Generating Functions

Tomasz Brengos
Committers : Mykhailo Moroz, Mihail Orlov

1 Generating Series

Instead of viewing a sequence as a function that returns its nth term, a *generating series* packages all of its terms into a single power series whose coefficients are exactly the sequence entries. Concretely, the sequence

$$2,\ 3,\ 5,\ 8,\ 12,\ \ldots$$

is encoded by the generating series

$$2 + 3x + 5x^2 + 8x^3 + 12x^4 + \cdots .$$

In general, given any sequence $\{c_n\}_{n\geq 0}$, its generating series is the formal power series

$$G(x) \ = \ \sum_{n=0}^{\infty} c_n x^n \ = \ c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots .$$

We say that $G(x)$ "generates" the sequence $\{c_n\}$ because each coefficient of x^n in $G(x)$ is precisely c_n. Generating series turn sequence-based problems into algebraic manipulations of power series, a technique we will exploit heavily in what follows.

Recall of the Basic Series

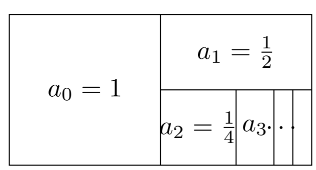

Figure 1: A geometric interpretation of the binary series, showing how $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$.

A Geometric View of the Binary Series For $|x| < 1$, we have the infinite geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n.$$

We now present a quick proof of this result by performing long division of 1 by $1 - x$.

$$\begin{array}{r|l} & 1+x+x^2+x^3+\cdots \\ 1-x & 1 \\ \hline & \underline{1-x} \\ & x \\ & \underline{x-x^2} \\ & x^2 \\ & \underline{x^2-x^3} \\ & x^3 \\ & \vdots \end{array}$$

The process works as follows: The long-division proceeds by repeatedly dividing the current remainder by the leading term of the divisor, producing one new power of x at each step:

1. Divide 1 by $1-x$. The multiplier needed to eliminate the constant term is 1, so
$$1 - 1 \cdot (1-x) = x.$$
Thus the first summand is 1, leaving a remainder of x.
2. Divide the remainder x by $1-x$. The multiplier is x, so
$$x - x \cdot (1-x) = x^2.$$
Hence the second summand is x, leaving a remainder of x^2.
3. Divide x^2 by $1-x$. The multiplier is x^2, giving
$$x^2 - x^2 \cdot (1-x) = x^3.$$
Therefore the third summand is x^2, with remainder x^3.
4. Continuing in this fashion produces the infinite series
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$$

Continuing indefinitely produces

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n,$$

as claimed.
We will use this fact in further examples throughout the notes.

2 Building Generating Functions

The simplest (or “basic”) generating function is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots,$$

which generates the constant sequence $1, 1, 1, \ldots$.

Replacing x with $-x$:

$$\frac{1}{1-(-x)} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots\,,$$

generating $1, -1, 1, -1, \ldots$.

Replacing x with $3x$:

$$\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots\,,$$

generating $1, 3, 9, 27, \ldots$.

Scaling a sequence by 3:

$$\frac{3}{1-3x} = 3 + 9x + 27x^2 + 81x^3 + \cdots\,,$$

generating $3, 9, 27, 81, \ldots$.

Termwise addition of sequences:
Adding the generating functions for $1, 1, 1, \ldots$ and $1, 3, 9, \ldots$ gives

$$\frac{1}{1-x} + \frac{1}{1-3x} = 2 + 4x + 10x^2 + 28x^3 + \cdots\,,$$

which generates $2, 4, 10, 28, \ldots$.

Replacing x with x^2:

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots\,,$$

generating $1, 0, 1, 0, 1, 0, \ldots$.

Shifting a sequence:
Multiplying by x shifts all coefficients right by one:

$$\frac{x}{1-3x} = 0 + x + 3x^2 + 9x^3 + \cdots\,,$$

generating $0, 1, 3, 9, \ldots$, and

$$\frac{x}{1-x^2} = 0 + x + 0x^2 + x^3 + \cdots\,,$$

generating $0, 1, 0, 1, \ldots$.

Combining shifted sequences:
Adding the two "even-odd" generating functions recovers

$$\frac{1}{1-x^2} + \frac{x}{1-x^2} = \frac{1+x}{1-x^2} = \frac{1}{1-x},$$

which generates $1, 1, 1, 1, \ldots$.

