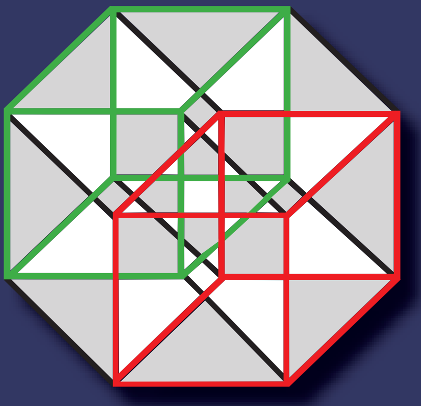
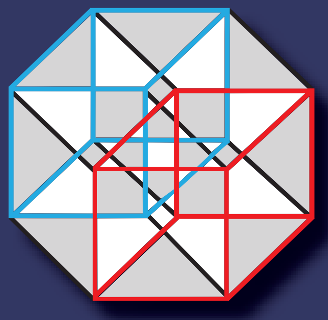
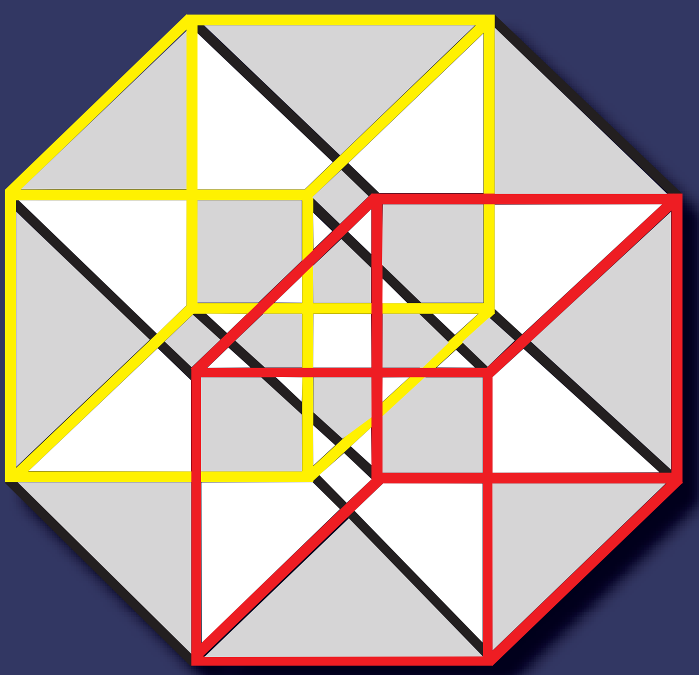




5th Edition

Introduction to Graph Theory

Robin J. Wilson



Introduction to Graph Theory

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Fifth Edition

Robin J. Wilson

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Contents

Go forth, my little book! pursue thy way!
Go forth, and please the gentle and the good.

William Wordsworth

Preface

In recent years, graph theory has established itself as an important mathematical tool in a wide variety of subjects, ranging from operational research and chemistry to genetics and linguistics, and from computer science and geography to sociology and architecture. At the same time it has also emerged as a worthwhile mathematical discipline in its own right.

In view of this, there has been a need for an inexpensive introductory text on the subject, suitable both for mathematicians taking courses in graph theory and for non-specialists wishing to learn the subject as quickly as possible. It is my hope that this latest edition continues to go some way towards filling this need. The only prerequisites to reading it are a basic knowledge of elementary set theory and matrix theory, although a further knowledge of abstract algebra and topology is needed for a few of the more difficult exercises.

The contents of this book may be conveniently divided into four parts. The first of these (Chapters 1–3) provides a basic foundation course, containing definitions and examples of graphs and digraphs, connectedness, Eulerian and Hamiltonian paths and cycles, and trees. This is followed by two chapters (Chapters 4 and 5) on planarity and colouring, with special reference to the four-colour theorem. The third part (Chapter 6) deals with transversal theory and connectivity, with applications to network flows. The book ends with a chapter on matroids (Chapter 7), which ties together material from the previous chapters and introduces some recent developments.

Throughout the book I have attempted to restrict the text to basic material. Of the 300 exercises, many are routine examples designed to test understanding of the text, while others will introduce you to new results and ideas. You should read each exercise, whether or not you work through it in detail, as some are referred to later in the book. Solutions are given for some of the exercises; these exercises are indicated by the symbol ^{s} next to the exercise number. More difficult exercises (called Challenge problems) appear at the end of each chapter.

I have used the symbol ■ to indicate the end of a proof, and bold-face type is used for definitions. The number of elements in a set S is denoted by $|S|$, and the empty set is denoted by Ø.

A substantial number of changes have been made in this edition. The text has been revised throughout and several sections have been rearranged and renumbered. Some new material has been added – notably on the four-colour theorem and on algorithms – and other material has been removed. Several further changes have arisen as a result of comments by a number of people, and I should like to thank them for their helpful remarks.

Finally, I wish to express my thanks to my former students, but for whom this book would have been completed earlier, to Mr William Shakespeare and others for their apt and witty comments at the beginning of each chapter, and most of all to my wife Joy for many things that have nothing to do with graph theory.

R.J.W.
November 2009

Introduction

The last thing one discovers in writing a book is what to put first.
Blaise Pascal

In this Introduction we provide an intuitive background to the material that we present more formally later on. Terms that appear here in bold-face type are to be thought of as descriptions rather than as definitions; having met them here in an informal setting, you should find them more familiar when you meet them later. So read this Introduction quickly, and then forget all about it!

What is a graph?

We begin by considering Figs 0.1 and 0.2, which depict part of a road map and part of an electrical network.

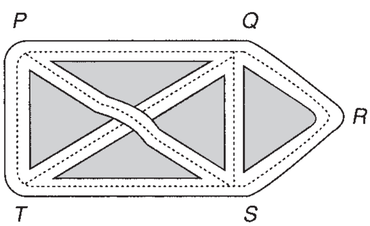


Figure 0.1

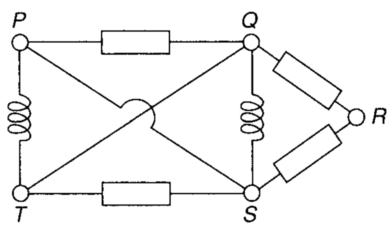


Figure 0.2

Either of these situations can be represented diagrammatically by means of points and lines, as in Fig. 0.3. The points P, Q, R, S and T are called **vertices**, the lines are called **edges**, and the whole diagram is called a **graph**. Note that the intersection of the lines PS and QT is not a vertex, since it does not correspond to a crossroads or to the meeting of two wires. The **degree** of a vertex is the number of edges with that vertex as an end-point; it corresponds in Fig. 0.1 to the number of roads at an intersection. For example, the degree of the vertex P is 3 and the degree of the vertex Q is 4.

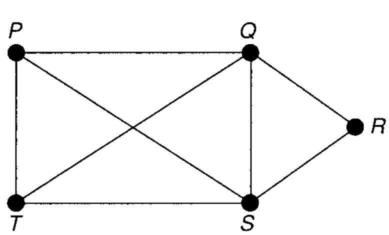


Figure 0.3

The graph in Fig. 0.3 can also represent other situations. For example, if *P*, *Q*, *R*, *S* and *T* represent football teams, then the existence of an edge might correspond to the playing of a game between the teams at its end-points. Thus, in Fig. 0.3, team *P* has played against teams *Q*, *S* and *T*, but not against team *R*. In this representation, the degree of a vertex is the number of games played by that team.

Another way of depicting these situations is to use the graph in Fig. 0.4. Here we have removed the 'crossing' of the lines *PS* and *QT* by redrawing the line *PS* outside the rectangle *PQST*. The resulting graph still tells us whether there is a direct road from one intersection to another, how the electrical network is wired up, and which football teams have played which. The only information we have lost concerns 'metrical' properties, such as the length of a road and the straightness of a wire.

Thus, a graph is a representation of a set of points and of how they are joined up, and any metrical properties are irrelevant. From this point of view, any graphs that represent the same situation, such as those of Figs 0.3 and 0.4, are regarded as the same graph.

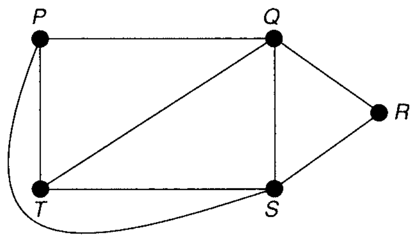


Figure 0.4

More generally, two graphs are the same if two vertices are joined by an edge in one graph if and only if the corresponding vertices are joined by an edge in the other. Another graph that is the same as those in Figs 0.3 and 0.4 is shown in Fig. 0.5. Here all idea of space and distance has gone, but we can still tell at a glance which points are joined by a road or a wire.

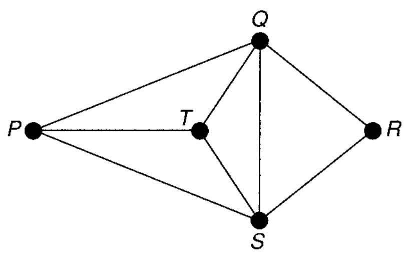


Figure 0.5

In this graph there is at most one edge joining each pair of vertices. Suppose now that in Fig. 0.5 the roads joining *Q* and *S*, and *S* and *T*, have too much traffic to carry. Then we can ease the situation by building extra roads joining these points, and the resulting diagram looks like Fig. 0.6. The edges joining *Q* and *S*, or *S* and *T*, are called **multiple edges**. If, in addition, we need a car park at *P*, then we indicate this by drawing an edge from *P* to itself, called a **loop** (see Fig. 0.7). In this book, graphs may

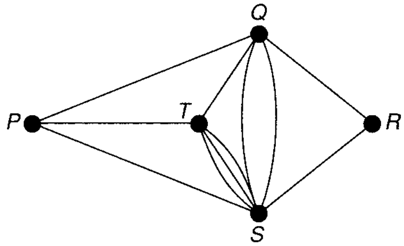


Figure 0.6

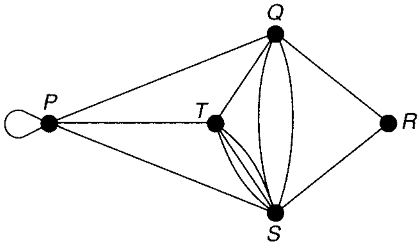


Figure 0.7

contain loops and multiple edges; graphs with no loops or multiple edges, such as the graph in Fig. 0.5, are called **simple graphs**.

The study of **directed graphs** (or **digraphs**, as we abbreviate them) arises from making the roads into one-way streets. An example of a digraph is given in Fig. 0.8, with the directions of the one-way streets indicated by arrows; such a 'directed edge' is called an **arc**. (In this example, there would be chaos at T, but that does not stop us from studying such situations!)

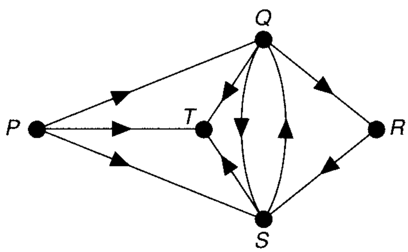


Figure 0.8

Much of graph theory is devoted to 'walks' of various kinds. A **walk** is a 'way of getting from one vertex to another' in a graph or digraph, and consists of a sequence of edges or arcs, one following after another. For example, in Fig. 0.5, $P \to Q \to R$ is a walk of length 2, and $P \to S \to Q \to T \to S \to R$ is a walk of length 5. A walk in which no vertex appears more than once is called a **path**; for example, $P \to T \to S \to R$ is a path. A walk of the form $Q \to S \to T \to Q$, in which no vertex appears more than once, except for the beginning and end vertices which coincide, is called a **cycle**.

In Chapter 2 we also consider walks with some extra property. In particular, we discuss graphs and digraphs containing walks that include every edge exactly once and end back at the initial vertex; such graphs and digraphs are called **Eulerian**. The graph in Fig. 0.5 is not Eulerian, since any walk that includes each edge exactly once (such as $P \to Q \to R \to S \to T \to P \to S \to Q \to T$) must end at a vertex different from the initial one. We also discuss graphs and digraphs containing cycles that pass through every vertex; these are called **Hamiltonian**. For example, the graph in Fig. 0.5 is Hamiltonian; a suitable cycle is $P \to Q \to R \to S \to T \to P$.

