) *x*) **by** (*rule cauchy-L1-AE-cauchy-subseq*[*OF integrable-cond-exp*], *auto*)
hence *ae-lim-cond-exp*: *AE x in M*. (λn. *cond-exp M F* (*s* (*r n*)) *x*) $\longrightarrow$ *lim* (λn. *cond-exp M F* (*s* (*r n*)) *x*) **using** *Cauchy-convergent-iff convergent-LIMSEQ-iff* **by** *fastforce*

have *cond-exp-bounded*: *AE x in M*. *norm* (*cond-exp M F* (*s* (*r n*)) *x*) $\leq$ *cond-exp M F* (λx. *2* $*$ *norm* (*f x*)) *x* **for** *n*
proof $-$
have *AE x in M*. *norm* (*cond-exp M F* (*s* (*r n*)) *x*) $\leq$ *cond-exp M F* (λx. *norm* (*s* (*r n*) *x*)) *x* **by** (*rule cond-exp-contraction-simple*[*OF assms*(*2*,*3*)])
moreover have *AE x in M*. *real-cond-exp M F* (λx. *norm* (*s* (*r n*) *x*)) *x* $\leq$ *real-cond-exp M F* (λx. *2* $*$ *norm* (*f x*)) *x* **using** *integrable-s integrable-2f assms*(*5*) **by** (*intro real-cond-exp-mono*, *auto*)
ultimately show *?thesis* **using** *cond-exp-real*[*OF integrable-norm*, *OF integrable-s*, *of r n*] *cond-exp-real*[*OF integrable-2f*] **by** *force*
qed
have *lim-integrable*: *integrable M* (λx. *lim* (λi. *cond-exp M F* (*s* (*r i*)) *x*)) **by** (*intro integrable-dominated-convergence*[*OF - borel-measurable-cond-exp' integrable-cond-exp ae-lim-cond-exp cond-exp-bounded*], *simp*)

{
fix *A* **assume** *A-in-sets-F*: *A* $\in$ *sets F*
have *AE x in M*. *norm* (*indicator A x* $*_R$ *cond-exp M F* (*s* (*r n*)) *x*) $\leq$ *cond-exp M F* (λx. *2* $*$ *norm* (*f x*)) *x* **for** *n*
proof $-$
have *AE x in M*. *norm* (*indicator A x* $*_R$ *cond-exp M F* (*s* (*r n*)) *x*) $\leq$ *norm* (*cond-exp M F* (*s* (*r n*)) *x*) **unfolding** *indicator-def* **by** *simp*
thus *?thesis* **using** *cond-exp-bounded*[*of n*] **by** *force*
qed
hence *lim-cond-exp-int*: (λn. *LINT x*:*A*|*M*. *cond-exp M F* (*s* (*r n*)) *x*) $\longrightarrow$ *LINT x*:*A*|*M*. *lim* (λn. *cond-exp M F* (*s* (*r n*)) *x*)
using *ae-lim-cond-exp measurable-from-subalg*[*OF subalg borel-measurable-indicator*, *OF A-in-sets-F*] *cond-exp-bounded*
unfolding *set-lebesgue-integral-def*
by (*intro integral-dominated-convergence*[*OF borel-measurable-scaleR borel-measurable-scaleR integrable-cond-exp*]) (*fastforce simp add*: *tendsto-scaleR*)+

have *AE x in M*. *norm* (*indicator A x* $*_R$ *s* (*r n*) *x*) $\leq$ *2* $*$ *norm* (*f x*) **for** *n*
proof $-$
have *AE x in M*. *norm* (*indicator A x* $*_R$ *s* (*r n*) *x*) $\leq$ *norm* (*s* (*r n*) *x*)

unfolding *indicator-def* **by** *simp*
thus *?thesis* **using** *assms*(*5*)[*of - r n*] **by** *fastforce*
qed
hence *lim-s-int*: $(\lambda n.\ LINT\ x{:}A|M.\ s\ (r\ n)\ x) \longrightarrow LINT\ x{:}A|M.\ f\ x$
using *measurable-from-subalg*[*OF subalg borel-measurable-indicator*, *OF A-in-sets-F*] *LIMSEQ-subseq-LIMSEQ*[*OF assms*(*4*) *strict-mono-r*] *assms*(*5*)
unfolding *set-lebesgue-integral-def comp-def*
by (*intro integral-dominated-convergence*[*OF borel-measurable-scaleR borel-measurable-scaleR integrable-2f*]) (*fastforce simp add*: *tendsto-scaleR*)+

have $LINT\ x{:}A|M.\ lim\ (\lambda n.\ cond\text{-}exp\ M\ F\ (s\ (r\ n))\ x) = lim\ (\lambda n.\ LINT\ x{:}A|M.\ cond\text{-}exp\ M\ F\ (s\ (r\ n))\ x)$ **using** *limI*[*OF lim-cond-exp-int*] **by** *argo*
also have ... $= lim\ (\lambda n.\ LINT\ x{:}A|M.\ s\ (r\ n)\ x)$ **using** *has-cond-expD*(*1*)[*OF has-cond-exp-simple*[*OF assms*(*2*,*3*)] *A-in-sets-F*, *symmetric*] **by** *presburger*
also have ... $= LINT\ x{:}A|M.\ f\ x$ **using** *limI*[*OF lim-s-int*] **by** *argo*
finally have $LINT\ x{:}A|M.\ lim\ (\lambda n.\ cond\text{-}exp\ M\ F\ (s\ (r\ n))\ x) = LINT\ x{:}A|M.\ f\ x$ **.**
}
hence *has-cond-exp M F f* $(\lambda x.\ lim\ (\lambda i.\ cond\text{-}exp\ M\ F\ (s\ (r\ i))\ x))$ **using** *assms*(*1*) *lim-integrable* **by** (*intro has-cond-expI′*, *auto*)
thus *thesis* **using** *AE-Cauchy Cauchy-convergent strict-mono-r* **by** (*auto intro*!: *that*)
qed

corollary *has-cond-expI*:
fixes $f :: 'a \Rightarrow 'b{::}\{second\text{-}countable\text{-}topology,banach\}$
assumes *integrable M f*
shows *has-cond-exp M F f* (*cond-exp M F f*)
proof −
obtain *s* **where** *s-is*: $\bigwedge i.\ simple\text{-}function\ M\ (s\ i)\ \bigwedge i.\ emeasure\ M\ \{y \in space\ M.\ s\ i\ y \neq 0\} \neq \infty\ \bigwedge x.\ x \in space\ M \Longrightarrow (\lambda i.\ s\ i\ x) \longrightarrow f\ x\ \bigwedge x\ i.\ x \in space\ M \Longrightarrow norm\ (s\ i\ x) \leq 2 * norm\ (f\ x)$ **using** *integrable-implies-simple-function-sequence*[*OF assms*] **by** *blast*
show *?thesis* **using** *has-cond-exp-simple-lim*[*OF assms s-is*] *has-cond-exp-charact*(*1*) **by** *metis*
qed

4.2 Properties

lemma *cond-exp-nested-subalg*:
fixes $f :: 'a \Rightarrow 'b{::}\{second\text{-}countable\text{-}topology,banach\}$
assumes *integrable M f subalgebra M G subalgebra G F*
shows $AE\ \xi\ in\ M.\ cond\text{-}exp\ M\ F\ f\ \xi = cond\text{-}exp\ M\ F\ (cond\text{-}exp\ M\ G\ f)\ \xi$
using *has-cond-expI assms sigma-finite-subalgebra-def* **by** (*auto intro*!: *has-cond-exp-nested-subalg*[*THEN has-cond-exp-charact*(*2*), *THEN AE-symmetric*] *sigma-finite-subalgebra.has-cond-expI*[*OF sigma-finite-subalgebra.intro*[*OF assms*(*2*)]] *nested-subalg-is-sigma-finite*)

lemma *cond-exp-set-integral*:
fixes $f :: 'a \Rightarrow 'b{::}\{second\text{-}countable\text{-}topology,banach\}$

assumes *integrable M f A* $\in$ *sets F*
shows $(\int x \in A.\ f\ x\ \partial M) = (\int x \in A.\ \textit{cond-exp}\ M\ F\ f\ x\ \partial M)$
using *has-cond-expD*(*1*)[*OF has-cond-expI*, *OF assms*] **by** *argo*

lemma *cond-exp-add*:
fixes $f :: {'a} \Rightarrow {'b}::\{\textit{second-countable-topology},\textit{banach}\}$
assumes *integrable M f integrable M g*
shows *AE x in M*. *cond-exp M F* ($\lambda x.\ f\ x + g\ x$) x = *cond-exp M F f x* + *cond-exp M F g x*
using *has-cond-exp-add*[*OF has-cond-expI*(*1*,*1*), *OF assms*, *THEN has-cond-exp-charact*(*2*)]
.

lemma *cond-exp-diff*:
fixes $f :: {'a} \Rightarrow {'b} :: \{\textit{second-countable-topology}, \textit{banach}\}$
assumes *integrable M f integrable M g*
shows *AE x in M*. *cond-exp M F* ($\lambda x.\ f\ x - g\ x$) x = *cond-exp M F f x* $-$ *cond-exp M F g x*
using *has-cond-exp-add*[*OF - has-cond-exp-scaleR-right*, *OF has-cond-expI*(*1*,*1*), *OF assms*, *THEN has-cond-exp-charact*(*2*), *of* -1] **by** *simp*

lemma *cond-exp-diff'*:
fixes $f :: {'a} \Rightarrow {'b} :: \{\textit{second-countable-topology}, \textit{banach}\}$
assumes *integrable M f integrable M g*
shows *AE x in M*. *cond-exp M F* ($f - g$) x = *cond-exp M F f x* $-$ *cond-exp M F g x*
unfolding *fun-diff-def* **using** *assms* **by** (*rule cond-exp-diff*)

lemma *cond-exp-scaleR-left*:
fixes $f :: {'a} \Rightarrow \textit{real}$
assumes *integrable M f*
shows *AE x in M*. *cond-exp M F* ($\lambda x.\ f\ x *_R c$) x = *cond-exp M F f x* $*_R\ c$
using *cond-exp-set-integral*[*OF assms*] *subalg assms* **unfolding** *subalgebra-def*
by (*intro cond-exp-charact*,
subst set-integral-scaleR-left, *blast*, *intro assms*,
subst set-integral-scaleR-left, *blast*, *intro integrable-cond-exp*)
auto

lemma *cond-exp-contraction*:
fixes $f :: {'a} \Rightarrow {'b}::\{\textit{second-countable-topology}, \textit{banach}\}$
assumes *integrable M f*
shows *AE x in M*. *norm* (*cond-exp M F f x*) $\leq$ *cond-exp M F* ($\lambda x.$ *norm* ($f\ x$)) x
proof $-$
obtain *s* **where** *s*: $\bigwedge i.$ *simple-function M* ($s\ i$) $\bigwedge i.$ *emeasure M* $\{y \in$ *space M*. $s\ i\ y \neq 0\} \neq \infty$ $\bigwedge x.\ x \in$ *space M* $\Longrightarrow (\lambda i.\ s\ i\ x) \longrightarrow f\ x$ $\bigwedge i\ x.\ x \in$ *space M*

$\Longrightarrow$ *norm* (*s i x*) $\leq$ *2* $*$ *norm* (*f x*)
  by (*blast intro*: *integrable-implies-simple-function-sequence*[*OF assms*])

obtain *r* **where** *r*: *strict-mono r* **and** *has-cond-exp M F f* (λx. *lim* (λi. *cond-exp M F* (*s* (*r i*)) *x*)) *AE x in M*. (λi. *cond-exp M F* (*s* (*r i*)) *x*) $\longrightarrow$ *lim* (λi. *cond-exp M F* (*s* (*r i*)) *x*)
  using *has-cond-exp-simple-lim*[*OF assms s*] **unfolding** *convergent-LIMSEQ-iff* **by** *blast*
hence *r-tendsto*: *AE x in M*. (λi. *cond-exp M F* (*s* (*r i*)) *x*) $\longrightarrow$ *cond-exp M F f x* **using** *has-cond-exp-charact*(*2*) **by** *force*

have *norm-s-r*: $\bigwedge i$. *simple-function M* (λx. *norm* (*s* (*r i*) *x*)) $\bigwedge i$. *emeasure M* $\{y \in$ *space M*. *norm* (*s* (*r i*) *y*) $\neq$ *0*$\}$ $\neq \infty$ $\bigwedge x$. $x \in$ *space M* $\Longrightarrow$ (λi. *norm* (*s* (*r i*) *x*)) $\longrightarrow$ *norm* (*f x*) $\bigwedge i\ x$. $x \in$ *space M* $\Longrightarrow$ *norm* (*norm* (*s* (*r i*) *x*)) $\leq$ *2* $*$ *norm* (*norm* (*f x*))
  using *s* **by** (*auto intro*: *LIMSEQ-subseq-LIMSEQ*[*OF tendsto-norm r*, *unfolded comp-def*] *simple-function-compose1*)

obtain *r'* **where** *r'*: *strict-mono r'* **and** *has-cond-exp M F* (λx. *norm* (*f x*)) (λx. *lim* (λi. *cond-exp M F* (λx. *norm* (*s* (*r* (*r' i*)) *x*)) *x*)) *AE x in M*. (λi. *cond-exp M F* (λx. *norm* (*s* (*r* (*r' i*)) *x*)) *x*) $\longrightarrow$ *lim* (λi. *cond-exp M F* (λx. *norm* (*s* (*r* (*r' i*)) *x*)) *x*) **using** *has-cond-exp-simple-lim*[*OF integrable-norm norm-s-r*, *OF assms*] **unfolding** *convergent-LIMSEQ-iff* **by** *blast*
hence *r'-tendsto*: *AE x in M*. (λi. *cond-exp M F* (λx. *norm* (*s* (*r* (*r' i*)) *x*)) *x*) $\longrightarrow$ *cond-exp M F* (λx. *norm* (*f x*)) *x* **using** *has-cond-exp-charact*(*2*) **by** *force*

have *AE x in M*. $\forall$ *i*. *norm* (*cond-exp M F* (*s* (*r* (*r' i*))) *x*) $\leq$ *cond-exp M F* (λx. *norm* (*s* (*r* (*r' i*)) *x*)) *x* **using** *s* **by** (*auto intro*: *cond-exp-contraction-simple simp add*: *AE-all-countable*)
moreover have *AE x in M*. (λi. *norm* (*cond-exp M F* (*s* (*r* (*r' i*))) *x*)) $\longrightarrow$ *norm* (*cond-exp M F f x*) **using** *r-tendsto LIMSEQ-subseq-LIMSEQ*[*OF tendsto-norm r'*, *unfolded comp-def*] **by** *fast*
ultimately show *?thesis* **using** *LIMSEQ-le r'-tendsto* **by** *fast*
qed

lemma *cond-exp-measurable-mult*:
  fixes *f g* :: *'a* $\Rightarrow$ *real*
  assumes [*measurable*]: *integrable M* (λx. *f x* $*$ *g x*) *integrable M g f* $\in$ *borel-measurable F*
  shows *integrable M* (λx. *f x* $*$ *cond-exp M F g x*)
    AE x in M. *cond-exp M F* (λx. *f x* $*$ *g x*) *x* = *f x* $*$ *cond-exp M F g x*
proof−
show *integrable*: *integrable M* (λx. *f x* $*$ *cond-exp M F g x*) **using** *cond-exp-real*[*OF assms*(*2*)] **by** (*intro integrable-cong-AE-imp*[*OF real-cond-exp-intg*(*1*), *OF assms*(*1*,*3*) *assms*(*2*)[*THEN borel-measurable-integrable*]] *measurable-from-subalg*[*OF subalg*]) *auto*
interpret *sigma-finite-measure restr-to-subalg M F* **by** (*rule sigma-fin-subalg*)

{
fix *A* **assume** *asm*: $A \in$ *sets F*
hence *asm'*: $A \in$ *sets M* **using** *subalg* **by** (*fastforce simp add*: *subalgebra-def*)
have *set-lebesgue-integral M A* (*cond-exp M F* ($\lambda x.$ *f x* $*$ *g x*)) = *set-lebesgue-integral M A* ($\lambda x.$ *f x* $*$ *g x*) **by** (*simp add*: *cond-exp-set-integral*[*OF assms*(*1*) *asm*])
also have ... = *set-lebesgue-integral M A* ($\lambda x.$ *f x* $*$ *real-cond-exp M F g x*) **using** *borel-measurable-times*[*OF borel-measurable-indicator*[*OF asm*] *assms*(*3*)] *borel-measurable-integrable*[*OF assms*(*2*)] *integrable-mult-indicator*[*OF asm' assms*(*1*)] **by** (*fastforce simp add*: *set-lebesgue-integral-def mult.assoc*[*symmetric*] *intro*: *real-cond-exp-intg*(*2*)[*symmetric*])
also have ... = *set-lebesgue-integral M A* ($\lambda x.$ *f x* $*$ *cond-exp M F g x*) **using** *cond-exp-real*[*OF assms*(*2*)] *asm' borel-measurable-cond-exp' borel-measurable-cond-exp2 measurable-from-subalg*[*OF subalg assms*(*3*)] **by** (*auto simp add*: *set-lebesgue-integral-def intro*: *integral-cong-AE*)
finally have *set-lebesgue-integral M A* (*cond-exp M F* ($\lambda x.$ *f x* $*$ *g x*)) = $\int x \in A.$ (*f x* $*$ *cond-exp M F g x*)∂M **.**
}
hence *AE x in restr-to-subalg M F. cond-exp M F* ($\lambda x.$ *f x* $*$ *g x*) *x* = *f x* $*$ *cond-exp M F g x* **by** (*intro density-unique-banach integrable-cond-exp integrable integrable-in-subalg subalg*, *measurable*, *simp add*: *set-lebesgue-integral-def integral-subalgebra2*[*OF subalg*] *sets-restr-to-subalg*[*OF subalg*])
thus *AE x in M. cond-exp M F* ($\lambda x.$ *f x* $*$ *g x*) *x* = *f x* $*$ *cond-exp M F g x* **by** (*rule AE-restr-to-subalg*[*OF subalg*])
qed

lemma *cond-exp-measurable-scaleR*:
fixes *f* :: *'a* $\Rightarrow$ *real* **and** *g* :: *'a* $\Rightarrow$ *'b* :: {*second-countable-topology*, *banach*}
assumes [*measurable*]: *integrable M* ($\lambda x.$ *f x* $*_R$ *g x*) *integrable M g f* $\in$ *borel-measurable F*
shows *integrable M* ($\lambda x.$ *f x* $*_R$ *cond-exp M F g x*)
AE x in M. cond-exp M F ($\lambda x.$ *f x* $*_R$ *g x*) *x* = *f x* $*_R$ *cond-exp M F g x*
proof −
let *?F* = *restr-to-subalg M F*
have *subalg'*: *subalgebra M* (*restr-to-subalg M F*) **by** (*metis sets-eq-imp-space-eq sets-restr-to-subalg subalg subalgebra-def*)
{
fix *z* **assume** *asm*[*measurable*]: *integrable M* ($\lambda x.$ *z x* $*_R$ *g x*) *z* $\in$ *borel-measurable ?F*
hence *asm'*[*measurable*]: *z* $\in$ *borel-measurable F* **using** *measurable-in-subalg' subalg* **by** *blast*
have *integrable M* ($\lambda x.$ *z x* $*_R$ *cond-exp M F g x*) *LINT x|M. z x* $*_R$ *g x* = *LINT x|M. z x* $*_R$ *cond-exp M F g x*
proof −
obtain *s* **where** *s-is*: $\bigwedge i.$ *simple-function ?F* (*s i*) $\bigwedge x.$ $x \in$ *space ?F* $\Longrightarrow$ ($\lambda i.$ *s i x*) $\longrightarrow$ *z x* $\bigwedge i\ x.$ $x \in$ *space ?F* $\Longrightarrow$ *norm* (*s i x*) $\leq$ *2* $*$ *norm* (*z x*) **using** *borel-measurable-implies-sequence-metric*[*OF asm*(*2*), *of 0*] **by** *force*

have *s-scaleR-g-tendsto*: *AE x in M.* ($\lambda i.$ *s i x* $*_R$ *g x*) $\longrightarrow$ *z x* $*_R$ *g x*

using *s-is*(*2*) **by** (*simp add*: *space-restr-to-subalg tendsto-scaleR*)

have *s-scaleR-cond-exp-g-tendsto*: *AE x in ?F*. (λi. *s i x* $*_R$ *cond-exp M F g x*) $\longrightarrow$ *z x* $*_R$ *cond-exp M F g x* **using** *s-is*(*2*) **by** (*simp add*: *tendsto-scaleR*)

have *s-scaleR-g-meas*: (λx. *s i x* $*_R$ *g x*) $\in$ *borel-measurable M* **for** *i* **using** *s-is*(*1*)[*THEN borel-measurable-simple-function*, *THEN subalg*′[*THEN measurable-from-subalg*]] **by** *simp*

have *s-scaleR-cond-exp-g-meas*: (λx. *s i x* $*_R$ *cond-exp M F g x*) $\in$ *borel-measurable ?F* **for** *i* **using** *s-is*(*1*)[*THEN borel-measurable-simple-function*] *measurable-in-subalg*[*OF subalg borel-measurable-cond-exp*] **by** (*fastforce intro*: *borel-measurable-scaleR*)

have *s-scaleR-g-AE-bdd*: *AE x in M*. *norm* (*s i x* $*_R$ *g x*) $\leq$ *2* $*$ *norm* (*z x* $*_R$ *g x*) **for** *i* **using** *s-is*(*3*) **by** (*fastforce simp add*: *space-restr-to-subalg mult.assoc*[*symmetric*] *mult-right-mono*)

{

fix *i*

have *asm*: *integrable M* (λx. *norm* (*z x*) $*$ *norm* (*g x*)) **using** *asm*(*1*)[*THEN integrable-norm*] **by** *simp*

have *AE x in ?F*. *norm* (*s i x* $*_R$ *cond-exp M F g x*) $\leq$ *2* $*$ *norm* (*z x*) $*$ *norm* (*cond-exp M F g x*) **using** *s-is*(*3*) **by** (*fastforce simp add*: *mult-mono*)

moreover have *AE x in ?F*. *norm* (*z x*) $*$ *cond-exp M F* (λx. *norm* (*g x*)) *x* = *cond-exp M F* (λx. *norm* (*z x*) $*$ *norm* (*g x*)) *x* **by** (*rule cond-exp-measurable-mult*(*2*)[*THEN AE-symmetric*, *OF asm integrable-norm*, *OF assms*(*2*), *THEN AE-restr-to-subalg2*[*OF subalg*]], *auto*)

ultimately have *AE x in ?F*. *norm* (*s i x* $*_R$ *cond-exp M F g x*) $\leq$ *2* $*$ *cond-exp M F* (λx. *norm* (*z x* $*_R$ *g x*)) *x* **using** *cond-exp-contraction*[*OF assms*(*2*), *THEN AE-restr-to-subalg2*[*OF subalg*]] *order-trans*[*OF - mult-mono*] **by** *fastforce*

}

note *s-scaleR-cond-exp-g-AE-bdd* = *this*

{

fix *i*

have *s-meas-M*[*measurable*]: *s i* $\in$ *borel-measurable M* **by** (*meson borel-measurable-simple-function measurable-from-subalg s-is*(*1*) *subalg*′)

have *s-meas-F*[*measurable*]: *s i* $\in$ *borel-measurable F* **by** (*meson borel-measurable-simple-function measurable-in-subalg*′ *s-is*(*1*) *subalg*)

have *s-scaleR-eq*: *s i x* $*_R$ *h x* = ($\sum$ *y*$\in$*s i* ‘ *space M*. (*indicator* (*s i* −‘ {*y*} $\cap$ *space M*) *x* $*_R$ *y*) $*_R$ *h x*) **if** *x* $\in$ *space M* **for** *x* **and** *h* :: ′*a* $\Rightarrow$ ′*b* **using** *simple-function-indicator-representation*[*OF s-is*(*1*), *of x i*] *that* **unfolding** *space-restr-to-subalg scaleR-left.sum*[*of - - h x*, *symmetric*] **by** *presburger*

have *LINT x*|*M*. *s i x* $*_R$ *g x* = *LINT x*|*M*. ($\sum$ *y*$\in$*s i* ‘ *space M*. *indicator* (*s i* −‘ {*y*} $\cap$ *space M*) *x* $*_R$ *y* $*_R$ *g x*) **using** *s-scaleR-eq* **by** (*intro Bochner-Integration.integral-cong*) *auto*

also have ... = ($\sum$ *y*$\in$*s i* ‘ *space M*. *LINT x*|*M*. *indicator* (*s i* −‘ {*y*} $\cap$ *space M*) *x* $*_R$ *y* $*_R$ *g x*) **by** (*intro Bochner-Integration.integral-sum in-*

tegrable-mult-indicator[*OF* - *integrable-scaleR-right*] *assms*(*2*)) *simp*
also have ... = ($\sum$ *y*∈*s i* ' *space M*. *y* $*_R$ *set-lebesgue-integral M* (*s i* −' {*y*} ∩ *space M*) *g*) **by** (*simp only*: *set-lebesgue-integral-def*[*symmetric*]) *simp*
also have ... = ($\sum$ *y*∈*s i* ' *space M*. *y* $*_R$ *set-lebesgue-integral M* (*s i* −' {*y*} ∩ *space M*) (*cond-exp M F g*)) **using** *assms*(*2*) *subalg borel-measurable-vimage*[*OF s-meas-F*] **by** (*subst cond-exp-set-integral*, *auto simp add*: *subalgebra-def*)
also have ... = ($\sum$ *y*∈*s i* ' *space M*. *LINT x*|*M*. *indicator* (*s i* −' {*y*} ∩ *space M*) *x* $*_R$ *y* $*_R$ *cond-exp M F g x*) **by** (*simp only*: *set-lebesgue-integral-def*[*symmetric*]) *simp*
also have ... = *LINT x*|*M*. ($\sum$ *y*∈*s i* ' *space M*. *indicator* (*s i* −' {*y*} ∩ *space M*) *x* $*_R$ *y* $*_R$ *cond-exp M F g x*) **by** (*intro Bochner-Integration.integral-sum*[*symmetric*] *integrable-mult-indicator*[*OF* - *integrable-scaleR-right*]) *auto*
also have ... = *LINT x*|*M*. *s i x* $*_R$ *cond-exp M F g x* **using** *s-scaleR-eq* **by** (*intro Bochner-Integration.integral-cong*) *auto*
finally have *LINT x*|*M*. *s i x* $*_R$ *g x* = *LINT x*|*?F*. *s i x* $*_R$ *cond-exp M F g x* **by** (*simp add*: *integral-subalgebra2*[*OF subalg*])
}
note *integral-s-eq* = *this*

show *integrable M* (λ*x*. *z x* $*_R$ *cond-exp M F g x*) **using** *s-scaleR-cond-exp-g-meas asm*(*2*) *borel-measurable-cond-exp*′ **by** (*intro integrable-from-subalg*[*OF subalg*] *integrable-cond-exp integrable-dominated-convergence*[*OF* - - - *s-scaleR-cond-exp-g-tendsto s-scaleR-cond-exp-g-AE-bdd*]) (*auto intro*: *measurable-from-subalg*[*OF subalg*] *integrable-in-subalg measurable-in-subalg subalg*)

have (λ*i*. *LINT x*|*M*. *s i x* $*_R$ *g x*) ⟶ *LINT x*|*M*. *z x* $*_R$ *g x* **using** *s-scaleR-g-meas asm*(*1*)[*THEN integrable-norm*] *asm*′ *borel-measurable-cond-exp*′ **by** (*intro integral-dominated-convergence*[*OF* - - - *s-scaleR-g-tendsto s-scaleR-g-AE-bdd*]) (*auto intro*: *measurable-from-subalg*[*OF subalg*])
moreover have (λ*i*. *LINT x*|*?F*. *s i x* $*_R$ *cond-exp M F g x*) ⟶ *LINT x*|*?F*. *z x* $*_R$ *cond-exp M F g x* **using** *s-scaleR-cond-exp-g-meas asm*(*2*) *borel-measurable-cond-exp*′ **by** (*intro integral-dominated-convergence*[*OF* - - - *s-scaleR-cond-exp-g-tendsto s-scaleR-cond-exp-g-AE-bdd*]) (*auto intro*: *measurable-from-subalg*[*OF subalg*] *integrable-in-subalg measurable-in-subalg subalg*)
ultimately show *LINT x*|*M*. *z x* $*_R$ *g x* = *LINT x*|*M*. *z x* $*_R$ *cond-exp M F g x* **using** *integral-s-eq* **using** *subalg* **by** (*simp add*: *LIMSEQ-unique integral-subalgebra2*)
qed
}
note * = *this*

show *integrable M* (λ*x*. *f x* $*_R$ *cond-exp M F g x*) **using** * *assms measurable-in-subalg*[*OF subalg*] **by** *blast*

