Discrete Mathematics 1 Lectures, Part 1

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1 Permutations

A *permutation* of n elements is an arrangement (ordering) of those elements. For example, there are 6 permutations of the set $\{a, b, c\}$:

$$(a, b, c), \quad (a, c, b), \quad (b, a, c), \quad (b, c, a), \quad (c, a, b), \quad (c, b, a).$$

Since there are 3 choices for the first element, 2 for the second (once the first is chosen), and 1 for the last, by the multiplicative principle there are $3 \cdot 2 \cdot 1 = 3! = 6$ permutations in total.

Factorials and counting. In general, the number of permutations of n (distinct) elements is given by

$$n! \;=\; n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

Partial permutations (k-permutations). Sometimes we only permute k of the n elements, where $1 \leq k \leq n$. The number of ways to do this is denoted $P(n, k)$ and can be found by thinking:

$$P(n, k) \;=\; n \times (n-1) \times \cdots \times (n-k+1).$$

There are k factors in that product. Using factorial notation, we can write

$$P(n, k) \;=\; \frac{n!}{(n-k)!}.$$

Relationship to combinations. An alternate derivation uses combinations: first *choose* which k elements from the n will appear (that can be done in $\binom{n}{k}$ ways), then *arrange* those k in order (which can be done in $k!$ ways). Hence,

$$P(n, k) \;=\; \binom{n}{k} k!.$$

Since $\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$, multiplying by $k!$ yields exactly $\dfrac{n!}{(n-k)!}$, consistent with the direct counting approach.

2 Derangements

A *derangement* of n elements is a permutation where no element remains in its original position. More precisely, if we think of a permutation as a bijection θ on the set $\{1, 2, \ldots, n\}$, then θ is a derangement if and only if

$$\theta(k) \;\neq\; k \quad \text{for all } k \in \{1, 2, \ldots, n\}.$$

Equivalently, a derangement has no fixed points.

For example, for $n = 3$, the permutations of $\{1, 2, 3\}$ are:

$$(1, 2, 3), \quad (1, 3, 2), \quad (2, 1, 3), \quad (2, 3, 1), \quad (3, 1, 2), \quad (3, 2, 1).$$

Among these, the derangements are $(2, 3, 1)$ and $(3, 1, 2)$; the other permutations fix at least one of the elements.

Counting Derangements via Inclusion-Exclusion

Let $D(n)$ denote the number of derangements of n elements. We will use the principle of inclusion-exclusion. Suppose we label the elements as $1, 2, \ldots, n$, and define A_i to be the set of permutations that fix the element i (i.e. $\theta(i) = i$). Then any derangement is a permutation that lies in none of the sets A_i (for $1 \leq i \leq n$). We have

$$\left|A_i\right| \;=\; (n-1)!,$$

since if we fix one position i, then we permute the remaining $n - 1$ elements freely. In general,

$$\left|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right| \;=\; (n-k)!.$$

By inclusion-exclusion, the size of the union $A_1 \cup A_2 \cup \cdots \cup A_n$ is

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)!.$$

Hence the number of permutations that do not lie in this union—i.e. the number of derangements—is

$$D(n) \;=\; n! \;-\; \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n}(n-n)!.$$

$$D(n) \;=\; \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! \;=\; n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Thus, a concise closed-form for the number of derangements is

$$D(n) \;=\; n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Note on the series for e^{-1}:
In Calculus, one learns that the exponential function has a power series expansion

$$e^x \;=\; \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Setting $x = -1$ gives

$$e^{-1} \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

Hence,

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} \;\xrightarrow[n \to \infty]{}\; e^{-1}.$$

If you have not taken (or do not recall) a full course in Calculus, think of this as a special case of a well-known infinite series expansion for the exponential function.

Since the finite sum $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$ converges to e^{-1} as $n \to \infty$, we conclude that

$$\lim_{n\to\infty} \frac{D(n)}{n!} \;=\; \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \;=\; e^{-1}.$$

Numerically, this means that for large n, about $1/e \approx 36.8\%$ of all permutations of $\{1, \dots, n\}$ are derangements (i.e. have no fixed points).

A Recurrence Relation

We can also show that $D(n)$ satisfies the recurrence

$$D(n) \;=\; (n-1)\big(D(n-1) + D(n-2)\big), \quad \text{with } D(1) = 0,\, D(2) = 1.$$

One way to see this: consider where 1 goes in a derangement of $\{1, 2, \dots, n\}$. It can go to any of $n-1$ positions. If 1 goes to position j, then either (i) the element j goes to position 1 (a swap), which reduces the problem to deranging the remaining $n-2$ elements, or (ii) the element j does *not* go to position 1, effectively reducing the problem to deranging $n-1$ elements. This yields the above recurrence.