Some graphs or digraphs are in two or more parts. For example, consider the graph whose vertices are the stations of the London Underground and the New York Subway, and whose edges are the lines joining adjacent stations. It is clearly impossible to travel from Trafalgar Square to Grand Central Station using only edges of this graph, but if we confine our attention to the London Underground part only, then we can travel from any station to any other. A graph or digraph that is 'in one piece', so that any two vertices are connected by a path, is **connected**; a graph or digraph that is in more than one piece is **disconnected** (see Fig. 0.9). We discuss connectedness in Chapters 1, 2 and 6.

We are sometimes interested in connected graphs with only one path between each pair of vertices. These graphs contain no cycles and are called **trees** (see Fig. 0.10). They generalize the idea of a family tree, and are considered in Chapter 3.

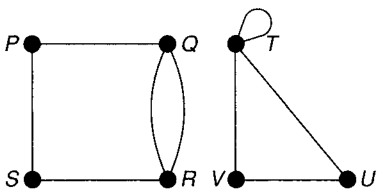


Figure 0.9

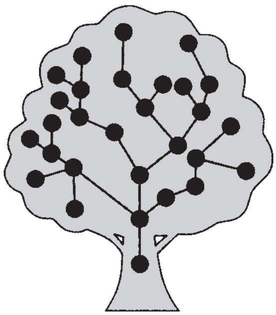


Figure 0.10

Earlier we saw how the graph of Fig. 0.3 can be redrawn as in Figs 0.4 and 0.5 so as to avoid crossings of edges. Any graph that can be redrawn without crossings in this way is called a **planar graph**. In Chapter 4 we give several criteria for planarity. Some of these involve the properties of particular 'subgraphs' of the graph in question; others involve the fundamental notion of duality.

Planar graphs also play an important role in colouring problems. In our 'road-map' graph, let us suppose that Shell, Esso, BP and Gulf wish to erect five garages between them, and that for economic reasons no company wishes to erect two garages at neighbouring corners. Then Shell can build at P, Esso can build at Q, BP can build at S, and Gulf can build at T, leaving either Shell or Gulf to build at R (see Fig. 0.11). However, if Gulf backs out of the agreement, then the other three companies cannot erect their garages in the specified manner.

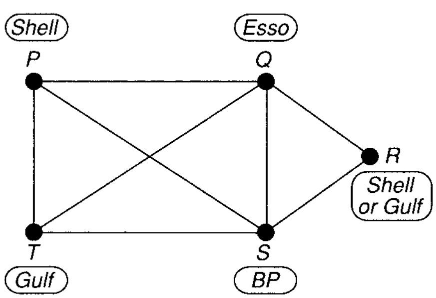


Figure 0.11

We discuss such problems in Chapter 5, where we try to colour the vertices of a simple graph with a given number of colours so that each edge of the graph joins vertices of different colours. If the graph is planar, then we can always colour its vertices in this way with only four colours – this is the celebrated **four-colour theorem**. Another version of this theorem is that we can always colour the countries of any map with four colours so that no two neighbouring countries share the same colour (see Fig. 0.12).

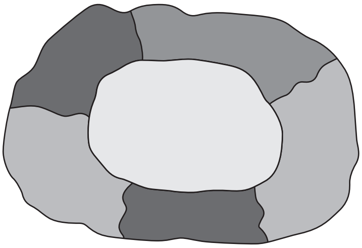


Figure 0.12

In Chapter 6 we investigate the celebrated **marriage problem**, which asks under what conditions a collection of girls, each of whom knows several boys, can be married so that each girl marries a boy she knows. This problem can be expressed in the language of a branch of set theory called 'transversal theory', and is related to problems of finding disjoint paths connecting two given vertices in a graph or digraph.

Chapter 6 concludes with a discussion of network flows and transportation problems. Suppose that we have a transportation network such as in Fig. 0.13, in which P is a factory, R is a market, and the edges of the graph are channels through which goods can be sent. Each channel has a capacity, indicated by a number next to the edge, representing the maximum amount of a commodity that can pass through that channel. The problem is to determine how much of the commodity can be sent from the factory to the market without exceeding the capacity of any channel.

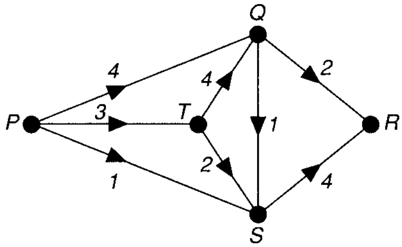


Figure 0.13

We conclude with a chapter on matroids. This ties together the material of the previous chapters, and follows the maxim 'be wise – generalize!' Matroid theory, the study of sets with 'independence structures' defined on them, generalizes both the linear independence of vectors and some results on graphs and transversals from earlier in the book. However, matroid theory is far from being 'generalization for generalization's sake'. On the contrary, it gives us clearer insights into several graphical results involving cycles and planar graphs, and provides simple proofs of results on transversals that are awkward to prove by more traditional methods. Matroids have played an important role in the development of combinatorial ideas in recent years.

We hope that this Introduction has been useful in setting the scene and describing some of the treats that lie ahead. We now embark upon a formal treatment of the subject.

Exercises

0.1$^{\mathrm{s}}$ Write down the number of vertices, the number of edges and the degree of each vertex in:
(i) the graph in Fig. 0.3;
(ii) the tree in Fig. 0.14.

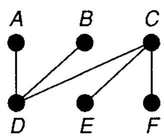


Figure 0.14

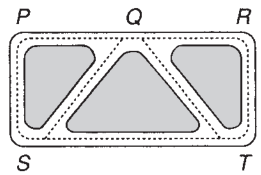


Figure 0.15

0.2 Draw the graph representing the road system in Fig. 0.15, and write down the number of vertices, the number of edges and the degree of each vertex.

0.3$^{\mathrm{s}}$ Figure 0.16 represents the chemical molecules of methane (CH_4) and propane (C_3H_8).
(i) Regarding these diagrams as trees, what can you say about the vertices representing carbon atoms (**C**) and hydrogen atoms (**H**)?
(ii) Draw a tree that represents the molecule with formula C_2H_6 (hexane).
(iii) There are two different chemical molecules with formula C_4H_{10}. Draw trees that represent these molecules.

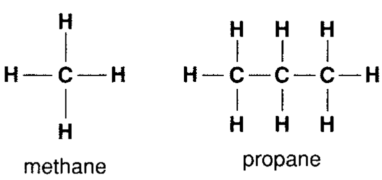


Figure 0.16

0.4 Draw a graph that represents the family tree in Fig. 0.17.

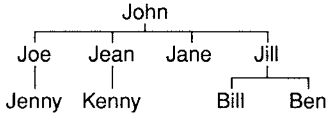


Figure 0.17

0.5[s] John likes Joan, Jean and Jane; Joe likes Jane and Joan; Jean and Joan like each other. Draw a digraph illustrating these relationships between John, Joan, Jean, Jane and Joe.

0.6 Snakes eat frogs and birds eat spiders; birds and spiders both eat insects; frogs eat snails, spiders and insects. Draw a digraph representing this predatory behaviour.

0.7 Draw a graph with vertices $A, B, \ldots, M$ that shows the various routes that one can take when tracing the Hampton Court maze in Fig. 0.18.

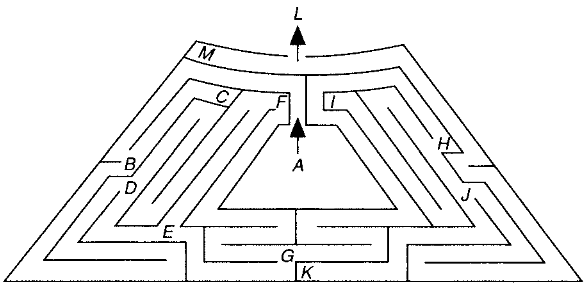


Figure 0.18

Chapter 1

Definitions and examples

I hate definitions!

Benjamin Disraeli

In this chapter, we lay the foundations for a proper study of graph theory. Section 1.1 formalizes some of the graph ideas in the Introduction, Section 1.2 provides a variety of examples, and Section 1.3 presents two variations on the basic idea. In Section 1.4 we show how graphs can be used to represent and solve three problems from recreational mathematics. More substantial applications are deferred until Chapters 2 and 3, when we have more machinery at our disposal.

1.1 Definitions

A **simple graph** G consists of a non-empty finite set $V(G)$ of elements called **vertices** (or **nodes** or **points**) and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called **edges** (or **lines**). We call $V(G)$ the **vertex-set** and $E(G)$ the **edge-set** of G. An edge $\{v, w\}$ is said to **join** the vertices v and w, and is usually abbreviated to vw. For example, Fig. 1.1 represents the simple graph G whose vertex-set $V(G)$ is $\{u, v, w, z\}$, and whose edge-set $E(G)$ consists of the edges uv, uw, vw and wz.

In any simple graph there is at most one edge joining a given pair of vertices. However, many results for simple graphs also hold for more general objects in which two vertices may have several edges joining them; such edges are called **multiple edges**. In addition, we may remove the restriction that an edge must join two *distinct* vertices, and allow **loops** – edges joining a vertex to itself. The resulting object, with loops and multiple edges allowed, is called a **general graph** – or, simply, a **graph** (see Fig. 1.2). Note that every simple graph is a graph, but not every graph is a simple graph.

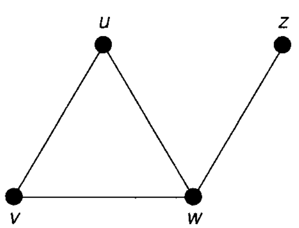


Figure 1.1

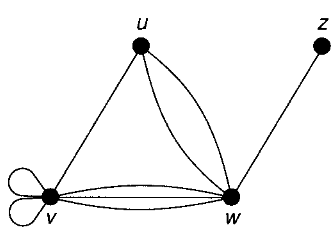


Figure 1.2

Thus, a **graph** G consists of a non-empty finite set $V(G)$ of elements called **vertices** and a finite family $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called **edges**; the use of the word 'family' permits the existence of multiple edges. We call $V(G)$ the **vertex-set** and $E(G)$ the **edge-family** of G. An edge $\{v, w\}$ is said to **join** the vertices v and w, and is again abbreviated to vw. Thus in Fig. 1.2, $V(G)$ is the set $\{u, v, w, z\}$ and $E(G)$ consists of the edges uv, vv (twice), vw (three times), uw (twice) and wz. Note that each loop vv joins the vertex v to itself. Although we sometimes need to restrict our attention to simple graphs, we shall prove our results for general graphs whenever possible.

Remark. The language of graph theory is not standard – all authors have their own terminology. Some use the term 'graph' for what we call a simple graph, while others use it for graphs with directed edges, or for graphs with infinitely many vertices or edges; we discuss these variations in Section 1.3. Any such definition is perfectly valid, provided that it is used consistently. In this book:

All graphs are finite and undirected, with loops and multiple edges allowed unless specifically excluded.

Isomorphism

Two graphs G_1 and G_2 are **isomorphic** if there is a one–one correspondence between the vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1 equals the number of edges joining the corresponding vertices of G_2. For example, the two graphs in Fig. 1.3 are isomorphic, under the correspondence

$$u \leftrightarrow l,\ v \leftrightarrow m,\ w \leftrightarrow n,\ x \leftrightarrow p,\ y \leftrightarrow q,\ z \leftrightarrow r.$$

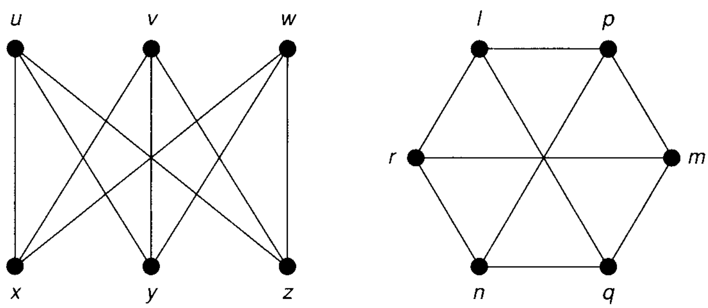


Figure 1.3

For many problems, the labels on the vertices are unnecessary and we drop them. We then say that two 'unlabelled graphs' are isomorphic if we can assign labels to their vertices so that the resulting 'labelled graphs' are isomorphic. For example, we regard the unlabelled graphs in Fig. 1.4 as isomorphic, since the labelled graphs in Fig. 1.3 are isomorphic.