{

fix A **assume** *asm*: $A \in F$
hence *integrable M* (λx. *indicat-real A x* $*_R$ *f x* $*_R$ *g x*) **using** *subalg* **by** (*fastforce simp add*: *subalgebra-def intro*!: *integrable-mult-indicator assms*(*1*))
hence *set-lebesgue-integral M A* (λx. *f x* $*_R$ *g x*) = *set-lebesgue-integral M A* (λx. *f x* $*_R$ *cond-exp M F g x*) **unfolding** *set-lebesgue-integral-def* **using** *asm* **by** (*auto intro*!: $*$ *measurable-in-subalg*[*OF subalg*])
}
thus *AE x in M*. *cond-exp M F* (λx. *f x* $*_R$ *g x*) *x* = *f x* $*_R$ *cond-exp M F g x* **using** *borel-measurable-cond-exp* **by** (*intro cond-exp-charact*, *auto intro*!: $*$ *assms measurable-in-subalg*[*OF subalg*])
qed

lemma *cond-exp-sum* [*intro*, *simp*]:
fixes *f* :: $'t \Rightarrow 'a \Rightarrow 'b$ :: {*second-countable-topology*,*banach*}
assumes [*measurable*]: $\bigwedge i$. *integrable M* (*f i*)
shows *AE x in M*. *cond-exp M F* (λx. $\sum i{\in}I$. *f i x*) *x* = ($\sum i{\in}I$. *cond-exp M F* (*f i*) *x*)
proof (*rule has-cond-exp-charact*, *intro has-cond-expI'*)
fix A **assume** [*measurable*]: $A \in$ *sets F*
then have *A-meas* [*measurable*]: $A \in$ *sets M* **by** (*meson subsetD subalg subalgebra-def*)

have ($\int x{\in}A$. ($\sum i{\in}I$. *f i x*)∂M) = ($\int x$. ($\sum i{\in}I$. *indicator A x* $*_R$ *f i x*)∂M) **unfolding** *set-lebesgue-integral-def* **by** (*simp add*: *scaleR-sum-right*)
also have ... = ($\sum i{\in}I$. ($\int x$. *indicator A x* $*_R$ *f i x* ∂M)) **using** *assms* **by** (*auto intro*!: *Bochner-Integration.integral-sum integrable-mult-indicator*)
also have ... = ($\sum i{\in}I$. ($\int x$. *indicator A x* $*_R$ *cond-exp M F* (*f i*) *x* ∂M)) **using** *cond-exp-set-integral*[*OF assms*] **by** (*simp add*: *set-lebesgue-integral-def*)
also have ... = ($\int x$. ($\sum i{\in}I$. *indicator A x* $*_R$ *cond-exp M F* (*f i*) *x*)∂M) **using** *assms* **by** (*auto intro*!: *Bochner-Integration.integral-sum*[*symmetric*] *integrable-mult-indicator*)
also have ... = ($\int x{\in}A$. ($\sum i{\in}I$. *cond-exp M F* (*f i*) *x*)∂M) **unfolding** *set-lebesgue-integral-def* **by** (*simp add*: *scaleR-sum-right*)
finally show ($\int x{\in}A$. ($\sum i{\in}I$. *f i x*)∂M) = ($\int x{\in}A$. ($\sum i{\in}I$. *cond-exp M F* (*f i*) *x*)∂M) **by** *auto*
qed (*auto simp add*: *assms integrable-cond-exp*)

4.3 Linearly Ordered Banach Spaces

In this subsection we show monotonicity results concerning the conditional expectation operator.

lemma *cond-exp-gr-c*:
fixes *f* :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes *integrable M f* *AE x in M*. *f x* > *c*
shows *AE x in M*. *cond-exp M F f x* > *c*
proof −
define X **where** X = {$x \in$ *space M*. *cond-exp M F f x* $\leq$ *c*}
have [*measurable*]: $X \in$ *sets F* **unfolding** *X-def* **by** *measurable* (*metis sets.top*

subalg subalgebra-def)
hence *X-in-M*: $X \in sets\ M$ **using** *sets-restr-to-subalg subalg subalgebra-def* **by** *blast*
have $emeasure\ M\ X = 0$
proof (*rule ccontr*)
assume $emeasure\ M\ X \neq 0$
have $emeasure\ (restr\text{-}to\text{-}subalg\ M\ F)\ X = emeasure\ M\ X$ **by** (*simp add*: *emeasure-restr-to-subalg subalg*)
hence $emeasure\ (restr\text{-}to\text{-}subalg\ M\ F)\ X > 0$ **using** ‹¬(*emeasure M X*) = *0*› *gr-zeroI* **by** *auto*
then obtain *A* **where** *A*: $A \in sets\ (restr\text{-}to\text{-}subalg\ M\ F)\ A \subseteq X\ emeasure\ (restr\text{-}to\text{-}subalg\ M\ F)\ A > 0\ emeasure\ (restr\text{-}to\text{-}subalg\ M\ F)\ A < \infty$
using *sigma-fin-subalg* **by** (*metis emeasure-notin-sets ennreal-0 infinity-ennreal-def le-less-linear neq-top-trans not-gr-zero order-refl sigma-finite-measure.approx-PInf-emeasure-with-finite*)
hence [*simp*]: $A \in sets\ F$ **using** *subalg sets-restr-to-subalg* **by** *blast*
hence *A-in-sets-M*[*simp*]: $A \in sets\ M$ **using** *sets-restr-to-subalg subalg subalgebra-def* **by** *blast*
have [*simp*]: *set-integrable M A* ($\lambda x.\ c$) **using** *A subalg* **by** (*auto simp add*: *set-integrable-def emeasure-restr-to-subalg*)
have [*simp*]: *set-integrable M A f* **unfolding** *set-integrable-def* **by** (*rule integrable-mult-indicator*, *auto simp add*: *assms*(*1*))
have *AE x in M*. $indicator\ A\ x *_R c = indicator\ A\ x *_R f\ x$
proof (*rule integral-eq-mono-AE-eq-AE*)
have $(\int x{\in}A.\ c\ \partial M) \leq (\int x{\in}A.\ f\ x\ \partial M)$ **using** *assms*(*2*) **by** (*intro set-integral-mono-AE-banach*) *auto*
moreover
{
have $(\int x{\in}A.\ f\ x\ \partial M) = (\int x{\in}A.\ cond\text{-}exp\ M\ F\ f\ x\ \partial M)$ **by** (*rule cond-exp-set-integral*, *auto simp add*: *assms*)
also have ... $\leq (\int x{\in}A.\ c\ \partial M)$ **using** *A* **by** (*auto intro*!: *set-integral-mono-banach simp add*: *X-def*)
finally have $(\int x{\in}A.\ f\ x\ \partial M) \leq (\int x{\in}A.\ c\ \partial M)$ **by** *simp*
}
ultimately show $LINT\ x|M.\ indicator\ A\ x *_R c = LINT\ x|M.\ indicator\ A\ x *_R f\ x$ **unfolding** *set-lebesgue-integral-def* **by** *simp*
show *AE x in M*. $indicator\ A\ x *_R c \leq indicator\ A\ x *_R f\ x$ **using** *assms* **by** (*auto simp add*: *X-def indicator-def*)
qed (*auto simp add*: *set-integrable-def*[*symmetric*])
hence *AE* $x{\in}A$ *in M*. $c = f\ x$ **by** *auto*
hence *AE* $x{\in}A$ *in M*. *False* **using** *assms*(*2*) **by** *auto*
hence $A \in null\text{-}sets\ M$ **using** *AE-iff-null-sets A-in-sets-M* **by** *metis*
thus *False* **using** *A*(*3*) **by** (*simp add*: *emeasure-restr-to-subalg null-setsD1 subalg*)
qed
thus *?thesis* **using** *AE-iff-null-sets*[*OF X-in-M*] **unfolding** *X-def* **by** *auto*
qed

corollary *cond-exp-less-c*:
fixes *f* :: $'a \Rightarrow {}'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *or-*

dered-real-vector}
assumes *integrable M f AE x in M. f x* < *c*
shows *AE x in M. cond-exp M F f x* < *c*
proof −
have *AE x in M. cond-exp M F f x* = − *cond-exp M F* (λ*x.* − *f x*) *x* **using** *cond-exp-uminus*[*OF assms*(*1*)] **by** *auto*
moreover have *AE x in M. cond-exp M F* (λ*x.* − *f x*) *x* > − *c* **using** *assms* **by** (*intro cond-exp-gr-c*) *auto*
ultimately show *?thesis* **by** (*force simp add*: *minus-less-iff*)
qed

lemma *cond-exp-mono-strict*:
fixes *f* :: ′*a* ⇒ ′*b* :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes *integrable M f integrable M g AE x in M. f x* < *g x*
shows *AE x in M. cond-exp M F f x* < *cond-exp M F g x*
using *cond-exp-less-c*[*OF Bochner-Integration.integrable-diff*, *OF assms*(*1*,*2*), *of 0*]
cond-exp-diff[*OF assms*(*1*,*2*)] *assms*(*3*) **by** *auto*

lemma *cond-exp-ge-c*:
fixes *f* :: ′*a* ⇒ ′*b* :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes [*measurable*]: *integrable M f*
and *AE x in M. f x* ≥ *c*
shows *AE x in M. cond-exp M F f x* ≥ *c*
proof −
let *?F* = *restr-to-subalg M F*
interpret *sigma-finite-measure restr-to-subalg M F* **using** *sigma-fin-subalg* **by** *auto*
{
fix *A* **assume** *asm*: *A* ∈ *sets ?F 0* < *measure ?F A*
have [*simp*]: *sets ?F* = *sets F measure ?F A* = *measure M A* **using** *asm* **by** (*auto simp add*: *measure-def sets-restr-to-subalg*[*OF subalg*] *emeasure-restr-to-subalg*[*OF subalg*])
have *M-A*: *emeasure M A* < ∞ **using** *measure-zero-top asm* **by** (*force simp add*: *top.not-eq-extremum*)
hence *F-A*: *emeasure ?F A* < ∞ **using** *asm*(*1*) *emeasure-restr-to-subalg subalg* **by** *fastforce*
have *set-lebesgue-integral M A* (λ-. *c*) ≤ *set-lebesgue-integral M A f* **using** *assms asm M-A subalg* **by** (*intro set-integral-mono-AE-banach*, *auto simp add*: *set-integrable-def integrable-mult-indicator subalgebra-def sets-restr-to-subalg*)
also have ... = *set-lebesgue-integral M A* (*cond-exp M F f*) **using** *cond-exp-set-integral*[*OF assms*(*1*)] *asm* **by** *auto*
also have ... = *set-lebesgue-integral ?F A* (*cond-exp M F f*) **unfolding** *set-lebesgue-integral-def* **using** *asm borel-measurable-cond-exp* **by** (*intro integral-subalgebra2*[*OF subalg*, *symmetric*], *simp*)
finally have (*1* / *measure ?F A*) $*_R$ *set-lebesgue-integral ?F A* (*cond-exp M F f*) ∈ {*c*..} **using** *asm subalg M-A* **by** (*auto simp add*: *set-integral-const subalgebra-def*

intro!: *pos-divideR-le-eq*[*THEN iffD1*])
}
thus *?thesis* **using** *AE-restr-to-subalg*[*OF subalg*] *averaging-theorem*[*OF integrable-in-subalg closed-atLeast*, *OF subalg borel-measurable-cond-exp integrable-cond-exp*] **by** *auto*
qed

corollary *cond-exp-le-c*:
fixes f :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes *integrable M f*
and *AE x in M*. $f\ x \leq c$
shows *AE x in M*. *cond-exp M F f x* $\leq c$
proof −
have *AE x in M*. *cond-exp M F f x* = − *cond-exp M F* (λx. − *f x*) *x* **using** *cond-exp-uminus*[*OF assms*(*1*)] **by** *force*
moreover have *AE x in M*. *cond-exp M F* (λx. − *f x*) *x* $\geq$ − *c* **using** *assms* **by** (*intro cond-exp-ge-c*) *auto*
ultimately show *?thesis* **by** (*force simp add*: *minus-le-iff*)
qed

corollary *cond-exp-mono*:
fixes f :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes *integrable M f integrable M g AE x in M*. $f\ x \leq g\ x$
shows *AE x in M*. *cond-exp M F f x* $\leq$ *cond-exp M F g x*
using *cond-exp-le-c*[*OF Bochner-Integration.integrable-diff*, *OF assms*(*1*,*2*), *of 0*]
cond-exp-diff[*OF assms*(*1*,*2*)] *assms*(*3*) **by** *auto*

corollary *cond-exp-min*:
fixes f :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes *integrable M f integrable M g*
shows *AE* ξ *in M*. *cond-exp M F* (λx. *min* (*f x*) (*g x*)) $\xi \leq$ *min* (*cond-exp M F f* ξ) (*cond-exp M F g* ξ)
proof −
have *AE* ξ *in M*. *cond-exp M F* (λx. *min* (*f x*) (*g x*)) $\xi \leq$ *cond-exp M F f* ξ **by** (*intro cond-exp-mono integrable-min assms*, *simp*)
moreover have *AE* ξ *in M*. *cond-exp M F* (λx. *min* (*f x*) (*g x*)) $\xi \leq$ *cond-exp M F g* ξ **by** (*intro cond-exp-mono integrable-min assms*, *simp*)
ultimately show *AE* ξ *in M*. *cond-exp M F* (λx. *min* (*f x*) (*g x*)) $\xi \leq$ *min* (*cond-exp M F f* ξ) (*cond-exp M F g* ξ) **by** *fastforce*
qed

corollary *cond-exp-max*:
fixes f :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*}
assumes *integrable M f integrable M g*

shows *AE* ξ *in* M. *cond-exp M F* (λx. *max* (*f x*) (*g x*)) $\xi \geq$ *max* (*cond-exp M F f* ξ) (*cond-exp M F g* ξ)
proof −
have *AE* ξ *in* M. *cond-exp M F* (λx. *max* (*f x*) (*g x*)) $\xi \geq$ *cond-exp M F f* ξ **by** (*intro cond-exp-mono integrable-max assms*, *simp*)
moreover have *AE* ξ *in* M. *cond-exp M F* (λx. *max* (*f x*) (*g x*)) $\xi \geq$ *cond-exp M F g* ξ **by** (*intro cond-exp-mono integrable-max assms*, *simp*)
ultimately show *AE* ξ *in* M. *cond-exp M F* (λx. *max* (*f x*) (*g x*)) $\xi \geq$ *max* (*cond-exp M F f* ξ) (*cond-exp M F g* ξ) **by** *fastforce*
qed

corollary *cond-exp-inf*:
fixes f :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*, *lattice*}
assumes *integrable M f integrable M g*
shows *AE* ξ *in* M. *cond-exp M F* (λx. *inf* (*f x*) (*g x*)) $\xi \leq$ *inf* (*cond-exp M F f* ξ) (*cond-exp M F g* ξ)
unfolding *inf-min* **using** *assms* **by** (*rule cond-exp-min*)

corollary *cond-exp-sup*:
fixes f :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *linorder-topology*, *ordered-real-vector*, *lattice*}
assumes *integrable M f integrable M g*
shows *AE* ξ *in* M. *cond-exp M F* (λx. *sup* (*f x*) (*g x*)) $\xi \geq$ *sup* (*cond-exp M F f* ξ) (*cond-exp M F g* ξ)
unfolding *sup-max* **using** *assms* **by** (*rule cond-exp-max*)

end

4.4 Probability Spaces

lemma (**in** *prob-space*) *sigma-finite-subalgebra-restr-to-subalg*:
assumes *subalgebra M F*
shows *sigma-finite-subalgebra M F*
proof (*intro sigma-finite-subalgebra.intro*)
interpret *F*: *prob-space restr-to-subalg M F* **using** *assms prob-space-restr-to-subalg prob-space-axioms* **by** *blast*
show *sigma-finite-measure* (*restr-to-subalg M F*) **by** (*rule F.sigma-finite-measure-axioms*)
qed (*rule assms*)

lemma (**in** *prob-space*) *cond-exp-trivial*:
fixes f :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*}
assumes *integrable M f*
shows *AE x in* M. *cond-exp M* (*sigma* (*space M*) {}) *f x* = *expectation f*
proof −
interpret *sigma-finite-subalgebra M sigma* (*space M*) {} **by** (*auto intro*: *sigma-finite-subalgebra-restr-to-subalg simp add*: *subalgebra-def sigma-sets-empty-eq*)
show *?thesis* **using** *assms* **by** (*intro cond-exp-charact*) (*auto simp add*: *sigma-sets-empty-eq set-lebesgue-integral-def prob-space cong*: *Bochner-Integration.integral-cong*)

qed

The following lemma shows that independent sigma algebras don't matter for the conditional expectation.

lemma (**in** *prob-space*) *cond-exp-indep-subalgebra*:
fixes *f* :: *'a* ⇒ *'b* :: {*second-countable-topology*, *banach*, *real-normed-field*}
assumes *subalgebra*: *subalgebra M F subalgebra M G*
and *independent*: *indep-set G* (*sigma* (*space M*) (*F* ∪ *vimage-algebra* (*space M*) *f borel*))
assumes [*measurable*]: *integrable M f*
shows *AE x in M*. *cond-exp M* (*sigma* (*space M*) (*F* ∪ *G*)) *f x* = *cond-exp M F f x*
proof −
interpret *Un-sigma*: *sigma-finite-subalgebra M sigma* (*space M*) (*F* ∪ *G*) **using** *assms*(*1*,*2*) **by** (*auto intro*!: *sigma-finite-subalgebra-restr-to-subalg sets.sigma-sets-subset simp add*: *subalgebra-def space-measure-of-conv sets-measure-of-conv*)
interpret *sigma-finite-subalgebra M F* **using** *assms* **by** (*auto intro*: *sigma-finite-subalgebra-restr-to-subalg*)
{
fix *A*
assume *asm*: *A* ∈ *sigma* (*space M*) {*a* ∩ *b* | *a b*. *a* ∈ *F* ∧ *b* ∈ *G*}
have *in-events*: *sigma-sets* (*space M*) {*a* ∩ *b* |*a b*. *a* ∈ *sets F* ∧ *b* ∈ *sets G*} ⊆ *events* **using** *subalgebra* **by** (*intro sets.sigma-sets-subset*, *auto simp add*: *subalgebra-def*)
have *Int-stable* {*a* ∩ *b* | *a b*. *a* ∈ *F* ∧ *b* ∈ *G*}
proof −
{
fix *af bf ag bg*
assume *F*: *af* ∈ *F bf* ∈ *F* **and** *G*: *ag* ∈ *G bg* ∈ *G*
have *af* ∩ *bf* ∈ *F* **by** (*intro sets.Int F*)
moreover have *ag* ∩ *bg* ∈ *G* **by** (*intro sets.Int G*)
ultimately have ∃ *a b*. *af* ∩ *ag* ∩ (*bf* ∩ *bg*) = *a* ∩ *b* ∧ *a* ∈ *sets F* ∧ *b* ∈ *sets G* **by** (*metis inf-assoc inf-left-commute*)
}
thus *?thesis* **by** (*force intro*!: *Int-stableI*)
qed
moreover have {*a* ∩ *b* | *a b*. *a* ∈ *F* ∧ *b* ∈ *G*} ⊆ *Pow* (*space M*) **using** *subalgebra* **by** (*force simp add*: *subalgebra-def dest*: *sets.sets-into-space*)
moreover have *A* ∈ *sigma-sets* (*space M*) {*a* ∩ *b* | *a b*. *a* ∈ *F* ∧ *b* ∈ *G*} **using** *calculation asm* **by** *force*
ultimately have *set-lebesgue-integral M A f* = *set-lebesgue-integral M A* (*cond-exp M F f*)
proof (*induction rule*: *sigma-sets-induct-disjoint*)
case (*basic A*)
then obtain *a b* **where** *A*: *A* = *a* ∩ *b a* ∈ *F b* ∈ *G* **by** *blast*

hence *events*[*measurable*]: *a* ∈ *events b* ∈ *events* **using** *subalgebra* **by** (*auto simp add*: *subalgebra-def*)

have [*simp*]: *sigma-sets* (*space M*) {*indicator b* −‘ *A* ∩ *space M* |*A*. *A* ∈ *borel*}

$\subseteq G$

using *borel-measurable-indicator*[*OF A*(*3*), *THEN measurable-sets*] *sets.top subalgebra*

by (*intro sets.sigma-sets-subset'*) (*fastforce simp add*: *subalgebra-def*)+

have *Un-in-sigma*: $F \cup$ *vimage-algebra* (*space M*) *f borel* $\subseteq$ *sigma* (*space M*) ($F \cup$ *vimage-algebra* (*space M*) *f borel*) **by** (*metis equalityE le-supI sets.space-closed sigma-le-sets space-vimage-algebra subalg subalgebra-def*)

have [*intro*]: *indep-var borel* (*indicator b*) *borel* ($\lambda\omega$. *indicator a* $\omega *_R f\ \omega$)

proof −

have [*simp*]: *sigma-sets* (*space M*) $\{(\lambda\omega.$ *indicator a* $\omega *_R f\ \omega) -` A \cap$ *space M* $|A.\ A \in$ *borel*$\}$ $\subseteq$ *sigma* (*space M*) ($F \cup$ *vimage-algebra* (*space M*) *f borel*)

proof −

have $*$: ($\lambda\omega$. *indicator a* $\omega *_R f\ \omega$) $\in$ *borel-measurable* (*sigma* (*space M*) ($F \cup$ *vimage-algebra* (*space M*) *f borel*))

using *borel-measurable-indicator*[*OF A*(*2*), *THEN measurable-sets*, *OF borel-open*] *subalgebra*

by (*intro borel-measurable-scaleR borel-measurableI Un-in-sigma*[*THEN subsetD*])

(*auto simp add*: *space-measure-of-conv subalgebra-def sets-vimage-algebra2*)

thus *?thesis* **using** *measurable-sets*[*OF* $*$] **by** (*intro sets.sigma-sets-subset'*, *auto simp add*: *space-measure-of-conv*)

qed

have *indep-set* (*sigma-sets* (*space M*) $\{$*indicator b* $-` A \cap$ *space M* $|A.\ A \in$ *borel*$\}$) (*sigma-sets* (*space M*) $\{(\lambda\omega.$ *indicator a* $\omega *_R f\ \omega) -` A \cap$ *space M* $|A.\ A \in$ *borel*$\}$)

using *independent* **unfolding** *indep-set-def* **by** (*rule indep-sets-mono-sets*, *auto split*: *bool.split*)

thus *?thesis* **by** (*subst indep-var-eq*, *auto intro*!: *borel-measurable-scaleR*)

qed

have [*intro*]: *indep-var borel* (*indicator b*) *borel* ($\lambda\omega$. *indicat-real a* $\omega *_R$ *cond-exp M F f* ω)

proof −

have [*simp*]:*sigma-sets* (*space M*) $\{(\lambda\omega.$ *indicator a* $\omega *_R$ *cond-exp M F f* $\omega) -` A \cap$ *space M* $|A.\ A \in$ *borel*$\}$ $\subseteq$ *sigma* (*space M*) ($F \cup$ *vimage-algebra* (*space M*) *f borel*)

proof −

have $*$: ($\lambda\omega$. *indicator a* $\omega *_R$ *cond-exp M F f* ω) $\in$ *borel-measurable* (*sigma* (*space M*) ($F \cup$ *vimage-algebra* (*space M*) *f borel*))

using *borel-measurable-indicator*[*OF A*(*2*), *THEN measurable-sets*, *OF borel-open*] *subalgebra*

borel-measurable-cond-exp[*THEN measurable-sets*, *OF borel-open*, *of - M F f*]

by (*intro borel-measurable-scaleR borel-measurableI Un-in-sigma*[*THEN subsetD*])

