

# Generating Functions

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## 1 Generating Series

Instead of viewing a sequence as a function that returns its  $n$ th term, a *generating series* packages all of its terms into a single power series whose coefficients are exactly the sequence entries. Concretely, the sequence

$$2, 3, 5, 8, 12, \dots$$

is encoded by the generating series

$$2 + 3x + 5x^2 + 8x^3 + 12x^4 + \dots.$$

In general, given any sequence  $\{c_n\}_{n \geq 0}$ , its generating series is the formal power series

$$G(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots.$$

We say that  $G(x)$  “generates” the sequence  $\{c_n\}$  because each coefficient of  $x^n$  in  $G(x)$  is precisely  $c_n$ . Generating series turn sequence-based problems into algebraic manipulations of power series, a technique we will exploit heavily in what follows.

### Recall of the Basic Series

$a_0 = 1$	$a_1 = \frac{1}{2}$			
	$a_2 = \frac{1}{4}$	$a_3 = \frac{1}{8}$	$a_4 = \frac{1}{16}$	$\dots$

Figure 1: A geometric interpretation of the binary series, showing how  $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$ .

**A Geometric View of the Binary Series** For  $|x| < 1$ , we have the infinite geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n.$$

We now present a quick proof of this result by performing long division of 1 by  $1 - x$ .

$$\begin{array}{r|l}
& 1 + x + x^2 + x^3 + \cdots \\
1 - x & 1 \\
\hline
& \underline{1 - x} \\
& x \\
& \underline{x - x^2} \\
& x^2 \\
& \underline{x^2 - x^3} \\
& x^3 \\
& \vdots
\end{array}$$

The process works as follows: The long-division proceeds by repeatedly dividing the current remainder by the leading term of the divisor, producing one new power of  $x$  at each step:

1. Divide 1 by  $1 - x$ . The multiplier needed to eliminate the constant term is 1, so

$$1 - 1 \cdot (1 - x) = x.$$

Thus the first summand is 1, leaving a remainder of  $x$ .

2. Divide the remainder  $x$  by  $1 - x$ . The multiplier is  $x$ , so

$$x - x \cdot (1 - x) = x^2.$$

Hence the second summand is  $x$ , leaving a remainder of  $x^2$ .

3. Divide  $x^2$  by  $1 - x$ . The multiplier is  $x^2$ , giving

$$x^2 - x^2 \cdot (1 - x) = x^3.$$

Therefore the third summand is  $x^2$ , with remainder  $x^3$ .

4. Continuing in this fashion produces the infinite series

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots.$$

Continuing indefinitely produces

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n,$$

as claimed.

We will use this fact in further examples throughout the notes.

## 2 Building Generating Functions

The simplest (or “basic”) generating function is

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots,$$

which generates the constant sequence  $1, 1, 1, \dots$

**Replacing  $x$  with  $-x$ :**

$$\frac{1}{1 - (-x)} = \frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots ,$$

generating  $1, -1, 1, -1, \dots$

**Replacing  $x$  with  $3x$ :**

$$\frac{1}{1 - 3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots ,$$

generating  $1, 3, 9, 27, \dots$

**Scaling a sequence by 3:**

$$\frac{3}{1 - 3x} = 3 + 9x + 27x^2 + 81x^3 + \cdots ,$$

generating  $3, 9, 27, 81, \dots$

**Termwise addition of sequences:**

Adding the generating functions for  $1, 1, 1, \dots$  and  $1, 3, 9, \dots$  gives

$$\frac{1}{1 - x} + \frac{1}{1 - 3x} = 2 + 4x + 10x^2 + 28x^3 + \cdots ,$$

which generates  $2, 4, 10, 28, \dots$

**Replacing  $x$  with  $x^2$ :**

$$\frac{1}{1 - x^2} = 1 + x^2 + x^4 + x^6 + \cdots ,$$

generating  $1, 0, 1, 0, 1, 0, \dots$

**Shifting a sequence:**

Multiplying by  $x$  shifts all coefficients right by one:

$$\frac{x}{1 - 3x} = 0 + x + 3x^2 + 9x^3 + \cdots ,$$

generating  $0, 1, 3, 9, \dots$ , and

$$\frac{x}{1 - x^2} = 0 + x + 0x^2 + x^3 + \cdots ,$$

generating  $0, 1, 0, 1, \dots$

**Combining shifted sequences:**

Adding the two “even-odd” generating functions recovers

$$\frac{1}{1 - x^2} + \frac{x}{1 - x^2} = \frac{1 + x}{1 - x^2} = \frac{1}{1 - x},$$