The difference between labelled and unlabelled graphs becomes more apparent when we try to count them. For example, if we restrict ourselves to graphs with three vertices, then there are eight different labelled graphs (see Fig. 1.5), but only four

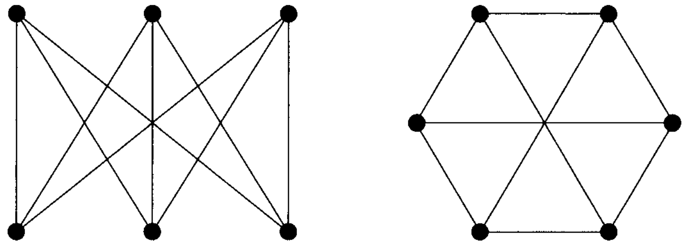


Figure 1.4

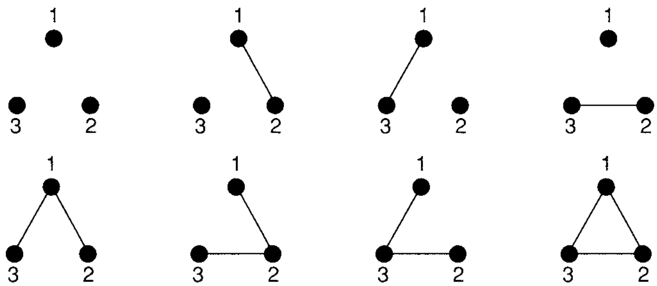


Figure 1.5

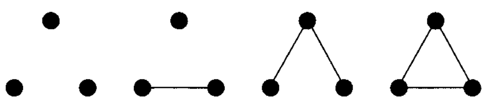


Figure 1.6

unlabelled ones (see Fig. 1.6). It is usually clear from the context whether we are referring to labelled or unlabelled graphs.

Connected graphs

We can combine two graphs to make a larger graph. If the two graphs are G_1 and G_2 and their vertex-sets $V(G_1)$ and $V(G_2)$ are disjoint, then their **union** $G_1 \cup G_2$ is the graph with vertex-set $V(G_1) \cup V(G_2)$ and edge-family $E(G_1) \cup E(G_2)$ (see Fig. 1.7).

Most of the graphs discussed so far have been 'in one piece'. A graph is **connected** if it cannot be expressed as a union of graphs, and **disconnected** otherwise. Clearly, any disconnected graph G can be expressed as the union of connected graphs, each of which is called a **component** of G; a disconnected graph with three components is shown in Fig. 1.8.

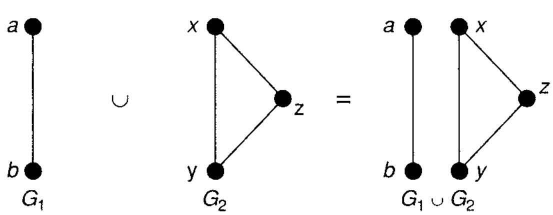


Figure 1.7

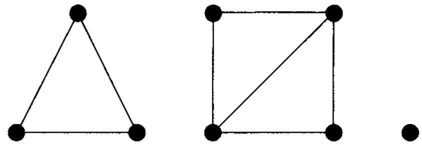


Figure 1.8

When proving results about graphs in general, we can often obtain the corresponding results for connected graphs and then apply them to each component separately. A table of all the unlabelled connected simple graphs with up to five vertices is given in Fig. 1.9.

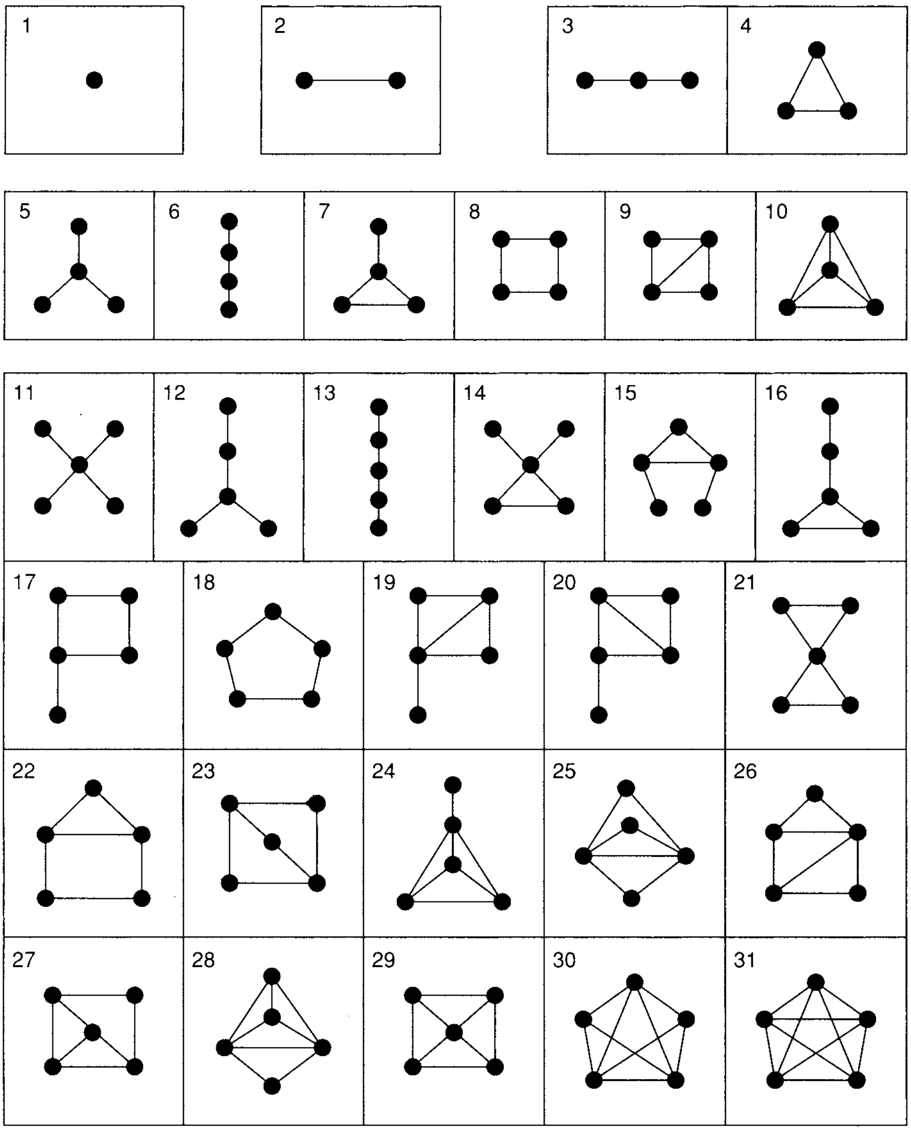


Figure 1.9

Adjacency and degrees

We say that two vertices v and w of a graph are **adjacent** if there is an edge vw joining them, and the vertices v and w are then **incident** with such an edge. We also say that two distinct edges e and f are **adjacent** if they have a vertex in common (see Fig. 1.10).

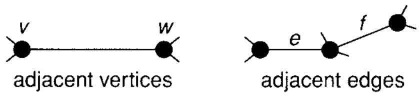


Figure 1.10

The **degree** of a vertex v is the number of edges incident with v, and is written $\deg(v)$; when calculating the degree of v, we usually make the convention that a loop at v contributes 2 (rather than 1) to $\deg(v)$. A vertex of degree 0 is an **isolated vertex** and a vertex of degree 1 is an **end-vertex**. Thus each of the two graphs in Fig. 1.11 has two end-vertices and three vertices of degree 2, while the graph in Fig. 1.12 has one end-vertex, one vertex of degree 3, one of degree 6 and one of degree 8.

The **degree sequence** of a graph consists of the degrees written in increasing order, with repeats where necessary. For example, the degree sequences of the graphs in Figs 1.11 and 1.12 are (1, 1, 2, 2, 2) and (1, 3, 6, 8).

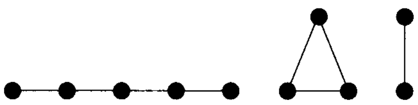


Figure 1.11

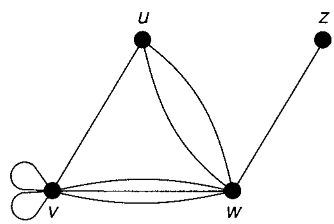


Figure 1.12

The earliest result on graph theory is essentially due to Leonhard Euler in 1735 (although he did not express it in the language of graphs). It is sometimes called the **handshaking lemma**.

THEOREM 1.1 (Handshaking lemma) *In any graph the sum of all the vertex-degrees is an even number.*

Proof. The sum of all the vertex-degrees is equal to twice the number of edges, since each edge contributes exactly 2 to the sum. It is thus an even number. ■

The handshaking lemma is so called because it tells us that if several people shake hands, then the total number of hands shaken must be even – this is precisely because just two hands are involved in each handshake. A useful corollary of the handshaking lemma is the following:

COROLLARY 1.2 *In any graph the number of vertices of odd degree is even.*

Proof. If the number of vertices of odd degree were odd, then the sum of the vertex-degrees would also be odd, contradicting Theorem 1.1. So the number is even. ■

Subgraphs

A graph H is a **subgraph** of a graph G if each of its vertices belongs to $V(G)$ and each of its edges belongs to $E(G)$. Thus the graph in Fig. 1.13 is a subgraph of the graph in Fig. 1.14, but is not a subgraph of the graph in Fig. 1.15, since the latter graph contains no 'triangles'.

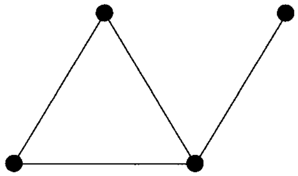


Figure 1.13

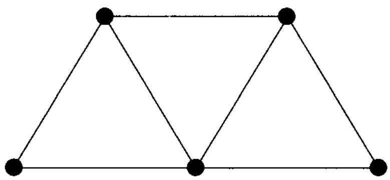


Figure 1.14

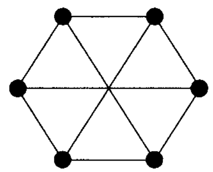


Figure 1.15

We can obtain subgraphs of a graph by deleting edges and vertices. If e is an edge of a graph G, we denote by $G - e$ the graph obtained from G by deleting the edge e; more generally, if F is any set of edges in G, we denote by $G - F$ the graph obtained by deleting the edges in F. Similarly, if v is a vertex of G, we denote by $G - v$ the graph obtained from G by deleting the vertex v together with the edges incident with v; more generally, if S is any set of vertices in G, we denote by $G - S$ the graph obtained by deleting the vertices in S and all edges incident with any of them. An example is shown in Fig. 1.16.

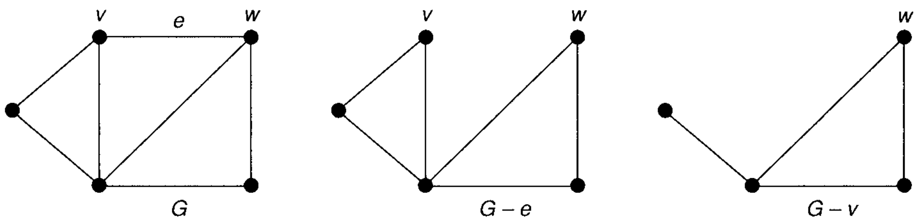


Figure 1.16

We also denote by $G \setminus e$ the graph obtained by taking an edge e and 'contracting' it – that is, removing it and identifying its ends v and w so that the resulting vertex is incident with all those edges (other than e) that were originally incident with v or w. An example is shown in Fig. 1.17.

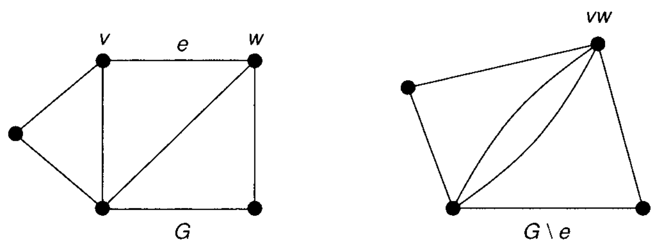


Figure 1.17

The complement of a simple graph

If G is a simple graph with vertex-set $V(G)$, its **complement** $\bar{G}$ is the simple graph with vertex-set $V(G)$ in which two vertices are adjacent if and only if they are *not* adjacent in G; Fig. 1.18 shows a graph and its complement. Note that the complement of $\bar{G}$ is G.