Differentiation:
Differentiating the basic Generating Function

$$\frac{d}{dx}\Big(\frac{1}{1-x}\Big) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots\,,$$

yields the generating function for $1, 2, 3, 4, \ldots$.

3 Recurrence Relations & Generating Functions

We conclude with an example of one of the many reasons studying generating functions is helpful: solving recurrence relations via algebraic manipulation of power series.

Example: Tower of Hanoi The minimum number of moves required to transfer n disks satisfies

$$a_0 = 0, \quad a_1 = 1, \quad a_n = 2\,a_{n-1} + 1 \quad (n \geq 1),$$

giving the sequence

$$0,\ 1,\ 3,\ 7,\ 15,\ 31, \ldots$$

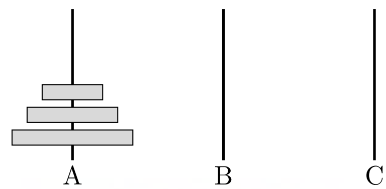

Figure 2: Initial configuration for Tower of Hanoi (3 disks).

Define the generating function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Using the recurrence for $n \geq 1$:

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \big(2a_{n-1} + 1\big) x^n = 2x \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} x^n,$$

so

$$f(x) - a_0 = 2x\, f(x) + \frac{x}{1-x},$$

and since $a_0 = 0$,

$$f(x) = \frac{x}{(1-x)(1-2x)}.$$

Performing partial fractions:

$$\frac{x}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{1}{1-2x},$$

hence

$$f(x) = -\frac{1}{1-x} + \frac{1}{1-2x}.$$

Extracting coefficients yields the closed-form solution

$$a_n = 2^n - 1,$$

confirming the well-known formula for the Tower of Hanoi moves.

4 Introduction to the Fibonacci Sequence

The Fibonacci sequence famously arises from a puzzle involving rabbit populations. Imagine starting with a single pair of rabbits that takes one month to mature. After maturing, each pair produces a new pair of rabbits every month. Mathematically, if F_n represents the number of rabbit pairs in month n, the sequence satisfies the initial conditions

$$F_0 = 0, \quad F_1 = 1,$$

and the recurrence

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0.$$

Q: is there a non-recursive (closed-form) formula for F_n ?

1) || (2 small rabbits)

2) || + || (1 big pair + 1 small pair)

3) || + || + || (2 big pairs + 1 small pair)

4) || + || + || + || + || (3 big pairs + 2 small pairs)

Figure 3: Illustration of rabbit pairs over successive months. Blue bars represent small rabbits; orange bars represent big (mature) rabbits.

Idea: consider and calculate it

5 Deriving the Closed-Form for the Fibonacci Sequence

Step 1: Define the generating function. Let $\{F_n\}_{n=0}^{\infty}$ be the Fibonacci sequence with

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0).$$

Define the generating function

$$f(x) \;=\; \sum_{n=0}^{\infty} F_n \, x^n.$$

We aim to find a closed-form expression for $f(x)$, and then extract a formula for F_n.

Step 2: Use the Fibonacci recurrence in $f(x)$. Starting from

$$f(x) \;=\; F_0 + F_1 x + \sum_{n=2}^{\infty} F_n \, x^n,$$

and noting $F_0 = 0$, $F_1 = 1$, we have

$$f(x) \;=\; x + \sum_{n=2}^{\infty} \bigl(F_{n-1} + F_{n-2}\bigr) \, x^n$$

because $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Separate the sums:

$$f(x) = x + \sum_{n=2}^{\infty} F_{n-1} \, x^n + \sum_{n=2}^{\infty} F_{n-2} \, x^n.$$

Shift indices to factor out $f(x)$:

$$\sum_{n=2}^{\infty} F_{n-1} \, x^n = x \sum_{n=2}^{\infty} F_{n-1} \, x^{n-1} = x \sum_{m=1}^{\infty} F_m \, x^m = x\bigl(f(x) - F_0\bigr) = x \, f(x),$$

since $F_0 = 0$. Similarly,

$$\sum_{n=2}^{\infty} F_{n-2} \, x^n = x^2 \sum_{n=2}^{\infty} F_{n-2} \, x^{n-2} = x^2 \sum_{k=0}^{\infty} F_k \, x^k = x^2 \, f(x).$$

Hence,

$$f(x) = x + x \, f(x) + x^2 \, f(x) \quad \Longrightarrow \quad f(x)\bigl(1 - x - x^2\bigr) = x.$$