(*auto simp add*: *space-measure-of-conv subalgebra-def*)

thus *?thesis* **using** *measurable-sets*[*OF* $*$] **by** (*intro sets.sigma-sets-subset'*,

auto simp add: *space-measure-of-conv*)
qed
have *indep-set* (*sigma-sets* (*space M*) {*indicator b* −' *A* ∩ *space M* |*A*. *A* ∈ *borel*}) (*sigma-sets* (*space M*) {(λω. *indicator a* ω $*_R$ *cond-exp M F f* ω) −' *A* ∩ *space M* |*A*. *A* ∈ *borel*})
using *independent* **unfolding** *indep-set-def* **by** (*rule indep-sets-mono-sets*, *auto split*: *bool.split*)
thus *?thesis* **by** (*subst indep-var-eq*, *auto intro*!: *borel-measurable-scaleR*)
qed

have *set-lebesgue-integral M A f* = (*LINT x*|*M*. *indicator b x* ∗ (*indicator a x* $*_R$ *f x*))
unfolding *set-lebesgue-integral-def A indicator-inter-arith*
by (*intro Bochner-Integration.integral-cong*, *auto simp add*: *scaleR-scaleR*[*symmetric*] *indicator-times-eq-if*(*1*))
also have ... = (*LINT x*|*M*. *indicator b x*) ∗ (*LINT x*|*M*. *indicator a x* $*_R$ *f x*)
by (*intro indep-var-lebesgue-integral*
Bochner-Integration.integrable-bound[*OF integrable-const*[*of 1* :: '*b*] *borel-measurable-indicator*]
integrable-mult-indicator[*OF - assms*(*4*)], *blast*) (*auto simp add*: *indicator-def*)
also have ... = (*LINT x*|*M*. *indicator b x*) ∗ (*LINT x*|*M*. *indicator a x* $*_R$ *cond-exp M F f x*)
using *cond-exp-set-integral*[*OF assms*(*4*) *A*(*2*)] **unfolding** *set-lebesgue-integral-def* **by** *argo*
also have ... = (*LINT x*|*M*. *indicator b x* ∗ (*indicator a x* $*_R$ *cond-exp M F f x*))
by (*intro indep-var-lebesgue-integral*[*symmetric*]
Bochner-Integration.integrable-bound[*OF integrable-const*[*of 1* :: '*b*] *borel-measurable-indicator*]
integrable-mult-indicator[*OF - integrable-cond-exp*], *blast*) (*auto simp add*: *indicator-def*)
also have ... = *set-lebesgue-integral M A* (*cond-exp M F f*)
unfolding *set-lebesgue-integral-def A indicator-inter-arith*
by (*intro Bochner-Integration.integral-cong*, *auto simp add*: *scaleR-scaleR*[*symmetric*] *indicator-times-eq-if*(*1*))
finally show *?case* **.**
next
case *empty*
then show *?case* **unfolding** *set-lebesgue-integral-def* **by** *simp*
next
case (*compl A*)
have *A-in-space*: *A* ⊆ *space M* **using** *compl* **using** *in-events sets.sets-into-space* **by** *blast*
have *set-lebesgue-integral M* (*space M* − *A*) *f* = *set-lebesgue-integral M* (*space M* − *A* ∪ *A*) *f* − *set-lebesgue-integral M A f*
using *compl*(*1*) *in-events*
by (*subst set-integral-Un*[*of space M* − *A A*], *blast*)

(*simp* | *intro integrable-mult-indicator*[*folded set-integrable-def*, *OF - assms*(*4*)], *fast*)+
also have ... = *set-lebesgue-integral M* (*space M* − *A* ∪ *A*) (*cond-exp M F f*) − *set-lebesgue-integral M A* (*cond-exp M F f*)
using *cond-exp-set-integral*[*OF assms*(*4*) *sets.top*] *compl subalgebra* **by** (*simp add*: *subalgebra-def Un-absorb2*[*OF A-in-space*])
also have ... = *set-lebesgue-integral M* (*space M* − *A*) (*cond-exp M F f*)
using *compl*(*1*) *in-events*
by (*subst set-integral-Un*[*of space M* − *A A*], *blast*)
(*simp* | *intro integrable-mult-indicator*[*folded set-integrable-def*, *OF - integrable-cond-exp*], *fast*)+
finally show *?case* **.**
next
case (*union A*)
have *set-lebesgue-integral M* ($\bigcup$ (*range A*)) *f* = ($\sum$ *i*. *set-lebesgue-integral M* (*A i*) *f*)
using *union in-events*
by (*intro lebesgue-integral-countable-add*) (*auto simp add*: *disjoint-family-onD intro*!: *integrable-mult-indicator*[*folded set-integrable-def*, *OF - assms*(*4*)])
also have ... = ($\sum$ *i*. *set-lebesgue-integral M* (*A i*) (*cond-exp M F f*)) **using** *union* **by** *presburger*
also have ... = *set-lebesgue-integral M* ($\bigcup$ (*range A*)) (*cond-exp M F f*)
using *union in-events*
by (*intro lebesgue-integral-countable-add*[*symmetric*]) (*auto simp add*: *disjoint-family-onD intro*!: *integrable-mult-indicator*[*folded set-integrable-def*, *OF - integrable-cond-exp*])
finally show *?case* **.**
qed
}
moreover have *sigma* (*space M*) {*a* ∩ *b* | *a b*. *a* ∈ *F* ∧ *b* ∈ *G*} = *sigma* (*space M*) (*F* ∪ *G*)
proof −
have *sigma-sets* (*space M*) {*a* ∩ *b* |*a b*. *a* ∈ *sets F* ∧ *b* ∈ *sets G*} = *sigma-sets* (*space M*) (*sets F* ∪ *sets G*)
proof −
{
fix *a b* **assume** *asm*: *a* ∈ *F b* ∈ *G*
hence *a* ∩ *b* ∈ *sigma-sets* (*space M*) (*F* ∪ *G*) **using** *subalgebra* **unfolding** *Int-range-binary* **by** (*intro sigma-sets-Inter*[*OF - binary-in-sigma-sets*]) (*force simp add*: *subalgebra-def dest*: *sets.sets-into-space*)+
}
moreover
{
fix *a*
assume *a* ∈ *sets F*
hence *a* ∈ *sigma-sets* (*space M*) {*a* ∩ *b* |*a b*. *a* ∈ *sets F* ∧ *b* ∈ *sets G*}
using *subalgebra sets.top*[*of G*] *sets.sets-into-space*[*of - F*]
by (*intro sigma-sets.Basic*, *auto simp add*: *subalgebra-def*)
}

moreover
{
fix *a* **assume** $a \in$ *sets F* $\vee$ $a \in$ *sets G* $a \notin$ *sets F*
hence $a \in$ *sets G* **by** *blast*
hence $a \in$ *sigma-sets* (*space M*) $\{a \cap b \mid a\ b.\ a \in$ *sets F* $\wedge$ $b \in$ *sets G*$\}$
using *subalgebra sets.top*[*of F*] *sets.sets-into-space*[*of - G*]
by (*intro sigma-sets.Basic*, *auto simp add*: *subalgebra-def*)
}
ultimately show *?thesis* **by** (*intro sigma-sets-eqI*) *auto*
qed
thus *?thesis* **using** *subalgebra* **by** (*intro sigma-eqI*) (*force simp add*: *subalgebra-def dest*: *sets.sets-into-space*)+
qed
moreover have (*cond-exp M F f*) $\in$ *borel-measurable* (*sigma* (*space M*) (*sets F* $\cup$ *sets G*))
proof $-$
have $F \subseteq$ *sigma* (*space M*) ($F \cup G$) **by** (*metis Un-least Un-upper1 measure-of-of-measure sets.space-closed sets-measure-of sigma-sets-subseteq subalg subalgebra*(*2*) *subalgebra-def*)
thus *?thesis* **using** *borel-measurable-cond-exp*[*THEN measurable-sets*, *OF borel-open*, *of - M F f*] *subalgebra* **by** (*intro borel-measurableI*, *force simp only*: *space-measure-of-conv subalgebra-def*)
qed
ultimately show *?thesis* **using** *assms*(*4*) *integrable-cond-exp* **by** (*intro Un-sigma.cond-exp-charact*) *presburger*+
qed

lemma (**in** *prob-space*) *cond-exp-indep*:
fixes *f* :: $'a \Rightarrow 'b$ :: {*second-countable-topology*, *banach*, *real-normed-field*}
assumes *subalgebra*: *subalgebra M F*
and *independent*: *indep-set F* (*vimage-algebra* (*space M*) *f borel*)
and *integrable*: *integrable M f*
shows *AE x in M*. *cond-exp M F f x* = *expectation f*
proof $-$
have *indep-set F* (*sigma* (*space M*) (*sigma* (*space M*) {} $\cup$ (*vimage-algebra* (*space M*) *f borel*)))
using *independent* **unfolding** *indep-set-def*
by (*rule indep-sets-mono-sets*, *simp add*: *bool.split*)
(*metis bot.extremum dual-order.refl sets.sets-measure-of-eq sets.sigma-sets-subset′ sets-vimage-algebra-space space-vimage-algebra sup.absorb-iff2*)
hence *cond-exp-indep*: *AE x in M*. *cond-exp M* (*sigma* (*space M*) (*sigma* (*space M*) {} $\cup$ *F*)) *f x* = *expectation f*
using *cond-exp-indep-subalgebra*[*OF - subalgebra - integrable*, *of sigma* (*space M*) {}] *cond-exp-trivial*[*OF integrable*]
by (*auto simp add*: *subalgebra-def sigma-sets-empty-eq*)
have *sets* (*sigma* (*space M*) (*sigma* (*space M*) {} $\cup$ *F*)) = *F*
using *subalgebra sets.top*[*of F*] **unfolding** *subalgebra-def*
by (*simp add*: *sigma-sets-empty-eq*, *subst insert-absorb*[*of space M F*], *blast*)
(*metis insert-absorb*[*OF sets.empty-sets*] *sets.sets-measure-of-eq*)

hence *AE x in M. cond-exp M* (*sigma* (*space M*) (*sigma* (*space M*) {} ∪ *F*)) *f* *x* = *cond-exp M F f x* **by** (*rule cond-exp-sets-cong*)
thus *?thesis* **using** *cond-exp-indep* **by** *force*
qed

end

theory *Filtered-Measure*
imports *HOL−Probability.Conditional-Expectation*
begin

5 Filtered Measure Spaces

5.1 Filtered Measure

locale *filtered-measure* =
fixes *M F* **and** t_0 :: *'b* :: {*second-countable-topology*, *order-topology*, *t2-space*}
assumes *subalgebras*: $\bigwedge i.\ t_0 \le i \Longrightarrow$ *subalgebra M* (*F i*)
and *sets-F-mono*: $\bigwedge i\ j.\ t_0 \le i \Longrightarrow i \le j \Longrightarrow$ *sets* (*F i*) ≤ *sets* (*F j*)
begin

lemma *space-F*[*simp*]:
assumes $t_0 \le i$
shows *space* (*F i*) = *space M*
using *subalgebras assms* **by** (*simp add*: *subalgebra-def*)

lemma *subalgebra-F*[*intro*]:
assumes $t_0 \le i$ $i \le j$
shows *subalgebra* (*F j*) (*F i*)
unfolding *subalgebra-def* **using** *assms* **by** (*simp add*: *sets-F-mono*)

lemma *borel-measurable-mono*:
assumes $t_0 \le i$ $i \le j$
shows *borel-measurable* (*F i*) ⊆ *borel-measurable* (*F j*)
unfolding *subset-iff* **by** (*metis assms subalgebra-F measurable-from-subalg*)

end

locale *linearly-filtered-measure* = *filtered-measure M F* t_0 **for** *M* **and** *F* :: - :: {*linorder-topology*} ⇒ - **and** t_0

locale *nat-filtered-measure* = *linearly-filtered-measure M F 0* **for** *M* **and** *F* :: *nat* ⇒ -
locale *real-filtered-measure* = *linearly-filtered-measure M F 0* **for** *M* **and** *F* :: *real* ⇒ -

5.2 Sigma Finite Filtered Measure

The locale presented here is a generalization of the *sigma-finite-subalgebra* for a particular filtration.

locale *sigma-finite-filtered-measure* = *filtered-measure* +
  assumes *sigma-finite-initial*: *sigma-finite-subalgebra M* $(F\ t_0)$

lemma (**in** *sigma-finite-filtered-measure*) *sigma-finite-subalgebra-F*[*intro*]:
  assumes $t_0 \leq i$
  shows *sigma-finite-subalgebra M* $(F\ i)$
  using *assms* **by** (*metis dual-order.refl sets-F-mono sigma-finite-initial sigma-finite-subalgebra.nested-subalg-is subalgebras subalgebra-def*)

locale *nat-sigma-finite-filtered-measure* = *sigma-finite-filtered-measure M F 0* :: *nat* **for** *M F*
locale *real-sigma-finite-filtered-measure* = *sigma-finite-filtered-measure M F 0* :: *real* **for** *M F*

sublocale *nat-sigma-finite-filtered-measure* ⊆ *sigma-finite-subalgebra M F i* **by** *blast*
sublocale *real-sigma-finite-filtered-measure* ⊆ *sigma-finite-subalgebra M F* $|i|$ **by** *fastforce*

5.3 Finite Filtered Measure

locale *finite-filtered-measure* = *filtered-measure* + *finite-measure*

sublocale *finite-filtered-measure* ⊆ *sigma-finite-filtered-measure*
  using *subalgebras* **by** (*unfold-locales*, *blast*, *meson dual-order.refl finite-measure-axioms finite-measure-def finite-measure-restr-to-subalg sigma-finite-measure.sigma-finite-countable*)

locale *nat-finite-filtered-measure* = *finite-filtered-measure M F 0* :: *nat* **for** *M F*
locale *real-finite-filtered-measure* = *finite-filtered-measure M F 0* :: *real* **for** *M F*

sublocale *nat-finite-filtered-measure* ⊆ *nat-sigma-finite-filtered-measure* **..**
sublocale *real-finite-filtered-measure* ⊆ *real-sigma-finite-filtered-measure* **..**

5.4 Constant Filtration

lemma *filtered-measure-constant-filtration*:
  assumes *subalgebra M F*
  shows *filtered-measure M* (λ-. *F*) t_0
  using *assms* **by** (*unfold-locales*) *blast*+

sublocale *sigma-finite-subalgebra* ⊆ *constant-filtration*: *sigma-finite-filtered-measure M* λ- :: *'t* :: {*second-countable-topology*, *linorder-topology*}. *F* t_0
  using *subalg* **by** (*unfold-locales*) *blast*+

lemma (**in** *finite-measure*) *filtered-measure-constant-filtration*:

assumes *subalgebra M F*
shows *finite-filtered-measure M* (λ-. *F*) t_0
using *assms* **by** (*unfold-locales*) *blast*+

end

theory *Stochastic-Process*
imports *Filtered-Measure Measure-Space-Supplement HOL−Probability.Independent-Family*
begin

6 Stochastic Processes

6.1 Stochastic Process

A stochastic process is a collection of random variables, indexed by a type $'b$.

locale *stochastic-process* =
fixes *M* t_0 **and** *X* :: $'b$ :: {*second-countable-topology*, *order-topology*, *t2-space*} $\Rightarrow$ $'a \Rightarrow {}'c$ :: {*second-countable-topology*, *banach*}
assumes *random-variable*[*measurable*]: $\bigwedge i.\ t_0 \le i \Longrightarrow X\ i \in$ *borel-measurable M*
begin

definition *left-continuous* **where** *left-continuous* = (*AE* ξ *in M*. $\forall t$. *continuous* (*at-left t*) ($\lambda i.\ X\ i\ \xi$))
definition *right-continuous* **where** *right-continuous* = (*AE* ξ *in M*. $\forall t$. *continuous* (*at-right t*) ($\lambda i.\ X\ i\ \xi$))

end

locale *nat-stochastic-process* = *stochastic-process M 0* :: *nat X* **for** *M X*
locale *real-stochastic-process* = *stochastic-process M 0* :: *real X* **for** *M X*

lemma *stochastic-process-const-fun*:
assumes $f \in$ *borel-measurable M*
shows *stochastic-process M* t_0 (λ-. *f*) **using** *assms* **by** (*unfold-locales*)

lemma *stochastic-process-const*:
shows *stochastic-process M* t_0 (λi -. *c i*) **by** (*unfold-locales*) *simp*

context *stochastic-process*
begin

lemma *compose-stochastic*:
assumes $\bigwedge i.\ t_0 \le i \Longrightarrow f\ i \in$ *borel-measurable borel*
shows *stochastic-process M* t_0 ($\lambda i\ \xi$. ($f\ i$) ($X\ i\ \xi$))
by (*unfold-locales*) (*intro measurable-compose*[*OF random-variable assms*])

lemma *norm-stochastic*: *stochastic-process M* t_0 ($\lambda i\ \xi.$ *norm* ($X\ i\ \xi$)) **by** (*fastforce intro*: *compose-stochastic*)

lemma *scaleR-right-stochastic*:
assumes *stochastic-process M* t_0 Y
shows *stochastic-process M* t_0 ($\lambda i\ \xi.$ ($Y\ i\ \xi$) $*_R$ ($X\ i\ \xi$))
using *stochastic-process.random-variable*[*OF assms*] *random-variable* **by** (*unfold-locales*) *simp*

lemma *scaleR-right-const-fun-stochastic*:
assumes $f \in$ *borel-measurable M*
shows *stochastic-process M* t_0 ($\lambda i\ \xi.\ f\ \xi *_R$ ($X\ i\ \xi$))
by (*unfold-locales*) (*intro borel-measurable-scaleR assms random-variable*)

lemma *scaleR-right-const-stochastic*: *stochastic-process M* t_0 ($\lambda i\ \xi.\ c\ i *_R$ ($X\ i\ \xi$))
by (*unfold-locales*) *simp*

lemma *add-stochastic*:
assumes *stochastic-process M* t_0 Y
shows *stochastic-process M* t_0 ($\lambda i\ \xi.\ X\ i\ \xi + Y\ i\ \xi$)
using *stochastic-process.random-variable*[*OF assms*] *random-variable* **by** (*unfold-locales*) *simp*

lemma *diff-stochastic*:
assumes *stochastic-process M* t_0 Y
shows *stochastic-process M* t_0 ($\lambda i\ \xi.\ X\ i\ \xi - Y\ i\ \xi$)
using *stochastic-process.random-variable*[*OF assms*] *random-variable* **by** (*unfold-locales*) *simp*

lemma *uminus-stochastic*: *stochastic-process M* t_0 ($-X$) **using** *scaleR-right-const-stochastic*[*of* λ-. -1] **by** (*simp add*: *fun-Compl-def*)

lemma *partial-sum-stochastic*: *stochastic-process M* t_0 ($\lambda n\ \xi.\ \sum i\in\{t_0..n\}.\ X\ i\ \xi$) **by** (*unfold-locales*) *simp*

lemma *partial-sum′-stochastic*: *stochastic-process M* t_0 ($\lambda n\ \xi.\ \sum i\in\{t_0..<n\}.\ X\ i\ \xi$) **by** (*unfold-locales*) *simp*

end

lemma *stochastic-process-sum*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *stochastic-process M* t_0 ($X\ i$)
shows *stochastic-process M* t_0 ($\lambda k\ \xi.\ \sum i \in I.\ X\ i\ k\ \xi$) **using** *assms*[*THEN stochastic-process.random-variable*] **by** (*unfold-locales*, *auto*)

6.1.1 Natural Filtration

The natural filtration induced by a stochastic process X is the filtration generated by all events involving the process up to the time index t, i.e. Σ

$t = \sigma \{X\ s\ |s.\ s \leq t\}$.

definition *natural-filtration* :: *'a measure* ⇒ *'b* ⇒ (*'b* ⇒ *'a* ⇒ *'c* :: *topological-space*) ⇒ *'b* :: {*second-countable-topology*, *order-topology*} ⇒ *'a measure* **where**
natural-filtration M t_0 *Y* = (λt. *family-vimage-algebra* (*space M*) {*Y i* | *i*. *i* ∈ {t_0..*t*}} *borel*)

abbreviation *nat-natural-filtration* ≡ λM. *natural-filtration M* (*0* :: *nat*)
abbreviation *real-natural-filtration* ≡ λM. *natural-filtration M* (*0* :: *real*)

lemma *space-natural-filtration*[*simp*]: *space* (*natural-filtration M* t_0 *X t*) = *space M* **unfolding** *natural-filtration-def space-family-vimage-algebra* **..**

lemma *sets-natural-filtration*: *sets* (*natural-filtration M* t_0 *X t*) = *sigma-sets* (*space M*) (⋃*i*∈{t_0..*t*}. {*X i* −' *A* ∩ *space M* | *A*. *A* ∈ *borel*})
unfolding *natural-filtration-def sets-family-vimage-algebra* **by** (*intro sigma-sets-eqI*) *blast+*

lemma *sets-natural-filtration'*:
assumes *borel* = *sigma UNIV S*
shows *sets* (*natural-filtration M* t_0 *X t*) = *sigma-sets* (*space M*) (⋃*i*∈{t_0..*t*}. {*X i* −' *A* ∩ *space M* | *A*. *A* ∈ *S*})
proof (*subst sets-natural-filtration*, *intro sigma-sets-eqI*, *clarify*)
fix *i* **and** *A* :: *'a set* **assume** *asm*: *i* ∈ {t_0..*t*} *A* ∈ *sets borel*
hence *A* ∈ *sigma-sets UNIV S* **unfolding** *assms* **by** *simp*
thus *X i* −' *A* ∩ *space M* ∈ *sigma-sets* (*space M*) (⋃*i*∈{t_0..*t*}. {*X i* −' *A* ∩ *space M* |*A*. *A* ∈ *S*})
proof (*induction*)
case (*Compl a*)
have *X i* −' (*UNIV* − *a*) ∩ *space M* = *space M* − (*X i* −' *a* ∩ *space M*) **by** *blast*
then show *?case* **using** *Compl*(*2*)[*THEN sigma-sets.Compl*] **by** *presburger*
next
case (*Union a*)
have *X i* −' ⋃ (*range a*) ∩ *space M* = ⋃ (*range* (λj. *X i* −' *a j* ∩ *space M*)) **by** *blast*
then show *?case* **using** *Union*(*2*)[*THEN sigma-sets.Union*] **by** *presburger*
qed (*auto intro*: *asm sigma-sets.Empty*)
qed (*intro sigma-sets.Basic*, *force simp add*: *assms*)

lemma *sets-natural-filtration-open*:
sets (*natural-filtration M* t_0 *X t*) = *sigma-sets* (*space M*) (⋃*i*∈{t_0..*t*}. {*X i* −' *A* ∩ *space M* | *A*. *open A*})
using *sets-natural-filtration'* **by** (*force simp only*: *borel-def mem-Collect-eq*)

lemma *sets-natural-filtration-oi*:
sets (*natural-filtration M* t_0 *X t*) = *sigma-sets* (*space M*) (⋃*i*∈{t_0..*t*}. {*X i* −' *A* ∩ *space M* | *A* :: - :: {*linorder-topology*, *second-countable-topology*} *set*. *A* ∈ *range greaterThan*})
by (*rule sets-natural-filtration'*[*OF borel-Ioi*])

lemma *sets-natural-filtration-io*:
sets (*natural-filtration M* t_0 *X t*) = *sigma-sets* (*space M*) ($\bigcup$ *i*∈{t_0..*t*}. {*X i* −' *A* ∩ *space M* | *A* :: - :: {*linorder-topology*, *second-countable-topology*} *set*. *A* ∈ *range lessThan*})
by (*rule sets-natural-filtration*′[*OF borel-Iio*])

lemma *sets-natural-filtration-ci*:
sets (*natural-filtration M* t_0 *X t*) = *sigma-sets* (*space M*) ($\bigcup$ *i*∈{t_0..*t*}. {*X i* −' *A* ∩ *space M* | *A* :: *real set*. *A* ∈ *range atLeast*})
by (*rule sets-natural-filtration*′[*OF borel-Ici*])

context *stochastic-process*
begin

lemma *subalgebra-natural-filtration*:
shows *subalgebra M* (*natural-filtration M* t_0 *X i*)
unfolding *subalgebra-def* **using** *measurable-family-iff-sets* **by** (*force simp add*: *natural-filtration-def*)

lemma *filtered-measure-natural-filtration*:
shows *filtered-measure M* (*natural-filtration M* t_0 *X*) t_0
by (*unfold-locales*) (*intro subalgebra-natural-filtration*, *simp only*: *sets-natural-filtration*, *intro sigma-sets-subseteq*, *force*)

In order to show that the natural filtration constitutes a filtered sigma finite measure, we need to provide a countable exhausting set in the preimage of $X\ t_0$.

lemma *sigma-finite-filtered-measure-natural-filtration*:
assumes *exhausting-set*: *countable A* ($\bigcup$ *A*) = *space M* $\bigwedge$*a*. *a* ∈ *A* ⟹ *emeasure M a* ≠ ∞ $\bigwedge$*a*. *a* ∈ *A* ⟹ ∃ *b* ∈ *borel*. *a* = *X* t_0 −' *b* ∩ *space M*
shows *sigma-finite-filtered-measure M* (*natural-filtration M* t_0 *X*) t_0
proof (*unfold-locales*)
have *A* ⊆ *sets* (*restr-to-subalg M* (*natural-filtration M* t_0 *X* t_0)) **using** *exhausting-set* **by** (*simp add*: *sets-restr-to-subalg*[*OF subalgebra-natural-filtration*] *sets-natural-filtration*) *fast*
moreover have $\bigcup$ *A* = *space* (*restr-to-subalg M* (*natural-filtration M* t_0 *X* t_0)) **unfolding** *space-restr-to-subalg* **using** *exhausting-set* **by** *simp*
moreover have ∀ *a*∈*A*. *emeasure* (*restr-to-subalg M* (*natural-filtration M* t_0 *X* t_0)) *a* ≠ ∞ **using** *calculation*(*1*) *exhausting-set*(*3*)
by (*auto simp add*: *sets-restr-to-subalg*[*OF subalgebra-natural-filtration*] *emeasure-restr-to-subalg*[*OF subalgebra-natural-filtration*])
ultimately show ∃ *A*. *countable A* ∧ *A* ⊆ *sets* (*restr-to-subalg M* (*natural-filtration M* t_0 *X* t_0)) ∧ $\bigcup$ *A* = *space* (*restr-to-subalg M* (*natural-filtration M* t_0 *X* t_0)) ∧ (∀ *a*∈*A*. *emeasure* (*restr-to-subalg M* (*natural-filtration M* t_0 *X* t_0)) *a* ≠ ∞) **using** *exhausting-set* **by** *blast*
show $\bigwedge$*i j*. ⟦t_0 ≤ *i*; *i* ≤ *j*⟧ ⟹ *sets* (*natural-filtration M* t_0 *X i*) ⊆ *sets* (*natural-filtration M* t_0 *X j*) **using** *filtered-measure.subalgebra-F*[*OF filtered-measure-natural-filtration*]
by (*simp add*: *subalgebra-def*)

qed (*auto intro*: *subalgebra-natural-filtration*)

lemma *finite-filtered-measure-natural-filtration*:
  assumes *finite-measure M*
  shows *finite-filtered-measure M* (*natural-filtration M* t_0 *X*) t_0
  using *finite-measure.axioms*[*OF assms*] *filtered-measure-natural-filtration* **by** *intro-locales*

end

Filtration generated by independent variables.