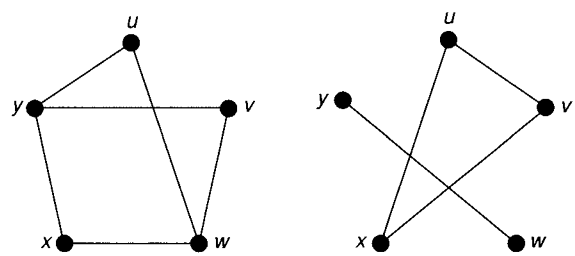


Figure 1.18

Matrix representations

Although it is convenient to represent a graph by a diagram of points joined by lines, such a representation may be unsuitable if we wish to store a large graph in a computer. One way of storing a simple graph is by listing the vertices adjacent to each vertex of the graph. An example of this is given in Fig. 1.19.

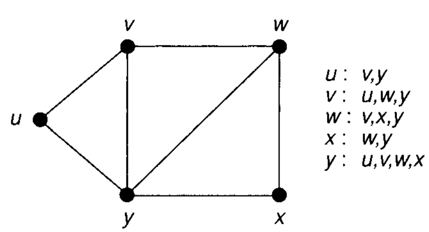
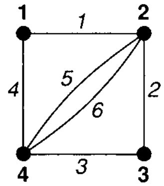


Figure 1.19

Other useful representations involve matrices. If G is a graph without loops, with vertices labelled $\{\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}\}$, its **adjacency matrix** $\mathbf{A}$ is the $n \times n$ matrix whose ijth entry is the number of edges joining vertex $\boldsymbol{i}$ and vertex $\boldsymbol{j}$. If, in addition, the edges are labelled $\{1, 2, \ldots, m\}$, its **incidence matrix** $\mathbf{M}$ is the $n \times m$ matrix whose ijth entry is 1 if vertex $\boldsymbol{i}$ is incident to edge j, and is 0 otherwise. Figure 1.20 shows a labelled graph G with its adjacency and incidence matrices.



$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 1.20

Exercises

1.1[s] Write down the vertex-set and edge-set of each graph in Fig. 1.3.

1.2 Write down the vertex-set and edge-family of the graph in Fig. 1.21.

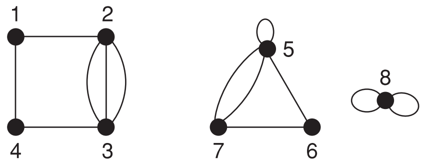


Figure 1.21

1.3 Draw
(i) a simple graph,
(ii) a non-simple graph with no loops,
(iii) a non-simple graph with no multiple edges,
each with five vertices and eight edges.

1.4[s] By suitably labelling the vertices, show that the two graphs in Fig. 1.22 are isomorphic.

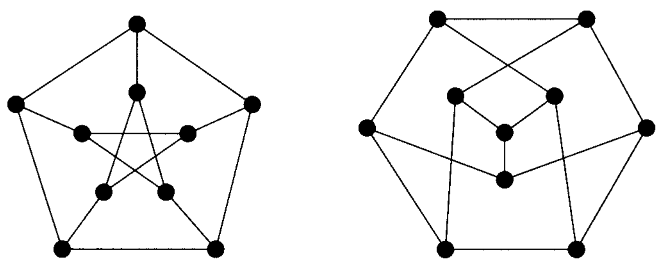


Figure 1.22

1.5[s] Explain why the two graphs in Fig. 1.23 are not isomorphic.

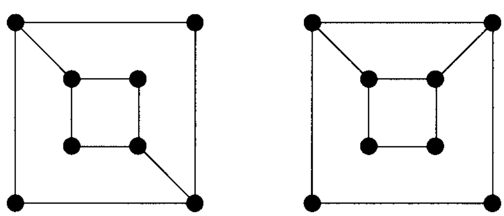


Figure 1.23

1.6 Are the graphs in Fig. 1.24 isomorphic?

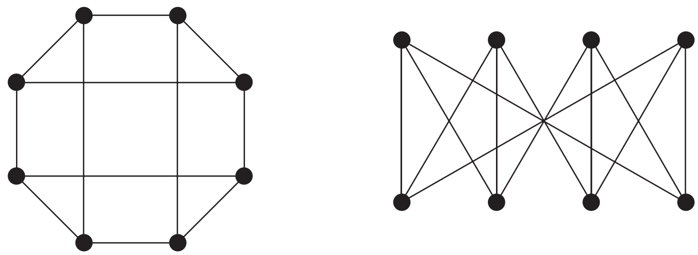


Figure 1.24

1.7 Classify the following statements as *true* or *false*:
(i) any two isomorphic graphs have the same degree sequence;
(ii) any two graphs with the same degree sequence are isomorphic.

1.8[s] Locate each of the graphs in Fig. 1.25 in the table of Fig. 1.9.

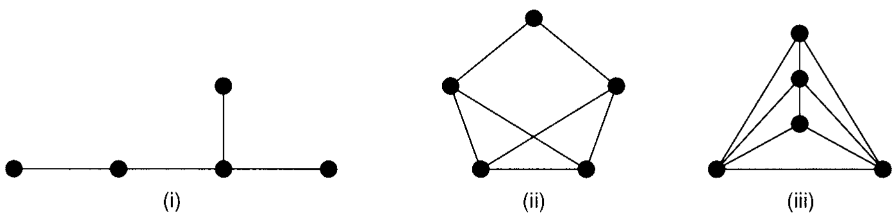


Figure 1.25

1.9 Locate each of the graphs in Fig. 1.26 in the table of Fig. 1.9.

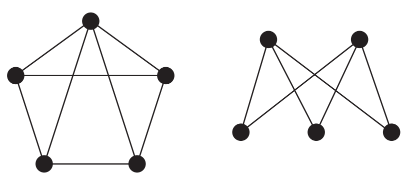


Figure 1.26

1.10 (i) Show that there are exactly $2^{n(n-1)/2}$ labelled simple graphs on n vertices.
(ii) How many of these have exactly m edges?

1.11[s] Write down the degree sequence of each graph with four vertices in Fig. 1.9, and verify that the handshaking lemma holds for each graph.

1.12 Write down the degree sequence of each graph with five vertices in Fig. 1.9, and verify that the handshaking lemma holds for each graph.

1.13 (i) Draw a graph on six vertices with degree sequence (3, 3, 5, 5, 5, 5); does there exist a *simple* graph with these degrees?
(ii) How are your answers to part (i) changed if the degree sequence is (2, 3, 3, 4, 5, 5)?

1.14 (i) Let G be a graph with four vertices and degree sequence (1, 2, 3, 4). Write down the number of edges of G, and construct such a graph.
(ii) Are there any simple graphs with four vertices and degree sequence (1, 2, 3, 4)?

1.15 If G is a simple graph with at least two vertices, prove that G must contain two or more vertices of the same degree.

1.16[s] Which graphs in Fig. 1.27 are subgraphs of the graphs in Fig. 1.22?

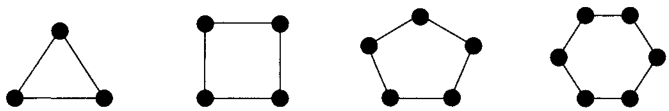


Figure 1.27

1.17 Let G be the graph of Fig. 1.3 and let e be the edge ux. Draw the graphs $G - e$ and $G \setminus e$.

1.18 Let G be a graph with n vertices and m edges, and let v be a vertex of G of degree k and e be an edge of G. How many vertices and edges have $G - e$, $G - v$ and $G \setminus e$?

1.19 Draw the complements of the graphs in Figs. 1.13, 1.14 and 1.15.

1.20[s] Write down the adjacency and incidence matrices of the graph in Fig. 1.28.

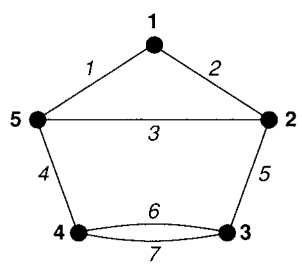


Figure 1.28

1.21 Write down the adjacency and incidence matrices of the graph in Fig. 1.2.

1.22 Draw the graph whose adjacency matrix is given in Fig. 1.29.

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Figure 1.29

1.23 Draw the graph whose incidence matrix is given in Fig. 1.30.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 1.30

1.24 If G is a graph without loops, what can you say about the sum of the entries in
(i) any row or column of the adjacency matrix of G?
(ii) any row of the incidence matrix of G?
(iii) any column of the incidence matrix of G?

1.2 Examples

In this section we examine some important types of graphs. You should become familiar with them, as they will appear frequently in examples and exercises.

Null graphs

A graph whose edge-set is empty is a **null graph**; note that each vertex of a null graph is isolated. We denote the null graph on n vertices by N_n: the graph N_4 is shown in Fig. 1.31. Null graphs are not very interesting.

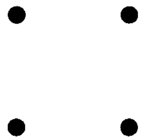


Figure 1.31

Complete graphs

A simple graph in which each pair of distinct vertices are adjacent is a **complete graph**. We denote the complete graph on n vertices by K_n: the graphs K_4 and K_5 are shown in Fig. 1.32. You should check that K_n has $\frac{1}{2}n(n-1)$ edges.

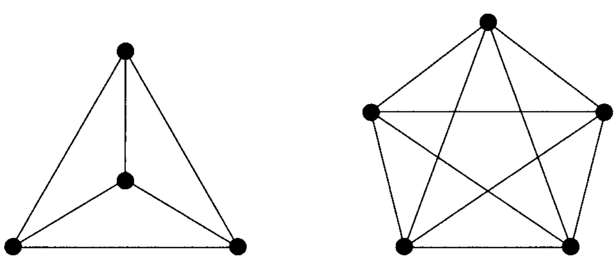


Figure 1.32

Cycle graphs, path graphs and wheels

A connected graph in which each vertex has degree 2 is a **cycle graph**. We denote the cycle graph on n vertices by C_n.

The graph obtained from C_n by removing an edge is the **path graph** on n vertices, denoted by P_n.

The graph obtained from C_{n-1} by joining each vertex to a new vertex v is the **wheel** on n vertices, denoted by W_n.

The graphs C_6, P_6 and W_6 are shown in Fig. 1.33.

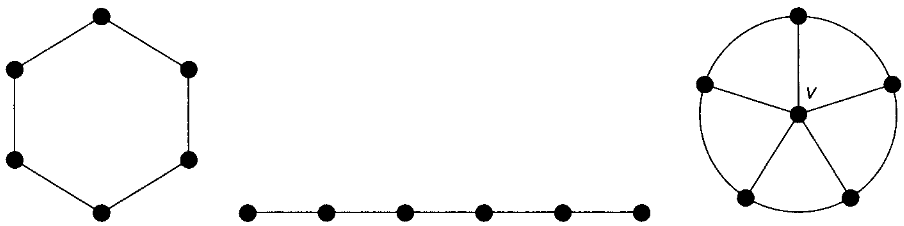


Figure 1.33

Regular graphs

A graph in which each vertex has the same degree is a **regular graph**. If each vertex has degree r, the graph is **regular of degree *r*** or ***r*-regular**. Note that the null graph N_n is regular of degree 0, the cycle graph C_n is regular of degree 2, and the complete graph K_n is regular of degree $n - 1$.

Of special importance are the **cubic graphs**, which are regular of degree 3; an example of a cubic graph is the **Petersen graph**, shown in Fig. 1.34.

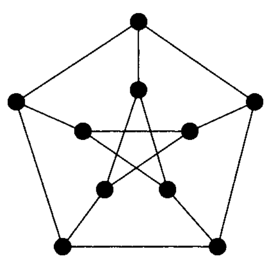


Figure 1.34

Platonic graphs

Of interest among the regular graphs are the **Platonic graphs**, formed from the vertices and edges of the five regular (Platonic) solids – the tetrahedron, octahedron, cube, icosahedron and dodecahedron (see Fig. 1.35).

