Quantum Annealing: Unveiling the Quantum Solution to Optimization

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Introduction

Quantum annealing is a heuristic quantum optimization algorithm that can be used to solve combinatorial optimization problems. In quantum annealers, each state can be represented as an energy level. These states are simulated in a short time by taking advantage of the superposition and entanglement properties of qubits and the lowest energy result is obtained. The lowest energy state gives the optimal solution or the most likely solution.

By encoding the problem and the solution into their respective Hamiltonians, quantum annealing harnesses quantum tunneling to enable states to traverse high-energy barriers and settle into minima, thereby generating optimal solutions.

In our work, we aim to elucidate the mathematical and physical foundations that underpin this approach, demonstrating how quantum mechanics contributes to speed and accuracy enhancements. We compare these quantum methods with classical approaches, assess the current state of the field, and discuss prospects and applications.

The Quantum Advantage

Quantum annealing starts with qubits in superposition, representing all possible answers. As the anneal progresses, biases and couplers reshape the energy landscape, forming a series of potential wells that correlate with solution states. The application of an external magnetic field (the bias) tilts the landscape, making certain states energetically favorable. Couplers create entanglement, enabling complex multi-qubit interactions that define the problem's constraints.

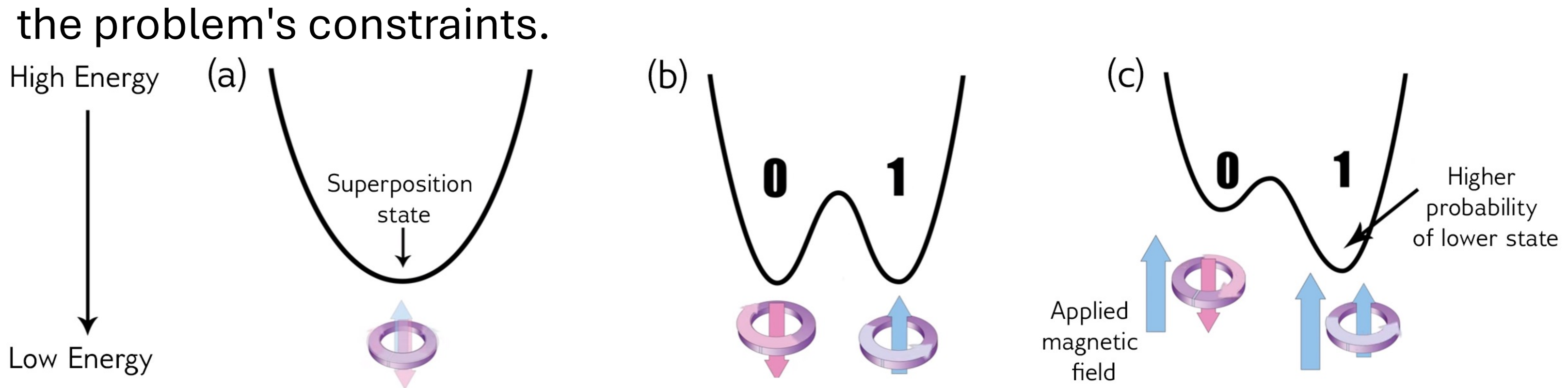

Figure 1. Source: D-Wave

The quantum system explores this landscape through superposition and quantum tunneling, seeking the lowest energy state. Initially, qubits can tunnel through barriers representing conflicting solutions. Over time, as the annealing process carefully adjusts the landscape, the system's evolution is guided by the adiabatic theorem towards the ground state, corresponding to the optimal solution.

Finally, each qubit settles into a classical state that represents the best answer found in this quantum exploration. This unique combination of quantum phenomena enables quantum annealers to solve complex optimization tasks that are challenging for classical computers.

