# Permutations and Derangements

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### 1 Permutations

A permutation of n elements is an arrangement (ordering) of those elements. For example, there are 6 permutations of the set  $\{a, b, c\}$ :

$$(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a).$$

Since there are 3 choices for the first element, 2 for the second (once the first is chosen), and 1 for the last, by the multiplicative principle there are  $3 \cdot 2 \cdot 1 = 3! = 6$  permutations in total.

**Factorials and counting.** In general, the number of permutations of n (distinct) elements is given by

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

**Partial permutations (k-permutations).** Sometimes we only permute k of the n elements, where  $1 \le k \le n$ . The number of ways to do this is denoted P(n,k) and can be found by thinking:

$$P(n,k) = n \times (n-1) \times \cdots \times (n-k+1).$$

There are k factors in that product. Using factorial notation, we can write

$$P(n,k) = \frac{n!}{(n-k)!}.$$

**Relationship to combinations.** An alternate derivation uses combinations: first *choose* which k elements from the n will appear (that can be done in  $\binom{n}{k}$  ways), then *arrange* those k in order (which can be done in k! ways). Hence,

$$P(n,k) = \binom{n}{k} k!.$$

Since  $\binom{n}{k} = \frac{n!}{(n-k)! \, k!}$ , multiplying by k! yields exactly  $\frac{n!}{(n-k)!}$ , consistent with the direct counting approach.

## 2 Derangements

A derangement of n elements is a permutation where no element remains in its original position. More precisely, if we think of a permutation as a bijection  $\theta$  on the set  $\{1, 2, ..., n\}$ , then  $\theta$  is a derangement if and only if

$$\theta(k) \neq k$$
 for all  $k \in \{1, 2, \dots, n\}$ .

Equivalently, a derangement has no fixed points.

For example, for n = 3, the permutations of  $\{1, 2, 3\}$  are:

$$(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1).$$

Among these, the derangements are (2,3,1) and (3,1,2); the other permutations fix at least one of the elements.

## Counting Derangements via Inclusion-Exclusion

Let D(n) denote the number of derangements of n elements. We will use the principle of inclusion-exclusion. Suppose we label the elements as 1, 2, ..., n, and define  $A_i$  to be the set of permutations that fix the element i (i.e.  $\theta(i) = i$ ). Then any derangement is a permutation that lies in none of the sets  $A_i$  (for  $1 \le i \le n$ ). We have

$$|A_i| = (n-1)!,$$

since if we fix one position i, then we permute the remaining n-1 elements freely. In general,

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!.$$

By inclusion-exclusion, the size of the union  $A_1 \cup A_2 \cup \cdots \cup A_n$  is

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)!.$$

Hence the number of permutations that do not lie in this union—i.e. the number of derangements—is

$$D(n) = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \dots + (-1)^n \binom{n}{n}(n-n)!.$$

$$D(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Thus, a concise closed-form for the number of derangements is

$$D(n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

#### Note on the series for $e^{-1}$ :

In Calculus, one learns that the exponential function has a power series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Setting x = -1 gives

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

Hence,

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} \xrightarrow[n \to \infty]{} e^{-1}.$$

If you have not taken (or do not recall) a full course in Calculus, think of this as a special case of a well-known infinite series expansion for the exponential function.

Since the finite sum  $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$  converges to  $e^{-1}$  as  $n \to \infty$ , we conclude that

$$\lim_{n \to \infty} \frac{D(n)}{n!} \; = \; \lim_{n \to \infty} \sum_{k=0}^n \frac{(-1)^k}{k!} \; = \; e^{-1}.$$

Numerically, this means that for large n, about  $1/e \approx 36.8\%$  of all permutations of  $\{1, \ldots, n\}$  are derangements (i.e. have no fixed points).

### A Recurrence Relation

We can also show that D(n) satisfies the recurrence

$$D(n) = (n-1)(D(n-1) + D(n-2)), \text{ with } D(1) = 0, D(2) = 1.$$

One way to see this: consider where 1 goes in a derangement of  $\{1, 2, ..., n\}$ . It can go to any of n-1 positions. If 1 goes to position j, then either (i) the element j goes to position 1 (a swap), which reduces the problem to deranging the remaining n-2 elements, or (ii) the element j does not go to position 1, effectively reducing the problem to deranging n-1 elements. This yields the above recurrence.