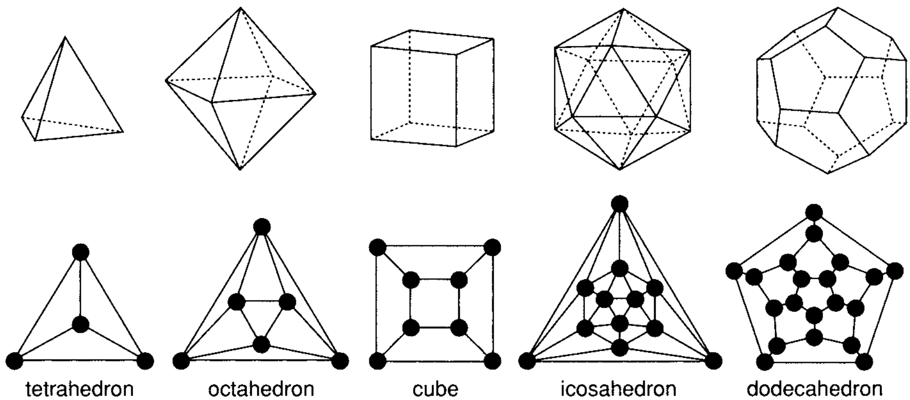


Figure 1.35

Bipartite graphs

If the vertex-set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B, then G is a **bipartite graph** (see Fig. 1.36). Alternatively, a bipartite graph is one whose vertices can be coloured black and white in such a way that each edge joins a black vertex (in A) and a white vertex (in B). We sometimes write $G = G(A, B)$ when we wish to specify the sets A and B.

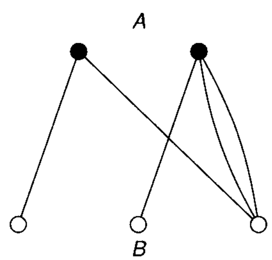


Figure 1.36

A **complete bipartite graph** is a bipartite graph in which each vertex in A is joined to each vertex in B by just one edge. We denote the complete bipartite graph with r black vertices and s white vertices by $K_{r,s}$: the graphs $K_{1,3}$, $K_{2,3}$, $K_{3,3}$ and $K_{4,3}$ are shown in Fig. 1.37. You should check that $K_{r,s}$ has $r + s$ vertices and rs edges.

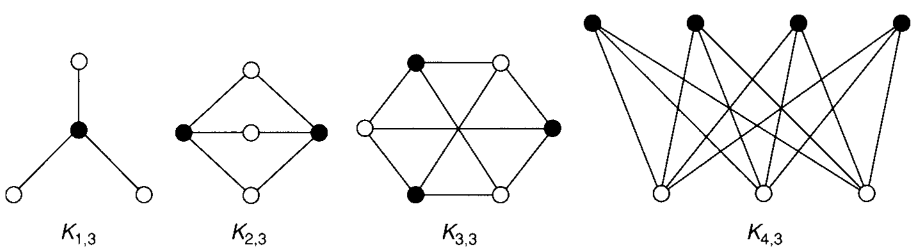


Figure 1.37

Cubes

Of interest among the regular bipartite graphs are the cubes. The ***k*-cube** Q_k is the graph whose vertices correspond to the sequences $(a_1, a_2, \ldots, a_k)$, where each $a_i = 0$ or 1, and whose edges join those sequences that differ in just one place. The graph Q_3 (the graph of the cube) is shown in Fig. 1.38 and Q_4 appears on the cover. Note that Q_k has 2^k vertices and is regular of degree k.

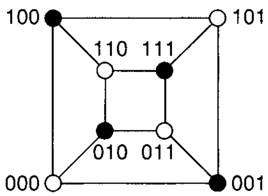


Figure 1.38

Exercises

1.25[s] Draw the following graphs:
(i) the null graph N_5;
(ii) the complete graph K_6;
(iii) the complete bipartite graph $K_{2,4}$;
(iv) the union of $K_{1,3}$ and W_4;
(v) the complement of the cycle graph C_4.

1.26[s] In the table of Fig. 1.9, locate all the regular graphs and the bipartite graphs.

1.27[s] How many edges has each of the following graphs:
(i) K_{10}; (ii) $K_{5,7}$; (iii) Q_4; (iv) W_8; (v) the Petersen graph?

1.28 How many edges has each of the following graphs:
(i) K_{12}; (ii) $K_{6,8}$; (iii) Q_5; (iv) W_{10}; (v) the complement of C_8?

1.29 If G has n vertices and is regular of degree r, how many edges has G? Use your answer to check the number of edges in the Petersen graph and the k-cube Q_k.

1.30 How many vertices and edges has each of the Platonic graphs?

1.31 Give an example (if it exists) of each of the following:
(i) a bipartite graph that is regular of degree 5;
(ii) a bipartite Platonic graph;
(iii) a complete graph that is a wheel;
(iv) a cubic graph with 11 vertices;
(v) a graph (other than K_5, $K_{4,4}$ or Q_4) that is regular of degree 4.

1.32[s] Draw all the simple cubic graphs with at most eight vertices.

1.33 What can you say about the complement of a complete graph, and the complement of a complete bipartite graph?

1.34 The **complete tripartite graph** $K_{r,s,t}$ consists of three sets of vertices (of sizes r, s and t), with an edge joining two vertices if and only if they lie in different sets. Draw the graphs $K_{2,2,2}$ and $K_{3,3,2}$ and find the number of edges of $K_{3,4,5}$.

1.3 Variations on a theme

In this section we consider two variations on the definition of a graph. In the first of these, each edge is given a particular direction, as in a one-way street. In the second, we allow our vertex-set and/or edge-set to be infinite.

Digraphs

A **directed graph**, or **digraph**, D consists of a non-empty finite set $V(D)$ of elements called **vertices** and a finite family $A(D)$ of *ordered* pairs of elements of $V(D)$ called **arcs** (or **directed edges**). We call $V(D)$ the **vertex-set** and $A(D)$ the **arc-family** of D. An arc (v, w) is usually abbreviated to vw. Thus in Fig. 1.39, $V(D)$ is the set $\{u, v, w, z\}$ and $A(D)$ consists of the arcs uv, vv, vw (twice), wv, wu and zw, with the ordering of the vertices in an arc indicated by an arrow.

If D is a digraph, the graph obtained from D by 'removing the arrows' (that is, by replacing each *arc* of the form vw by a corresponding *edge* vw) is the **underlying graph** of D (see Fig. 1.40).

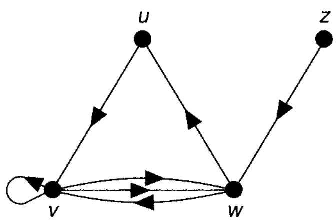


Figure 1.39

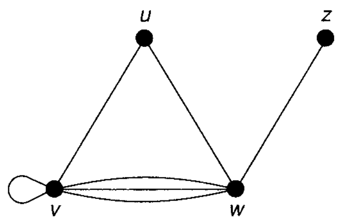


Figure 1.40

D is a **simple digraph** if the arcs of D are all distinct, and if there are no 'loops' (arcs of the form vv). Note that the underlying graph of a simple digraph need not be a simple graph (see Fig. 1.41).

We can imitate many of the definitions given in Section 1.1 for graphs. For example, two digraphs are **isomorphic** if there is an isomorphism between their underlying graphs that preserves the ordering of the vertices in each arc. Note that the digraphs in Figs 1.39 and 1.42 are not isomorphic.

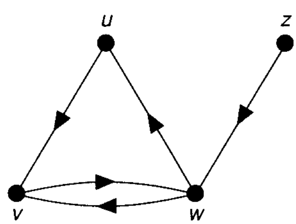


Figure 1.41

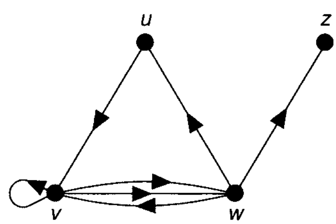


Figure 1.42

We can also define connectedness. A digraph D is (weakly) **connected** if it cannot be expressed as the union of two digraphs, defined in the obvious way. This is equivalent to saying that the underlying graph of D is a connected graph. A stronger definition of connectedness for digraphs will be given in Section 2.1.

Two vertices v and w of a digraph D are **adjacent** if there is an arc in $A(D)$ of the form vw or wv, and the vertices v and w are **incident** with such an arc.

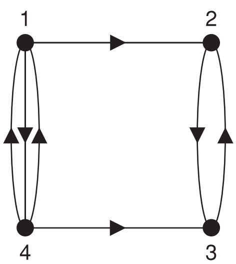
The **out-degree** of a vertex v of D is the number of arcs of the form vw, and is denoted by outdeg(v). Similarly, the **in-degree** of v is the number of arcs of D of the form wv, and is denoted by indeg(v).

There is a digraph version of Theorem 1.1, which we call, naturally enough, the **handshaking dilemma**!

THEOREM 1.3 (Handshaking dilemma) *In any digraph the sum of all the out-degrees is equal to the sum of all the in-degrees.*

***Proof*.** The sum of all the out-degrees is equal to the number of arcs, since each arc contributes exactly 1 to the sum. Similarly, the sum of all the in-degrees is equal to the number of arcs. ■

If D is a digraph without loops, with vertices labelled $\{\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}\}$, its **adjacency matrix A** is the $n \times n$ matrix whose ijth entry is the number of arcs from vertex $\boldsymbol{i}$ to vertex $\boldsymbol{j}$. Figure 1.43 shows a labelled digraph with its adjacency matrix.



$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

Figure 1.43

An important type of digraph is a **tournament**. This is a digraph in which any two vertices are joined by exactly one arc (see Fig. 1.44). Note that its underlying graph is a complete graph. Such a digraph can be used to record the result of a tennis tournament or of any other game in which draws are not allowed. In Fig. 1.44, for example, team z beats team w, but is beaten by team v, and so on.

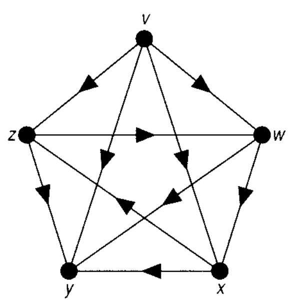


Figure 1.44

Infinite graphs

An **infinite graph** G consists of an infinite set $V(G)$ of elements called **vertices** and an infinite family $E(G)$ of unordered pairs of elements of $V(G)$ called **edges**. If $V(G)$ and $E(G)$ are both countably infinite (that is, they can be labelled 1, 2, 3, . . .), then G is a **countable graph**. For convenience, we exclude the possibility of $V(G)$ being infinite but $E(G)$ finite, as such objects are merely finite graphs together with infinitely many isolated vertices, or of $E(G)$ being infinite but $V(G)$ finite, as such objects are essentially finite graphs but with infinitely many loops or multiple edges.

Many of our earlier definitions extend immediately to infinite graphs. The **degree** of a vertex v of an infinite graph is the cardinality of the set of edges incident with v, and may be finite or infinite. An infinite graph is **locally finite** if each of its vertices has finite degree; two important examples are the infinite square lattice and the infinite triangular lattice, shown in Figs 1.45 and 1.46. We similarly define a **locally countable** infinite graph to be one in which each vertex has countable degree.

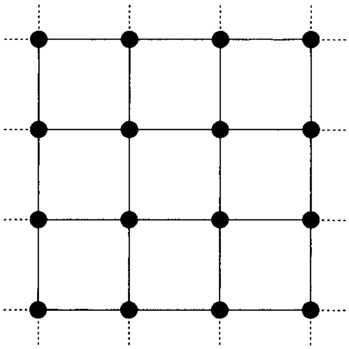


Figure 1.45

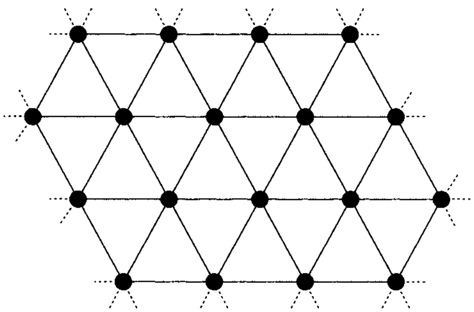


Figure 1.46

We can now prove the following simple, but fundamental, result.

THEOREM 1.4 *Every connected locally countable infinite graph is a countable graph.*

Proof. Let v be any vertex of such an infinite graph, and let A_1 be the set of vertices adjacent to v, A_2 be the set of all vertices adjacent to a vertex of A_1, and so on. By hypothesis, A_1 is countable, and hence so are $A_2, A_3, \ldots$, since the union of a countable collection of countable sets is countable. Hence $\{v\}, A_1\ A_2, \ldots$ is a sequence of sets whose union is countable and contains every vertex of the infinite graph, by connectedness. The result follows. ■

COROLLARY 1.5 *Every connected locally finite infinite graph is a countable graph.*

Exercises

1.35[s] Write down the vertex-set and arc-set of the digraph in Fig. 1.41.

