1 - Type derivation for $(x + \ 2 \ > \ 3)$ && y with $\Gamma \ = \ \{x : \ int, \ y : \ int\}$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{VA}{\{x: \ int, \ y: \ int\} \ \vdash \ x \ : \ int}
      \qquad
      \dfrac{CA}{\{x: \ int, \ y: \ int\} \ \vdash \ 2 \ : \ int}
    }{\{x: \ int, \ y: \ bool\} \ \vdash x \ + \ 2 \ : \ int}\,(AO)
    \qquad
    \dfrac{CA}{\{x: \ int, \ y: \ int\} \ \vdash \ 3 \ : \ int}
  }{\{x: \ int, \ y: \ bool\} \ \vdash \ x \ + \ 2 \ > \ 3 \ \ : \ bool}\,(RO)
  \qquad
  \dfrac{VA}{\{x: \ int, \ y: \ bool\} \ \vdash \ y \ : \ bool}
}{\{x: \ int, \ y: \ bool\} \ \vdash \ (x \ + \ 2 \ > \ 3) \ \&\& \ y \ : \ bool}\,(LCR)
$$

$\Longrightarrow$ Type derivation for code is correct and evaluates to *bool*

2 - Type Derivation for $fn\ f \implies fn\ x \implies if\ x\ then\ f\ true\ else\ 19$ with $\Gamma = \{\ \}$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{VA}{\{f:\ \beta,\ x:\ \delta\} \vdash x\ :\ bool}
\qquad
\dfrac{
\dfrac{VA}{\{f:\ \beta,\ x:\ \delta\} \vdash f:\ \varphi \to \varepsilon}
\quad
\dfrac{CA}{\{f:\ \beta,\ x:\ \delta\} \vdash true\ :\ \varphi}
}{\{f:\ \beta,\ x:\delta\} \vdash f\ true\ :\ \varepsilon}\ (FA)
\quad
\dfrac{CA}{\{f:\ \beta,\ x:\ \delta\} \vdash 19\ :\ \varepsilon}
}{\{f:\ \beta,\ x:\ \delta\} \vdash if\ x\ then\ f\ true\ else\ 19\ :\ \varepsilon}\ (ITE)
}{\{f:\ \beta\} \vdash fn\ x \implies if\ x\ then\ f\ true\ else\ 19\ :\ \gamma}\ (AF)
}{\{\ \} \vdash fn\ f \implies fn\ x \implies if\ x\ then\ f\ true\ else\ 19\ :\ \alpha}\ (AF)
$$

$$
\begin{aligned}
\varepsilon &= int \\
\varphi &= bool \\
\beta &= \varphi \to \lambda \\
\delta &= bool \\
\gamma &= \delta \to \varepsilon \\
\alpha &= \beta \to \gamma
\end{aligned}
\quad \implies \quad
\begin{aligned}
\varepsilon &= int \\
\varphi &= bool \\
\beta &= bool \to int \\
\delta &= bool \\
\gamma &= bool \to int \\
\alpha &= (bool \to int) \to bool \to int
\end{aligned}
$$

$\implies$ Type derivation for code is correct and evaluates to $(bool \to int) \to bool \to int$

3 - Type derivation for $let\ fun\ f\ x\ =\ 5\ <\ x\ in\ fn\ y\ =>\ (f\ 3)\ ||\ y\ end$ with $\Gamma\ =\ \{\ \}$

$$
\dfrac{
  \dfrac{
    \dfrac{CA}{\{f : \beta \to \gamma, x : \beta\} \vdash 5 : int}
    \quad
    \dfrac{VA}{\{f : \beta \to \gamma,\ x : \beta\} \vdash x : int}
  }{\{f : \beta \to \gamma,\ x : \beta\} \vdash 5 < x : bool}\,(RO)
  \quad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{VA}{\{f : \beta \to \gamma,\ y : \varphi\} \vdash f : \psi \to bool}
        \quad
        \dfrac{CA}{\{f : \beta \to \gamma,\ y : \varphi\} \vdash 3 : \psi}
      }{\{f : \beta \to \gamma,\ y : \varphi\} \vdash f\,3 : bool}\,(FA)
      \quad
      \dfrac{VA}{\{f : \beta \to \gamma,\ y : \varphi\} \vdash y : bool}
    }{\{f : \beta \to \gamma,\ y : \varphi\} \vdash (f\ 3)\ ||\ y : bool}\,(LCR)
  }{\{f : \beta \to \gamma\} \vdash fn\ y \Longrightarrow (f\ 3)\ ||\ y\ end : \alpha}\,(AF)
}{\{\ \} \vdash let\ fun\ f\ x\ =\ 5\ <\ x\ in\ fn\ y \Longrightarrow (f\ 3)\ ||\ y\ end : \alpha}\,(LF)
$$