which generates  $1, 1, 1, 1, \dots$

**Differentiation:**

Differentiating the basic Generating Function

$$\frac{d}{dx} \left( \frac{1}{1 - x} \right) = \frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots ,$$

yields the generating function for  $1, 2, 3, 4, \dots$

### 3 Recurrence Relations & Generating Functions

We conclude with an example of one of the many reasons studying generating functions is helpful: solving recurrence relations via algebraic manipulation of power series.

**Example: Tower of Hanoi** The minimum number of moves required to transfer  $n$  disks satisfies

$$a_0 = 0, \quad a_1 = 1, \quad a_n = 2a_{n-1} + 1 \quad (n \geq 1),$$

giving the sequence

$$0, 1, 3, 7, 15, 31, \dots$$

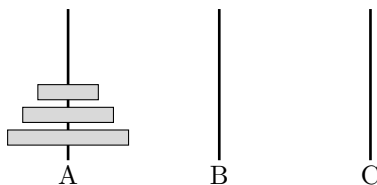


Figure 2: Initial configuration for Tower of Hanoi (3 disks).

Define the generating function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Using the recurrence for  $n \geq 1$ :

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (2a_{n-1} + 1) x^n = 2x \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} x^n,$$

so

$$f(x) - a_0 = 2x f(x) + \frac{x}{1-x},$$

and since  $a_0 = 0$ ,

$$f(x) = \frac{x}{(1-x)(1-2x)}.$$

Performing partial fractions:

$$\frac{x}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{1}{1-2x},$$

hence

$$f(x) = -\frac{1}{1-x} + \frac{1}{1-2x}.$$

Extracting coefficients yields the closed-form solution

$$a_n = 2^n - 1,$$

confirming the well-known formula for the Tower of Hanoi moves.

## 4 Introduction to the Fibonacci Sequence

The Fibonacci sequence famously arises from a puzzle involving rabbit populations. Imagine starting with a single pair of rabbits that takes one month to mature. After maturing, each pair produces a new pair of rabbits every month. Mathematically, if  $F_n$  represents the number of rabbit pairs in month  $n$ , the sequence satisfies the initial conditions

$$F_0 = 0, \quad F_1 = 1,$$

and the recurrence

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0.$$

Q: is there a non-recursive (closed-form) formula for  $F_n$  ?

$$\begin{aligned}
1) & \quad || \quad (2 \text{ small rabbits}) \\
2) & \quad || + || \quad (1 \text{ big pair} + 1 \text{ small pair}) \\
3) & \quad || + || + || \quad (2 \text{ big pairs} + 1 \text{ small pair}) \\
4) & \quad || + || + || + || + || \quad (3 \text{ big pairs} + 2 \text{ small pairs})
\end{aligned}$$

Figure 3: Illustration of rabbit pairs over successive months. Blue bars represent small rabbits; orange bars represent big (mature) rabbits.

**Idea: consider and calculate it**

## 5 Deriving the Closed-Form for the Fibonacci Sequence

**Step 1: Define the generating function.** Let  $\{F_n\}_{n=0}^{\infty}$  be the Fibonacci sequence with

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0).$$

Define the generating function

$$f(x) = \sum_{n=0}^{\infty} F_n x^n.$$

We aim to find a closed-form expression for  $f(x)$ , and then extract a formula for  $F_n$ .

**Step 2: Use the Fibonacci recurrence in  $f(x)$ .** Starting from

$$f(x) = F_0 + F_1 x + \sum_{n=2}^{\infty} F_n x^n,$$

and noting  $F_0 = 0$ ,  $F_1 = 1$ , we have

$$f(x) = x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) x^n$$

because  $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 2$ . Separate the sums:

$$f(x) = x + \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n.$$

Shift indices to factor out  $f(x)$ :

$$\sum_{n=2}^{\infty} F_{n-1} x^n = x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} = x \sum_{m=1}^{\infty} F_m x^m = x(f(x) - F_0) = x f(x),$$

since  $F_0 = 0$ . Similarly,

$$\sum_{n=2}^{\infty} F_{n-2} x^n = x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2} = x^2 \sum_{k=0}^{\infty} F_k x^k = x^2 f(x).$$