lemma (**in** *prob-space*) *indep-set-natural-filtration*:
  assumes $t_0 \leq s$ $s < t$ *indep-vars* (λ-. *borel*) *X* $\{t_0..\}$
  shows *indep-set* (*natural-filtration M* t_0 *X s*) (*vimage-algebra* (*space M*) (*X t*) *borel*)
proof −
  have *indep-sets* (λi. $\{X\ i -`\ A \cap$ *space M* $|A.\ A \in$ *sets borel*$\}$) ($\bigcup$ (*range* (*case-bool* $\{t_0..s\}$ $\{t\}$)))
    using *assms*
   by (*intro assms*(*3*)[*unfolded indep-vars-def*, *THEN conjunct2*, *THEN indep-sets-mono*]) (*auto simp add*: *case-bool-if*)
  thus *?thesis* **unfolding** *indep-set-def* **using** *assms*
   by (*intro indep-sets-cong*[*THEN iffD1*, *OF refl* - *indep-sets-collect-sigma*[*of* λi. $\{X\ i -`\ A \cap$ *space M* $|\ A.\ A \in$ *borel*$\}$ *case-bool* $\{t_0..s\}$ $\{t\}$]])
       (*simp add*: *sets-natural-filtration sets-vimage-algebra split*: *bool.split*, *simp*, *intro Int-stableI*, *clarsimp*, *metis sets.Int vimage-Int Int-commute Int-left-absorb Int-left-commute*, *force simp add*: *disjoint-family-on-def split*: *bool.split*)
qed

6.2 Adapted Process

We call a collection a stochastic process X adapted if $X\ i$ is $F\ i$-borel-measurable for all indices i.

locale *adapted-process* = *filtered-measure M F* t_0 **for** *M F* t_0 **and** *X* :: - ⇒ - ⇒ - :: {*second-countable-topology*, *banach*} +
  assumes *adapted*[*measurable*]: $\bigwedge i.\ t_0 \leq i \Longrightarrow X\ i \in$ *borel-measurable* (*F i*)
begin

lemma *adaptedE*[*elim*]:
  assumes $[\![\bigwedge j\ i.\ t_0 \leq j \Longrightarrow j \leq i \Longrightarrow X\ j \in$ *borel-measurable* $(F\ i)]\!] \Longrightarrow P$
  shows *P*
 using *assms* **using** *adapted* **by** (*metis dual-order.trans borel-measurable-subalgebra sets-F-mono space-F*)

lemma *adaptedD*:
  assumes $t_0 \leq j$ $j \leq i$
  shows $X\ j \in$ *borel-measurable* (*F i*) **using** *assms adaptedE* **by** *meson*

end

locale *nat-adapted-process* = *adapted-process* M F *0* :: *nat* X **for** M F X
locale *real-adapted-process* = *adapted-process* M F *0* :: *real* X **for** M F X

sublocale *nat-adapted-process* ⊆ *nat-filtered-measure* **..**
sublocale *real-adapted-process* ⊆ *real-filtered-measure* **..**

lemma (**in** *filtered-measure*) *adapted-process-const-fun*:
  assumes $f \in$ *borel-measurable* $(F\ t_0)$
  shows *adapted-process* M F t_0 $(\lambda$-. $f)$
  using *measurable-from-subalg subalgebra-F assms* **by** (*unfold-locales*) *blast*

lemma (**in** *filtered-measure*) *adapted-process-const*:
  shows *adapted-process* M F t_0 $(\lambda i$ -. $c\ i)$ **by** (*unfold-locales*) *simp*

context *adapted-process*
begin

lemma *compose-adapted*:
  assumes $\bigwedge i.\ t_0 \leq i \Longrightarrow f\ i \in$ *borel-measurable borel*
  shows *adapted-process* M F t_0 $(\lambda i\ \xi.\ (f\ i)\ (X\ i\ \xi))$
  by (*unfold-locales*) (*intro measurable-compose*[*OF adapted assms*])

lemma *norm-adapted*: *adapted-process* M F t_0 $(\lambda i\ \xi.$ *norm* $(X\ i\ \xi))$ **by** (*fastforce intro*: *compose-adapted*)

lemma *scaleR-right-adapted*:
  assumes *adapted-process* M F t_0 R
  shows *adapted-process* M F t_0 $(\lambda i\ \xi.\ (R\ i\ \xi) *_R (X\ i\ \xi))$
  using *adapted-process.adapted*[*OF assms*] *adapted* **by** (*unfold-locales*) *simp*

lemma *scaleR-right-const-fun-adapted*:
  assumes $f \in$ *borel-measurable* $(F\ t_0)$
  shows *adapted-process* M F t_0 $(\lambda i\ \xi.\ f\ \xi *_R (X\ i\ \xi))$
  using *assms* **by** (*fast intro*: *scaleR-right-adapted adapted-process-const-fun*)

lemma *scaleR-right-const-adapted*: *adapted-process* M F t_0 $(\lambda i\ \xi.\ c\ i *_R (X\ i\ \xi))$
by (*unfold-locales*) *simp*

lemma *add-adapted*:
  assumes *adapted-process* M F t_0 Y
  shows *adapted-process* M F t_0 $(\lambda i\ \xi.\ X\ i\ \xi + Y\ i\ \xi)$
  using *adapted-process.adapted*[*OF assms*] *adapted* **by** (*unfold-locales*) *simp*

lemma *diff-adapted*:
  assumes *adapted-process* M F t_0 Y
  shows *adapted-process* M F t_0 $(\lambda i\ \xi.\ X\ i\ \xi - Y\ i\ \xi)$
  using *adapted-process.adapted*[*OF assms*] *adapted* **by** (*unfold-locales*) *simp*

lemma *uminus-adapted*: *adapted-process* M F t_0 $(-X)$ **using** *scaleR-right-const-adapted*[*of* λ-. -1] **by** (*simp add*: *fun-Compl-def*)

lemma *partial-sum-adapted*: *adapted-process* M F t_0 $(\lambda n\ \xi.\ \sum i{\in}\{t_0..n\}.\ X\ i\ \xi)$
proof (*unfold-locales*)
  fix i :: $'b$
  have $X\ j \in$ *borel-measurable* $(F\ i)$ **if** $t_0 \leq j\ j \leq i$ **for** j **using** *that adaptedE* **by** *meson*
  thus $(\lambda\xi.\ \sum i{\in}\{t_0..i\}.\ X\ i\ \xi) \in$ *borel-measurable* $(F\ i)$ **by** *simp*
qed

lemma *partial-sum′-adapted*: *adapted-process* M F t_0 $(\lambda n\ \xi.\ \sum i{\in}\{t_0..{<}n\}.\ X\ i\ \xi)$

proof (*unfold-locales*)
  fix i :: $'b$
  have $X\ j \in$ *borel-measurable* $(F\ i)$ **if** $t_0 \leq j\ j < i$ **for** j **using** *that adaptedE* **by** *fastforce*
  thus $(\lambda\xi.\ \sum i{\in}\{t_0..{<}i\}.\ X\ i\ \xi) \in$ *borel-measurable* $(F\ i)$ **by** *simp*
qed

end

lemma (**in** *nat-adapted-process*) *partial-sum-Suc-adapted*: *nat-adapted-process* M F $(\lambda n\ \xi.\ \sum i{<}n.\ X\ (Suc\ i)\ \xi)$
proof (*unfold-locales*)
  fix i
  have $X\ j \in$ *borel-measurable* $(F\ i)$ **if** $j \leq i$ **for** j **using** *that adaptedD* **by** *blast*
  thus $(\lambda\xi.\ \sum i{<}i.\ X\ (Suc\ i)\ \xi) \in$ *borel-measurable* $(F\ i)$ **by** *auto*
qed

lemma (**in** *filtered-measure*) *adapted-process-sum*:
  assumes $\bigwedge i.\ i \in I \Longrightarrow$ *adapted-process* M F t_0 $(X\ i)$
  shows *adapted-process* M F t_0 $(\lambda k\ \xi.\ \sum i \in I.\ X\ i\ k\ \xi)$
proof −
  {
    fix i k **assume** $i \in I$ **and** *asm*: $t_0 \leq k$
    then interpret *adapted-process* M F t_0 $X\ i$ **using** *assms* **by** *simp*
    have $X\ i\ k \in$ *borel-measurable* M $X\ i\ k \in$ *borel-measurable* $(F\ k)$ **using** *measurable-from-subalg subalgebras adapted asm* **by** (*blast*, *simp*)
  }
  thus *?thesis* **by** (*unfold-locales*) *simp*
qed

An adapted process is necessarily a stochastic process.

sublocale *adapted-process* $\subseteq$ *stochastic-process* **using** *measurable-from-subalg subalgebras adapted* **by** (*unfold-locales*) *blast*

sublocale *nat-adapted-process* $\subseteq$ *nat-stochastic-process* **..**

sublocale *real-adapted-process* $\subseteq$ *real-stochastic-process* ..

A stochastic process is always adapted to the natural filtration it generates.

lemma (**in** *stochastic-process*) *adapted-process-natural-filtration*: *adapted-process M* (*natural-filtration M* t_0 *X*) t_0 *X*
 using *filtered-measure-natural-filtration*
 by (*intro-locales*) (*auto simp add*: *natural-filtration-def intro*!: *adapted-process-axioms.intro measurable-family-vimage-algebra*)

6.3 Progressively Measurable Process

locale *progressive-process* = *filtered-measure M F* t_0 **for** *M F* t_0 **and** *X* :: - ⇒ - ⇒ - :: {*second-countable-topology*, *banach*} +
 assumes *progressive*[*measurable*]: $\bigwedge t.\ t_0 \le t \Longrightarrow (\lambda(i, x).\ X\, i\, x) \in$ *borel-measurable* (*restrict-space borel* $\{t_0..t\} \bigotimes_M F\ t$)
begin

lemma *progressiveD*:
 assumes $S \in$ *borel*
 shows $(\lambda(j, \xi).\ X\ j\ \xi)$ −‘ $S \cap (\{t_0..i\} \times$ *space M*$) \in$ (*restrict-space borel* $\{t_0..i\}$ $\bigotimes_M F\ i$)
 using *measurable-sets*[*OF progressive*, *OF - assms*, *of i*]
 by (*cases* $t_0 \le i$) (*auto simp add*: *space-restrict-space sets-pair-measure space-pair-measure*)

end

locale *nat-progressive-process* = *progressive-process M F 0* :: *nat X* **for** *M F X*
locale *real-progressive-process* = *progressive-process M F 0* :: *real X* **for** *M F X*

lemma (**in** *filtered-measure*) *progressive-process-const-fun*:
 assumes $f \in$ *borel-measurable* $(F\ t_0)$
 shows *progressive-process M F* t_0 (λ-. *f*)
proof (*unfold-locales*)
 fix *i* **assume** *asm*: $t_0 \le i$
 have $f \in$ *borel-measurable* (*F i*) **using** *borel-measurable-mono*[*OF order.refl asm*] *assms* **by** *blast*
 thus *case-prod* (λ-. *f*) $\in$ *borel-measurable* (*restrict-space borel* $\{t_0..i\} \bigotimes_M F\ i$) **using** *measurable-compose*[*OF measurable-snd*] **by** *simp*
qed

lemma (**in** *filtered-measure*) *progressive-process-const*:
 assumes $c \in$ *borel-measurable borel*
 shows *progressive-process M F* t_0 (λ*i* -. *c i*)
 using *assms* **by** (*unfold-locales*) (*auto simp add*: *measurable-split-conv intro*!: *measurable-compose*[*OF measurable-fst*] *measurable-restrict-space1*)

context *progressive-process*
begin

lemma *compose-progressive*:
assumes *case-prod f* $\in$ *borel-measurable borel*
shows *progressive-process M F* t_0 $(\lambda i\ \xi.\ (f\ i)\ (X\ i\ \xi))$
proof
fix i **assume** *asm*: $t_0 \leq i$
have $(\lambda(j,\ \xi).\ (j,\ X\ j\ \xi)) \in$ (*restrict-space borel* $\{t_0..i\} \bigotimes_M F\ i) \to_M$ *borel* $\bigotimes_M$ *borel*
using *progressive*[*OF asm*] *measurable-fst''*[*OF measurable-restrict-space1*, *OF measurable-id*]
by (*auto simp add*: *measurable-pair-iff measurable-split-conv*)
moreover have $(\lambda(j,\ \xi).\ f\ j\ (X\ j\ \xi)) =$ *case-prod f o* $((\lambda(j,\ y).\ (j,\ y))$ *o* $(\lambda(j,\ \xi).\ (j,\ X\ j\ \xi)))$ **by** *fastforce*
ultimately show $(\lambda(j,\ \xi).\ (f\ j)\ (X\ j\ \xi)) \in$ *borel-measurable* (*restrict-space borel* $\{t_0..i\} \bigotimes_M F\ i)$ **using** *assms* **by** (*simp add*: *borel-prod*)
qed

lemma *norm-progressive*: *progressive-process M F* t_0 $(\lambda i\ \xi.$ *norm* $(X\ i\ \xi))$ **using** *measurable-compose*[*OF progressive borel-measurable-norm*] **by** (*unfold-locales*) *simp*

lemma *scaleR-right-progressive*:
assumes *progressive-process M F* t_0 R
shows *progressive-process M F* t_0 $(\lambda i\ \xi.\ (R\ i\ \xi) *_R (X\ i\ \xi))$
using *progressive-process.progressive*[*OF assms*] **by** (*unfold-locales*) (*simp add*: *progressive assms*)

lemma *scaleR-right-const-fun-progressive*:
assumes $f \in$ *borel-measurable* $(F\ t_0)$
shows *progressive-process M F* t_0 $(\lambda i\ \xi.\ f\ \xi *_R (X\ i\ \xi))$
using *assms* **by** (*fast intro*: *scaleR-right-progressive progressive-process-const-fun*)

lemma *scaleR-right-const-progressive*:
assumes $c \in$ *borel-measurable borel*
shows *progressive-process M F* t_0 $(\lambda i\ \xi.\ c\ i *_R (X\ i\ \xi))$
using *assms* **by** (*fastforce intro*: *scaleR-right-progressive progressive-process-const*)

lemma *add-progressive*:
assumes *progressive-process M F* t_0 Y
shows *progressive-process M F* t_0 $(\lambda i\ \xi.\ X\ i\ \xi + Y\ i\ \xi)$
using *progressive-process.progressive*[*OF assms*] **by** (*unfold-locales*) (*simp add*: *progressive assms*)

lemma *diff-progressive*:
assumes *progressive-process M F* t_0 Y
shows *progressive-process M F* t_0 $(\lambda i\ \xi.\ X\ i\ \xi - Y\ i\ \xi)$
using *progressive-process.progressive*[*OF assms*] **by** (*unfold-locales*) (*simp add*: *progressive assms*)

lemma *uminus-progressive*: *progressive-process M F* t_0 $(-X)$ **using** *scaleR-right-const-progressive*[*of*

λ-. -1] **by** (*simp add*: *fun-Compl-def*)

end

A progressively measurable process is also adapted.

sublocale *progressive-process* $\subseteq$ *adapted-process* **using** *measurable-compose-rev*[*OF progressive measurable-Pair1'*]
  unfolding *prod.case space-restrict-space*
  by *unfold-locales simp*

sublocale *nat-progressive-process* $\subseteq$ *nat-adapted-process* **..**
sublocale *real-progressive-process* $\subseteq$ *real-adapted-process* **..**

In the discrete setting, adaptedness is equivalent to progressive measurability.

theorem *nat-progressive-iff-adapted*: *nat-progressive-process M F X* $\longleftrightarrow$ *nat-adapted-process M F X*
proof (*intro iffI*)
  assume *asm*: *nat-progressive-process M F X*
  interpret *nat-progressive-process M F X* **by** (*rule asm*)
  show *nat-adapted-process M F X* **..**
next
  assume *asm*: *nat-adapted-process M F X*
  interpret *nat-adapted-process M F X* **by** (*rule asm*)
  show *nat-progressive-process M F X*
  proof (*unfold-locales*, *intro borel-measurableI*)
    fix *S* :: *'b set* **and** *i* :: *nat* **assume** *open-S*: *open S*
    {
      fix *j* **assume** *asm*: $j \leq i$
     hence *X j* $-$‘ *S* $\cap$ *space M* $\in$ *F i* **using** *adaptedD*[*of j*, *THEN measurable-sets*] *space-F open-S* **by** *fastforce*
      moreover have *case-prod X* $-$‘ *S* $\cap$ $\{j\}$ $\times$ *space M* = $\{j\}$ $\times$ (*X j* $-$‘ *S* $\cap$ *space M*) **for** *j* **by** *fast*
      moreover have $\{j$ :: *nat*$\}$ $\in$ *restrict-space borel* $\{0..i\}$ **using** *asm* **by** (*simp add*: *sets-restrict-space-iff*)
      ultimately have *case-prod X* $-$‘ *S* $\cap$ $\{j\}$ $\times$ *space M* $\in$ *restrict-space borel* $\{0..i\}$ $\bigotimes_M$ *F i* **by** *simp*
    }
    hence (λj. ($\lambda(x, y)$. *X x y*) $-$‘ *S* $\cap$ $\{j\}$ $\times$ *space M*) ‘ $\{..i\}$ $\subseteq$ *restrict-space borel* $\{0..i\}$ $\bigotimes_M$ *F i* **by** *blast*
    moreover have *case-prod X* $-$‘ *S* $\cap$ *space* (*restrict-space borel* $\{0..i\}$ $\bigotimes_M$ *F i*) = ($\bigcup j \leq i$. *case-prod X* $-$‘ *S* $\cap$ $\{j\}$ $\times$ *space M*) **unfolding** *space-pair-measure space-restrict-space space-F* **by** *force*
    ultimately show *case-prod X* $-$‘ *S* $\cap$ *space* (*restrict-space borel* $\{0..i\}$ $\bigotimes_M$ *F i*) $\in$ *restrict-space borel* $\{0..i\}$ $\bigotimes_M$ *F i* **by** (*metis sets.countable-UN*)
  qed
qed

6.4 Predictable Process

We introduce the constant Σ_P to denote the predictable sigma algebra.

context *linearly-filtered-measure*
begin

definition $\Sigma_P :: ('b \times 'a)$ *measure* **where** *predictable-sigma*: $\Sigma_P \equiv$ *sigma* $(\{t_0..\} \times$ *space* $M)$ $(\{\{s{<}..t\} \times A \mid A\ s\ t.\ A \in F\ s \wedge t_0 \le s \wedge s < t\} \cup \{\{t_0\} \times A \mid A.\ A \in F\ t_0\})$

lemma *space-predictable-sigma*[*simp*]: *space* $\Sigma_P = (\{t_0..\} \times$ *space* $M)$ **unfolding** *predictable-sigma space-measure-of-conv* **by** *blast*

lemma *sets-predictable-sigma*: *sets* $\Sigma_P =$ *sigma-sets* $(\{t_0..\} \times$ *space* $M)$ $(\{\{s{<}..t\} \times A \mid A\ s\ t.\ A \in F\ s \wedge t_0 \le s \wedge s < t\} \cup \{\{t_0\} \times A \mid A.\ A \in F\ t_0\})$
unfolding *predictable-sigma* **using** *space-F sets.sets-into-space* **by** (*subst sets-measure-of*) *fastforce+*

lemma *measurable-predictable-sigma-snd*:
assumes *countable* $\mathcal{I}$ $\mathcal{I} \subseteq \{\{s{<}..t\} \mid s\ t.\ t_0 \le s \wedge s < t\}$ $\{t_0{<}..\} \subseteq (\bigcup \mathcal{I})$
shows *snd* $\in \Sigma_P \rightarrow_M F\ t_0$
proof (*intro measurableI*)
fix $S :: 'a$ *set* **assume** *asm*: $S \in F\ t_0$
have *countable*: *countable* $((\lambda I.\ I \times S)$ ‘ $\mathcal{I})$ **using** *assms*(*1*) **by** *blast*
have $(\lambda I.\ I \times S)$ ‘ $\mathcal{I} \subseteq \{\{s{<}..t\} \times A \mid A\ s\ t.\ A \in F\ s \wedge t_0 \le s \wedge s < t\}$ **using** *sets-F-mono*[*OF order-refl*, *THEN subsetD*, *OF - asm*] *assms*(*2*) **by** *blast*
hence $(\bigcup I{\in}\mathcal{I}.\ I \times S) \cup \{t_0\} \times S \in \Sigma_P$ **unfolding** *sets-predictable-sigma* **using** *asm* **by** (*intro sigma-sets-Un*[*OF sigma-sets-UNION*[*OF countable*] *sigma-sets.Basic*] *sigma-sets.Basic*) *blast+*
moreover have *snd* $-$‘ $S \cap$ *space* $\Sigma_P = \{t_0..\} \times S$ **using** *sets.sets-into-space*[*OF asm*] **by** *fastforce*
moreover have $\{t_0\} \cup \{t_0{<}..\} = \{t_0..\}$ **by** *auto*
moreover have $(\bigcup I{\in}\mathcal{I}.\ I \times S) \cup \{t_0\} \times S = \{t_0..\} \times S$ **using** *assms*(*2*,*3*) *calculation*(*3*) **by** *fastforce*
ultimately show *snd* $-$‘ $S \cap$ *space* $\Sigma_P \in \Sigma_P$ **by** *argo*
qed (*auto*)

lemma *measurable-predictable-sigma-fst*:
assumes *countable* $\mathcal{I}$ $\mathcal{I} \subseteq \{\{s{<}..t\} \mid s\ t.\ t_0 \le s \wedge s < t\}$ $\{t_0{<}..\} \subseteq (\bigcup \mathcal{I})$
shows *fst* $\in \Sigma_P \rightarrow_M$ *borel*
proof $-$
have $A \times$ *space* $M \in$ *sets* Σ_P **if** $A \in$ *sigma-sets* $\{t_0..\}$ $\{\{s{<}..t\} \mid s\ t.\ t_0 \le s \wedge s < t\}$ **for** A **unfolding** *sets-predictable-sigma* **using** *that*
proof (*induction rule*: *sigma-sets.induct*)
case (*Basic a*)
thus *?case* **using** *space-F sets.top* **by** *blast*
next
case (*Compl a*)
have $(\{t_0..\} - a) \times$ *space* $M = \{t_0..\} \times$ *space* $M - a \times$ *space* M **by** *blast*

then show *?case* **using** *Compl*(*2*)[*THEN sigma-sets.Compl*] **by** *presburger*
next
case (*Union a*)
have $\bigcup$ (*range a*) $\times$ *space M* $=$ $\bigcup$ (*range* ($\lambda i.$ *a i* $\times$ *space M*)) **by** *blast*
then show *?case* **using** *Union*(*2*)[*THEN sigma-sets.Union*] **by** *presburger*
qed (*auto*)
moreover have *restrict-space borel* $\{t_0..\}$ $=$ *sigma* $\{t_0..\}$ $\{\{s<..t\} \mid s\ t.\ t_0 \leq s \wedge s < t\}$
proof −
have *sigma-sets* $\{t_0..\}$ $((\cap)\ \{t_0..\}$ ‘ *sigma-sets UNIV* (*range greaterThan*)) $=$ *sigma-sets* $\{t_0..\}$ $\{\{s<..t\} \mid s\ t.\ t_0 \leq s \wedge s < t\}$
proof (*intro sigma-sets-eqI* ; *clarify*)
fix *A* :: *'b set* **assume** *asm*: $A \in$ *sigma-sets UNIV* (*range greaterThan*)
thus $\{t_0..\} \cap A \in$ *sigma-sets* $\{t_0..\}$ $\{\{s<..t\} \mid s\ t.\ t_0 \leq s \wedge s < t\}$
proof (*induction rule*: *sigma-sets.induct*)
case (*Basic a*)
then obtain *s* **where** *s*: $a = \{s<..\}$ **by** *blast*
show *?case*
proof (*cases* $t_0 \leq s$)
case *True*
hence $*$: $\{t_0..\} \cap a = (\bigcup i \in \mathcal{I}.\ \{s<..\} \cap i)$ **using** *s assms*(*3*) **by** *force*
have $((\cap)\ \{s<..\}$ ‘ $\mathcal{I}) \subseteq$ *sigma-sets* $\{t_0..\}$ $\{\{s<..t\} \mid s\ t.\ t_0 \leq s \wedge s < t\}$
proof (*clarify*)
fix *A* **assume** $A \in \mathcal{I}$
then obtain s' t' **where** *A*: $A = \{s'<..t'\}$ $t_0 \leq s'$ $s' < t'$ **using** *assms*(*2*) **by** *blast*
hence $\{s<..\} \cap A = \{max\ s\ s'<..t'\}$ **by** *fastforce*
moreover have $t_0 \leq max\ s\ s'$ **using** *A True* **by** *linarith*
moreover have $max\ s\ s' < t'$ **if** $s < t'$ **using** *A that* **by** *linarith*
moreover have $\{s<..\} \cap A = \{\}$ **if** $\neg\ s < t'$ **using** *A that* **by** *force*
ultimately show $\{s<..\} \cap A \in$ *sigma-sets* $\{t_0..\}$ $\{\{s<..t\} \mid s\ t.\ t_0 \leq s \wedge s < t\}$ **by** (*cases* $s < t'$) (*blast*, *simp add*: *sigma-sets.Empty*)
qed
thus *?thesis* **unfolding** $*$ **using** *assms*(*1*) **by** (*intro sigma-sets-UNION*) *auto*
next
case *False*
hence $\{t_0..\} \cap a = \{t_0..\}$ **using** *s* **by** *force*
thus *?thesis* **using** *sigma-sets-top* **by** *auto*
qed
next
case (*Compl a*)
have $\{t_0..\} \cap (UNIV - a) = \{t_0..\} - (\{t_0..\} \cap a)$ **by** *blast*
then show *?case* **using** *Compl*(*2*)[*THEN sigma-sets.Compl*] **by** *presburger*
next
case (*Union a*)
have $\{t_0..\} \cap \bigcup$ (*range a*) $=$ $\bigcup$ (*range* ($\lambda i.$ $\{t_0..\} \cap a\ i$)) **by** *blast*
then show *?case* **using** *Union*(*2*)[*THEN sigma-sets.Union*] **by** *presburger*
qed (*simp add*: *sigma-sets.Empty*)