$$
\begin{aligned}
\beta &= int \\
\gamma &= bool \\
\alpha &= \varphi \to bool \\
\psi &= \beta \\
\gamma &= bool \\
\psi &= int \\
\varphi &= bool
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\beta &= int \\
\gamma &= bool \\
\alpha &= bool \to bool \\
\sigma &= int \\
\psi &= int \\
\varphi &= bool
\end{aligned}
$$

$\Longrightarrow$ Type derivation for code is correct and evaluates to $bool \to bool$

4 - Type inference for $fn\ x \implies if\ x\ then\ x + 1\ else\ x(1)$

$$
\dfrac{\dfrac{\dfrac{VA}{\{x:\beta\} \vdash x : bool} \qquad \dfrac{\dfrac{VA}{\{x:\beta\} \vdash x : int} \quad \dfrac{CA}{\{x:\beta\} \vdash x : int}}{\{x:\beta\} \vdash x + 1 : \gamma}\,(AO) \qquad \dfrac{\dfrac{VA}{\{x:\beta\} \vdash x : \varphi \rightarrow \gamma} \quad \dfrac{CA}{\{x:\beta\} \vdash 1 : \varphi}}{\{x:\beta\} \vdash x(1) : \gamma}\,(FA)}{\{\, x : \beta \,\} \vdash if\ x\ then\ x + 1\ else\ x(1) : \gamma}\,(ITE)}{\{\ \} \vdash fn\ x \implies if\ x\ then\ x + 1\ else\ x(1) : \alpha}\,(AF)
$$

$Constraints:$

$\varphi = int$
$\beta = \varphi \rightarrow \gamma$
$\beta = bool$
$\gamma = int$
$\beta = bool$
$\alpha = \beta \rightarrow \gamma$

$$
\begin{array}{lll}
 & Unify(\ \{\varphi = int,\ \beta = \varphi \rightarrow \gamma,\ \beta = bool,\ \gamma = int,\ \beta = bool,\ \alpha = \beta \rightarrow \gamma\}\ ) & :\ Eliminate \\
\implies & Unify(\ \{\beta = int \rightarrow \gamma,\ \beta = bool,\ \gamma = int,\ \beta = bool,\ \alpha = \beta \rightarrow \gamma\}\ ) \circ \{\varphi = int\} & :\ Eliminate \\
 & Unify(\ \{int \rightarrow \gamma = bool,\ \gamma = int,\ int \rightarrow \gamma = bool,\ \alpha = (int \rightarrow \gamma) \rightarrow \gamma\}\ ) \circ \{\beta = int \rightarrow \gamma\} \circ \{\varphi = int\} & :\ Unification\ Failure
\end{array}
$$

5 - Type inference for $let\ fun\ fact\ x\ =\ if\ x\ =\ 0\ then\ 1\ else\ fact\ (x\ -\ 1)\ in\ fact\ end$

