Discrete Mathematics 1 Lectures, Part 1

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1 Counting (Combinatorics)

Counting forms the basis of combinatorics. In these lectures we explore several counting rules, examples, and proofs.

1.1 Rule of Sum (Addition Principle)

If a set S is partitioned into disjoint subsets,

$$S = S_1 \cup S_2 \cup \cdots \cup S_k,$$

then the total number of elements in S is the sum of the number of elements in each subset:

$$|S| = |S_1| + |S_2| + \cdots + |S_k|.$$

Example: Suppose we wish to count the number of ways to choose a subset of a set X of size u, but we only consider subsets of a fixed size k. If we let S be the family of all such subsets, then using the rule of sum by dividing the choices according to a distinguished element (say, whether a chosen element is included or not) we can count the subsets by summing over the possibilities. (This idea is used later in proofs for binomial coefficients and the power set.)

Theorem:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof: Consider $S = \binom{X}{k}$, the set of all subsets of X of size k. Take any element $a \in X$. Define: S_1 as the subsets in S that contain a and S_2 as the subsets in S that do not contain a.

Since every subset of S either contains a or does not, we see that S_1 and S_2 are disjoint and their union forms S, i.e.,

$$S_1 \cup S_2 = S.$$

By the rule of sum, we get:

$$|S| = |S_1| + |S_2|.$$

Now, each subset in S_1 must contain a, so we choose the remaining $k-1$ elements from $X \setminus \{a\}$, which has $n-1$ elements. Thus, $|S_1| = \binom{n-1}{k-1}$. Each subset in S_2 does not contain a, so we choose all k elements from $X \setminus \{a\}$. Thus, $|S_2| = \binom{n-1}{k}$.

Therefore,

$$\binom{n}{k} = |S| = |S_1| + |S_2| = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Example: Let $S = \{\triangle, \square, \circ\}$ and $k = 2$, choosing $a = \circ$. Fixing $\circ$ as one of the elements in the subset of size k, we get:

$$S_1 = \{\{\circ, \triangle\}, \{\circ, \square\}\}.$$

Taking all subsets of size k without $\circ$:

$$S_2 = \{\{\triangle, \square\}\}.$$

We have $|S_1| = 2$ and $|S_2| = 1$, so $|S_1| + |S_2| = 3$.
On the other hand,

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3.$$

Thus, $|S| = |S_1| + |S_2| = 3$, verifying the identity.

1.2 Rule of Product (Multiplication Principle)

When an object is constructed by a sequence of choices, where:

- The first choice can be made in a ways,
- The second in b ways,
- ...

the total number of objects is the product:

$$a \times b \times \cdots .$$

Example: A word of length n over the binary alphabet $\{0, 1\}$ is formed by choosing one of 2 possibilities for each position. Hence, there are

$$2^n$$

possible words.

1.3 Rule of Bijection

If there exists a bijection (a one-to-one and onto mapping) between two sets S and T, then they have the same number of elements:

$$|S| = |T|.$$

Example: Consider the power set of a set X, denoted by $\mathcal{P}(X)$. There is a natural bijection between $\mathcal{P}(X)$ and the set of binary sequences of length $|X|$: for each subset $A \subseteq X$, assign the sequence $(a_1, a_2, \ldots, a_n)$ where

$$a_i = \begin{cases} 1, & \text{if } x_i \in A, \\ 0, & \text{if } x_i \notin A. \end{cases}$$

This shows that

$$|\mathcal{P}(X)| = 2^{|X|}.$$

1.4 Counting in Two Ways

Rule of Counting in Two Ways When two formulae enumerate the same quantity, they must be equal.

Example:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof: Consider a lattice grid of size $(n+1) \times (n+1)$, defined as:

$$X = \{(i,j) \mid i,j \in \{1,2,\dots,n+1\}\}.$$

Clearly, $|X| = (n+1)^2$.

Now, partition X into three subsets: - X_1, the points strictly below the secondary diagonal. - X_2, the points strictly above the secondary diagonal. - X_3, the points on the secondary diagonal itself.