1.36 Write down the vertex-set and arc-family of the graph in Fig. 1.42.

1.37[s] Two of the digraphs in Fig. 1.47 are isomorphic. Which two are they?

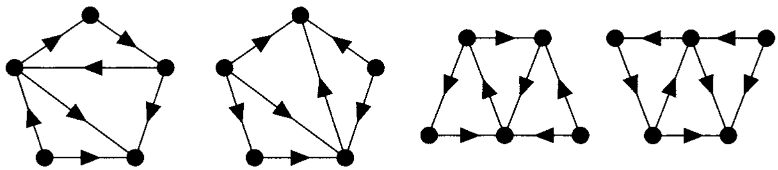


Figure 1.47

1.38 Two of the digraphs in Fig. 1.48 are isomorphic. Which two are they?

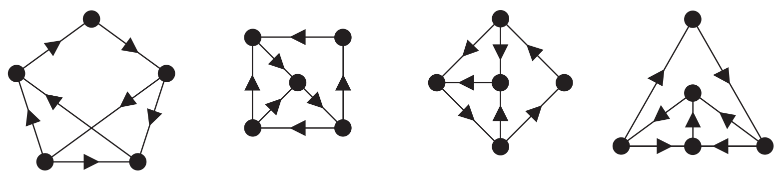


Figure 1.48

1.39[s] Verify the handshaking dilemma for the digraph of Fig. 1.39.

1.40 Verify the handshaking dilemma for the tournament of Fig. 1.49.

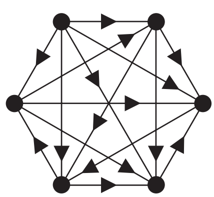


Figure 1.49

1.41[s] Write down the adjacency matrix of the digraph in Fig. 1.39.

1.42 Write down the adjacency matrix of the tournament in Fig 1.44.

1.43 The **converse** D' of a digraph D is obtained from D by reversing the direction of each arc.
(i) Give an example of a digraph that is isomorphic to its converse.
(ii) What is the connection between the adjacency matrices of D and D'?

1.44 Let T be a tournament on n vertices. If $\sum$ denotes a summation over all the vertices of T, prove that $\sum \text{outdeg}(v)^2 = \sum \text{indeg}(v)^2$.

1.45[s] Give an example of each of the following:
(i) an infinite bipartite graph;
(ii) an infinite connected cubic graph.

1.46 Give an example of each of the following:
(i) an infinite graph with infinitely many end-vertices;
(ii) an infinite graph with uncountably many vertices and edges.

1.4 Three puzzles

In this section we present three recreational puzzles that can be solved using ideas relating to graphs. In each puzzle, notice how the use of a graph diagram makes the problem much easier to understand and to solve.

The eight-circles problem

Place the letters A, B, C, D, E, F, G, H into the eight circles in Fig. 1.50, in such a way that no letter is adjacent to a letter that is next to it in the alphabet.

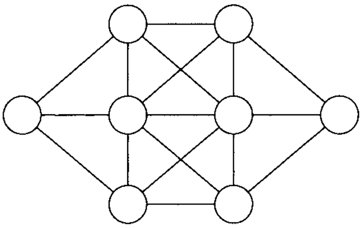


Figure 1.50

First note that trying all the possibilities is not feasible, as there are $8! = 40{,}320$ ways of placing eight letters into eight circles. We therefore need a more systematic approach.

Note that:

(i) the easiest letters to place are A and H, because each has only one letter to which it cannot be adjacent (B and G, respectively);
(ii) the hardest circles to fill are those in the middle, as each is adjacent to six others.

This suggests that we place A and H in the middle circles. If we place A to the left of H, then the only possible positions for B and G are as shown in Fig. 1.51.

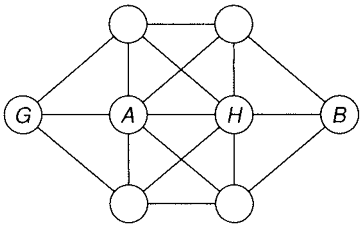


Figure 1.51

The letter C must now be placed on the left-hand side of the diagram, and F must be placed on the right-hand side. It is then a simple matter to place the remaining letters, as shown in Fig. 1.52.

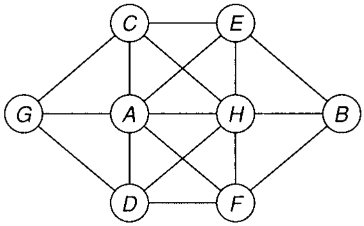


Figure 1.52

Six people at a party

Show that, in any gathering of six people, there are either three people who all know each other, or three people none of whom knows either of the other two.

To solve this, we draw a graph in which we represent each person by a vertex, and join two vertices by a solid edge if the corresponding people know each other and by a dotted edge if not. We must show that there is always a solid triangle or a dotted triangle.

Let v be any vertex. Then there must be exactly five edges incident with v, either solid or dotted, and so at least three of these edges must be of the same type. Let us assume that there are three solid edges (see Fig. 1.53); the case of at least three dotted edges is similar.

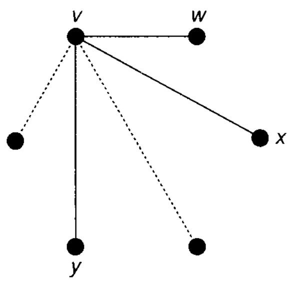


Figure 1.53

If the people corresponding to the vertices w and x know each other, then v, w and x form a solid triangle, as required. Similarly, if the people corresponding to the vertices w and y, or to the vertices x and y, know each other, then we again obtain a solid triangle. These three cases are shown in Fig. 1.54.

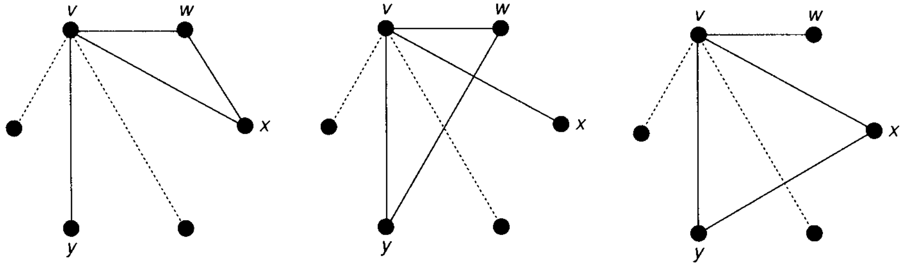


Figure 1.54

Finally, if no two of the people corresponding to the vertices w, x and y know each other, then w, x and y form a dotted triangle, as required (see Fig. 1.55).

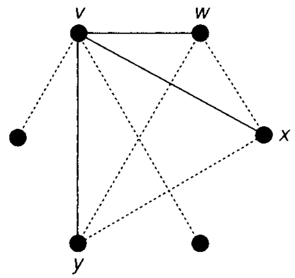


Figure 1.55

Since this exhausts all possibilities, the result follows.

The four-cubes problem

We conclude this section with a puzzle that has long been popular under the name of 'Instant Insanity'.

Given four cubes whose faces are coloured red, blue, green and yellow, as in Fig. 1.56, can we pile them up so that all four colours appear on each side of the resulting 4×1 stack?

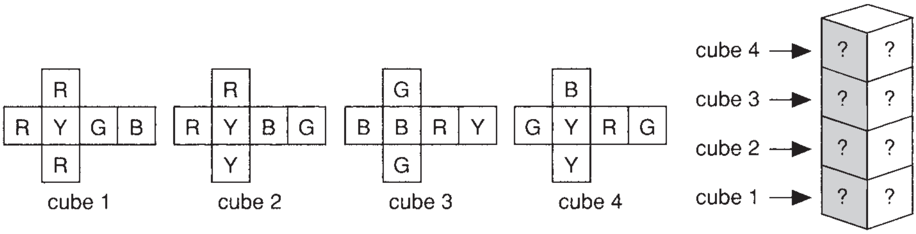


Figure 1.56

Although these cubes can be stacked in thousands of different ways, there is essentially only one way that gives a solution.

To solve this problem, we represent each cube by a graph with four vertices, R, B, G and Y, one for each colour. In each of these graphs, two vertices are adjacent if and only if the cube in question has the corresponding colours on *opposite* faces. The graphs corresponding to the cubes of Fig. 1.56 are shown in Fig. 1.57.

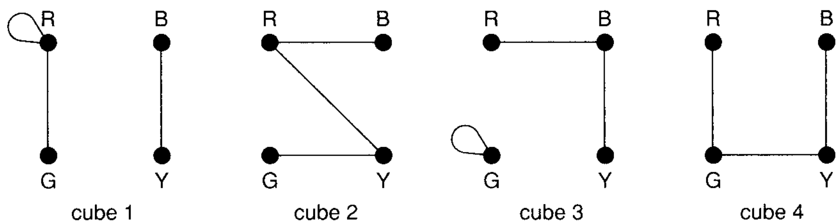


Figure 1.57

We next superimpose these graphs to form a new graph G (see Fig. 1.58).

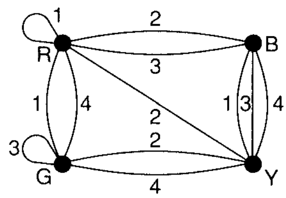


Figure 1.58

A solution of the puzzle is obtained by finding two subgraphs, H_1 and H_2, of G. The subgraph H_1 tells us which pair of colours appear on the front and back of each cube, and the subgraph H_2 tells us which pair of colours appear on the left and right. To this end, the subgraphs H_1 and H_2 must have the following properties:

(a) *Each subgraph contains exactly one edge from each cube*: this ensures that each cube has a front and back, and a left and right, and the subgraphs tell us which pairs of colours appear on these faces.
(b) *The subgraphs have no edges in common*: this ensures that the faces on the front and back are different from those on the sides.
(c) *Each subgraph is regular of degree* 2: this tells us that each colour appears exactly twice on the sides of the stack (once on each side) and exactly twice on the front and back (once on the front and once on the back).

Using these observations, we can easily check that neither loop can be included in the subgraphs. It then follows, after a little experimentation, that the subgraphs are as shown in Fig. 1.59, and the solution can then be read from them, as in Fig. 1.60.

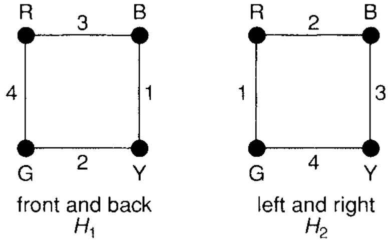


Figure 1.59

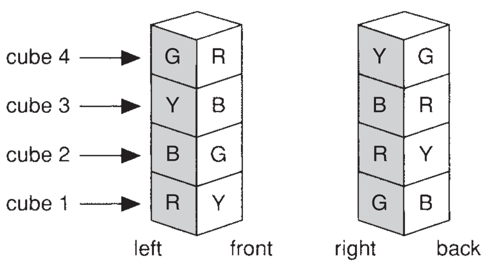


Figure 1.60

Exercises

1.47[s] Find another solution of the eight-circles problem.

1.48[s] Show that there is a gathering of five people in which there are no three people who all know each other, and no three people none of whom knows either of the other two.

1.49[s] Find a solution of the four-cubes problem for the set of cubes in Fig. 1.61.

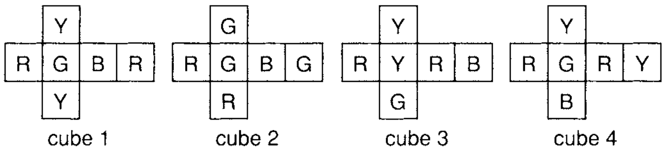


Figure 1.61

1.50 Show that the four-cubes problem in Fig. 1.62 has no solution.

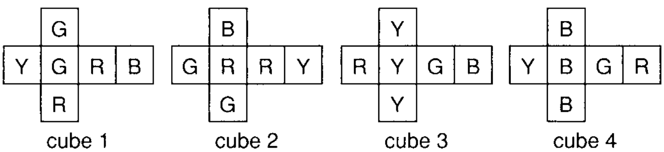


Figure 1.62

Challenge problems

1.51 A simple graph that is isomorphic to its complement is **self-complementary**.
(i) Prove that, if G is self-complementary, then G has $4k$ or $4k + 1$ vertices, where k is an integer.
(ii) Find all self-complementary graphs with four and five vertices.
(iii) Find a self-complementary graph with eight vertices.

