

# Preparation Tasks 2

Tomasz Brengos  
Committers : Aliaksei Kudzelka

## 1 Problem 1

Prove that the num. of partitions of a positive int  $n$  into  $k$  *even* parts is eq. to the num. of partitions of  $n$  into  $k$  odd parts.

- **Idea:** First, express  $n$  in two ways by decomposing it into  $k$  even parts and into  $k$  odd parts. We can write

$$\begin{aligned}n &= 2a_1 + 2a_2 + \cdots + 2a_k, \\n - k &= 2a_1 + 2a_2 + \cdots + 2a_k - k, \\n - k &= (2a_1 - 1) + (2a_2 - 1) + \cdots + (2a_k - 1).\end{aligned}$$

The first line expresses  $n$  as a sum of  $k$  even numbers (each  $2a_i$ ). Subtracting  $k$  from both sides shows that  $n - k$  can be expressed as a sum of  $k$  odd numbers (each  $2a_i - 1$ ).

From this idea, we expect a correspondence between partitions of  $n$  into  $k$  even parts and partitions of  $n - k$  into  $k$  odd parts.

- **Formal proof:** Define the sets

$$P_e(n, k) := \{\text{partitions of } n \text{ into } k \text{ even parts}\},$$

and

$$P_o(n, k) := \{\text{partitions of } n \text{ into } k \text{ odd parts}\}.$$

For  $n$  even, it turns out these two sets have the same size:

$$|P_e(n, k)| = |P_o(n - k, k)|.$$

To show this, we construct an explicit bijection. Define a function

$$f : P_e(n, k) \rightarrow P_o(n - k, k),$$

by

$$f(b_1, b_2, \dots, b_k) = (b_1 - 1, b_2 - 1, \dots, b_k - 1).$$

Here  $(b_1, \dots, b_k) \in P_e(n, k)$  means  $b_1 + \cdots + b_k = n$  with each  $b_i$  even, so  $b_i \geq 2$  for all  $i$ . Thus, each  $b_i - 1 \geq 1$  and is odd, which ensures  $f(b_1, \dots, b_k) \in P_o(n - k, k)$  is a partition of  $n - k$  into  $k$  odd parts.

The function  $f$  is well-defined. Moreover, it is invertible by simply adding 1 to each part. In fact, define

$$g : P_o(n - k, k) \rightarrow P_e(n, k)$$

as

$$g(c_1, c_2, \dots, c_k) = (c_1 + 1, c_2 + 1, \dots, c_k + 1).$$

If  $(c_1, \dots, c_k)$  is a partition of  $n - k$  into odd parts, then each  $c_i$  is odd and  $c_i \geq 1$ , so  $c_i + 1$  is even and  $\geq 2$ , and  $\sum_{i=1}^k (c_i + 1) = (n - k) + k = n$ . Thus  $g(c_1, \dots, c_k) \in P_e(n, k)$ . It is easy to check that  $g$  is indeed the inverse of  $f$ : we have  $f(g(c_1, \dots, c_k)) = (c_1, \dots, c_k)$  and  $g(f(b_1, \dots, b_k)) = (b_1, \dots, b_k)$ . Therefore,  $f$  is bijective. ■

## 2 Problem 2

Show that the number of partitions of a positive int  $n$  with at most  $k$  components is eq. to the num. of partitions of  $2n$  with at most  $k$  even components.

- **Idea:** We consider partitions of  $n$  that have at most  $k$  parts. Let

$$n = a_1 + a_2 + \cdots + a_L,$$

where  $L \leq k$  and  $a_1 \geq a_2 \geq \cdots \geq a_L > 0$ . (In other words,  $a_1, \dots, a_L$  are the parts of a partition of  $n$ , listed in non-increasing order, with at most  $k$  parts.) For example, if  $n = 5$  and  $k = 3$ , the partitions of 5 with at most 3 parts can be represented (padding with zeros up to 3 parts) as:

$$(2, 2, 1), \quad (3, 2, 0), \quad (5, 0, 0),$$

where we use 0 to indicate an empty part (no number in that position).

Notice that doubling each part in these examples produces a partition of  $2n = 10$  with only even parts (and still at most 3 components). For instance,  $(2, 2, 1)$  doubles to  $(4, 4, 2)$ ,  $(3, 2, 0)$  doubles to  $(6, 4, 0)$ , and  $(5, 0, 0)$  doubles to  $(10, 0, 0)$ . This suggests a direct correspondence between partitions of  $n$  (up to  $k$  parts) and partitions of  $2n$  into even parts (up to  $k$  parts).

- **Formal proof:** Let us define the relevant sets in words (as suggested in the notes):
  - $P(n, \leq k)$  — the set of all partitions of  $n$  with *at most*  $k$  components (parts).
  - $P_e(2n, \leq k)$  — the set of all partitions of  $2n$  with at most  $k$  *even* components.

We aim to show that

$$|P(n, \leq k)| = |P_e(2n, \leq k)|,$$

i.e. the two sets have equal cardinality. To prove this, we construct a bijection  $f : P(n, \leq k) \rightarrow P_e(2n, \leq k)$ . Given any partition  $(a_1, a_2, \dots, a_L)$  of  $n$  (with  $L \leq k$ ), map it to

$$f(a_1, a_2, \dots, a_L) = (2a_1, 2a_2, \dots, 2a_L).$$

In other words,  $f$  doubles each part of the partition of  $n$ . If the partition of  $n$  has fewer than  $k$  parts, we may imagine that it is padded with zeros (as above) which double to zeros, so the resulting partition of  $2n$  still has at most  $k$  parts. By construction,  $f(a_1, \dots, a_L)$  is a partition of  $2n$  in which every part is even, so indeed  $f(a_1, \dots, a_L) \in P_e(2n, \leq k)$ . The function  $f$  is invertible by halving each even part: for any partition  $(b_1, b_2, \dots, b_M) \in P_e(2n, \leq k)$  (each  $b_i$  even), the inverse map  $f^{-1}$  gives

$$f^{-1}(b_1, b_2, \dots, b_M) = \left( \frac{b_1}{2}, \frac{b_2}{2}, \dots, \frac{b_M}{2} \right),$$

which is a partition of  $2n/2 = n$  with at most  $k$  parts. Thus  $f$  is a bijection between  $P(n, \leq k)$  and  $P_e(2n, \leq k)$ , and consequently  $|P(n, \leq k)| = |P_e(2n, \leq k)|$ . ■