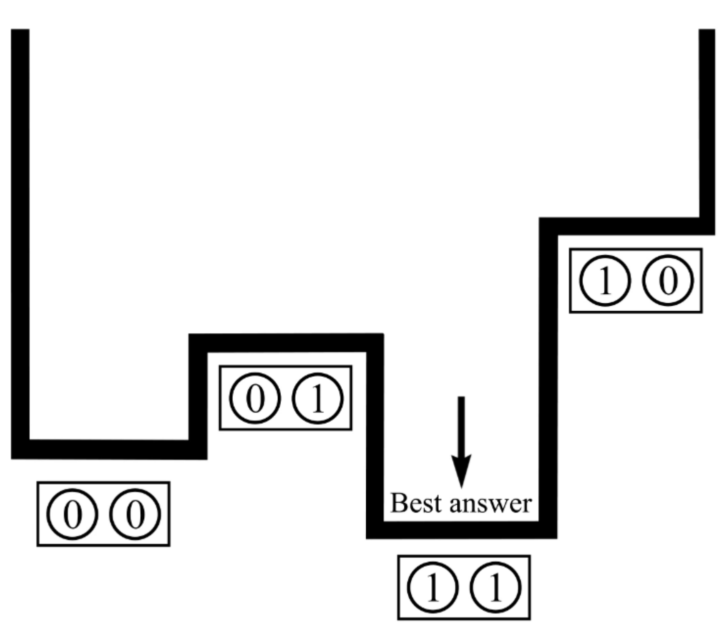

Figure 2. Source: D-Wave

Mathematical Core

In quantum annealing, the system's behavior is governed by a Hamiltonian, which dictates how it evolves over time towards the lowest energy state. The Hamiltonian comprises two terms: the initial Hamiltonian, H_{init}, and the final, or problem Hamiltonian H_{prob}.

H_{init}: sets the qubits in a superposition of all states

H_{prob} :encodes the specific optimization problem

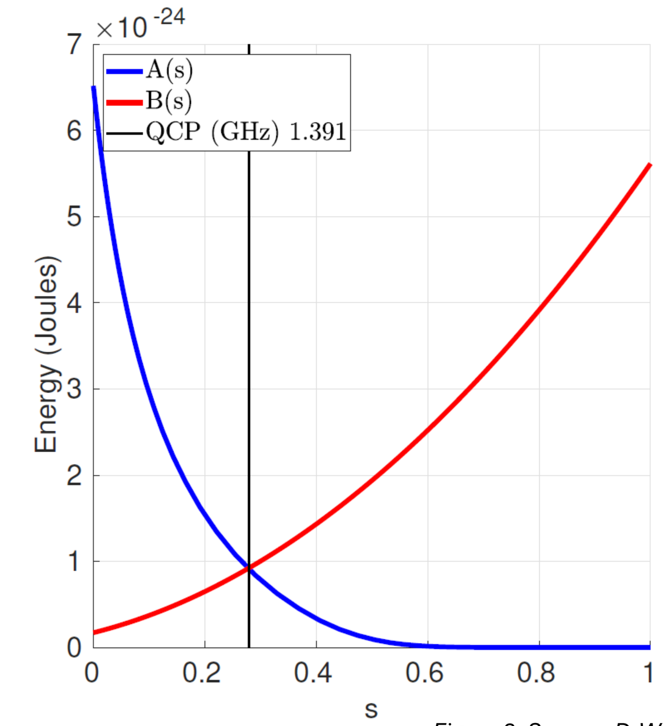

Figure 3. Source: D-Wave

$$H(s) = A(s)H_{init} + B(s)H_{prob}$$
$$H_{init} = -A(s)\sum_i \sigma_i^x$$
$$H_{prob} = B(s)(\sum_i h_i\sigma_i^z + \sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z)$$

Here, $A(s)$ is a time-dependent coefficient that decreases as the annealing process progresses, and σ_i^x is the Pauli-X matrix affecting the i-th qubit. This Hamiltonian encodes the optimization problem with $B(s)$ as a time-dependent coefficient that increases with time. The terms h_i represent the biases (or external magnetic fields) on individual qubits, while J_{ij} are the couplings or interactions between qubit i and j, and σ_i^z and σ_j^z are the Pauli-Z matrices for qubits i and j.

As the annealing process unfolds, $A(s)$ diminishes, reducing the influence of the initial Hamiltonian and the superposition state, while $B(s)$ grows, increasing the problem Hamiltonian's influence and steering the system towards a state that represents the problem's solution. The interplay between H_{init} and H_{prob} through the annealing schedule s, which typically moves from 0 at the start to 1 at the end of the anneal, controls the quantum system's evolution to find the lowest energy state and the solution to the optimization problem.

Quantum Tunneling: The Heart of Quantum Annealing

Quantum tunneling allows qubits to bypass energy barriers, facilitating transitions between states. This effect is captured by the formula:

$$P = \exp(-\frac{2}{\hbar}\sqrt{2m(V-E)d})$$

where P is the tunneling probability, enhancing the system's ability to find the global minimum.