Thus,

$$f(x) = \frac{x}{1 - x - x^2}.$$

Step 3: Partial-Fraction Decomposition (as in the images). First, rewrite

$$\frac{1}{1-x-x^2} \;=\; \frac{1}{-\bigl(x^2+x-1\bigr)} \;=\; -\,\frac{1}{x^2+x-1}.$$

Next, factor x^2+x-1. Observe that the roots of

$$x^2+x-1 \;=\; 0$$

are

$$x \;=\; -\,\frac{1+\sqrt{5}}{2} \quad \text{and} \quad x \;=\; -\,\frac{1-\sqrt{5}}{2}.$$

Hence,

$$x^2+x-1 \;=\; \Bigl(x+\tfrac{1+\sqrt{5}}{2}\Bigr)\cdot\Bigl(x+\tfrac{1-\sqrt{5}}{2}\Bigr).$$

Therefore,

$$-\,\frac{1}{x^2+x-1} \;=\; -\,\frac{1}{\Bigl(x+\tfrac{1+\sqrt{5}}{2}\Bigr)\Bigl(x+\tfrac{1-\sqrt{5}}{2}\Bigr)}.$$

We look for constants A and B such that

$$-\,\frac{1}{\Bigl(x+\tfrac{1+\sqrt{5}}{2}\Bigr)\Bigl(x+\tfrac{1-\sqrt{5}}{2}\Bigr)} \;=\; \frac{A}{x+\tfrac{1+\sqrt{5}}{2}} \;+\; \frac{B}{x+\tfrac{1-\sqrt{5}}{2}}.$$

Step 4: Solve for A and B. Comparing coefficients of x and the constant term in

$$-1 \;=\; A\Bigl(x+\beta\Bigr) \;+\; B\Bigl(x+\alpha\Bigr),$$

we obtain the system

$$\begin{cases} A+B=0, \\ A\,\beta+B\,\alpha=-\,1. \end{cases}$$

It follows that

$$B=-\,A, \quad A\,(\beta-\alpha)=-\,1 \;\Longrightarrow\; A=\frac{1}{\alpha-\beta} \quad \text{and} \quad B=-\,\frac{1}{\alpha-\beta}.$$

Hence,

$$-\,\frac{1}{(x+\alpha)(x+\beta)} \;=\; \frac{1}{\alpha-\beta}\,\frac{1}{x+\alpha} \;-\; \frac{1}{\alpha-\beta}\,\frac{1}{x+\beta}.$$

Step 5: Combine with the earlier factor -1 and rewrite. Recalling that

$$\frac{1}{1-x-x^2} \;=\; -\,\frac{1}{x^2+x-1} \;=\; -\,\frac{1}{(x+\alpha)\,(x+\beta)},$$

we combine the above result to conclude

$$\frac{1}{1-x-x^2} \;=\; \frac{1}{\alpha-\beta}\left(\frac{1}{x+\alpha} \;-\; \frac{1}{x+\beta}\right).$$

Step 6: Expand each term in a power series. Notice that

$$\frac{1}{x+\alpha} \;=\; \frac{1}{\alpha}\,\frac{1}{1+\frac{x}{\alpha}} \;=\; \frac{1}{\alpha}\sum_{n=0}^{\infty}\big(-\tfrac{x}{\alpha}\big)^{n} \;=\; \sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha^{\,n+1}}\,x^n,$$

valid for $\left|\frac{x}{\alpha}\right| < 1$. Similarly,

$$\frac{1}{x+\beta} \;=\; \sum_{n=0}^{\infty}\frac{(-1)^n}{\beta^{\,n+1}}\,x^n.$$

Hence,

$$\frac{1}{1-x-x^2} \;=\; \frac{1}{\alpha-\beta}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha^{\,n+1}}\,x^n \;-\; \sum_{n=0}^{\infty}\frac{(-1)^n}{\beta^{\,n+1}}\,x^n\right] = \sum_{n=0}^{\infty}\left[\frac{1}{\alpha-\beta}\Big(\frac{(-1)^n}{\alpha^{n+1}} \;-\; \frac{(-1)^n}{\beta^{n+1}}\Big)\right]x^n.$$

Step 7: Identify Fibonacci numbers. Recall that $\alpha-\beta=\sqrt{5}$, and

$$F_n \;=\; \frac{\alpha^n-\beta^n}{\alpha-\beta} \;=\; \frac{\alpha^n-\beta^n}{\sqrt{5}}.$$

One checks (or uses known identities) to see that the coefficient of x^n in the above power series is exactly F_n. Consequently,

$$\sum_{n=0}^{\infty}F_n\,x^n \;=\; \frac{1}{1-x-x^2},$$

which is the generating function for the Fibonacci sequence.