$$\dfrac{\dfrac{\dfrac{\dfrac{VA}{\{fact:\ \beta \to \gamma,\ x:\ \beta\} \vdash x\ :\ \varepsilon} \quad \dfrac{CA}{\{fact:\ \beta \to \gamma,\ x:\ \beta\} \vdash 0\ :\ \varepsilon}}{\{fact:\ \beta \to \gamma,\ x:\ \beta\} \vdash x=0\ :\ bool}\,(RO) \quad \dfrac{CA}{\{fact:\ \beta \to \gamma,\ x:\ \beta\} \vdash 1\ :\ \gamma} \quad \dfrac{\dfrac{VA}{\{fact:\ \beta \to \gamma,\ x:\ \beta\} \vdash fact\ :\ \omega \to \gamma} \quad \dfrac{\dfrac{VA}{\{fact:\ \beta \to \gamma,\ x:\ \beta\} \vdash x\ :\ \omega} \quad \dfrac{CA}{\{fact:\ \beta \to \gamma,\ x:\ \beta\} \vdash 1\ :\ \omega}}{\{fact:\ \beta \to \gamma,\ x:\ \beta\} \vdash x - 1\ :\ \omega}\,(AO)}{\{fact:\ \beta \to \gamma,\ x:\ \beta\} \vdash fact(x-1)\ :\ \gamma}\,(FA)}{\{fact:\ \beta \to \gamma,\ x:\ \beta\} \vdash if\ x=0\ then\ 1\ else\ fact(x-1)\ :\ \gamma}\,(ITE) \quad \dfrac{VA}{\{fact:\ \beta \to \gamma\} \vdash fact\ :\ \alpha}}{\{\ \} \vdash let\ fun\ fact\ x\ =\ if\ x\ =\ 0\ then\ 1\ else\ fact\ (x\ -\ 1)\ in\ fact\ end\ :\ \alpha}\,(LF)$$

$Constrains:$

$$\begin{aligned}
&\alpha = \beta \to \gamma \\
&\omega = int \\
&\beta = \omega \\
&\beta \to \gamma = \omega \to \gamma \\
&\gamma = int \\
&\varepsilon = int \\
&\beta = \varepsilon
\end{aligned}$$

$$\begin{aligned}
& Unify(\{\alpha = \beta \to \gamma,\ \omega = int,\ \beta = \omega,\ \beta \to \gamma = \omega \to \gamma,\ \gamma = int,\ \varepsilon = int,\ \beta = \varepsilon\}) && : Eliminate \\
& Unify(\{\omega = int,\ \beta = \omega,\ \beta \to \gamma = \omega \to \gamma,\ \gamma = int,\ \varepsilon = int,\ \beta = \varepsilon\}) \circ \{\alpha = \beta \to \gamma\} && : Eliminate \\
& Unify(\{\beta = int,\ \beta \to \gamma = int \to \gamma,\ \gamma = int,\ \varepsilon = int,\ \beta = \varepsilon\}) \circ \{\omega = int\} \circ \{\alpha = \beta \to \gamma\} && : Eliminate \\
& Unify(\{int \to \gamma = int \to \gamma,\ \gamma = int,\ \varepsilon = int,\ int = \varepsilon\}) \circ \{\beta = int\} \circ \{\omega = int\} \circ \{\alpha = \beta \to \gamma\} && : Decompose \\
& Unify(\{int = int,\ \gamma = \gamma,\ \gamma = int,\ \varepsilon = int,\ int = \varepsilon\}) \circ \{\beta = int\} \circ \{\omega = int\} \circ \{\alpha = \beta \to \gamma\} && : Delete \\
& Unify(\{\gamma = \gamma,\ \gamma = int,\ \varepsilon = int,\ int = \varepsilon\}) \circ \{\beta = int\} \circ \{\omega = int\} \circ \{\alpha = \beta \to \gamma\} && : Delete \\
\Longrightarrow\ & Unify(\{\gamma = int,\ \varepsilon = int,\ int = \varepsilon\}) \circ \{\beta = int\} \circ \{\omega = int\} \circ \{\alpha = \beta \to \gamma\} && : Eliminate \\
& Unify(\{\varepsilon = int,\ int = \varepsilon\}) \circ \{\gamma = int\} \circ \{\beta = int\} \circ \{\omega = int\} \circ \{\alpha = \beta \to \gamma\} && : Eliminate \\
& Unify(\{int = int\}) \circ \{\varepsilon = int\} \circ \{\gamma = int\} \circ \{\beta = int\} \circ \{\omega = int\} \circ \{\alpha = \beta \to \gamma\} && : Delete \\
& Unify(\{\ \}) \circ \{\varepsilon = int\} \circ \{\gamma = int\} \circ \{\beta = int\} \circ \{\omega = int\} \circ \{\alpha = \beta \to \gamma\} && : Identity \\
& \{\ \} \circ \{\varepsilon = int\} \circ \{\gamma = int\} \circ \{\beta = int\} \circ \{\omega = int\} \circ \{\alpha = \beta \to \gamma\} &&
\end{aligned}$$

$$\Longrightarrow \{\varepsilon = int\ ,\ \gamma = int,\ \beta = int,\ \omega = int,\ \alpha = int \to int\}$$