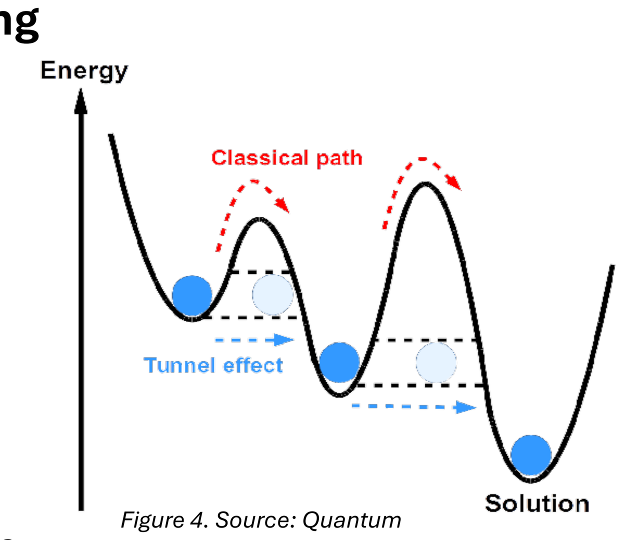

Figure 4. Source: Quantum World Association

Adiabatic Theorem: Ensuring Ground State Evolution

One of the fundamental reasons why quantum annealing is feasible is due to the adiabatic/Kato theorem. According to this theorem, a quantum system governed by the time-dependent Schrödinger equation will remain in its corresponding eigenstate if the Hamiltonian that describes the system changes slowly enough.

For quantum annealing, this means if the system starts in the ground state of an initial Hamiltonian and this Hamiltonian is changed very slowly to a problem Hamiltonian, the system is likely to stay in the ground state of the evolving Hamiltonian. This process ensures that the system can evolve toward the ground state of the problem Hamiltonian.

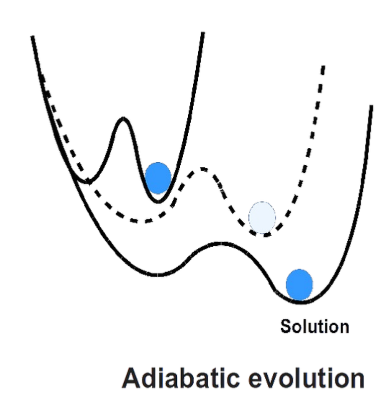

Figure 5. Source: Quantum World Association

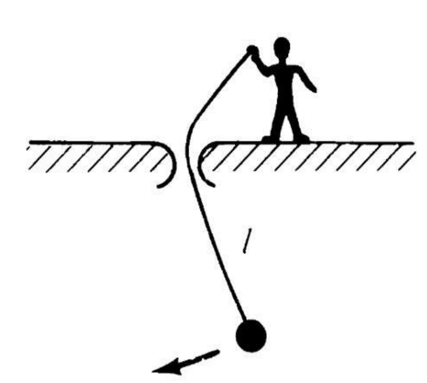

Figure 6. Source: Marco Frasca on The Gauge Connection

Theorem 17.3 (Kato dynamics): Let $W(s)$ solve the differential equation:

$$\frac{d}{ds}W(s) = iA(s)W(s), \quad 0 \le s \le 1; \quad W(0) = I$$

where

$$iA(s) \equiv [P'(s), P(s)]$$

Then $W(s)$ is unitary and obeys:

$$W(s)P(0)W(s)^{-1} = P(s)$$

Applications in the Real World

The practical of capitalizing on quantum annealing extends across all domains that deal with Quadratic Unconstrained Binary Optimization (QUBO) problems.

For example, in computational biology, QUBO can map problems such as sampling valid ensemble configurations of dense polymer mixtures. In finance, questions related to portfolio optimization or decision-making have been the focus of research aiming for potential speedups.

The practical tech stack for quantum annealing is dominated by the company D-Wave systems. From hardware level to cloud deployment, they are offering a vertical stack for developers and researchers to build and explore.

Challenges, Shortcomings, and Critique

The discussion regarding potential quantum speedups varies significantly. Although the field frequently receives papers claiming speedups with systems fine-tuned for established NP-Complete problems like 3-SAT, TSP, and Ising Spin, the current consensus acknowledges the legitimacy of quantum annealing (QA) in providing speedups with the note that the applicability of these speedups is limited to specific cases.

Main critiques appear because of the necessity and overhead of the fine-tuning and specialization needed for each different question to enable further constructive interference to enhance the system's probability to reach the solution.

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