Hence,

$$f(x) = x + x f(x) + x^2 f(x) \implies f(x)(1 - x - x^2) = x.$$

Thus,

$$f(x) = \frac{x}{1 - x - x^2}.$$

**Step 3: Partial-Fraction Decomposition (as in the images).** First, rewrite

$$\frac{1}{1-x-x^2} = \frac{1}{-(x^2+x-1)} = -\frac{1}{x^2+x-1}.$$

Next, factor  $x^2 + x - 1$ . Observe that the roots of

$$x^2 + x - 1 = 0$$

are

$$x = -\frac{1+\sqrt{5}}{2} \quad \text{and} \quad x = -\frac{1-\sqrt{5}}{2}.$$

Hence,

$$x^2 + x - 1 = \left(x + \frac{1+\sqrt{5}}{2}\right) \cdot \left(x + \frac{1-\sqrt{5}}{2}\right).$$

Therefore,

$$-\frac{1}{x^2+x-1} = -\frac{1}{\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)}.$$

We look for constants  $A$  and  $B$  such that

$$-\frac{1}{\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)} = \frac{A}{x + \frac{1+\sqrt{5}}{2}} + \frac{B}{x + \frac{1-\sqrt{5}}{2}}.$$

**Step 4: Solve for  $A$  and  $B$ .** Comparing coefficients of  $x$  and the constant term in

$$-1 = A(x + \beta) + B(x + \alpha),$$

we obtain the system

$$\begin{cases} A + B = 0, \\ A\beta + B\alpha = -1. \end{cases}$$

It follows that

$$B = -A, \quad A(\beta - \alpha) = -1 \implies A = \frac{1}{\alpha - \beta} \quad \text{and} \quad B = -\frac{1}{\alpha - \beta}.$$

Hence,

$$-\frac{1}{(x + \alpha)(x + \beta)} = \frac{1}{\alpha - \beta} \frac{1}{x + \alpha} - \frac{1}{\alpha - \beta} \frac{1}{x + \beta}.$$

**Step 5: Combine with the earlier factor  $-1$  and rewrite.** Recalling that

$$\frac{1}{1-x-x^2} = -\frac{1}{x^2+x-1} = -\frac{1}{(x + \alpha)(x + \beta)},$$

we combine the above result to conclude

$$\frac{1}{1-x-x^2} = \frac{1}{\alpha - \beta} \left( \frac{1}{x + \alpha} - \frac{1}{x + \beta} \right).$$

**Step 6: Expand each term in a power series.** Notice that

$$\frac{1}{x + \alpha} = \frac{1}{\alpha} \frac{1}{1 + \frac{x}{\alpha}} = \frac{1}{\alpha} \sum_{n=0}^{\infty} \left(-\frac{x}{\alpha}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha^{n+1}} x^n,$$

valid for  $\left|\frac{x}{\alpha}\right| < 1$ . Similarly,

$$\frac{1}{x + \beta} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\beta^{n+1}} x^n.$$

Hence,

$$\frac{1}{1 - x - x^2} = \frac{1}{\alpha - \beta} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha^{n+1}} x^n - \sum_{n=0}^{\infty} \frac{(-1)^n}{\beta^{n+1}} x^n \right] = \sum_{n=0}^{\infty} \left[ \frac{1}{\alpha - \beta} \left( \frac{(-1)^n}{\alpha^{n+1}} - \frac{(-1)^n}{\beta^{n+1}} \right) \right] x^n.$$

**Step 7: Identify Fibonacci numbers.** Recall that  $\alpha - \beta = \sqrt{5}$ , and

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

One checks (or uses known identities) to see that the coefficient of  $x^n$  in the above power series is exactly  $F_n$ . Consequently,

$$\sum_{n=0}^{\infty} F_n x^n = \frac{1}{1 - x - x^2},$$

which is the generating function for the Fibonacci sequence.

**Conclusion.** We have shown that the generating function for the Fibonacci sequence is  $\frac{x}{1-x-x^2}$ . Through partial fractions and comparing coefficients, we deduced that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

This gives a non-recursive (closed-form) expression for  $F_n$ , completing the derivation.

