# Inclusion-Exclusion Principle

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**Very often, we need to calculate the number of elements in the union of certain sets.** Assuming that we know the sizes of these sets, and their mutual intersections, the principle of inclusion and exclusion allows us to do exactly that.

Suppose you have two sets $A$ and $B$. The size of the union is certainly at most $|A| + |B|$. However, in doing so we count each element of $A \cap B$ twice. To correct for this, we subtract $|A \cap B|$ to obtain

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

In general, the formula gets more complicated because we must take into account intersections of multiple sets. The following statement is what we call the *principle of inclusion and exclusion*:

**Lemma 1.** *For any collection of finite sets $A_1, A_2, \ldots, A_n$, we have*

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|.$$

*Equivalently,*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

**Proof Outline (informal):** *Each element that belongs to exactly $t$ of the sets $A_i$ is counted $\binom{t}{1}$ times in the first summation, then subtracted $\binom{t}{2}$ times in the second summation, added $\binom{t}{3}$ times in the third, and so on. In other words, its total contribution is*

$$\binom{t}{1} - \binom{t}{2} + \binom{t}{3} - \cdots + (-1)^{t-1} \binom{t}{t},$$

*which equals 1. This alternating sum ensures that each element is ultimately counted exactly once, thereby correcting for any overcounting.*