Since these three sets form a partition, we have:

$$|X| = |X_1| + |X_2| + |X_3|.$$

Observing their sizes:

$$|X_1| = |X_2| = 1 + 2 + \cdots + n, \quad |X_3| = n+1.$$

Thus,

$$(n+1)^2 = 2(1 + 2 + \cdots + n) + (n+1).$$

Rearranging, we get:

$$1 + 2 + \cdots + n = \frac{(n+1)^2 - (n+1)}{2} = \frac{n(n+1)}{2}.$$

Hence, we have proven the formula:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

1.5 Binomial Coefficients and Permutations

Let X be a set with $|X| = n$.

Subsets: The number of ways to choose a k-subset of X is given by the binomial coefficient

$$\binom{n}{k}.$$

Permutations: A k-permutation of a set X of size n is a k-word over the alphabet X whose entries are distinct.

Theorem: There are exactly

$$n(n-1)(n-2)\dots(n-k+1)$$

k-permutations of an n-set.

Question: How are k-permutations of an n-set related to k-subsets of an n-set?

Answer: The difference between a k-permutation and a k-subset is that a permutation is ordered, while a subset is not. To express a k-permutation in terms of a k-subset, we need to account for all possible arrangements of the elements, which is $k!$. Thus,

$$k\text{-permutation} = \binom{n}{k} \cdot k!$$

Expressing $\binom{n}{k}$ as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

we obtain:

$$k\text{-permutation} = \frac{n!}{(n-k)!}.$$

Proof by Counting in Two Ways: Count the number of k-permutations of an n-set in two ways:

(1) Directly, by applying the rule of product:

$$n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}.$$

(2) First choose a k-subset (in $\binom{n}{k}$ ways) and then arrange it (in $k!$ ways), giving

$$\binom{n}{k} \cdot k!.$$

Equate these two counts to obtain the relation.

1.6 Binomial Theorem

For any x, y in a field and nonnegative integer n, the binomial theorem states:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Explanation: This theorem is a direct consequence of counting the number of ways to choose k copies of x (and the remaining $n-k$ copies of y) when expanding the product.

1.7 Multisets

Definition: A multiset of a set X of size n is a function

$$m : X \to \mathbb{N}$$

that assigns a non-negative integer to each element of X, representing its multiplicity in the multiset.

Example: Let $X = \{a, b, c\}$, and consider the multiset $\{a, a, b\}$. Then, the function m is given by:

$$m(a) = 2, \quad m(b) = 1, \quad m(c) = 0.$$

Question: What is the number of k-multisets of a set of size n?

Theorem: The number of all k-multisets of an n-set is

$$\binom{n+k-1}{k}.$$

Proof: Let X be the set of all k-multisets of an n-set. Let Y be the set of all distributions of k identical objects into n buckets.

Claim 1: There is a bijection from X to Y. Thus, by the rule of bijection, we have

$$|X| = |Y|.$$

Claim 2: Let Z be the set of all binary sequences of length $n+k-1$ with exactly $n-1$ ones (or equivalently, k zeros). There is a bijection from Y to Z. Hence,

$$|Y| = |Z| \Rightarrow |X| = |Z|.$$

Since the number of such binary sequences is given by

$$\binom{n+k-1}{k},$$

we conclude that

$$|X| = \binom{n+k-1}{k}.$$

1.8 Lattice Paths

Consider an $m \times n$ grid with lattice points at the intersections.

Problem: How many paths are there from $(0,0)$ to (m,n) if one may only move right or up?

Solution: Every path consists of exactly m right moves and n up moves. Thus, a path can be represented as a sequence of $m+n$ moves, where we choose n positions (out of $m+n$) for the up moves. Hence, the number of paths is:

$$\binom{m+n}{n}.$$

Bijection Explanation: There is a bijection between the set of such lattice paths and the set of binary sequences of length $m+n$ with exactly n ones (representing the up moves).

2 Partitions and Stirling Numbers

2.1 Set Partitions

A *partition* of a set N is a way to write N as a union of disjoint nonempty subsets (called blocks).

Example: For $N = \{1,2,3,4\}$, one partition into 2 blocks could be $\{\{1,3\},\{2,4\}\}$.