1.52 (*For those who know linear algebra*) If G is a simple graph with edge-set $E(G)$, the **vector space of** $\boldsymbol{G}$ is the vector space over the field $\mathbf{Z}_2 = \{0, 1\}$ of integers modulo 2, whose elements are subsets of $E(G)$. The sum $E + F$ of two such subsets E and F is the set of edges in E or F but not both, and scalar multiplication is defined by $1.E = E$ and $0.E = \varnothing$. Show that this defines a vector space over $\mathbf{Z}_2$, and find a basis for it.

1.53 The **line graph** $L(G)$ of a simple graph G is the graph whose vertices are in one–one correspondence with the *edges* of G, with two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G are adjacent.
(i) Show that K_3 and $K_{1,3}$ have the same line graph.
(ii) Show that the line graph of the tetrahedron graph is the octahedron graph.
(iii) Prove that, if G is regular of degree k, then $L(G)$ is regular of degree $2k - 2$.
(iv) Find an expression for the number of edges of $L(G)$ in terms of the degrees of the vertices of G.
(v) Show that $L(K_5)$ is the complement of the Petersen graph.

1.54 (For those who know group theory) An **automorphism** φ of a simple graph G is a one–one mapping of the vertex-set of G onto itself with the property that $\varphi(v)$ and $\varphi(w)$ are adjacent whenever v and w are. The **automorphism group** $\Gamma(G)$ of G is the group of automorphisms of G under composition.

(i) Prove that the groups $\Gamma(G)$ and $\Gamma(\bar{G})$ are isomorphic.

(ii) Find the groups $\Gamma(K_n)$, $\Gamma(K_{r,s})$ and $\Gamma(C_n)$.

(iii) Use the results of parts (i) and (ii) and of Exercise 1.53(v) to find the automorphism group of the Petersen graph.

1.55 Show that an infinite graph G can be drawn in Euclidean 3-space if $V(G)$ and $E(G)$ can each be put in one–one correspondence with a subset of the set of real numbers.

1.56 Prove that the solution of the four-cubes problem in the text is the only solution for that set of cubes.

Chapter 2

Paths and cycles

So many paths that wind and wind,
While just the art of being kind
Is all the sad world needs.
Ella Wheeler Wilcox

Now that we have a reasonable armoury of graphs, we can look at their properties. To do this, we need some definitions that describe ways of 'going from one vertex to another', in both graphs and digraphs. We give these definitions in Section 2.1 and present some results on connectivity. In Sections 2.2 and 2.3 we study two particular types of graph and digraph: those with trails that include every edge, and those with cycles that pass through every vertex. We conclude this chapter, in Section 2.4, with some applications of paths and cycles.

2.1 Connectivity

Given a graph G, a **walk** in G is a finite sequence of edges of the form

$$v_0v_1, v_1v_2, \ldots, v_{m-1}v_m, \text{ also denoted by } v_0 \to v_1 \to v_2 \to \cdots \to v_m,$$

in which any two consecutive edges are adjacent or identical. Such a walk determines a sequence of vertices $v_0, v_1, \ldots, v_m$. We call v_0 the **initial vertex** and v_m the **final vertex** of the walk, and speak of a **walk from v_0 to v_m**. The number of edges in a walk is called its **length**; for example, in Fig. 2.1,

$$v \to w \to x \to y \to z \to z \to y \to w$$

is a walk of length 7 from v to w.

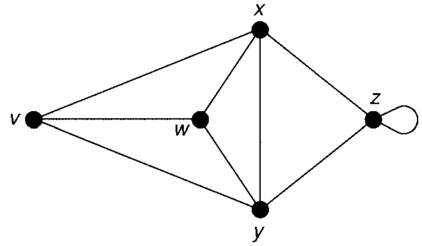


Figure 2.1

The concept of a walk is usually too general for our purposes, so we impose some restrictions. A walk in which all the edges are distinct is a **trail**. If, in addition, the vertices $v_0, v_1, \ldots, v_m$ are distinct (except, possibly, $v_0 = v_m$), then the trail is a **path**. A walk, path or trail is **closed** if $v_0 = v_m$, and a closed path with at least one edge is a **cycle**. For example,

$v \to w \to x \to y \to z \to z \to x$ is a trail,
$v \to w \to x \to y \to z$ is a path,
$v \to w \to x \to y \to z \to x \to v$ is a closed trail,
and $v \to w \to x \to y \to v$ is a cycle.

Note that a loop is a cycle of length 1, and a pair of multiple edges is a cycle of length 2. A cycle of length 3, such as

$$v \to w \to x \to v,$$

is called a **triangle**.

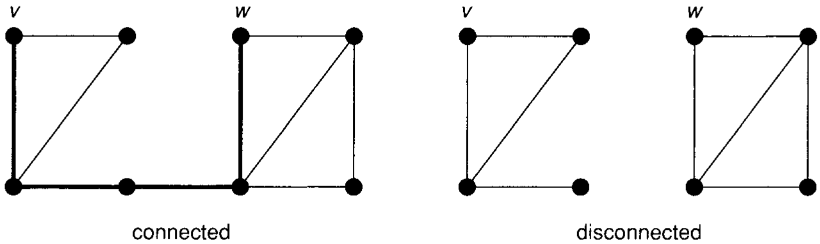


Figure 2.2

We observe that a graph is connected if and only if there is a path between each pair of vertices (see Fig. 2.2). We can also prove the following results on bipartite graphs:

THEOREM 2.1 *A graph G is bipartite if and only if every cycle of G has even length.*

Proof. $\Rightarrow$ If G is bipartite, we can split its vertex-set into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B. Let

$$v_0 \to v_1 \to \cdots \to v_m \to v_0$$

be a cycle in G, and assume (without loss of generality) that v_0 is in A. Then v_1 is in B, v_2 is in A, v_3 is in B, and so on. Since v_m must be in B, the cycle has even length.

$\Leftarrow$ Conversely, assume that every cycle of G has even length. We may assume that G is connected. Choose any vertex v, let A be the set of vertices w for which the shortest path from v to w has even length, and let B be the set of vertices not in A. If any two vertices of A (or of B) were adjacent, then the shortest paths from these vertices to v would include a cycle of odd length. Thus each edge of G must join a vertex of A and a vertex of B, so G is bipartite. ■

We next investigate bounds on the number of edges of a simple connected graph with n vertices. Such a graph has fewest edges when it has no cycles, and has most edges when it is a complete graph; this implies that the number of edges must lie between $n - 1$ and $\frac{1}{2}n(n - 1)$. In fact, we can prove a stronger result that includes this as a special case.

THEOREM 2.2 *Let G be a simple graph on n vertices. If G has k components, then the number m of edges of G satisfies*

$$n - k \leq m \leq \tfrac{1}{2}(n - k)(n - k + 1).$$

Proof. We prove the lower bound $m \geq n - k$ by induction on the number of edges of G, the result being trivial if G is a null graph. If G contains as few edges as possible (say m_0), then the removal of any edge of G must increase the number of components by 1, and the graph that remains has n vertices, $k + 1$ components and $m_0 - 1$ edges. It follows from the induction hypothesis that $m_0 - 1 \geq n - (k + 1)$, giving $m_0 \geq n - k$, as required.

To prove the upper bound, we can assume that each component of G is a complete graph. Suppose, then, that there are two components C_i and C_j with n_i and n_j vertices, respectively, where $n_i \geq n_j > 1$. If we now replace C_i and C_j by complete graphs on $n_i + 1$ and $n_j - 1$ vertices, then the total number of vertices remains unchanged, but the number of edges is changed by

$$\tfrac{1}{2}\{(n_i + 1)n_i - n_i(n_i - 1)\} - \tfrac{1}{2}\{n_j(n_j - 1) - (n_j - 1)(n_j - 2)\} = n_i - n_j + 1,$$

which is positive. It follows that, in order to attain the maximum number of edges, G must consist of a complete graph with $n - k + 1$ vertices and $k - 1$ isolated vertices. The result now follows. ■

We deduce the following corollary (see Exercise 2.4):

COROLLARY 2.3 *Any simple graph with n vertices and more than $\frac{1}{2}(n - 1)(n - 2)$ edges is connected.*

Connectivity

Another approach used in the study of connected graphs is to ask 'how connected is a given graph?'. Such questions have arisen in connection with the vulnerability of certain networks, such as a rail or telecommunication network. One interpretation of this is to ask how many edges (railway lines or telephone cables) or vertices (stations or telephone exchanges) need to be out of action for the network to become disconnected. We now introduce some terms that are useful when discussing such questions.

A **disconnecting set** in a connected graph G is a set of edges whose deletion disconnects G. For example, in the graph of Fig. 2.3, the sets $\{e_1, e_2, e_5\}$ and $\{e_3, e_6, e_7, e_8\}$ are disconnecting sets of G; the disconnected graph left after deletion of the second is shown in Fig. 2.4.

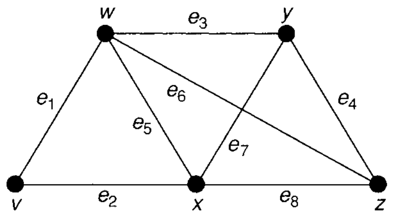


Figure 2.3

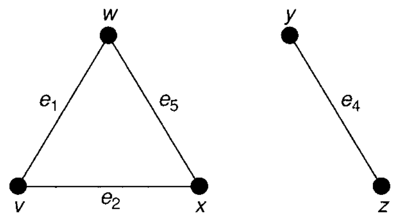


Figure 2.4

We further define a **cutset** to be a minimal disconnecting set – that is, a disconnecting set, no proper subset of which is a disconnecting set; in the above example, only the second disconnecting set is a cutset. Note that the deletion of the edges in a cutset always leaves a graph with exactly two components. If a cutset has only one edge e, we call e a **bridge** (see Fig. 2.5).

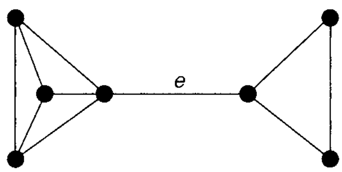


Figure 2.5

These definitions can easily be extended to disconnected graphs. If G is any such graph, a **disconnecting set** of G is a set of edges whose removal increases the number of components of G, and a **cutset** of G is a minimal disconnecting set.

If G is connected, its **edge-connectivity** $\lambda(G)$ is the size of the smallest cutset in G. Thus $\lambda(G)$ is the smallest number of edges that we need to delete in order to disconnect G. For example, if G is the graph of Fig. 2.3, then $\lambda(G) = 2$, corresponding to the cutset $\{e_1, e_2\}$. We also say that G is ***k*-edge-connected** if $\lambda(G) \geq k$. Thus the graph of Fig. 2.3 is 1-edge-connected and 2-edge-connected, but is not 3-edge-connected.

It can be proved that a graph is 2-edge-connected if and only if any two distinct vertices are joined by at least two paths with no edges in common (see Exercise 2.8(i)); for example, any two distinct vertices of Fig. 2.3 are joined by at least two such paths. More generally we have the following celebrated theorem of K. Menger. It will be proved in Chapter 6.

THEOREM 2.4 (Menger, 1927) *A graph G is k-edge-connected if and only if any two distinct vertices of G are joined by at least k paths, no two of which have any edges in common.*

We can also define the analogous concepts for the removal of vertices. A **separating set** in a connected graph G is a set of vertices whose deletion disconnects G; recall that when we delete a vertex, we must also remove its incident edges. For example, in the graph of Fig. 2.3, the sets $\{w, x\}$ and $\{w, x, y\}$ are separating sets of G; the disconnected graph left after removal of the first is shown in Fig. 2.6. If a separating set contains only one vertex v, we call v a **cut-vertex** (see Fig. 2.7). These definitions extend immediately to disconnected graphs, as above.

Figure 2.6

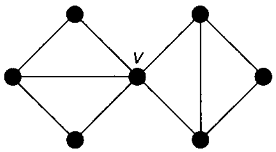


Figure 2.7

If G is connected and not a complete graph, its (vertex) **connectivity** $\kappa(G)$ is the size of the smallest separating set in G. Thus $\kappa(G)$ is the smallest number of vertices that we need to delete in order to disconnect G. For example, if G is the graph of Fig. 2.3, then $\kappa(G) = 2$, corresponding to the separating set $\{w, x\}$. We also say that G is ***k*-connected** if $\kappa(G) \geq k$. Thus the graph of Fig. 2.3 is 1-connected and 2-connected, but is not 3-connected.