next
fix *s t* **assume** *asm*: $t_0 \le s$ $s < t$
hence $*$: $\{s{<}..t\} = \{s{<}..\} \cap (\{t_0..\} - \{t{<}..\})$ **by** *force*
have $\{s{<}..\} \in$ *sigma-sets* $\{t_0..\}$ $((\cap)$ $\{t_0..\}$ *‘ sigma-sets UNIV* (*range greaterThan*)) **using** *asm* **by** (*intro sigma-sets.Basic*) *auto*
moreover have $\{t_0..\} - \{t{<}..\} \in$ *sigma-sets* $\{t_0..\}$ $((\cap)$ $\{t_0..\}$ *‘ sigma-sets UNIV* (*range greaterThan*)) **using** *asm* **by** (*intro sigma-sets.Compl sigma-sets.Basic*) *auto*
ultimately show $\{s{<}..t\} \in$ *sigma-sets* $\{t_0..\}$ $((\cap)$ $\{t_0..\}$ *‘ sigma-sets UNIV* (*range greaterThan*)) **unfolding** $*$ *Int-range-binary*[*of* $\{s{<}..\}$] **by** (*intro sigma-sets-Inter*[*OF - binary-in-sigma-sets*]) *auto*
qed
thus *?thesis* **unfolding** *borel-Ioi restrict-space-def emeasure-sigma* **by** (*force intro*: *sigma-eqI*)
qed
ultimately have *restrict-space borel* $\{t_0..\} \bigotimes_M$ *sigma* (*space M*) $\{\} \subseteq$ *sets* Σ_P

unfolding *sets-pair-measure space-restrict-space space-measure-of-conv*
using *space-predictable-sigma sets.sigma-algebra-axioms*[*of* Σ_P]
by (*intro sigma-algebra.sigma-sets-subset*) (*auto simp add*: *sigma-sets-empty-eq sets-measure-of-conv*)
moreover have *space* (*restrict-space borel* $\{t_0..\} \bigotimes_M$ *sigma* (*space M*) $\{\}$) = *space* Σ_P **by** (*simp add*: *space-pair-measure*)
moreover have *fst* $\in$ *restrict-space borel* $\{t_0..\} \bigotimes_M$ *sigma* (*space M*) $\{\} \rightarrow_M$ *borel* **by** (*fastforce intro*: *measurable-fst''*[*OF measurable-restrict-space1*, *of* $\lambda x.\ x$])

ultimately show *?thesis* **by** (*meson borel-measurable-subalgebra*)
qed

end

locale *predictable-process* = *linearly-filtered-measure M F* t_0 **for** *M F* t_0 **and** *X* :: *-* $\Rightarrow$ *-* $\Rightarrow$ *-* :: {*second-countable-topology*, *banach*} +
assumes *predictable*: $(\lambda(t, x).\ X\ t\ x) \in$ *borel-measurable* Σ_P
begin

lemmas *predictableD* = *measurable-sets*[*OF predictable*, *unfolded space-predictable-sigma*]

end

locale *nat-predictable-process* = *predictable-process M F 0* :: *nat X* **for** *M F X*
locale *real-predictable-process* = *predictable-process M F 0* :: *real X* **for** *M F X*

lemma (**in** *nat-filtered-measure*) *measurable-predictable-sigma-snd'*:
shows *snd* $\in \Sigma_P \rightarrow_M$ *F 0*
by (*intro measurable-predictable-sigma-snd*[*of range* ($\lambda x.$ {*Suc x*})]) (*force* | *simp add*: *greaterThan-0*)+

lemma (**in** *nat-filtered-measure*) *measurable-predictable-sigma-fst'*:
shows *fst* $\in \Sigma_P \rightarrow_M$ *borel*
by (*intro measurable-predictable-sigma-fst*[*of range* (λx. {*Suc x*})]) (*force* | *simp add*: *greaterThan-0*)+

lemma (**in** *real-filtered-measure*) *measurable-predictable-sigma-snd'*:
shows *snd* $\in \Sigma_P \rightarrow_M F\ 0$
using *real-arch-simple* **by** (*intro measurable-predictable-sigma-snd*[*of range* (λx::*nat*. {*0*<..*real* (*Suc x*)})]) (*fastforce intro*: *add-increasing*)+

lemma (**in** *real-filtered-measure*) *measurable-predictable-sigma-fst'*:
shows *fst* $\in \Sigma_P \rightarrow_M$ *borel*
using *real-arch-simple* **by** (*intro measurable-predictable-sigma-fst*[*of range* (λx::*nat*. {*0*<..*real* (*Suc x*)})]) (*fastforce intro*: *add-increasing*)+

lemma (**in** *linearly-filtered-measure*) *predictable-process-const-fun*:
assumes *snd* $\in \Sigma_P \rightarrow_M F\ t_0$ *f* $\in$ *borel-measurable* ($F\ t_0$)
shows *predictable-process M F* t_0 (λ-. *f*)
using *measurable-compose-rev*[*OF assms*(*2*)] *assms*(*1*) **by** (*unfold-locales*) (*auto simp add*: *measurable-split-conv*)

lemma (**in** *nat-filtered-measure*) *predictable-process-const-fun'*[*intro*]:
assumes *f* $\in$ *borel-measurable* (*F 0*)
shows *nat-predictable-process M F* (λ-. *f*)
using *assms* **by** (*intro predictable-process-const-fun*[*OF measurable-predictable-sigma-snd'*, *THEN nat-predictable-process.intro*])

lemma (**in** *real-filtered-measure*) *predictable-process-const-fun'*[*intro*]:
assumes *f* $\in$ *borel-measurable* (*F 0*)
shows *real-predictable-process M F* (λ-. *f*)
using *assms* **by** (*intro predictable-process-const-fun*[*OF measurable-predictable-sigma-snd'*, *THEN real-predictable-process.intro*])

lemma (**in** *linearly-filtered-measure*) *predictable-process-const*:
assumes *fst* $\in$ *borel-measurable* Σ_P *c* $\in$ *borel-measurable borel*
shows *predictable-process M F* t_0 (λi -. *c i*)
using *assms* **by** (*unfold-locales*) (*simp add*: *measurable-split-conv*)

lemma (**in** *linearly-filtered-measure*) *predictable-process-const-const*[*intro*]:
shows *predictable-process M F* t_0 (λ- -. *c*)
by (*unfold-locales*) *simp*

lemma (**in** *nat-filtered-measure*) *predictable-process-const'*[*intro*]:
assumes *c* $\in$ *borel-measurable borel*
shows *nat-predictable-process M F* (λi -. *c i*)
using *assms* **by** (*intro predictable-process-const*[*OF measurable-predictable-sigma-fst'*,

THEN nat-predictable-process.intro])

lemma (**in** *real-filtered-measure*) *predictable-process-const'*[*intro*]:
assumes $c \in$ *borel-measurable borel*
shows *real-predictable-process M F* (λi -. c i)
using *assms* **by** (*intro predictable-process-const*[*OF measurable-predictable-sigma-fst'*, *THEN real-predictable-process.intro*])

context *predictable-process*
begin

lemma *compose-predictable*:
assumes *fst* $\in$ *borel-measurable* Σ_P *case-prod f* $\in$ *borel-measurable borel*
shows *predictable-process M F* t_0 ($\lambda i\ \xi.\ (f\ i)\ (X\ i\ \xi)$)
proof
have $(\lambda(i, \xi).\ (i, X\ i\ \xi)) \in \Sigma_P \to_M$ *borel* $\bigotimes_M$ *borel* **using** *predictable assms*(*1*) **by** (*auto simp add*: *measurable-pair-iff measurable-split-conv*)
moreover have $(\lambda(i, \xi).\ f\ i\ (X\ i\ \xi))$ = *case-prod f o* $(\lambda(i, \xi).\ (i, X\ i\ \xi))$ **by** *fastforce*
ultimately show $(\lambda(i, \xi).\ f\ i\ (X\ i\ \xi)) \in$ *borel-measurable* Σ_P **unfolding** *borel-prod* **using** *assms* **by** *simp*
qed

lemma *norm-predictable*: *predictable-process M F* t_0 ($\lambda i\ \xi.$ *norm* $(X\ i\ \xi)$) **using** *measurable-compose*[*OF predictable borel-measurable-norm*]
by (*unfold-locales*) (*simp add*: *prod.case-distrib*)

lemma *scaleR-right-predictable*:
assumes *predictable-process M F* t_0 R
shows *predictable-process M F* t_0 ($\lambda i\ \xi.\ (R\ i\ \xi) *_R (X\ i\ \xi)$)
using *predictable predictable-process.predictable*[*OF assms*] **by** (*unfold-locales*) (*auto simp add*: *measurable-split-conv*)

lemma *scaleR-right-const-fun-predictable*:
assumes *snd* $\in \Sigma_P \to_M F\ t_0$ $f \in$ *borel-measurable* $(F\ t_0)$
shows *predictable-process M F* t_0 ($\lambda i\ \xi.\ f\ \xi *_R (X\ i\ \xi)$)
using *assms* **by** (*fast intro*: *scaleR-right-predictable predictable-process-const-fun*)

lemma *scaleR-right-const-predictable*:
assumes *fst* $\in$ *borel-measurable* Σ_P $c \in$ *borel-measurable borel*
shows *predictable-process M F* t_0 ($\lambda i\ \xi.\ c\ i *_R (X\ i\ \xi)$)
using *assms* **by** (*fastforce intro*: *scaleR-right-predictable predictable-process-const*)

lemma *scaleR-right-const'-predictable*: *predictable-process M F* t_0 ($\lambda i\ \xi.\ c *_R (X\ i\ \xi)$)
by (*fastforce intro*: *scaleR-right-predictable*)

lemma *add-predictable*:
assumes *predictable-process M F* t_0 Y

shows *predictable-process M F* t_0 $(\lambda i\ \xi.\ X\ i\ \xi + Y\ i\ \xi)$
using *predictable predictable-process.predictable*[*OF assms*] **by** (*unfold-locales*) (*auto simp add*: *measurable-split-conv*)

lemma *diff-predictable*:
assumes *predictable-process M F* t_0 *Y*
shows *predictable-process M F* t_0 $(\lambda i\ \xi.\ X\ i\ \xi - Y\ i\ \xi)$
using *predictable predictable-process.predictable*[*OF assms*] **by** (*unfold-locales*) (*auto simp add*: *measurable-split-conv*)

lemma *uminus-predictable*: *predictable-process M F* t_0 $(-X)$ **using** *scaleR-right-const′-predictable*[*of* -1] **by** (*simp add*: *fun-Compl-def*)

end

Every predictable process is also progressively measurable.

sublocale *predictable-process* $\subseteq$ *progressive-process*
proof (*unfold-locales*)
fix i :: $'b$ **assume** *asm*: $t_0 \leq i$
{
fix S :: $('b \times 'a)$ *set* **assume** $S \in \{\{s{<}..t\} \times A \mid A\ s\ t.\ A \in F\ s \wedge t_0 \leq s \wedge s < t\} \cup \{\{t_0\} \times A \mid A.\ A \in F\ t_0\}$
hence $(\lambda x.\ x)$ $-$‘ $S \cap (\{t_0..i\} \times$ *space M*$) \in$ *restrict-space borel* $\{t_0..i\} \bigotimes_M F\ i$
proof
assume $S \in \{\{s{<}..t\} \times A \mid A\ s\ t.\ A \in F\ s \wedge t_0 \leq s \wedge s < t\}$
then obtain $s\ t\ A$ **where** *S-is*: $S = \{s{<}..t\} \times A$ $t_0 \leq s$ $s < t$ $A \in F\ s$ **by** *blast*
hence $(\lambda x.\ x)$ $-$‘ $S \cap (\{t_0..i\} \times$ *space M*$) = \{s{<}..min\ i\ t\} \times A$ **using** *sets.sets-into-space*[*OF S-is*(*4*)] **by** *auto*
then show *?thesis* **using** *S-is sets-F-mono*[*of s i*] **by** (*cases* $s \leq i$) (*fastforce simp add*: *sets-restrict-space-iff*)+
next
assume $S \in \{\{t_0\} \times A \mid A.\ A \in F\ t_0\}$
then obtain A **where** *S-is*: $S = \{t_0\} \times A$ $A \in F\ t_0$ **by** *blast*
hence $(\lambda x.\ x)$ $-$‘ $S \cap (\{t_0..i\} \times$ *space M*$) = \{t_0\} \times A$ **using** *asm sets.sets-into-space*[*OF S-is*(*2*)] **by** *auto*
thus *?thesis* **using** *S-is*(*2*) *sets-F-mono*[*OF order-refl asm*] *asm* **by** (*fastforce simp add*: *sets-restrict-space-iff*)
qed
hence $(\lambda x.\ x)$ $-$‘ $S \cap$ *space* (*restrict-space borel* $\{t_0..i\} \bigotimes_M F\ i) \in$ *restrict-space borel* $\{t_0..i\} \bigotimes_M F\ i$ **by** (*simp add*: *space-pair-measure space-F*[*OF asm*])
}
moreover have $\{\{s{<}..t\} \times A \mid A\ s\ t.\ A \in$ *sets* $(F\ s) \wedge t_0 \leq s \wedge s < t\} \cup \{\{t_0\} \times A \mid A.\ A \in$ *sets* $(F\ t_0)\} \subseteq$ *Pow* $(\{t_0..\} \times$ *space M*$)$ **using** *sets.sets-into-space* **by** *force*
ultimately have $(\lambda x.\ x) \in$ *restrict-space borel* $\{t_0..i\} \bigotimes_M F\ i \rightarrow_M \Sigma_P$ **using** *space-F*[*OF asm*] **by** (*intro measurable-sigma-sets*[*OF sets-predictable-sigma*]) (*fast*, *force simp add*: *space-pair-measure*)

thus *case-prod X* $\in$ *borel-measurable* (*restrict-space borel* $\{t_0..i\}$ $\bigotimes_M$ *F i*) **using** *predictable* **by** *simp*
qed

sublocale *nat-predictable-process* $\subseteq$ *nat-progressive-process* **..**
sublocale *real-predictable-process* $\subseteq$ *real-progressive-process* **..**

The following lemma characterizes predictability in a discrete-time setting.

lemma (**in** *nat-filtered-measure*) *sets-in-filtration*:
assumes $(\bigcup i.\ \{i\} \times A\ i) \in \Sigma_P$
shows $A\ (Suc\ i) \in F\ i\ A\ 0 \in F\ 0$
using *assms* **unfolding** *sets-predictable-sigma*
proof (*induction* $(\bigcup i.\ \{i\} \times A\ i)$ *arbitrary*: *A*)
case *Basic*
{
assume $\exists S.\ (\bigcup i.\ \{i\} \times A\ i) = \{0\} \times S$
then obtain *S* **where** *S*: $(\bigcup i.\ \{i\} \times A\ i) = \{bot\} \times S$ **unfolding** *bot-nat-def* **by** *blast*
hence $S \in F\ bot$ **using** *Basic* **by** (*fastforce simp add*: *times-eq-iff bot-nat-def*)
moreover have $A\ i = \{\}$ **if** $i \neq bot$ **for** *i* **using** *that S* **by** *blast*
moreover have $A\ bot = S$ **using** *S* **by** *blast*
ultimately have $A\ (Suc\ i) \in F\ i\ A\ 0 \in F\ 0$ **for** *i* **unfolding** *bot-nat-def* **by** (*auto simp add*: *bot-nat-def*)
}
note $* = this$
{
assume $\nexists S.\ (\bigcup i.\ \{i\} \times A\ i) = \{0\} \times S$
then obtain *s t B* **where** *B*: $(\bigcup i.\ \{i\} \times A\ i) = \{s{<}..t\} \times B\ B \in sets\ (F\ s)$ $s < t$ **using** *Basic* **by** *auto*
hence $A\ i = B$ **if** $i \in \{s{<}..t\}$ **for** *i* **using** *that* **by** *fast*
moreover have $A\ i = \{\}$ **if** $i \notin \{s{<}..t\}$ **for** *i* **using** *B that* **by** *fastforce*
ultimately have $A\ (Suc\ i) \in F\ i\ A\ 0 \in F\ 0$ **for** *i* **unfolding** *bot-nat-def* **using** *B sets-F-mono* **by** (*auto simp add*: *bot-nat-def*) (*metis less-Suc-eq-le sets.empty-sets subset-eq*)
}
note $** = this$
show $A\ (Suc\ i) \in sets\ (F\ i)\ A\ 0 \in sets\ (F\ 0)$ **using** $*(1)[of\ i]$ $*(2)$ $**(1)[of\ i]$ $**(2)$ **by** *blast+*
next
case *Empty*
{
case *1*
then show *?case* **using** *Empty* **by** *simp*
next
case *2*
then show *?case* **using** *Empty* **by** *simp*
}
next
case (*Compl a*)

have *a-in*: $a \subseteq \{0..\} \times$ *space M* **using** *Compl(1) sets.sets-into-space sets-predictable-sigma space-predictable-sigma* **by** *metis*
hence *A-in*: $A\ i \subseteq$ *space M* **for** *i* **using** *Compl(4)* **by** *blast*
have *a*: $a = \{0..\} \times$ *space M* $-$ $(\bigcup i.\ \{i\} \times A\ i)$ **using** *a-in Compl(4)* **by** *blast*
also have ... $= -\ (\bigcap j.\ -\ (\{j\} \times (\mathit{space}\ M - A\ j)))$ **by** *blast*
also have ... $= (\bigcup j.\ \{j\} \times (\mathit{space}\ M - A\ j))$ **by** *blast*
finally have $*$: $(\mathit{space}\ M - A\ (\mathit{Suc}\ i)) \in F\ i$ $(\mathit{space}\ M - A\ 0) \in F\ 0$ **using** *Compl(2,3)* **by** *auto*
{
case *1*
then show *?case* **using** $*$ *A-in* **by** (*metis bot-nat-0.extremum double-diff sets.Diff sets.top sets-F-mono sets-le-imp-space-le space-F*)
next
case *2*
then show *?case* **using** $*$ *A-in* **by** (*metis bot-nat-0.extremum double-diff sets.Diff sets.top sets-F-mono sets-le-imp-space-le space-F*)
}
next
case (*Union a*)
have *a-in*: $a\ i \subseteq \{0..\} \times$ *space M* **for** *i* **using** *Union(1) sets.sets-into-space sets-predictable-sigma space-predictable-sigma* **by** *metis*
hence *A-in*: $A\ i \subseteq$ *space M* **for** *i* **using** *Union(4)* **by** *blast*
have $\mathit{snd}\ x \in \mathit{snd}$ ‘ $(a\ i \cap (\{\mathit{fst}\ x\} \times \mathit{space}\ M))$ **if** $x \in a\ i$ **for** *i x* **using** *that a-in* **by** *fastforce*
hence *a-i*: $a\ i = (\bigcup j.\ \{j\} \times (\mathit{snd}$ ‘ $(a\ i \cap (\{j\} \times \mathit{space}\ M))))$ **for** *i* **by** *force*
have *A-i*: $A\ i = \mathit{snd}$ ‘ $(\bigcup\ (\mathit{range}\ a) \cap (\{i\} \times \mathit{space}\ M))$ **for** *i* **unfolding** *Union(4)* **using** *A-in* **by** *force*
have $*$: snd ‘ $(a\ j \cap (\{\mathit{Suc}\ i\} \times \mathit{space}\ M)) \in F\ i$ snd ‘ $(a\ j \cap (\{0\} \times \mathit{space}\ M)) \in F\ 0$ **for** *j* **using** *Union(2,3)[OF a-i]* **by** *auto*
{
case *1*
have $(\bigcup j.\ \mathit{snd}$ ‘ $(a\ j \cap (\{\mathit{Suc}\ i\} \times \mathit{space}\ M))) \in F\ i$ **using** $*$ **by** *fast*
moreover have $(\bigcup j.\ \mathit{snd}$ ‘ $(a\ j \cap (\{\mathit{Suc}\ i\} \times \mathit{space}\ M))) = \mathit{snd}$ ‘ $(\bigcup\ (\mathit{range}\ a) \cap (\{\mathit{Suc}\ i\} \times \mathit{space}\ M))$ **by** *fast*
ultimately show *?case* **using** *A-i* **by** *metis*
next
case *2*
have $(\bigcup j.\ \mathit{snd}$ ‘ $(a\ j \cap (\{0\} \times \mathit{space}\ M))) \in F\ 0$ **using** $*$ **by** *fast*
moreover have $(\bigcup j.\ \mathit{snd}$ ‘ $(a\ j \cap (\{0\} \times \mathit{space}\ M))) = \mathit{snd}$ ‘ $(\bigcup\ (\mathit{range}\ a) \cap (\{0\} \times \mathit{space}\ M))$ **by** *fast*
ultimately show *?case* **using** *A-i* **by** *metis*
}
qed

This leads to the following useful fact.

lemma (**in** *nat-predictable-process*) *adapted-Suc*: *nat-adapted-process M F* $(\lambda i.\ X\ (\mathit{Suc}\ i))$
proof (*unfold-locales*, *intro borel-measurableI*)
fix *S* :: *'b set* **and** *i* **assume** *open-S*: *open S*

have $\{Suc\ i\} = \{i{<}..Suc\ i\}$ **by** *fastforce*
hence $\{Suc\ i\} \times space\ M \in \Sigma_P$ **using** *space-F*[*symmetric*, *of i*] **unfolding** *sets-predictable-sigma* **by** (*intro sigma-sets.Basic*) *blast*
moreover have $case\text{-}prod\ X -` S \cap (UNIV \times space\ M) \in \Sigma_P$ **unfolding** *atLeast-0*[*symmetric*] **using** *open-S* **by** (*intro predictableD*, *simp add*: *borel-open*)
ultimately have $case\text{-}prod\ X -` S \cap (\{Suc\ i\} \times space\ M) \in \Sigma_P$ **unfolding** *sets-predictable-sigma* **using** *space-F sets.sets-into-space*
by (*subst Times-Int-distrib1*[*of* $\{Suc\ i\}$ *UNIV space M*, *simplified*], *subst inf.commute*, *subst Int-assoc*[*symmetric*], *subst Int-range-binary*)
(*intro sigma-sets-Inter binary-in-sigma-sets*, *fast*)+
moreover have $case\text{-}prod\ X -` S \cap (\{Suc\ i\} \times space\ M) = \{Suc\ i\} \times (X\ (Suc\ i) -` S \cap space\ M)$ **by** (*auto simp add*: *le-Suc-eq*)
moreover have ... $= (\bigcup j.\ \{j\} \times (if\ j = Suc\ i\ then\ (X\ (Suc\ i) -` S \cap space\ M)\ else\ \{\}))$ **by** (*force split*: *if-splits*)
ultimately have $(\bigcup j.\ \{j\} \times (if\ j = Suc\ i\ then\ (X\ (Suc\ i) -` S \cap space\ M)\ else\ \{\})) \in \Sigma_P$ **by** *argo*
thus $X\ (Suc\ i) -` S \cap space\ (F\ i) \in sets\ (F\ i)$ **using** *sets-in-filtration*[*of* $\lambda j.\ if\ j = Suc\ i\ then\ (X\ (Suc\ i) -` S \cap space\ M)\ else\ \{\}$] *space-F*[*OF zero-le*] **by** *presburger*
qed

theorem *nat-predictable-process-iff*: $nat\text{-}predictable\text{-}process\ M\ F\ X \longleftrightarrow nat\text{-}adapted\text{-}process\ M\ F\ (\lambda i.\ X\ (Suc\ i)) \wedge X\ 0 \in borel\text{-}measurable\ (F\ 0)$
proof (*intro iffI*)
assume *asm*: $nat\text{-}adapted\text{-}process\ M\ F\ (\lambda i.\ X\ (Suc\ i)) \wedge X\ 0 \in borel\text{-}measurable\ (F\ 0)$
interpret $nat\text{-}adapted\text{-}process\ M\ F\ \lambda i.\ X\ (Suc\ i)$ **using** *asm* **by** *blast*
have $(\lambda(x, y).\ X\ x\ y) \in borel\text{-}measurable\ \Sigma_P$
proof (*intro borel-measurableI*)
fix $S :: 'b\ set$ **assume** *open-S*: *open S*
have $\{i\} \times (X\ i -` S \cap space\ M) \in sets\ \Sigma_P$ **for** *i*
proof (*cases i*)
case *0*
then show *?thesis* **unfolding** *sets-predictable-sigma*
using *measurable-sets*[*OF - borel-open*[*OF open-S*], *of X 0 F 0*] *asm* **by** *auto*
next
case (*Suc i*)
have $\{Suc\ i\} = \{i{<}..Suc\ i\}$ **by** *fastforce*
then show *?thesis* **unfolding** *sets-predictable-sigma*
using *measurable-sets*[*OF adapted borel-open*[*OF open-S*], *of i*]
by (*intro sigma-sets.Basic*, *auto simp add*: *Suc*)
qed
moreover have $(\lambda(x, y).\ X\ x\ y) -` S \cap space\ \Sigma_P = (\bigcup i.\ \{i\} \times (X\ i -` S \cap space\ M))$ **by** *fastforce*
ultimately show $(\lambda(x, y).\ X\ x\ y) -` S \cap space\ \Sigma_P \in sets\ \Sigma_P$ **by** *simp*
qed
thus *nat-predictable-process M F X* **by** (*unfold-locales*)
next
assume *asm*: *nat-predictable-process M F X*

interpret *nat-predictable-process M F X* **by** (*rule asm*)
show *nat-adapted-process M F* ($\lambda i.$ *X* (*Suc i*)) $\wedge$ *X 0* $\in$ *borel-measurable* (*F 0*)
using *adapted-Suc* **by** *simp*
qed

end

theory *Martingale*
imports *Stochastic-Process Conditional-Expectation-Banach*
begin

7 Martingales

The following locales are necessary for defining martingales.