## 6 More examples

In earlier sections (see, e.g., *A Geometric View of the Binary Series* on page 14), we explored methods to solve recurrences and introduced generating functions as a tool to transform sequences into functions. In this section, we briefly reiterate these ideas and demonstrate, through several examples, how generating functions serve as a bridge between discrete mathematics and calculus.

**Example 1: Constant Sequence.** Consider the sequence defined by

$$a_n = 1 \quad \text{for all } n \geq 0,$$

so that the sequence is

$$1, 1, 1, \dots$$

By the geometric series formula (proved earlier), its generating function is given by

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

**Example 2: Exponential Sequence.** Now, let

$$a_n = \frac{1}{n!}.$$

Then the generating function is

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \quad x \in \mathbb{R}.$$

A partial justification of this result can be obtained by recalling the Taylor series expansion of the exponential function. Although a complete treatment of Taylor series is a topic in calculus (not yet covered in this course), note that differentiating the power series term-by-term confirms the identity.

**Example 3: Binomial Coefficient Sequence.** Consider the sequence defined by

$$a_n = \binom{n+k}{k}.$$

**Theorem.** The generating function for this sequence is

$$f(x) = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}, \quad |x| < 1.$$



*Proof.*

- For  $k = 1$ : Note that

$$a_n = \binom{n+1}{1} = n+1,$$

so that

$$f(x) = \sum_{n=0}^{\infty} \binom{n+1}{1} x^n = \sum_{n=0}^{\infty} (n+1)x^n.$$

Recall the geometric series,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

and observe that by differentiating both sides term-by-term with respect to  $x$ , we can derive the generating function for the sequence  $(n+1)$ . In detail, differentiate the left-hand side:

$$\frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{1}{(1-x)^2}.$$

On the right-hand side, notice that since

$$\frac{d}{dx} x^{n+1} = (n+1)x^n,$$

differentiating the series yields

$$\frac{d}{dx} \left( \sum_{n=0}^{\infty} x^n \right) = \sum_{n=0}^{\infty} (n+1)x^n.$$

Thus, we conclude that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

This recovers the generating function for  $k = 1$ . A less formal derivation was given in the subsection *Building Generating Functions* on page 16.

A complete inductive proof follows similar lines but is omitted here for brevity.

**Example 4: Alternating Factorial Sequence.** Define the sequence by

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{(-1)^{\frac{n-1}{2}}}{n!}, & \text{if } n \text{ is odd.} \end{cases}$$

Then the generating function is

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin(x), \quad x \in \mathbb{R}.$$

Even though a full treatment of the Taylor series for trigonometric functions is part of calculus (again, a topic not yet covered here), this example illustrates how generating functions capture nontrivial sequence behavior by connecting discrete structures with analytic functions.

## 7 Generating Function Applications

One key application is the multiplication (or convolution) of generating functions, which naturally arises when we combine two distinct combinatorial constructions into a single, more complex structure.

**Question:** If  $a_k$  counts all objects of type A of size  $k$  and  $b_k$  counts all objects of type B of size  $k$ , how many pairs of objects  $(A, B)$  have a total size of  $n$ ?

**Answer:** The number of such pairs is given by

$$\sum_{k=0}^n a_k b_{n-k}.$$

**Observation:** The generating function for the sequence

$$C_n = \sum_{k=0}^n a_k b_{n-k}$$

is

$$\sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) x^n.$$

It shows that multiplying the generating functions corresponding to  $\{a_n\}$  and  $\{b_n\}$  produces a new generating function whose coefficients are given by the convolution of the two original sequences.

**Example: Dice Sum Counting.** A classic example of this application is counting the number of ways to obtain a given sum when rolling two standard six-sided dice. For a single die, the generating function is:

$$D(x) = x + x^2 + x^3 + x^4 + x^5 + x^6,$$

where the term  $x^k$  corresponds to rolling a  $k$ . Since the two dice are independent, the generating function for the sum of the two dice is:

$$D(x)^2 = (x + x^2 + x^3 + x^4 + x^5 + x^6)^2.$$

Expanding this product, the coefficient of  $x^n$  in  $D(x)^2$  equals the number of ways to achieve a total sum of  $n$ . For instance, one can verify that the coefficient of  $x^7$  is 6, which corresponds to the six possible outcomes that sum to 7 (namely, the pairs  $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$ ).