Conclusion. We have shown that the generating function for the Fibonacci sequence is $\frac{x}{1-x-x^2}$. Through partial fractions and comparing coefficients, we deduced that

$$F_n = \frac{\alpha^n-\beta^n}{\sqrt{5}}.$$

This gives a non-recursive (closed-form) expression for F_n, completing the derivation.

6 More examples

In earlier sections (see, e.g., *A Geometric View of the Binary Series* on page 14), we explored methods to solve recurrences and introduced generating functions as a tool to transform sequences into functions. In this section, we briefly reiterate these ideas and demonstrate, through several examples, how generating functions serve as a bridge between discrete mathematics and calculus.

Example 1: Constant Sequence. Consider the sequence defined by

$$a_n = 1 \quad \text{for all } n \geq 0,$$

so that the sequence is

$$1, 1, 1, \ldots.$$

By the geometric series formula (proved earlier), its generating function is given by

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Example 2: Exponential Sequence. Now, let

$$a_n = \frac{1}{n!}.$$

Then the generating function is

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \quad x \in \mathbb{R}.$$

A partial justification of this result can be obtained by recalling the Taylor series expansion of the exponential function. Although a complete treatment of Taylor series is a topic in calculus (not yet covered in this course), note that differentiating the power series term-by-term confirms the identity.

Example 3: Binomial Coefficient Sequence. Consider the sequence defined by

$$a_n = \binom{n+k}{k}.$$

Theorem. The generating function for this sequence is

$$f(x) = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}, \quad |x| < 1.$$

Proof.

- For $k = 1$: Note that
$$a_n = \binom{n+1}{1} = n + 1,$$
so that
$$f(x) = \sum_{n=0}^{\infty} \binom{n+1}{1} x^n = \sum_{n=0}^{\infty} (n+1)x^n.$$
Recall the geometric series,
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$
and observe that by differentiating both sides term-by-term with respect to x, we can derive the generating function for the sequence $(n + 1)$. In detail, differentiate the left-hand side:
$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$
On the right-hand side, notice that since
$$\frac{d}{dx} x^{n+1} = (n+1)x^n,$$
differentiating the series yields
$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} (n+1)x^n.$$
Thus, we conclude that
$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$
This recovers the generating function for $k = 1$. A less formal derivation was given in the subsection *Building Generating Functions* on page 16.

A complete inductive proof follows similar lines but is omitted here for brevity.

Example 4: Alternating Factorial Sequence. Define the sequence by

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{(-1)^{\frac{n-1}{2}}}{n!}, & \text{if } n \text{ is odd.} \end{cases}$$

Then the generating function is

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin(x), \quad x \in \mathbb{R}.$$

Even though a full treatment of the Taylor series for trigonometric functions is part of calculus (again, a topic not yet covered here), this example illustrates how generating functions capture nontrivial sequence behavior by connecting discrete structures with analytic functions.

7 Generating Function Applications

One key application is the multiplication (or convolution) of generating functions, which naturally arises when we combine two distinct combinatorial constructions into a single, more complex structure.

Question: If a_k counts all objects of type A of size k and b_k counts all objects of type B of size k, how many pairs of objects (A, B) have a total size of n?

Answer: The number of such pairs is given by

$$\sum_{k=0}^{n} a_k\, b_{n-k}.$$

Observation: The generating function for the sequence

$$C_n = \sum_{k=0}^{n} a_k\, b_{n-k}$$

is

$$\sum_{n=0}^{\infty} C_n\, x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k\, b_{n-k} \right) x^n.$$

It shows that multiplying the generating functions corresponding to $\{a_n\}$ and $\{b_n\}$ produces a new generating function whose coefficients are given by the convolution of the two original sequences.

Example: Dice Sum Counting. A classic example of this application is counting the number of ways to obtain a given sum when rolling two standard six-sided dice. For a single die, the generating function is:

$$D(x) = x + x^2 + x^3 + x^4 + x^5 + x^6,$$

where the term x^k corresponds to rolling a k. Since the two dice are independent, the generating function for the sum of the two dice is:

$$D(x)^2 = \left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)^2.$$

Expanding this product, the coefficient of x^n in $D(x)^2$ equals the number of ways to achieve a total sum of n. For instance, one can verify that the coefficient of x^7 is 6, which corresponds to the six possible outcomes that sum to 7 (namely, the pairs $(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$).