7.1 Additional Locale Definitions

locale *sigma-finite-adapted-process* = *sigma-finite-filtered-measure* $M\ F\ t_0$ + *adapted-process* $M\ F\ t_0\ X$ **for** $M\ F\ t_0\ X$

locale *nat-sigma-finite-adapted-process* = *sigma-finite-adapted-process M F 0* :: *nat X* **for** *M F X*
locale *real-sigma-finite-adapted-process* = *sigma-finite-adapted-process M F 0* :: *real X* **for** *M F X*

sublocale *nat-sigma-finite-adapted-process* $\subseteq$ *nat-sigma-finite-filtered-measure* **..**
sublocale *real-sigma-finite-adapted-process* $\subseteq$ *real-sigma-finite-filtered-measure* **..**

locale *finite-adapted-process* = *finite-filtered-measure* $M\ F\ t_0$ + *adapted-process* $M\ F\ t_0\ X$ **for** $M\ F\ t_0\ X$

sublocale *finite-adapted-process* $\subseteq$ *sigma-finite-adapted-process* **..**

locale *nat-finite-adapted-process* = *finite-adapted-process M F 0* :: *nat X* **for** *M F X*
locale *real-finite-adapted-process* = *finite-adapted-process M F 0* :: *real X* **for** *M F X*

sublocale *nat-finite-adapted-process* $\subseteq$ *nat-sigma-finite-adapted-process* **..**
sublocale *real-finite-adapted-process* $\subseteq$ *real-sigma-finite-adapted-process* **..**

locale *sigma-finite-adapted-process-order* = *sigma-finite-adapted-process* $M\ F\ t_0\ X$ **for** $M\ F\ t_0$ **and** X :: - $\Rightarrow$ - $\Rightarrow$ - :: {*order-topology*, *ordered-real-vector*}

locale *nat-sigma-finite-adapted-process-order* = *sigma-finite-adapted-process-order* *M F 0* :: *nat X* **for** *M F X*
locale *real-sigma-finite-adapted-process-order* = *sigma-finite-adapted-process-order* *M F 0* :: *real X* **for** *M F X*

sublocale *nat-sigma-finite-adapted-process-order* ⊆ *nat-sigma-finite-adapted-process*
..
sublocale *real-sigma-finite-adapted-process-order* ⊆ *real-sigma-finite-adapted-process*
..

locale *finite-adapted-process-order* = *finite-adapted-process* $M\ F\ t_0\ X$ **for** $M\ F\ t_0$ **and** *X* :: - ⇒ - ⇒ - :: {*order-topology*, *ordered-real-vector*}

locale *nat-finite-adapted-process-order* = *finite-adapted-process-order* *M F 0* :: *nat X* **for** *M F X*
locale *real-finite-adapted-process-order* = *finite-adapted-process-order* *M F 0* :: *real X* **for** *M F X*

sublocale *nat-finite-adapted-process-order* ⊆ *nat-sigma-finite-adapted-process-order*
..
sublocale *real-finite-adapted-process-order* ⊆ *real-sigma-finite-adapted-process-order*
..

locale *sigma-finite-adapted-process-linorder* = *sigma-finite-adapted-process-order* $M\ F\ t_0\ X$ **for** $M\ F\ t_0$ **and** *X* :: - ⇒ - ⇒ - :: {*linorder-topology*}

locale *nat-sigma-finite-adapted-process-linorder* = *sigma-finite-adapted-process-linorder* *M F 0* :: *nat X* **for** *M F X*
locale *real-sigma-finite-adapted-process-linorder* = *sigma-finite-adapted-process-linorder* *M F 0* :: *real X* **for** *M F X*

sublocale *nat-sigma-finite-adapted-process-linorder* ⊆ *nat-sigma-finite-adapted-process-order*
..
sublocale *real-sigma-finite-adapted-process-linorder* ⊆ *real-sigma-finite-adapted-process-order*
..

locale *finite-adapted-process-linorder* = *finite-adapted-process-order* $M\ F\ t_0\ X$ **for** $M\ F\ t_0$ **and** *X* :: - ⇒ - ⇒ - :: {*linorder-topology*}

locale *nat-finite-adapted-process-linorder* = *finite-adapted-process-linorder* *M F 0* :: *nat X* **for** *M F X*
locale *real-finite-adapted-process-linorder* = *finite-adapted-process-linorder* *M F 0* :: *real X* **for** *M F X*

sublocale *nat-finite-adapted-process-linorder* ⊆ *nat-sigma-finite-adapted-process-linorder*
..
sublocale *real-finite-adapted-process-linorder* ⊆ *real-sigma-finite-adapted-process-linorder*

..

7.2 Martingale

locale *martingale* = *sigma-finite-adapted-process* +
assumes *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable* $M\ (X\ i)$
and *martingale-property*: $\bigwedge i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow AE\ \xi\ in\ M.\ X\ i\ \xi =$ *cond-exp* $M\ (F\ i)\ (X\ j)\ \xi$

locale *martingale-order* = *martingale* $M\ F\ t_0\ X$ **for** $M\ F\ t_0$ **and** X :: - ⇒ - ⇒ - :: {*order-topology*, *ordered-real-vector*}
locale *martingale-linorder* = *martingale* $M\ F\ t_0\ X$ **for** $M\ F\ t_0$ **and** X :: - ⇒ - ⇒ - :: {*linorder-topology*, *ordered-real-vector*}
sublocale *martingale-linorder* ⊆ *martingale-order* **..**

lemma (**in** *sigma-finite-filtered-measure*) *martingale-const-fun*[*intro*]:
assumes *integrable* $M\ f\ f \in$ *borel-measurable* $(F\ t_0)$
shows *martingale* $M\ F\ t_0\ (\lambda\text{-}.\ f)$
using *assms sigma-finite-subalgebra.cond-exp-F-meas*[*OF - assms*(*1*), *THEN AE-symmetric*] *borel-measurable-mono*
by (*unfold-locales*) *blast*+

lemma (**in** *sigma-finite-filtered-measure*) *martingale-cond-exp*[*intro*]:
assumes *integrable* $M\ f$
shows *martingale* $M\ F\ t_0\ (\lambda i.$ *cond-exp* $M\ (F\ i)\ f)$
using *sigma-finite-subalgebra.borel-measurable-cond-exp′ borel-measurable-cond-exp*

by (*unfold-locales*) (*auto intro*: *sigma-finite-subalgebra.cond-exp-nested-subalg*[*OF - assms*] *simp add*: *subalgebra-F subalgebras*)

corollary (**in** *sigma-finite-filtered-measure*) *martingale-zero*[*intro*]: *martingale* $M\ F\ t_0\ (\lambda\text{- -}.\ 0)$ **by** *fastforce*

corollary (**in** *finite-filtered-measure*) *martingale-const*[*intro*]: *martingale* $M\ F\ t_0\ (\lambda\text{- -}.\ c)$ **by** *fastforce*

7.3 Submartingale

locale *submartingale* = *sigma-finite-adapted-process-order* +
assumes *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable* $M\ (X\ i)$
and *submartingale-property*: $\bigwedge i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow AE\ \xi\ in\ M.\ X\ i\ \xi \leq$ *cond-exp* $M\ (F\ i)\ (X\ j)\ \xi$

locale *submartingale-linorder* = *submartingale* $M\ F\ t_0\ X$ **for** $M\ F\ t_0$ **and** X :: - ⇒ - ⇒ - :: {*linorder-topology*}

sublocale *martingale-order* ⊆ *submartingale* **using** *martingale-property* **by** (*unfold-locales*) (*force simp add*: *integrable*)+
sublocale *martingale-linorder* ⊆ *submartingale-linorder* **..**

7.4 Supermartingale

locale *supermartingale* = *sigma-finite-adapted-process-order* +
assumes *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable* M $(X\ i)$
and *supermartingale-property*: $\bigwedge i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow AE\ \xi\ in\ M.\ X\ i\ \xi \geq$ *cond-exp* M $(F\ i)$ $(X\ j)$ ξ

locale *supermartingale-linorder* = *supermartingale* $M\ F\ t_0\ X$ **for** $M\ F\ t_0$ **and** X :: - $\Rightarrow$ - $\Rightarrow$ - :: {*linorder-topology*}

sublocale *martingale-order* $\subseteq$ *supermartingale* **using** *martingale-property* **by** (*unfold-locales*) (*force simp add*: *integrable*)+
sublocale *martingale-linorder* $\subseteq$ *supermartingale-linorder* **..**

A stochastic process is a martingale, if and only if it is both a submartingale and a supermartingale.

lemma *martingale-iff*:
shows *martingale* $M\ F\ t_0\ X \longleftrightarrow$ *submartingale* $M\ F\ t_0\ X \wedge$ *supermartingale* $M\ F\ t_0\ X$
proof (*rule iffI*)
assume *asm*: *martingale* $M\ F\ t_0\ X$
interpret *martingale-order* $M\ F\ t_0\ X$ **by** (*intro martingale-order.intro asm*)
show *submartingale* $M\ F\ t_0\ X \wedge$ *supermartingale* $M\ F\ t_0\ X$ **using** *submartingale-axioms supermartingale-axioms* **by** *blast*
next
assume *asm*: *submartingale* $M\ F\ t_0\ X \wedge$ *supermartingale* $M\ F\ t_0\ X$
interpret *submartingale* $M\ F\ t_0\ X$ **by** (*simp add*: *asm*)
interpret *supermartingale* $M\ F\ t_0\ X$ **by** (*simp add*: *asm*)
show *martingale* $M\ F\ t_0\ X$ **using** *submartingale-property supermartingale-property* **by** (*unfold-locales*) (*intro integrable*, *blast*, *force*)
qed

7.5 Martingale Lemmas

context *martingale*
begin

lemma *cond-exp-diff-eq-zero*:
assumes $t_0 \leq i$ $i \leq j$
shows $AE\ \xi\ in\ M.$ *cond-exp* M $(F\ i)$ $(\lambda\xi.\ X\ j\ \xi - X\ i\ \xi)\ \xi = 0$
using *martingale-property*[*OF assms*] *assms*
sigma-finite-subalgebra.cond-exp-F-meas[*OF - integrable adapted*, *of i*]
sigma-finite-subalgebra.cond-exp-diff[*OF - integrable*(*1*,*1*), *of F i j i*] **by** *fastforce*

lemma *set-integral-eq*:
assumes $A \in F\ i$ $t_0 \leq i$ $i \leq j$
shows *set-lebesgue-integral* $M\ A\ (X\ i) =$ *set-lebesgue-integral* $M\ A\ (X\ j)$
proof −
interpret *sigma-finite-subalgebra* $M\ F\ i$ **using** *assms*(*2*) **by** *blast*

have $\int x \in A.\ X\ i\ x\ \partial M = \int x \in A.$ *cond-exp* M $(F\ i)$ $(X\ j)$ $x\ \partial M$ **using** *martingale-property*[*OF assms*(*2*,*3*)] *borel-measurable-cond-exp'* *assms subalgebras subalgebra-def* **by** (*intro set-lebesgue-integral-cong-AE*[*OF - random-variable*]) *fastforce+*
also have ... $= \int x \in A.\ X\ j\ x\ \partial M$ **using** *assms* **by** (*auto simp*: *integrable intro*: *cond-exp-set-integral*[*symmetric*])
finally show *?thesis* **.**
qed

lemma *scaleR-const*[*intro*]:
shows *martingale* $M\ F\ t_0$ $(\lambda i\ x.\ c *_R X\ i\ x)$
proof $-$
{
fix $i\ j$ :: *'b* **assume** *asm*: $t_0 \le i\ i \le j$
interpret *sigma-finite-subalgebra* $M\ F\ i$ **using** *asm* **by** *blast*
have *AE* x *in* $M.\ c *_R X\ i\ x =$ *cond-exp* M $(F\ i)$ $(\lambda x.\ c *_R X\ j\ x)$ x **using** *asm cond-exp-scaleR-right*[*OF integrable*, *of j*, *THEN AE-symmetric*] *martingale-property*[*OF asm*] **by** *force*
}
thus *?thesis* **by** (*unfold-locales*) (*auto simp add*: *integrable martingale.integrable*)
qed

lemma *uminus*[*intro*]:
shows *martingale* $M\ F\ t_0$ $(-\ X)$
using *scaleR-const*[*of* -1] **by** (*force intro*: *back-subst*[*of martingale* $M\ F\ t_0$])

lemma *add*[*intro*]:
assumes *martingale* $M\ F\ t_0\ Y$
shows *martingale* $M\ F\ t_0$ $(\lambda i\ \xi.\ X\ i\ \xi + Y\ i\ \xi)$
proof $-$
interpret Y: *martingale* $M\ F\ t_0\ Y$ **by** (*rule assms*)
{
fix $i\ j$ :: *'b* **assume** *asm*: $t_0 \le i\ i \le j$
hence *AE* ξ *in* $M.\ X\ i\ \xi + Y\ i\ \xi =$ *cond-exp* M $(F\ i)$ $(\lambda x.\ X\ j\ x + Y\ j\ x)$ ξ
using *sigma-finite-subalgebra.cond-exp-add*[*OF - integrable martingale.integrable*[*OF assms*], *of F i j j*, *THEN AE-symmetric*]
martingale-property[*OF asm*] *martingale.martingale-property*[*OF assms asm*] **by** *force*
}
thus *?thesis* **using** *assms*
by (*unfold-locales*) (*auto simp add*: *integrable martingale.integrable*)
qed

lemma *diff*[*intro*]:
assumes *martingale* $M\ F\ t_0\ Y$
shows *martingale* $M\ F\ t_0$ $(\lambda i\ x.\ X\ i\ x - Y\ i\ x)$
proof $-$
interpret Y: *martingale* $M\ F\ t_0\ Y$ **by** (*rule assms*)
{

fix $i\ j$ :: $'b$ **assume** *asm*: $t_0 \leq i\ i \leq j$
hence $AE\ \xi\ in\ M.\ X\ i\ \xi - Y\ i\ \xi = cond\text{-}exp\ M\ (F\ i)\ (\lambda x.\ X\ j\ x - Y\ j\ x)\ \xi$
using *sigma-finite-subalgebra.cond-exp-diff*[*OF - integrable martingale.integrable*[*OF assms*], *of F i j j*, *THEN AE-symmetric*]
martingale-property[*OF asm*] *martingale.martingale-property*[*OF assms asm*] **by** *fastforce*
}
thus *?thesis* **using** *assms* **by** (*unfold-locales*) (*auto simp add*: *integrable martingale.integrable*)
qed

end

lemma (**in** *sigma-finite-adapted-process*) *martingale-of-cond-exp-diff-eq-zero*:
assumes *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow integrable\ M\ (X\ i)$
and *diff-zero*: $\bigwedge i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow AE\ x\ in\ M.\ cond\text{-}exp\ M\ (F\ i)\ (\lambda\xi.\ X\ j\ \xi - X\ i\ \xi)\ x = 0$
shows *martingale* $M\ F\ t_0\ X$
proof
{
fix $i\ j$ :: $'b$ **assume** *asm*: $t_0 \leq i\ i \leq j$
thus $AE\ \xi\ in\ M.\ X\ i\ \xi = cond\text{-}exp\ M\ (F\ i)\ (X\ j)\ \xi$
using *diff-zero*[*OF asm*] *sigma-finite-subalgebra.cond-exp-diff*[*OF - integrable*(*1*,*1*), *of F i j i*]
sigma-finite-subalgebra.cond-exp-F-meas[*OF - integrable adapted*, *of i*] **by** *fastforce*
}
qed (*intro integrable*)

lemma (**in** *sigma-finite-adapted-process*) *martingale-of-set-integral-eq*:
assumes *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow integrable\ M\ (X\ i)$
and $\bigwedge A\ i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow A \in F\ i \Longrightarrow set\text{-}lebesgue\text{-}integral\ M\ A\ (X\ i) = set\text{-}lebesgue\text{-}integral\ M\ A\ (X\ j)$
shows *martingale* $M\ F\ t_0\ X$
proof (*unfold-locales*)
fix $i\ j$ :: $'b$ **assume** *asm*: $t_0 \leq i\ i \leq j$
interpret *sigma-finite-subalgebra M F i* **using** *asm* **by** *blast*
interpret *r*: *sigma-finite-measure restr-to-subalg M* (*F i*) **by** (*simp add*: *sigma-fin-subalg*)
{
fix *A* **assume** $A \in restr\text{-}to\text{-}subalg\ M\ (F\ i)$
hence $*$: $A \in F\ i$ **using** *sets-restr-to-subalg subalgebras asm* **by** *blast*
have *set-lebesgue-integral* (*restr-to-subalg M* (*F i*)) *A* (*X i*) = *set-lebesgue-integral M A* (*X i*) **using** $*$ *subalg asm* **by** (*auto simp*: *set-lebesgue-integral-def intro*: *integral-subalgebra2 borel-measurable-scaleR adapted borel-measurable-indicator*)
also have ... = *set-lebesgue-integral M A* (*cond-exp M* (*F i*) (*X j*)) **using** $*$ *assms*(*2*)[*OF asm*] *cond-exp-set-integral*[*OF integrable*] *asm* **by** *auto*
finally have *set-lebesgue-integral* (*restr-to-subalg M* (*F i*)) *A* (*X i*) = *set-lebesgue-integral* (*restr-to-subalg M* (*F i*)) *A* (*cond-exp M* (*F i*) (*X j*)) **using** $*$ *subalg* **by** (*auto simp*: *set-lebesgue-integral-def intro*!: *integral-subalgebra2*[*symmetric*] *borel-measurable-scaleR*

borel-measurable-cond-exp borel-measurable-indicator)
}
hence *AE* ξ *in restr-to-subalg M* (*F i*). *X i* ξ = *cond-exp M* (*F i*) (*X j*) ξ **using** *asm* **by** (*intro r.density-unique-banach*, *auto intro*: *integrable-in-subalg subalg borel-measurable-cond-exp integrable*)
thus *AE* ξ *in M*. *X i* ξ = *cond-exp M* (*F i*) (*X j*) ξ **using** *AE-restr-to-subalg*[*OF subalg*] **by** *blast*
qed (*simp add*: *integrable*)

7.6 Submartingale Lemmas

context *submartingale*
begin

lemma *cond-exp-diff-nonneg*:
assumes $t_0 \leq i$ $i \leq j$
shows *AE x in M*. *cond-exp M* (*F i*) ($\lambda\xi$. *X j* ξ − *X i* ξ) $x \geq 0$
using *submartingale-property*[*OF assms*] *assms sigma-finite-subalgebra.cond-exp-diff*[*OF - integrable*(*1*,*1*), *of - j i*] *sigma-finite-subalgebra.cond-exp-F-meas*[*OF - integrable adapted*, *of i*] **by** *fastforce*

lemma *add*[*intro*]:
assumes *submartingale M F* t_0 *Y*
shows *submartingale M F* t_0 (λi ξ. *X i* ξ + *Y i* ξ)
proof −
interpret *Y*: *submartingale M F* t_0 *Y* **by** (*rule assms*)
{
fix *i j* :: *'b* **assume** *asm*: $t_0 \leq i$ $i \leq j$
hence *AE* ξ *in M*. *X i* ξ + *Y i* ξ ≤ *cond-exp M* (*F i*) (λx. *X j x* + *Y j x*) ξ
using *sigma-finite-subalgebra.cond-exp-add*[*OF - integrable submartingale.integrable*[*OF assms*], *of F i j j*]
submartingale-property[*OF asm*] *submartingale.submartingale-property*[*OF assms asm*] *add-mono*[*of X i - - Y i -*] **by** *force*
}
thus *?thesis* **using** *assms* **by** (*unfold-locales*) (*auto simp add*: *borel-measurable-add random-variable adapted integrable Y.random-variable Y.adapted submartingale.integrable*)

qed

lemma *diff*[*intro*]:
assumes *supermartingale M F* t_0 *Y*
shows *submartingale M F* t_0 (λi ξ. *X i* ξ − *Y i* ξ)
proof −
interpret *Y*: *supermartingale M F* t_0 *Y* **by** (*rule assms*)
{
fix *i j* :: *'b* **assume** *asm*: $t_0 \leq i$ $i \leq j$
hence *AE* ξ *in M*. *X i* ξ − *Y i* ξ ≤ *cond-exp M* (*F i*) (λx. *X j x* − *Y j x*) ξ
using *sigma-finite-subalgebra.cond-exp-diff*[*OF - integrable supermartingale.integrable*[*OF assms*], *of F i j j*]

submartingale-property[*OF asm*] *supermartingale.supermartingale-property*[*OF assms asm*] *diff-mono*[*of X i - - - Y i -*] **by** *force*
}
thus *?thesis* **using** *assms* **by** (*unfold-locales*) (*auto simp add*: *borel-measurable-diff random-variable adapted integrable Y.random-variable Y.adapted supermartingale.integrable*)

qed

lemma *scaleR-nonneg*:
assumes $c \geq 0$
shows *submartingale M F* t_0 ($\lambda i\ \xi.\ c *_R X\ i\ \xi$)
proof
{
fix *i j* :: *'b* **assume** *asm*: $t_0 \leq i$ $i \leq j$
thus *AE* ξ *in M*. $c *_R X\ i\ \xi \leq$ *cond-exp M* (*F i*) ($\lambda\xi.\ c *_R X\ j\ \xi$) ξ
using *sigma-finite-subalgebra.cond-exp-scaleR-right*[*OF - integrable*, *of F i j c*] *submartingale-property*[*OF asm*] **by** (*fastforce intro*!: *scaleR-left-mono*[*OF - assms*])
}
qed (*auto simp add*: *borel-measurable-integrable borel-measurable-scaleR integrable random-variable adapted borel-measurable-const-scaleR*)

lemma *scaleR-le-zero*:
assumes $c \leq 0$
shows *supermartingale M F* t_0 ($\lambda i\ \xi.\ c *_R X\ i\ \xi$)
proof
{
fix *i j* :: *'b* **assume** *asm*: $t_0 \leq i$ $i \leq j$
thus *AE* ξ *in M*. $c *_R X\ i\ \xi \geq$ *cond-exp M* (*F i*) ($\lambda\xi.\ c *_R X\ j\ \xi$) ξ
using *sigma-finite-subalgebra.cond-exp-scaleR-right*[*OF - integrable*, *of F i j c*] *submartingale-property*[*OF asm*]
by (*fastforce intro*!: *scaleR-left-mono-neg*[*OF - assms*])
}
qed (*auto simp add*: *borel-measurable-integrable borel-measurable-scaleR integrable random-variable adapted borel-measurable-const-scaleR*)

lemma *uminus*[*intro*]:
shows *supermartingale M F* t_0 ($-$ *X*)
unfolding *fun-Compl-def* **using** *scaleR-le-zero*[*of* -1] **by** *simp*

end

context *submartingale-linorder*
begin

lemma *set-integral-le*:
assumes $A \in F\ i$ $t_0 \leq i$ $i \leq j$
shows *set-lebesgue-integral M A* (*X i*) $\leq$ *set-lebesgue-integral M A* (*X j*)
using *submartingale-property*[*OF assms*(*2*), *of j*] *assms subalgebras*

by (*subst sigma-finite-subalgebra.cond-exp-set-integral*[*OF* - *integrable assms*(*1*), *of j*])
(*auto intro*!: *scaleR-left-mono integral-mono-AE-banach integrable-mult-indicator integrable simp add*: *subalgebra-def set-lebesgue-integral-def*)

lemma *max*:
assumes *submartingale-linorder M F* t_0 *Y*
shows *submartingale-linorder M F* t_0 ($\lambda i\ \xi.\ max\ (X\ i\ \xi)\ (Y\ i\ \xi)$)
proof (*unfold-locales*)
interpret *Y*: *submartingale-linorder M F* t_0 *Y* **by** (*rule assms*)
{
fix *i j* :: *'b* **assume** *asm*: $t_0 \leq i$ $i \leq j$
have *AE* ξ *in M*. *max* ($X\ i\ \xi$) ($Y\ i\ \xi$) $\leq$ *max* (*cond-exp M* (*F i*) (*X j*) ξ) (*cond-exp M* (*F i*) (*Y j*) ξ) **using** *submartingale-property Y.submartingale-property asm* **unfolding** *max-def* **by** *fastforce*
thus *AE* ξ *in M*. *max* ($X\ i\ \xi$) ($Y\ i\ \xi$) $\leq$ *cond-exp M* (*F i*) ($\lambda\xi.\ max\ (X\ j\ \xi)\ (Y\ j\ \xi)$) ξ **using** *sigma-finite-subalgebra.cond-exp-max*[*OF* - *integrable Y.integrable*, *of F i j j*] *asm* **by** (*fast intro*: *order.trans*)
}
show $\bigwedge i.\ t_0 \leq i \Longrightarrow (\lambda\xi.\ max\ (X\ i\ \xi)\ (Y\ i\ \xi)) \in$ *borel-measurable* (*F i*) $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable M* ($\lambda\xi.\ max\ (X\ i\ \xi)\ (Y\ i\ \xi)$) **by** (*force intro*: *Y.integrable integrable assms*)+
qed

lemma *max-0*:
shows *submartingale-linorder M F* t_0 ($\lambda i\ \xi.\ max\ 0\ (X\ i\ \xi)$)
proof −
interpret *zero*: *martingale-linorder M F* t_0 λ- -. *0* **by** (*force intro*: *martingale-linorder.intro martingale-order.intro*)
show *?thesis* **by** (*intro zero.max submartingale-linorder.intro submartingale-axioms*)
qed

end

lemma (**in** *sigma-finite-adapted-process-order*) *submartingale-of-cond-exp-diff-nonneg*:
assumes *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable M* (*X i*)
and *diff-nonneg*: $\bigwedge i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow$ *AE x in M*. *cond-exp M* (*F i*) ($\lambda\xi.\ X\ j\ \xi - X\ i\ \xi$) $x \geq 0$
shows *submartingale M F* t_0 *X*
proof (*unfold-locales*)
{
fix *i j* :: *'b* **assume** *asm*: $t_0 \leq i$ $i \leq j$
thus *AE* ξ *in M*. $X\ i\ \xi \leq$ *cond-exp M* (*F i*) (*X j*) ξ
using *diff-nonneg*[*OF asm*] *sigma-finite-subalgebra.cond-exp-diff*[*OF* - *integrable*(*1*,*1*), *of F i j i*]
sigma-finite-subalgebra.cond-exp-F-meas[*OF* - *integrable adapted*, *of i*] **by** *fastforce*
}
qed (*intro integrable*)

lemma (**in** *sigma-finite-adapted-process-linorder*) *submartingale-of-set-integral-le*:
assumes *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable* M $(X\ i)$
and $\bigwedge A\ i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow A \in F\ i \Longrightarrow$ *set-lebesgue-integral* M A $(X\ i) \leq$ *set-lebesgue-integral* M A $(X\ j)$
shows *submartingale* M F t_0 X
proof (*unfold-locales*)
{
fix $i\ j :: 'b$ **assume** *asm*: $t_0 \leq i\ i \leq j$
interpret *r*: *sigma-finite-measure restr-to-subalg* M $(F\ i)$ **using** *asm sigma-finite-subalgebra.sigma-fin-subalg* **by** *blast*
{
fix A **assume** $A \in$ *restr-to-subalg* M $(F\ i)$
hence $*$: $A \in F\ i$ **using** *asm sets-restr-to-subalg subalgebras* **by** *blast*
have *set-lebesgue-integral* (*restr-to-subalg* M $(F\ i)$) A $(X\ i)$ = *set-lebesgue-integral* M A $(X\ i)$ **using** $*$ *asm subalgebras* **by** (*auto simp*: *set-lebesgue-integral-def intro*: *integral-subalgebra2 borel-measurable-scaleR adapted borel-measurable-indicator*)
also have ... $\leq$ *set-lebesgue-integral* M A (*cond-exp* M $(F\ i)$ $(X\ j)$) **using** $*$ *assms*(*2*)[*OF asm*] *asm sigma-finite-subalgebra.cond-exp-set-integral*[*OF - integrable*] **by** *fastforce*
also have ... = *set-lebesgue-integral* (*restr-to-subalg* M $(F\ i)$) A (*cond-exp* M $(F\ i)$ $(X\ j)$) **using** $*$ *asm subalgebras* **by** (*auto simp*: *set-lebesgue-integral-def intro*!: *integral-subalgebra2*[*symmetric*] *borel-measurable-scaleR borel-measurable-cond-exp borel-measurable-indicator*)
finally have *0* $\leq$ *set-lebesgue-integral* (*restr-to-subalg* M $(F\ i)$) A ($\lambda\xi$. *cond-exp* M $(F\ i)$ $(X\ j)$ ξ $-$ $X\ i\ \xi$) **using** $*$ *asm subalgebras* **by** (*subst set-integral-diff*, *auto simp add*: *set-integrable-def sets-restr-to-subalg intro*!: *integrable adapted integrable-in-subalg borel-measurable-scaleR borel-measurable-indicator borel-measurable-cond-exp integrable-mult-indicator*)
}
hence *AE* ξ *in restr-to-subalg* M $(F\ i)$. *0* $\leq$ *cond-exp* M $(F\ i)$ $(X\ j)$ ξ $-$ $X\ i\ \xi$
by (*intro r.density-nonneg integrable-in-subalg asm subalgebras borel-measurable-diff borel-measurable-cond-exp adapted Bochner-Integration.integrable-diff integrable-cond-exp integrable*)
thus *AE* ξ *in* M. $X\ i\ \xi \leq$ *cond-exp* M $(F\ i)$ $(X\ j)$ ξ **using** *AE-restr-to-subalg*[*OF subalgebras*] *asm* **by** *simp*
}
qed (*intro integrable*)

