## Functions between sets

Tomasz Brengos Committers : Mykhailo Moroz

Let N and R be sets with |N| = n and |R| = r.

(i) **Total Functions:** The number of functions from N to R is

$$r^n$$
.

Explanation: For every element in N, there are |R| = r possible values in R. Thus, for the first element, there are r choices, for the second element, there are r choices, and so on. Applying the rule of product, the total number of functions is  $r^n$ .

(ii) **Injective Functions:** When  $r \ge n$ , an injective function (one-to-one) from N to R can be chosen by assigning distinct images to the n elements.

If a function is injective, then for each value in the range there is only one corresponding argument. This means that function values cannot repeat, ensuring that  $x_1 \neq x_2$  implies  $f(x_1) \neq f(x_2)$ .

Since there are |R| = r choices for the first argument, r - 1 choices for the second, r - 2 for the third, and so on, applying the rule of product, the number of injective functions from N to R is:

$$r \cdot (r-1) \cdots (r-n+1) = \frac{r!}{(r-n)!}.$$

(iii) Surjective Functions: A function is surjective (onto) if every element in R has a pre-image in N, meaning every element in R is an image of some element in N. Consider a surjection  $f: N \to R = \{y_1, y_2, \ldots, y_r\}$ . We observe that the preimages  $f^{-1}(y_1), f^{-1}(y_2), \ldots, f^{-1}(y_r)$  form a partition of N into r non-empty subsets, as each element  $y_i$  in R corresponds to one or more elements from N. The number of ways to partition N into r parts is given by the Stirling number S(n, r), and since we can permute the r elements in R in r! ways, the total number of surjective functions from N to R is:

where S(n,r) is the Stirling number of the second kind, counting the ways to partition N into r non-empty subsets.

**Example:** For  $N = \{1, 2, 3\}$  and  $R = \{y_1, y_2\}$ :

Here |N| = 3 and |R| = 2.

- Total functions:  $2^3 = 8$ .
- Injective functions: Not possible since |R| < |N|.
- Surjective functions: Consider all possible surjective functions:

 $f_1: \{1,2\} \mapsto y_1, 3 \mapsto y_2$  - Another possible permutation for this partition:  $f_2: \{1,2\} \mapsto y_2, 3 \mapsto y_1$   $f_3: \{2,3\} \mapsto y_1, 1 \mapsto y_2$  - Another possible permutation for this partition:  $f_4: 1 \mapsto y_1, \{2,3\} \mapsto y_2$   $f_5: \{1,3\} \mapsto y_1, 2 \mapsto y_2$  - Another possible permutation for this partition:  $f_6: 2 \mapsto y_1, \{1,3\} \mapsto y_2$ So, we have 6 surjective functions. Using the formula for surjective functions, we first find the Stirling number S(3,2)=3, which corresponds to the number of partitions without considering permutations. Then, accounting for the permutations of the r=2 elements in R, we compute:

$$2! \cdot S(3,2) = 2! \cdot 3 = 6,$$

which matches the number of surjective functions we listed.