2.2 Stirling Numbers of the Second Kind

The number of ways to partition an n-element set into k nonempty blocks is denoted by $S(n,k)$.

Recurrence Relation: These numbers satisfy the recurrence:

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k).$$

Proof Idea: When adding a new element to an $(n-1)$-element set:
- It can form a new block (contributing $S(n-1,k-1)$),
- Or it can join one of the k existing blocks (contributing $k\,S(n-1,k)$).

Bell Numbers: The total number of partitions of an n-set is given by the Bell number:

$$B_n = \sum_{k=1}^{n} S(n,k).$$

3 Inclusion-Exclusion Principle

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then the size of their union is given by:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Proof Outline: Each element that belongs to exactly t of the sets A_i is counted $\binom{t}{1}$ times in the first summation, subtracted $\binom{t}{2}$ times in the second, and so on. The alternating sum ensures that each element is counted exactly once.

4 Permutations and Derangements

4.1 Permutations

A permutation of an n-set is an arrangement of its elements. There are

$$n!$$

possible permutations.

4.2 Derangements

A *derangement* is a permutation in which no element appears in its original position.

Counting Derangements: The number of derangements of an n-set, denoted $D(n)$, can be computed by:

$$D(n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Proof using Inclusion-Exclusion: Define, for each i, the set

$$A_i = \{\text{permutations in which the } i\text{-th element is fixed}\}.$$

Then, the number of derangements is the total number of permutations minus those that have at least one fixed point. Applying the inclusion-exclusion principle yields:

$$D(n) = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0!.$$

This simplifies to the formula above.

5 Functions Between Sets

Let N and R be sets with $|N| = n$ and $|R| = r$.

(i) **Total Functions:** The number of functions from N to R is

$$r^n.$$

(ii) **Injective Functions:** When $r \geq n$, an injective function (one-to-one) from N to R can be chosen by assigning distinct images to the n elements. Thus, the number is:

$$r \cdot (r-1) \cdots (r-n+1) = \frac{r!}{(r-n)!}.$$

(iii) **Surjective Functions:** A function is surjective (onto) if every element in R is an image. Using partitions of the domain and applying the Stirling numbers, the number of surjective functions is:

$$r!\, S(n,r),$$

where $S(n,r)$ is the Stirling number of the second kind.

Example: For $N = \{1,2,3\}$ and $R = \{a,b\}$:
- Total functions: $2^3 = 8$.
- Injective functions: Not possible since $|R| < |N|$.
- Surjective functions: Here $r = 2$, and one can verify directly by listing functions that cover both a and b.

Additional Examples and Proofs from the Notes

Example on Counting Words

Let S be the set of all words of length n over the alphabet $\{0,1\}$. By the rule of product, each letter has 2 choices, and hence

$$|S| = 2^n.$$

Furthermore, if one wants to count the number of words with a given number of zeros and ones, one uses the binomial coefficient.

Example on Bijections for Power Sets

Consider a set $X = \{x_1, x_2, \ldots, x_n\}$. Each subset of X can be represented by an n-tuple of 0's and 1's. The mapping that sends each subset to its corresponding binary vector is a bijection. This proves that

$$|\mathcal{P}(X)| = 2^n.$$

Proof of the Recurrence for Stirling Numbers

Given an n-set, consider the addition of a new element x. When partitioning the set into k blocks, either:
- x forms a block by itself (which gives $S(n-1,k-1)$ partitions), or
- x is added to one of the k blocks of a partition of the remaining $n-1$ elements (which gives $k\,S(n-1,k)$ partitions).

Thus, we obtain

$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k).$$

Proof of Derangements using Inclusion-Exclusion

For the set $N = \{1,2,\ldots,n\}$, define

$$A_i = \{\sigma \in S_n \mid \sigma(i) = i\}.$$

Then, by the inclusion-exclusion principle,

$$D(n) = n! - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.$$

Since $|A_i| = (n-1)!$, $|A_i \cap A_j| = (n-2)!$, and in general

$$|A_{i_1} \cap \cdots \cap A_{i_k}| = (n-k)!,$$

we have:

$$D(n) = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \right].$$