7.7 Supermartingale Lemmas

The following lemmas are exact duals of the ones for submartingales.

context *supermartingale*
begin

lemma *cond-exp-diff-nonneg*:
assumes $t_0 \leq i\ i \leq j$
shows *AE* x *in* M. *cond-exp* M $(F\ i)$ ($\lambda\xi$. $X\ i\ \xi - X\ j\ \xi$) $x \geq$ *0*
using *assms supermartingale-property*[*OF assms*] *sigma-finite-subalgebra.cond-exp-diff*[*OF*

- *integrable*(*1*,*1*), *of F i i j*]
sigma-finite-subalgebra.cond-exp-F-meas[*OF - integrable adapted*, *of i*] **by** *fastforce*

lemma *add*[*intro*]:
assumes *supermartingale M F* t_0 *Y*
shows *supermartingale M F* t_0 ($\lambda i\ \xi.\ X\ i\ \xi + Y\ i\ \xi$)
proof −
interpret *Y*: *supermartingale M F* t_0 *Y* **by** (*rule assms*)
{
fix *i j* :: *'b* **assume** *asm*: $t_0 \leq i$ $i \leq j$
hence *AE* ξ *in M*. $X\ i\ \xi + Y\ i\ \xi \geq$ *cond-exp M* (*F i*) ($\lambda x.\ X\ j\ x + Y\ j\ x$) ξ
using *sigma-finite-subalgebra.cond-exp-add*[*OF - integrable supermartingale.integrable*[*OF assms*], *of F i j j*]
supermartingale-property[*OF asm*] *supermartingale.supermartingale-property*[*OF assms asm*] *add-mono*[*of - X i - - Y i -*] **by** *force*
}
thus *?thesis* **using** *assms* **by** (*unfold-locales*) (*auto simp add*: *borel-measurable-add random-variable adapted integrable Y.random-variable Y.adapted supermartingale.integrable*)

qed

lemma *diff*[*intro*]:
assumes *submartingale M F* t_0 *Y*
shows *supermartingale M F* t_0 ($\lambda i\ \xi.\ X\ i\ \xi - Y\ i\ \xi$)
proof −
interpret *Y*: *submartingale M F* t_0 *Y* **by** (*rule assms*)
{
fix *i j* :: *'b* **assume** *asm*: $t_0 \leq i$ $i \leq j$
hence *AE* ξ *in M*. $X\ i\ \xi - Y\ i\ \xi \geq$ *cond-exp M* (*F i*) ($\lambda x.\ X\ j\ x - Y\ j\ x$) ξ
using *sigma-finite-subalgebra.cond-exp-diff*[*OF - integrable submartingale.integrable*[*OF assms*], *of F i j j*, *unfolded fun-diff-def*]
supermartingale-property[*OF asm*] *submartingale.submartingale-property*[*OF assms asm*] *diff-mono*[*of - X i - Y i -*] **by** *force*
}
thus *?thesis* **using** *assms* **by** (*unfold-locales*) (*auto simp add*: *borel-measurable-diff random-variable adapted integrable Y.random-variable Y.adapted submartingale.integrable*)

qed

lemma *scaleR-nonneg*:
assumes $c \geq 0$
shows *supermartingale M F* t_0 ($\lambda i\ \xi.\ c *_R X\ i\ \xi$)
proof
{
fix *i j* :: *'b* **assume** *asm*: $t_0 \leq i$ $i \leq j$
thus *AE* ξ *in M*. $c *_R X\ i\ \xi \geq$ *cond-exp M* (*F i*) ($\lambda\xi.\ c *_R X\ j\ \xi$) ξ
using *sigma-finite-subalgebra.cond-exp-scaleR-right*[*OF - integrable*, *of F i j c*] *supermartingale-property*[*OF asm*] **by** (*fastforce intro*!: *scaleR-left-mono*[*OF -*

assms])
  }
qed (*auto simp add*: *borel-measurable-integrable borel-measurable-scaleR integrable random-variable adapted borel-measurable-const-scaleR*)

lemma *scaleR-le-zero*:
  assumes $c \leq 0$
  shows *submartingale* M F t_0 ($\lambda i\ \xi.\ c *_R X\ i\ \xi$)
proof
  {
    fix i j :: $'b$ **assume** *asm*: $t_0 \leq i$ $i \leq j$
    thus $AE\ \xi$ *in* $M.\ c *_R X\ i\ \xi \leq$ *cond-exp* M ($F\ i$) ($\lambda\xi.\ c *_R X\ j\ \xi$) ξ
      using *sigma-finite-subalgebra.cond-exp-scaleR-right*[*OF - integrable*, *of F i j c*] *supermartingale-property*[*OF asm*] **by** (*fastforce intro*!: *scaleR-left-mono-neg*[*OF - assms*])
  }
qed (*auto simp add*: *borel-measurable-integrable borel-measurable-scaleR integrable random-variable adapted borel-measurable-const-scaleR*)

lemma *uminus*[*intro*]:
  shows *submartingale* M F t_0 ($-$ X)
  unfolding *fun-Compl-def* **using** *scaleR-le-zero*[*of* -1] **by** *simp*

end

context *supermartingale-linorder*
begin

lemma *set-integral-ge*:
  assumes $A \in F\ i$ $t_0 \leq i$ $i \leq j$
  shows *set-lebesgue-integral* M A ($X\ i$) $\geq$ *set-lebesgue-integral* M A ($X\ j$)
  using *supermartingale-property*[*OF assms*(*2*), *of j*] *assms subalgebras*
  by (*subst sigma-finite-subalgebra.cond-exp-set-integral*[*OF - integrable assms*(*1*), *of j*])
    (*auto intro*!: *scaleR-left-mono integral-mono-AE-banach integrable-mult-indicator integrable simp add*: *subalgebra-def set-lebesgue-integral-def*)

lemma *min*:
  assumes *supermartingale-linorder* M F t_0 Y
  shows *supermartingale-linorder* M F t_0 ($\lambda i\ \xi.$ *min* ($X\ i\ \xi$) ($Y\ i\ \xi$))
proof (*unfold-locales*)
  interpret Y: *supermartingale-linorder* M F t_0 Y **by** (*rule assms*)
  {
    fix i j :: $'b$ **assume** *asm*: $t_0 \leq i$ $i \leq j$
    have $AE\ \xi$ *in* M. *min* ($X\ i\ \xi$) ($Y\ i\ \xi$) $\geq$ *min* (*cond-exp* M ($F\ i$) ($X\ j$) ξ) (*cond-exp* M ($F\ i$) ($Y\ j$) ξ) **using** *supermartingale-property Y.supermartingale-property asm* **unfolding** *min-def* **by** *fastforce*
    thus $AE\ \xi$ *in* M. *min* ($X\ i\ \xi$) ($Y\ i\ \xi$) $\geq$ *cond-exp* M ($F\ i$) ($\lambda\xi.$ *min* ($X\ j\ \xi$) ($Y\ j\ \xi$)) ξ **using** *sigma-finite-subalgebra.cond-exp-min*[*OF - integrable Y.integrable*, *of*

$F\ i\ j\ j]$ *asm* **by** (*fast intro*: *order.trans*)
}
show $\bigwedge i.\ t_0 \leq i \Longrightarrow (\lambda\xi.\ min\ (X\ i\ \xi)\ (Y\ i\ \xi)) \in$ *borel-measurable* $(F\ i)$ $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable* M $(\lambda\xi.\ min\ (X\ i\ \xi)\ (Y\ i\ \xi))$ **by** (*force intro*: *Y.integrable integrable assms*)+
qed

lemma *min-0*:
shows *supermartingale-linorder* $M\ F\ t_0\ (\lambda i\ \xi.\ min\ 0\ (X\ i\ \xi))$
proof −
interpret *zero*: *martingale-linorder* $M\ F\ t_0\ \lambda$- -. *0* **by** (*force intro*: *martingale-linorder.intro*)
show *?thesis* **by** (*intro zero.min supermartingale-linorder.intro supermartingale-axioms*)
qed

end

lemma (**in** *sigma-finite-adapted-process-order*) *supermartingale-of-cond-exp-diff-le-zero*:
assumes *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable* $M\ (X\ i)$
and *diff-le-zero*: $\bigwedge i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow$ *AE x in M. cond-exp* $M\ (F\ i)\ (\lambda\xi.\ X\ j\ \xi - X\ i\ \xi)\ x \leq 0$
shows *supermartingale* $M\ F\ t_0\ X$
proof
{
fix $i\ j :: 'b$ **assume** *asm*: $t_0 \leq i\ i \leq j$
thus *AE* ξ *in M.* $X\ i\ \xi \geq$ *cond-exp* $M\ (F\ i)\ (X\ j)\ \xi$
using *diff-le-zero*[*OF asm*] *sigma-finite-subalgebra.cond-exp-diff*[*OF - integrable*(*1*,*1*), *of F i j i*]
sigma-finite-subalgebra.cond-exp-F-meas[*OF - integrable adapted*, *of i*] **by** *fastforce*
}
qed (*intro integrable*)

lemma (**in** *sigma-finite-adapted-process-linorder*) *supermartingale-of-set-integral-ge*:
assumes *integrable*: $\bigwedge i.\ t_0 \leq i \Longrightarrow$ *integrable* $M\ (X\ i)$
and $\bigwedge A\ i\ j.\ t_0 \leq i \Longrightarrow i \leq j \Longrightarrow A \in F\ i \Longrightarrow$ *set-lebesgue-integral* $M\ A\ (X\ j) \leq$ *set-lebesgue-integral* $M\ A\ (X\ i)$
shows *supermartingale* $M\ F\ t_0\ X$
proof −
interpret -: *adapted-process* $M\ F\ t_0\ -X$ **by** (*rule uminus-adapted*)
interpret *uminus-X*: *sigma-finite-adapted-process-linorder* $M\ F\ t_0\ -X$ **..**
note $* =$ *set-integral-uminus*[*unfolded set-integrable-def*, *OF integrable-mult-indicator*[*OF - integrable*]]
have *supermartingale* $M\ F\ t_0\ (-(-\ X))$
using *ord-eq-le-trans*[*OF* $*$ *ord-le-eq-trans*[*OF le-imp-neg-le*[*OF assms*(*2*)] $*$[*symmetric*]]] *subalgebras*
by (*intro submartingale.uminus uminus-X.submartingale-of-set-integral-le*)
(*clarsimp simp add*: *fun-Compl-def subalgebra-def integrable* | *fastforce*)+

thus *?thesis* **unfolding** *fun-Compl-def* **by** *simp*
qed

7.8 Discrete Time Martingales

locale *nat-martingale* = *martingale M F 0* :: *nat X* **for** *M F X*
locale *nat-submartingale* = *submartingale M F 0* :: *nat X* **for** *M F X*
locale *nat-supermartingale* = *supermartingale M F 0* :: *nat X* **for** *M F X*

locale *nat-submartingale-linorder* = *submartingale-linorder M F 0* :: *nat X* **for** *M F X*
locale *nat-supermartingale-linorder* = *supermartingale-linorder M F 0* :: *nat X* **for** *M F X*

sublocale *nat-submartingale-linorder* ⊆ *nat-submartingale* **..**
sublocale *nat-supermartingale-linorder* ⊆ *nat-supermartingale* **..**

lemma (**in** *nat-martingale*) *predictable-const*:
  assumes *nat-predictable-process M F X*
  shows *AE ξ in M. X i ξ = X j ξ*
proof −
  have ∗: *AE ξ in M. X i ξ = X 0 ξ* **for** *i*
  proof (*induction i*)
    case *0*
    then show *?case* **by** (*simp add*: *bot-nat-def*)
  next
    case (*Suc i*)
   interpret *S*: *nat-adapted-process M F λi. X* (*Suc i*) **by** (*intro nat-predictable-process.adapted-Suc assms*)
    show *?case* **using** *Suc S.adapted*[*of i*] *martingale-property*[*OF - le-SucI*, *of i*] *sigma-finite-subalgebra.cond-exp-F-meas*[*OF - integrable*, *of F i Suc i*] **by** *fastforce*
  qed
  show *?thesis* **using** ∗[*of i*] ∗[*of j*] **by** *force*
qed

lemma (**in** *nat-sigma-finite-adapted-process*) *martingale-of-set-integral-eq-Suc*:
  assumes *integrable*: ⋀*i. integrable M* (*X i*)
      and ⋀*A i. A* ∈ *F i* ⟹ *set-lebesgue-integral M A* (*X i*) = *set-lebesgue-integral M A* (*X* (*Suc i*))
    shows *nat-martingale M F X*
proof (*intro nat-martingale.intro martingale-of-set-integral-eq*)
  fix *i j A* **assume** *asm*: *i* ≤ *j A* ∈ *sets* (*F i*)
  show *set-lebesgue-integral M A* (*X i*) = *set-lebesgue-integral M A* (*X j*) **using** *asm*
  proof (*induction j* − *i arbitrary*: *i j*)
    case *0*
    then show *?case* **using** *asm* **by** *simp*
  next
    case (*Suc n*)

hence $*$: $n = j - Suc\ i$ **by** *linarith*
have $Suc\ i \le j$ **using** $Suc(2,3)$ **by** *linarith*
thus *?case* **using** *sets-F-mono*[*OF - le-SucI*] *Suc*(*4*) *Suc*(*1*)[*OF* $*$] **by** (*auto intro*: *assms*(*2*)[*THEN trans*])
qed
qed (*simp add*: *integrable*)

lemma (**in** *nat-sigma-finite-adapted-process*) *martingale-nat*:
assumes *integrable*: $\bigwedge i.$ *integrable M* ($X\ i$)
and $\bigwedge i.$ $AE\ \xi\ in\ M.\ X\ i\ \xi = cond\text{-}exp\ M\ (F\ i)\ (X\ (Suc\ i))\ \xi$
shows *nat-martingale M F X*
proof (*unfold-locales*)
fix $i\ j :: nat$ **assume** *asm*: $i \le j$
show $AE\ \xi\ in\ M.\ X\ i\ \xi = cond\text{-}exp\ M\ (F\ i)\ (X\ j)\ \xi$ **using** *asm*
proof (*induction* $j - i$ *arbitrary*: *i j*)
case *0*
hence $j = i$ **by** *simp*
thus *?case* **using** *sigma-finite-subalgebra.cond-exp-F-meas*[*OF - integrable adapted*, *THEN AE-symmetric*] **by** *blast*
next
case (*Suc n*)
have *j*: $j = Suc\ (n + i)$ **using** *Suc* **by** *linarith*
have *n*: $n = n + i - i$ **using** *Suc* **by** *linarith*
have $*$: $AE\ \xi\ in\ M.\ cond\text{-}exp\ M\ (F\ (n + i))\ (X\ j)\ \xi = X\ (n + i)\ \xi$ **unfolding** *j* **using** *assms*(*2*)[*THEN AE-symmetric*] **by** *blast*
have $AE\ \xi\ in\ M.\ cond\text{-}exp\ M\ (F\ i)\ (X\ j)\ \xi = cond\text{-}exp\ M\ (F\ i)\ (cond\text{-}exp\ M\ (F\ (n + i))\ (X\ j))\ \xi$ **by** (*intro cond-exp-nested-subalg integrable subalg*, *simp add*: *subalgebra-def sets-F-mono*)
hence $AE\ \xi\ in\ M.\ cond\text{-}exp\ M\ (F\ i)\ (X\ j)\ \xi = cond\text{-}exp\ M\ (F\ i)\ (X\ (n + i))\ \xi$ **using** *cond-exp-cong-AE*[*OF integrable-cond-exp integrable* $*$] **by** *force*
thus *?case* **using** *Suc*(*1*)[*OF n*] **by** *fastforce*
qed
qed (*simp add*: *integrable*)

lemma (**in** *nat-sigma-finite-adapted-process*) *martingale-of-cond-exp-diff-Suc-eq-zero*:
assumes *integrable*: $\bigwedge i.$ *integrable M* ($X\ i$)
and $\bigwedge i.$ $AE\ \xi\ in\ M.\ cond\text{-}exp\ M\ (F\ i)\ (\lambda\xi.\ X\ (Suc\ i)\ \xi - X\ i\ \xi)\ \xi = 0$
shows *nat-martingale M F X*
proof (*intro martingale-nat integrable*)
fix *i*
show $AE\ \xi\ in\ M.\ X\ i\ \xi = cond\text{-}exp\ M\ (F\ i)\ (X\ (Suc\ i))\ \xi$ **using** *cond-exp-diff*[*OF integrable*(*1*,*1*), *of i Suc i i*] *cond-exp-F-meas*[*OF integrable adapted*, *of i*] *assms*(*2*)[*of i*] **by** *fastforce*
qed

7.9 Discrete Time Submartingales

lemma (**in** *nat-submartingale*) *predictable-mono*:
assumes *nat-predictable-process M F X* $i \le j$

shows *AE ξ in M. X i ξ ≤ X j ξ*
using *assms(2)*
proof (*induction j − i arbitrary: i j*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
hence ∗: *n = j − Suc i* **by** *linarith*
interpret *S: nat-adapted-process M F λi. X (Suc i)* **by** (*intro nat-predictable-process.adapted-Suc assms*)
have *Suc i ≤ j* **using** *Suc(2,3)* **by** *linarith*
thus *?case* **using** *Suc(1)[OF ∗] S.adapted[of i] submartingale-property[OF - le-SucI, of i] sigma-finite-subalgebra.cond-exp-F-meas[OF - integrable, of F i Suc i]* **by** *fastforce*
qed

lemma (**in** *nat-sigma-finite-adapted-process-linorder*) *submartingale-of-set-integral-le-Suc*:
assumes *integrable*: ⋀*i. integrable M (X i)*
and ⋀*A i. A ∈ F i ⟹ set-lebesgue-integral M A (X i) ≤ set-lebesgue-integral M A (X (Suc i))*
shows *nat-submartingale M F X*
proof (*intro nat-submartingale.intro submartingale-of-set-integral-le*)
fix *i j A* **assume** *asm*: *i ≤ j A ∈ sets (F i)*
show *set-lebesgue-integral M A (X i) ≤ set-lebesgue-integral M A (X j)* **using** *asm*
proof (*induction j − i arbitrary: i j*)
case *0*
then show *?case* **using** *asm* **by** *simp*
next
case (*Suc n*)
hence ∗: *n = j − Suc i* **by** *linarith*
have *Suc i ≤ j* **using** *Suc(2,3)* **by** *linarith*
thus *?case* **using** *sets-F-mono[OF - le-SucI] Suc(4) Suc(1)[OF ∗]* **by** (*auto intro: assms(2)[THEN order-trans]*)
qed
qed (*simp add: integrable*)

lemma (**in** *nat-sigma-finite-adapted-process-linorder*) *submartingale-nat*:
assumes *integrable*: ⋀*i. integrable M (X i)*
and ⋀*i. AE ξ in M. X i ξ ≤ cond-exp M (F i) (X (Suc i)) ξ*
shows *nat-submartingale M F X*
using *subalg integrable assms(2)*
by (*intro submartingale-of-set-integral-le-Suc ord-le-eq-trans[OF set-integral-mono-AE-banach cond-exp-set-integral[symmetric]], simp*)
(*meson in-mono integrable-mult-indicator set-integrable-def subalgebra-def, meson integrable-cond-exp in-mono integrable-mult-indicator set-integrable-def subalgebra-def, fast+*)

lemma (**in** *nat-sigma-finite-adapted-process-linorder*) *submartingale-of-cond-exp-diff-Suc-nonneg*:

assumes *integrable*: $\bigwedge i.$ *integrable* M $(X\ i)$
and $\bigwedge i.$ *AE* ξ *in* $M.$ *cond-exp* M $(F\ i)$ $(\lambda\xi.\ X\ (Suc\ i)\ \xi - X\ i\ \xi)\ \xi \geq 0$
shows *nat-submartingale* $M\ F\ X$
proof (*intro submartingale-nat integrable*)
fix *i*
show *AE* ξ *in* $M.$ $X\ i\ \xi \leq$ *cond-exp* M $(F\ i)$ $(X\ (Suc\ i))\ \xi$ **using** *cond-exp-diff*[*OF integrable*(*1*,*1*), *of i Suc i i*] *cond-exp-F-meas*[*OF integrable adapted*, *of i*] *assms*(*2*)[*of i*] **by** *fastforce*
qed

lemma (**in** *nat-submartingale-linorder*) *partial-sum-scaleR*:
assumes *nat-adapted-process* $M\ F\ C$ $\bigwedge i.$ *AE* ξ *in* $M.$ $0 \leq C\ i\ \xi$ $\bigwedge i.$ *AE* ξ *in* $M.$ $C\ i\ \xi \leq R$
shows *nat-submartingale* $M\ F$ $(\lambda n\ \xi.\ \sum i{<}n.\ C\ i\ \xi *_R (X\ (Suc\ i)\ \xi - X\ i\ \xi))$
proof−
interpret *C*: *nat-adapted-process* $M\ F\ C$ **by** (*rule assms*)
interpret C': *nat-adapted-process* $M\ F$ $\lambda i\ \xi.\ C\ (i - 1)\ \xi *_R (X\ i\ \xi - X\ (i - 1)\ \xi)$ **by** (*intro nat-adapted-process.intro adapted-process.scaleR-right-adapted adapted-process.diff-adapted*, *unfold-locales*) (*auto intro*: *adaptedD C.adaptedD*)+
interpret C'': *nat-adapted-process* $M\ F$ $\lambda n\ \xi.\ \sum i{<}n.\ C\ i\ \xi *_R (X\ (Suc\ i)\ \xi - X\ i\ \xi)$ **by** (*rule* C'*.partial-sum-Suc-adapted*[*unfolded diff-Suc-1*])
interpret *S*: *nat-sigma-finite-adapted-process-linorder* $M\ F$ $(\lambda n\ \xi.\ \sum i{<}n.\ C\ i\ \xi *_R (X\ (Suc\ i)\ \xi - X\ i\ \xi))$ **..**
have *integrable* M $(\lambda x.\ C\ i\ x *_R (X\ (Suc\ i)\ x - X\ i\ x))$ **for** *i* **using** *assms*(*2*,*3*)[*of i*] **by** (*intro Bochner-Integration.integrable-bound*[*OF integrable-scaleR-right*, *OF Bochner-Integration.integrable-diff*, *OF integrable*(*1*,*1*), *of R Suc i i*]) (*auto simp add*: *mult-mono*)
moreover have *AE* ξ *in* $M.$ $0 \leq$ *cond-exp* M $(F\ i)$ $(\lambda\xi.\ (\sum i{<}Suc\ i.\ C\ i\ \xi *_R (X\ (Suc\ i)\ \xi - X\ i\ \xi)) - (\sum i{<}i.\ C\ i\ \xi *_R (X\ (Suc\ i)\ \xi - X\ i\ \xi)))\ \xi$ **for** *i*
using *sigma-finite-subalgebra.cond-exp-measurable-scaleR*[*OF - calculation - C.adapted*, *of i*]
cond-exp-diff-nonneg[*OF - le-SucI*, *OF - order.refl*, *of i*] *assms*(*2*,*3*)[*of i*]
by (*fastforce simp add*: *scaleR-nonneg-nonneg integrable*)
ultimately show *?thesis* **by** (*intro S.submartingale-of-cond-exp-diff-Suc-nonneg Bochner-Integration.integrable-sum*, *blast*+)
qed

lemma (**in** *nat-submartingale-linorder*) *partial-sum-scaleR'*:
assumes *nat-predictable-process* $M\ F\ C$ $\bigwedge i.$ *AE* ξ *in* $M.$ $0 \leq C\ i\ \xi$ $\bigwedge i.$ *AE* ξ *in* $M.$ $C\ i\ \xi \leq R$
shows *nat-submartingale* $M\ F$ $(\lambda n\ \xi.\ \sum i{<}n.\ C\ (Suc\ i)\ \xi *_R (X\ (Suc\ i)\ \xi - X\ i\ \xi))$
proof −
interpret *C*: *nat-predictable-process* $M\ F\ C$ **by** (*rule assms*)
interpret *Suc-C*: *nat-adapted-process* $M\ F$ $\lambda i.\ C\ (Suc\ i)$ **using** *C.adapted-Suc* **.**
show *?thesis* **by** (*intro partial-sum-scaleR*[*of - R*] *assms*) (*intro-locales*)
qed

7.10 Discrete Time Supermartingales

lemma (**in** *nat-supermartingale*) *predictable-mono*:
assumes *nat-predictable-process* $M\ F\ X\ i \leq j$
shows $AE\ \xi\ in\ M.\ X\ i\ \xi \geq X\ j\ \xi$
using *assms*(*2*)
proof (*induction* $j - i$ *arbitrary*: *i j*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
hence $*$: $n = j - Suc\ i$ **by** *linarith*
interpret *S*: *nat-adapted-process* $M\ F\ \lambda i.\ X\ (Suc\ i)$ **by** (*intro nat-predictable-process.adapted-Suc assms*)
have $Suc\ i \leq j$ **using** *Suc*(*2*,*3*) **by** *linarith*
thus *?case* **using** *Suc*(*1*)[*OF* $*$] *S.adapted*[*of i*] *supermartingale-property*[*OF - le-SucI*, *of i*] *sigma-finite-subalgebra.cond-exp-F-meas*[*OF - integrable*, *of F i Suc i*] **by** *fastforce*
qed

lemma (**in** *nat-sigma-finite-adapted-process-linorder*) *supermartingale-of-set-integral-ge-Suc*:
assumes *integrable*: $\bigwedge i.$ *integrable* $M\ (X\ i)$
and $\bigwedge A\ i.\ A \in F\ i \Longrightarrow$ *set-lebesgue-integral* $M\ A\ (X\ i) \geq$ *set-lebesgue-integral* $M\ A\ (X\ (Suc\ i))$
shows *nat-supermartingale* $M\ F\ X$
proof −
interpret -: *adapted-process* $M\ F\ 0\ -X$ **by** (*rule uminus-adapted*)
interpret *uminus-X*: *nat-sigma-finite-adapted-process-linorder* $M\ F\ -X$ **..**
note $*$ = *set-integral-uminus*[*unfolded set-integrable-def*, *OF integrable-mult-indicator*[*OF - integrable*]]
have *nat-supermartingale* $M\ F\ (-(-\ X))$
using *ord-eq-le-trans*[*OF* $*$ *ord-le-eq-trans*[*OF le-imp-neg-le*[*OF assms*(*2*)] $*$[*symmetric*]]] *subalgebras*
by (*intro nat-supermartingale.intro submartingale.uminus nat-submartingale.axioms uminus-X.submartingale-of-set-integral-le-Suc*)
(*clarsimp simp add*: *fun-Compl-def subalgebra-def integrable* | *fastforce*)+
thus *?thesis* **unfolding** *fun-Compl-def* **by** *simp*
qed

lemma (**in** *nat-sigma-finite-adapted-process-linorder*) *supermartingale-nat*:
assumes *integrable*: $\bigwedge i.$ *integrable* $M\ (X\ i)$
and $\bigwedge i.\ AE\ \xi\ in\ M.\ X\ i\ \xi \geq$ *cond-exp* $M\ (F\ i)\ (X\ (Suc\ i))\ \xi$
shows *nat-supermartingale* $M\ F\ X$
proof −
interpret -: *adapted-process* $M\ F\ 0\ -X$ **by** (*rule uminus-adapted*)
interpret *uminus-X*: *nat-sigma-finite-adapted-process-linorder* $M\ F\ -X$ **..**
have $AE\ \xi\ in\ M.\ -\ X\ i\ \xi \leq$ *cond-exp* $M\ (F\ i)\ (\lambda x.\ -\ X\ (Suc\ i)\ x)\ \xi$ **for** *i* **using** *assms*(*2*) *cond-exp-uminus*[*OF integrable*, *of i Suc i*] **by** *force*
hence *nat-supermartingale* $M\ F\ (-(-\ X))$ **by** (*intro nat-supermartingale.intro submartingale.uminus nat-submartingale.axioms uminus-X.submartingale-nat*) (*auto*

simp add: *fun-Compl-def integrable*)
  thus *?thesis* **unfolding** *fun-Compl-def* **by** *simp*
qed

lemma (**in** *nat-sigma-finite-adapted-process-linorder*) *supermartingale-of-cond-exp-diff-Suc-le-zero*:
  assumes *integrable*: $\bigwedge i.\ integrable\ M\ (X\ i)$
    and $\bigwedge i.\ AE\ \xi\ in\ M.\ cond\text{-}exp\ M\ (F\ i)\ (\lambda\xi.\ X\ (Suc\ i)\ \xi - X\ i\ \xi)\ \xi \leq 0$
   shows *nat-supermartingale M F X*
proof (*intro supermartingale-nat integrable*)
  fix *i*
  show $AE\ \xi\ in\ M.\ X\ i\ \xi \geq cond\text{-}exp\ M\ (F\ i)\ (X\ (Suc\ i))\ \xi$ **using** *cond-exp-diff*[*OF integrable*(*1*,*1*), *of i Suc i i*] *cond-exp-F-meas*[*OF integrable adapted*, *of i*] *assms*(*2*)[*of i*] **by** *fastforce*
qed

end

theory *Example-Coin-Toss*
 imports *Martingale HOL−Probability.Stream-Space HOL−Probability.Probability-Mass-Function*
begin

We consider a probability space consisting of infinite sequences of coin tosses.

definition *bernoulli-stream* :: *real* ⇒ (*bool stream*) *measure* **where**
  bernoulli-stream p = *stream-space* (*measure-pmf* (*bernoulli-pmf p*))

lemma *space-bernoulli-stream*[*simp*]: *space* (*bernoulli-stream p*) = *UNIV* **by** (*simp add*: *bernoulli-stream-def space-stream-space*)

We define the fortune of the player at time n to be the number of heads minus number of tails.

definition *fortune* :: *nat* ⇒ *bool stream* ⇒ *real* **where**
  fortune n = ($\lambda s.\ \sum b \leftarrow stake\ (Suc\ n)\ s.\ if\ b\ then\ 1\ else\ -1$)

definition *toss* :: *nat* ⇒ *bool stream* ⇒ *real* **where**
  toss n = ($\lambda s.\ if\ snth\ s\ n\ then\ 1\ else\ -1$)

lemma *toss-indicator-def*: $toss\ n = indicator\ \{s.\ s\ !!\ n\} - indicator\ \{s.\ \neg\ s\ !!\ n\}$
  unfolding *toss-def indicator-def* **by** *force*

lemma *range-toss*: $range\ (toss\ n) = \{-1,\ 1\}$
proof −
  have *sconst True* !! *n* **by** *simp*
  moreover have ¬*sconst False* !! *n* **by** *simp*
  ultimately have $\exists x.\ x\ !!\ n\ \exists x.\ \neg x\ !!\ n$ **by** *blast+*
  thus *?thesis* **unfolding** *toss-def image-def* **by** *auto*
qed

lemma *vimage-toss*: *toss n* $-`$ *A* = (*if 1* ∈ *A then* {*s. s* !! *n*} *else* {}) ∪ (*if* −*1* ∈ *A then* {*s.* ¬*s* !! *n*} *else* {})
  unfolding *vimage-def toss-def* **by** *auto*

lemma *fortune-Suc*: *fortune* (*Suc n*) *s* = *fortune n s* + *toss* (*Suc n*) *s*
  by (*induction n arbitrary*: *s*) (*simp add*: *fortune-def toss-def*)+

lemma *fortune-toss-sum*: *fortune n s* = (∑ *i* ∈ {..*n*}. *toss i s*)
  by (*induction n arbitrary*: *s*) (*simp add*: *fortune-def toss-def*, *simp add*: *fortune-Suc*)

lemma *fortune-bound*: *norm* (*fortune n s*) ≤ *Suc n* **by** (*induction n*) (*force simp add*: *fortune-toss-sum toss-def*)+

Our definition of *bernoulli-stream* constitutes a probability space.

interpretation *prob-space bernoulli-stream p* **unfolding** *bernoulli-stream-def* **by** (*simp add*: *measure-pmf.prob-space-axioms prob-space.prob-space-stream-space*)

abbreviation *toss-filtration p* ≡ *nat-natural-filtration* (*bernoulli-stream p*) *toss*

The stochastic process *toss* is adapted to the filtration it generates.

interpretation *toss*: *nat-adapted-process bernoulli-stream p nat-natural-filtration* (*bernoulli-stream p*) *toss toss*
  by (*intro nat-adapted-process.intro stochastic-process.adapted-process-natural-filtration*)
    (*unfold-locales*, *auto simp add*: *toss-def bernoulli-stream-def*)

Similarly, the stochastic process *fortune* is adapted to the filtration generated by the tosses.

interpretation *fortune*: *nat-finite-adapted-process-linorder bernoulli-stream p nat-natural-filtration* (*bernoulli-stream p*) *toss fortune*
proof −
  show *nat-finite-adapted-process-linorder* (*bernoulli-stream p*) (*toss-filtration p*) *fortune*
  unfolding *fortune-toss-sum*
  by (*intro nat-finite-adapted-process-linorder.intro*
      finite-adapted-process-linorder.intro
      finite-adapted-process-order.intro
      finite-adapted-process.intro
      toss.partial-sum-adapted[*folded atMost-atLeast0*]) *intro-locales*
qed

lemma *integrable-toss*: *integrable* (*bernoulli-stream p*) (*toss n*)
  using *toss.random-variable*
  by (*intro Bochner-Integration.integrable-bound*[*OF integrable-const*[*of - 1* :: *real*]]) (*auto simp add*: *toss-def*)

lemma *integrable-fortune*: *integrable* (*bernoulli-stream p*) (*fortune n*) **using** *fortune-bound*

by (*intro Bochner-Integration.integrable-bound*[*OF integrable-const*[*of - Suc n*] *fortune.random-variable*]) *auto*

We provide the following lemma to explicitly calculate the probability of events in this probability space.

lemma *measure-bernoulli-stream-snth-pred*:
assumes $0 \leq p$ **and** $p \leq 1$ **and** *finite J*
shows *prob p* $\{w \in$ *space* (*bernoulli-stream p*). $\forall j{\in}J.\ P\ j = w\ !!\ j\} = p$ ^ *card* ($J \cap$ *Collect P*) $*$ $(1 - p)$ ^(*card* ($J -$ *Collect P*))
proof −
let *?PiE* = (Π_E $i{\in}J$. *if P i then* {*True*} *else* {*False*})
have *product-prob-space* (λ-. *measure-pmf* (*bernoulli-pmf p*)) **by** *unfold-locales*

hence ∗: *to-stream* −‘ $\{s.\ \forall i{\in}J.\ P\ i = s\ !!\ i\} = \{s.\ \forall i{\in}J.\ P\ i = s\ i\}$ **using** *assms* **by** (*simp add*: *to-stream-def*)
also have ... = *prod-emb UNIV* (λ-. *measure-pmf* (*bernoulli-pmf p*)) *J ?PiE*
proof −
{
fix *s* **assume** ($\forall i{\in}J.\ P\ i = s\ i$)
hence ($\forall i{\in}J.\ P\ i = s\ i$) = ($s \in$ *prod-emb UNIV* (λ-. *measure-pmf* (*bernoulli-pmf p*)) *J ?PiE*)
by (*subst prod-emb-iff*[*of s*]) (*smt* (*verit*, *best*) *not-def assms*(*3*) *id-def PiE-eq-singleton UNIV-I extensional-UNIV insert-iff singletonD space-measure-pmf*)
}
moreover
{
fix *s* **assume** ¬($\forall i{\in}J.\ P\ i = s\ i$)
then obtain *i* **where** $i \in J$ $P\ i \neq s\ i$ **by** *blast*
hence ($\forall i{\in}J.\ P\ i = s\ i$) = ($s \in$ *prod-emb UNIV* (λ-. *measure-pmf* (*bernoulli-pmf p*)) *J ?PiE*)
by (*simp add*: *restrict-def prod-emb-iff*[*of s*]) (*smt* (*verit*, *ccfv-SIG*) *PiE-mem assms*(*3*) *id-def insert-iff singleton-iff*)
}
ultimately show *?thesis* **by** *auto*
qed
finally have *integ*: (*to-stream* −‘ $\{s.\ \forall i{\in}J.\ P\ i = s\ !!\ i\}$) = *prod-emb UNIV* (λ-. *measure-pmf* (*bernoulli-pmf p*)) *J ?PiE* **.**
let *?M* = (Pi_M *UNIV* (λ-. *measure-pmf* (*bernoulli-pmf p*)))
have *emeasure* (*bernoulli-stream p*) $\{s \in$ *space* (*bernoulli-stream p*). $\forall i{\in}J.\ P\ i = s\ !!\ i\}$ = *emeasure ?M* (*to-stream* −‘ $\{s.\ \forall i{\in}J.\ P\ i = s\ !!\ i\}$)
using *assms emeasure-distr*[*of to-stream ?M* (*vimage-algebra* (*streams* (*space* (*measure-pmf* (*bernoulli-pmf p*)))) (!!) *?M*) $\{s.\ \forall i{\in}J.\ P\ i = s\ !!\ i\}$, *symmetric*] *measurable-to-stream*[*of* (*measure-pmf* (*bernoulli-pmf p*))]
by (*simp only*: *bernoulli-stream-def stream-space-def* ∗, *simp add*: *space-PiM*) (*smt* (*verit*, *best*) *emeasure-notin-sets in-vimage-algebra inf-top.right-neutral sets-distr vimage-Collect*)
also have ... = *emeasure ?M* (*prod-emb UNIV* (λ-. *measure-pmf* (*bernoulli-pmf p*)) *J ?PiE*) **using** *integ* **by** (*simp add*: *space-PiM*)
also have ... = ($\prod i{\in}J$. *emeasure* (*measure-pmf* (*bernoulli-pmf p*)) (*if P i then*

$\{True\}$ *else* $\{False\}))$
  by (*subst emeasure-PiM-emb*) (*auto simp add*: *prob-space-measure-pmf assms*(*3*))
 also have ... $= (\prod i{\in}J \cap$ *Collect P*. *ennreal p*) $*$ $(\prod i{\in}J -$ *Collect P*. *ennreal* $(1 - p))$
  unfolding *emeasure-pmf-single*[*of bernoulli-pmf p True*, *unfolded pmf-bernoulli-True*[*OF assms*(*1*,*2*)], *symmetric*]
   emeasure-pmf-single[*of bernoulli-pmf p False*, *unfolded pmf-bernoulli-False*[*OF assms*(*1*,*2*)], *symmetric*]
   by (*simp add*: *prod.Int-Diff*[*OF assms*(*3*), *of - Collect P*])
 also have ... $= p$ ^ *card* ($J \cap$ *Collect P*) $*$ $(1 - p)$ ^ *card* ($J -$ *Collect P*) **using** *assms* **by** (*simp add*: *prod-ennreal ennreal-mult′ ennreal-power*)
 finally show *?thesis* **using** *assms* **by** (*intro measure-eq-emeasure-eq-ennreal*) *auto*
qed

lemma
 assumes $0 \leq p$ **and** $p \leq 1$
 shows *measure-bernoulli-stream-snth*: *prob p* $\{w \in$ *space* (*bernoulli-stream p*). *w* !! *i*$\} = p$
  and *measure-bernoulli-stream-neg-snth*: *prob p* $\{w \in$ *space* (*bernoulli-stream p*). $\neg$*w* !! *i*$\} = 1 - p$
 using *measure-bernoulli-stream-snth-pred*[*OF assms*, *of* $\{i\}$ λx. *True*]
   measure-bernoulli-stream-snth-pred[*OF assms*, *of* $\{i\}$ λx. *False*] **by** *auto*

Now we can express the expected value of a single coin toss.

lemma *integral-toss*:
 assumes $0 \leq p$ $p \leq 1$
 shows *expectation p* (*toss n*) $= 2 * p - 1$
proof $-$
 have [*simp*]:$\{s.$ *s* !! *n*$\} \in$ *events p* **using** *measurable-snth*[*THEN measurable-sets*, *of* $\{True\}$ *measure-pmf* (*bernoulli-pmf p*) *n*, *folded bernoulli-stream-def*]
   by (*simp add*: *vimage-def*)
 have *expectation p* (*toss n*) = *Bochner-Integration.simple-bochner-integral* (*bernoulli-stream p*) (*toss n*)
   using *toss.random-variable*[*of n*, *THEN measurable-sets*]
  by (*intro simple-bochner-integrable-eq-integral*[*symmetric*] *simple-bochner-integrable.intros*) (*auto simp add*: *toss-def simple-function-def image-def*)
 also have ... $= p -$ *prob p* $\{s. \neg$ *s* !! *n*$\}$ **unfolding** *simple-bochner-integral-def* **using** *measure-bernoulli-stream-snth*[*OF assms*]
   by (*simp add*: *range-toss*, *simp add*: *toss-def*)
 also have ... $= p - (1 -$ *prob p* $\{s.$ *s* !! *n*$\})$ **by** (*subst prob-compl*[*symmetric*], *auto simp add*: *Collect-neg-eq Compl-eq-Diff-UNIV*)
 finally show *?thesis* **using** *measure-bernoulli-stream-snth*[*OF assms*] **by** *simp*
qed

Now, we show that the tosses are independent from one another.

lemma *indep-vars-toss*:
 assumes $0 \leq p$ $p \leq 1$
 shows *indep-vars p* (λ-. *borel*) *toss* $\{0..\}$

proof ($subst$ $indep\text{-}vars\text{-}def$, $intro$ $conjI$ $indep\text{-}sets\text{-}sigma$)
{
fix A J **assume** asm: $J \neq \{\}$ $finite$ J $\forall j{\in}J.$ A $j \in \{toss\ j\ -`\ A \cap space$ $(bernoulli\text{-}stream\ p)$ $|A.\ A \in borel\}$
hence $\forall j{\in}J.\ \exists B \in borel.\ A\ j = toss\ j\ -`\ B \cap space\ (bernoulli\text{-}stream\ p)$ **by** $auto$
then obtain B **where** $B\text{-}is$: $A\ j = toss\ j\ -`\ B\ j \cap space\ (bernoulli\text{-}stream\ p)$ $B\ j \in borel$ **if** $j \in J$ **for** j **by** $metis$

have $prob\ p\ (\bigcap\ (A\ `\ J)) = (\prod j{\in}J.\ prob\ p\ (A\ j))$
proof $cases$

We consider the case where there is a zero probability event.

assume $\exists j \in J.\ 1 \notin B\ j \wedge -1 \notin B\ j$
then obtain j **where** $j\text{-}is$: $j \in J$ $1 \notin B\ j$ $-1 \notin B\ j$ **by** $blast$
hence $A\text{-}j\text{-}empty$: $A\ j = \{\}$ **using** $B\text{-}is$ **by** ($force$ $simp$ add: $toss\text{-}def$ $vimage\text{-}def$)
hence $\bigcap\ (A\ `\ J) = \{\}$ **using** $j\text{-}is$ **by** $blast$
moreover have $prob\ p\ (A\ j) = 0$ **using** $A\text{-}j\text{-}empty$ **by** $simp$
ultimately show $?thesis$ **using** $j\text{-}is$ $asm(2)$ **by** $auto$
next

We now assume all events have positive probability.

assume $\neg(\exists j \in J.\ 1 \notin B\ j \wedge -1 \notin B\ j)$
hence $*$: $1 \in B\ j \vee -1 \in B\ j$ **if** $j \in J$ **for** j **using** $that$ **by** $blast$

define J' **where** [$simp$]: $J' = \{j \in J.\ (1 \in B\ j) \longleftrightarrow (-1 \notin B\ j)\}$
hence $toss\ j\ w \in B\ j \longleftrightarrow (1 \in B\ j) = w\ !!\ j$ **if** $j \in J'$ **for** w j **using** $that$ **unfolding** $toss\text{-}def$ **by** $simp$
hence $(\bigcap\ (A\ `\ J')) = \{w \in space\ (bernoulli\text{-}stream\ p).\ \forall j{\in}J'.\ (1 \in B\ j) = w\ !!\ j\}$ **using** $B\text{-}is$ **by** $force$
hence $prob\text{-}J'$: $prob\ p\ (\bigcap\ (A\ `\ J')) = p\ \widehat{}\ card\ (J' \cap \{j.\ 1 \in B\ j\}) * (1 - p)\ \widehat{}\ card\ (J' - \{j.\ 1 \in B\ j\})$
using $measure\text{-}bernoulli\text{-}stream\text{-}snth\text{-}pred[OF\ assms\ finite\text{-}subset[OF\ \text{-}\ asm(2)],\ of\ J'\ \lambda j.\ 1 \in B\ j]$ **by** $auto$

The index set J' consists of the indices of all non-trivial events.

have $A\text{-}j\text{-}True$: $A\ j = \{w \in space\ (bernoulli\text{-}stream\ p).\ w\ !!\ j\}$ **if** $j \in J' \cap \{j.\ 1 \in B\ j\}$ **for** j
using $that$ **by** ($auto$ $simp$ add: $toss\text{-}def$ $B\text{-}is(1)$ $split$: $if\text{-}splits$)

have $A\text{-}j\text{-}False$: $A\ j = \{w \in space\ (bernoulli\text{-}stream\ p).\ \neg w\ !!\ j\}$ **if** $j \in J' - \{j.\ 1 \in B\ j\}$ **for** j
using $that$ $B\text{-}is$ **by** ($auto$ $simp$ add: $toss\text{-}def$)

have $A\text{-}j\text{-}top$: $A\ j = space\ (bernoulli\text{-}stream\ p)$ **if** $j \in J - J'$ **for** j **using** $that$ $*$ **by** ($auto$ $simp$ add: $B\text{-}is$ $toss\text{-}def$)
hence $\bigcap\ (A\ `\ J) = \bigcap\ (A\ `\ J')$ **by** $auto$
hence $prob\ p\ (\bigcap\ (A\ `\ J)) = prob\ p\ (\bigcap\ (A\ `\ J'))$ **by** $presburger$

also have ... = ($\prod j \in J' \cap \{j.\ 1 \in B\ j\}.\ prob\ p\ (A\ j)) * (\prod j \in J' - \{j.\ 1 \in B\ j\}.\ prob\ p\ (A\ j)$)
by (*simp only*: *prob-J′ A-j-True A-j-False measure-bernoulli-stream-snth*[*OF assms*] *measure-bernoulli-stream-neg-snth*[*OF assms*] *cong*: *prod.cong*) *simp*
also have ... = ($\prod j \in J'.\ prob\ p\ (A\ j)$) **using** *asm*(*2*) **by** (*intro prod.Int-Diff*[*symmetric*]) *auto*
also have ... = ($\prod j \in J'.\ prob\ p\ (A\ j)) * (\prod j \in J - J'.\ prob\ p\ (A\ j)$) **using** *A-j-top prob-space* **by** *simp*
also have ... = ($\prod j \in J.\ prob\ p\ (A\ j)$) **using** *asm*(*2*) **by** (*metis* (*no-types*, *lifting*) *J′-def mem-Collect-eq mult.commute prod.subset-diff subsetI*)
finally show *?thesis* **.**
qed
}
thus *indep-sets p* ($\lambda i.\ \{toss\ i$ −‘ $A \cap space\ (bernoulli\text{-}stream\ p)\ |A.\ A \in sets\ borel\})\ \{0..\}$ **using** *measurable-sets*[*OF toss.random-variable*]
by (*intro indep-setsI subsetI*) *fastforce*
qed (*simp*, *intro Int-stableI*, *simp*, *metis sets.Int vimage-Int*)

The fortune of a player is a martingale (resp. sub- or supermartingale) with respect to the filtration generated by the coin tosses.

theorem *fortune-martingale*:
assumes $p = 1/2$
shows *nat-martingale* (*bernoulli-stream p*) (*toss-filtration p*) *fortune*
using *cond-exp-indep*[*OF fortune.subalg indep-set-natural-filtration integrable-toss*, *OF zero-order*(*1*) *lessI indep-vars-toss*, *of p*]
integral-toss assms
by (*intro fortune.martingale-of-cond-exp-diff-Suc-eq-zero integrable-fortune*) (*force simp add*: *fortune-toss-sum*)

theorem *fortune-submartingale*:
assumes $1/2 \leq p\ p \leq 1$
shows *nat-submartingale* (*bernoulli-stream p*) (*toss-filtration p*) *fortune*
proof (*intro fortune.submartingale-of-cond-exp-diff-Suc-nonneg integrable-fortune*)
fix *n*
show *AE* ξ *in bernoulli-stream p*. $0 \leq$ *cond-exp* (*bernoulli-stream p*) (*toss-filtration p n*) ($\lambda\xi.\ fortune\ (Suc\ n)\ \xi - fortune\ n\ \xi)\ \xi$
using *cond-exp-indep*[*OF fortune.subalg indep-set-natural-filtration integrable-toss*, *OF zero-order*(*1*) *lessI indep-vars-toss*, *of p n*]
integral-toss[*of p Suc n*] *assms*
by (*force simp add*: *fortune-toss-sum*)
qed

theorem *fortune-supermartingale*:
assumes $0 \leq p\ p \leq 1/2$
shows *nat-supermartingale* (*bernoulli-stream p*) (*toss-filtration p*) *fortune*
proof (*intro fortune.supermartingale-of-cond-exp-diff-Suc-le-zero integrable-fortune*)
fix *n*
show *AE* ξ *in bernoulli-stream p*. $0 \geq$ *cond-exp* (*bernoulli-stream p*) (*toss-filtration p n*) ($\lambda\xi.\ fortune\ (Suc\ n)\ \xi - fortune\ n\ \xi)\ \xi$

using *cond-exp-indep*[*OF fortune.subalg indep-set-natural-filtration integrable-toss*, *OF zero-order*(*1*) *lessI indep-vars-toss*, *of p n*]
integral-toss[*of p Suc n*] *assms*
by (*force simp add*: *fortune-toss-sum*)
qed

end