It can be proved that a graph with at least three vertices is 2-connected if and only if any two distinct vertices are joined by at least two paths with no other vertices in common; for example, any two distinct vertices of Fig. 2.3 are joined by at least two such paths (see Exercise 2.8(ii)). More generally, we have another theorem of Menger, which will also be proved in Chapter 6.

THEOREM 2.5 (Menger, 1927) *A graph G with at least $k + 1$ vertices is k-connected if and only if any two vertices of G are joined by at least k paths, no two of which have any other vertices in common.*

These concepts are not unrelated. For example, it can be proved that, if G is any connected graph, then

$$\kappa(G) \leq \lambda(G) \leq \delta(G),$$

where $\delta(G)$ is the smallest vertex-degree in G.

As we shall see, there are striking and unexpected similarities between the properties of cycles and cutsets; for example, look at Exercises 2.46, 2.47, 2.48, 2.49 and Theorem 3.2. The reasons for these similarities will become clear in Chapter 7.

Digraphs

There are also natural generalizations to digraphs of the above definitions. A **walk** in a digraph D is a finite sequence of arcs of the form $v_0v_1, v_1v_2, \ldots, v_{m-1}v_m$. We sometimes write this sequence as

$$v_0 \to v_1 \to \cdots \to v_m,$$

and speak of a **walk from v_0 to v_m**. In an analogous way, we can define directed trails, directed paths and directed cycles (or, simply, trails, paths and cycles, when there is no possibility of confusion). Note that although a trail cannot contain a given arc vw more than once, it can contain both vw and wv; for example, in Fig. 2.8,

$$z \to w \to v \to w \to u$$

is a trail.

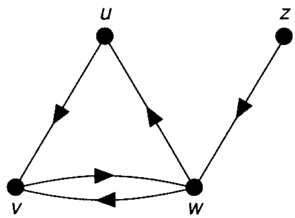


Figure 2.8

We can also define connectedness. The two most useful types of connected digraph correspond to whether or not we take account of the directions of the arcs: these definitions are the natural extensions to digraphs of the two descriptions of connected graphs in Sections 1.2 and 2.1.

We have already defined a digraph D to be **connected** if it cannot be expressed as the union of two digraphs; this is equivalent to saying that the underlying graph of D is a connected graph. We also say that D is **strongly connected** if, for any two vertices v and w of D, there is a directed path from v to w. Every strongly connected digraph is connected, but not all connected digraphs are strongly connected; for example, the connected digraph of Fig. 2.8 is not strongly connected, since there is no path from v to z.

The distinction between a connected digraph and a strongly connected one becomes more evident if we consider the road map of a city, all of whose streets are one way. If the road map is connected, then we can drive from any part of the city to any other, ignoring the directions of the one-way streets as we go. However, if the map is strongly connected, then we can drive from any part of the city to any other, always going the 'right way' down the one-way streets.

Since every one-way system needs to be strongly connected, it is natural to ask when we can impose a one-way system on a street map so that we can drive from any part of the city to any other. This is not always possible: for example, if the city consists of two parts connected by a single bridge, then we cannot impose such a one-way system on the city, since whatever direction we give to the bridge, one part of the city is cut off. But if there are no bridges, then we can always impose such a one-way system. This result is stated below in Theorem 2.6.

For convenience, we define a graph G to be **orientable** if each edge of G can be directed so that the resulting digraph is strongly connected; such a digraph is an **orientation** of G. For example, if G is the graph shown in Fig. 2.9, then G is orientable; an orientation is shown in Fig. 2.10.

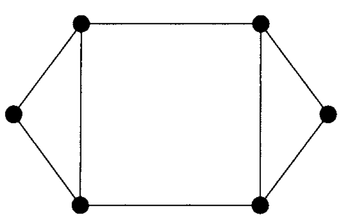


Figure 2.9

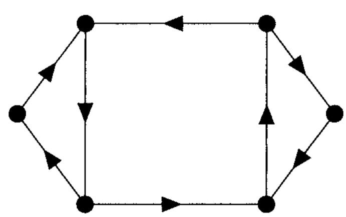


Figure 2.10

The following theorem gives a necessary and sufficient condition (due to H. E. Robbins) for a graph to be orientable.

THEOREM 2.6 *A connected graph G is orientable if and only if each edge of G lies in at least one cycle.*

Proof. The necessity of the condition is clear.

To prove the sufficiency, we choose any cycle C and direct its edges cyclically. If each edge of G is contained in C, then the proof is complete. If not, we choose any edge e that is not in C but which is adjacent to an edge of C. By hypothesis, e is contained in some cycle C' whose edges we may direct cyclically, except for those edges that have already been directed – that is, those edges of C' that also lie in C; the situation is illustrated in Fig. 2.11, with dotted lines denoting edges of C'. We proceed in this way, at each stage directing at least one new edge, until all edges have been directed. Since the digraph must remain strongly connected at each stage of the process, the result follows. ■

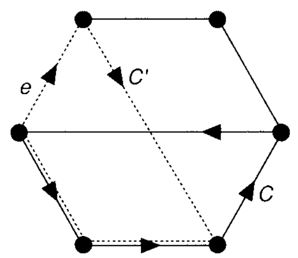


Figure 2.11

Infinite graphs

We can also extend the concept of a walk to an infinite graph G. There are essentially three different types of walk in G:

(i) a *finite walk* is defined exactly as above;

(ii) a *one-way infinite walk* with initial vertex v_0 is an infinite sequence of edges of the form

$$v_0 \to v_1 \to v_2 \to \cdots;$$

(iii) a *two-way infinite walk* is an infinite sequence of edges of the form

$$\cdots \to v_{-2} \to v_{-1} \to v_0 \to v_1 \to v_2 \to \cdots.$$

One-way and two-way infinite trails and paths are defined analogously.

The following result, known as **König's lemma**, tells us that infinite paths are not difficult to come by; we shall use this result in Chapter 4.

THEOREM 2.7 (König, 1927) *Let G be a connected locally finite infinite graph. Then, for any vertex v of G, there exists a one-way infinite path with initial vertex v.*

Proof. For each vertex z other than v, there is a non-trivial path from v to z. It follows that there are infinitely many paths in G with initial vertex v. Since the degree of v is finite, infinitely many of these paths must start with the same edge. If vv_1 is such an edge, then we can repeat this procedure for the vertex v_1 and thus obtain a new vertex v_2 and a corresponding edge v_1v_2. By carrying on in this way, we obtain the one-way infinite path

$$v \to v_1 \to v_2 \to \cdots . \qquad \blacksquare$$

Exercises

2.1[s] In the Petersen graph, find
(i) a trail of length 5;
(ii) a path of length 9;
(iii) cycles of lengths 5, 6, 8 and 9;
(iv) cutsets with three, four and five edges.

2.2 In the dodecahedron graph, find
(i) a trail of length 5;
(ii) a path of length 10;
(iii) cycles of lengths 5, 8 and 9;
(iv) cutsets with three, four and five edges.

2.3[s] The **girth** of a graph is the length of its shortest cycle. Write down the girths of the following graphs:
(i) K_9; (ii) $K_{5,7}$; (iii) C_8; (iv) W_8; (v) Q_5;
(vi) the Petersen graph; (vii) the graph of the dodecahedron.

2.4 Prove Corollary 2.3.

2.5[s] Prove that a simple graph and its complement cannot both be disconnected.

2.6[s] Write down $\kappa(G)$ and $\lambda(G)$ for each of the following graphs G:
(i) C_6; (ii) W_6; (iii) $K_{4,7}$; (iv) Q_4.

2.7 (i) Show that, if G is a connected graph with minimum degree k, then $\lambda(G) \le k$.
(ii) Draw a graph G with minimum degree k for which $\kappa(G) < \lambda(G) < k$.

2.8 (i) Prove that a graph is 2-edge-connected if and only if any two distinct vertices are joined by at least two paths with no edges in common.
(ii) Prove that a graph with at least three vertices is 2-connected if and only if any two distinct vertices are joined by at least two paths with no other vertices in common.

2.9 In a connected graph, the **distance** $d(v, w)$ between a vertex v and a vertex w is the length of the shortest path from v to w.
(i) If $d(v, w) \ge 2$, show that there exists a vertex z such that

$$d(v, z) + d(z, w) = d(v, w).$$

(ii) In the Petersen graph, show that $d(v, w) = 1$ or 2, for any distinct vertices v and w.

2.10 Show, by finding an orientation for each, that K_n ($n \geq 3$) and $K_{r,s}$ ($r, s \geq 2$) are orientable.

2.11 Find orientations for the Petersen graph and the graph of the dodecahedron.

2.12 A tournament is **transitive** if the existence of arcs uv and vw implies the existence of an arc uw.
(i) Give an example of a transitive tournament.
(ii) Show that in a transitive tournament the teams can be ranked so that each team beats all the teams which follow it in the ranking.
(iii) Deduce that a transitive tournament with at least two vertices cannot be strongly connected.

2.13 A tournament T is **irreducible** if it is impossible to split the set of vertices of T into two disjoint sets V_1 and V_2 so that each arc joining a vertex of V_1 and a vertex of V_2 is directed from V_1 to V_2.
(i) Give an example of an irreducible tournament.
(ii) Prove that a tournament is irreducible if and only if it is strongly connected.

2.14^s Give an example to show that the conclusion of König's lemma is false if we omit the condition that the infinite graph is locally finite.

2.2 Eulerian graphs and digraphs

In this section we consider Eulerian graphs and digraphs. The name 'Eulerian' arises from the fact that in 1735 the Swiss mathematician Leonhard Euler solved the famous **Königsberg bridges problem** which asks whether you can cross each of the seven bridges in Fig. 2.12 exactly once and return to your starting point. This is equivalent to asking whether the graph in Fig. 2.13 has an 'Eulerian trail' (as defined below), although Euler did not rephrase the problem in this way.

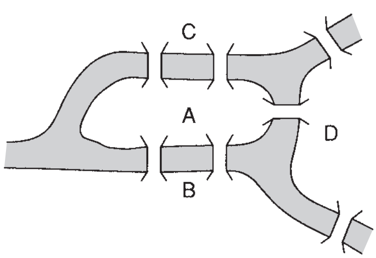


Figure 2.12

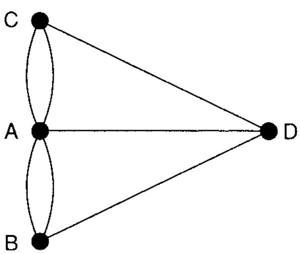


Figure 2.13

A translation of Euler's paper, and a discussion of various related topics, may be found in Biggs, Lloyd and Wilson [17]. Problems on Eulerian graphs frequently appear in books on recreational mathematics. A typical problem might ask whether a given diagram, such as that in Fig. 2.13, can be drawn without lifting one's pencil from the paper and without repeating any lines.

Eulerian graphs

A connected graph G is **Eulerian** if there exists a closed trail that includes every edge of G; such a trail is an **Eulerian trail**. Note that this definition requires you to traverse each edge once and once only, and to finish at your starting point. A non-Eulerian graph G is **semi-Eulerian** if there exists a (non-closed) trail that includes every edge of G. Figures 2.14, 2.15 and 2.16 show graphs that are Eulerian, semi-Eulerian and non-Eulerian, respectively.

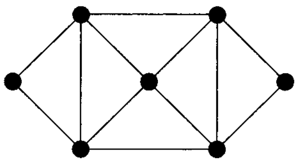


Figure 2.14

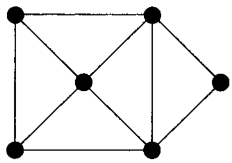


Figure 2.15

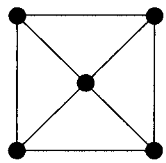


Figure 2.16

One question that immediately arises is 'are there necessary and sufficient conditions for a graph to be Eulerian?' Before answering this question in Theorem 2.9, we prove a simple lemma.

LEMMA 2.8 *If G is a graph in which the degree of each vertex is at least 2, then G contains a cycle.*

Proof. If G has any loops or multiple edges, the result is trivial. We can therefore suppose that G is a simple graph.

Let v be any vertex of G. We construct a walk

$$v \to v_1 \to v_2 \to \cdots$$

inductively, by choosing v_1 to be any vertex adjacent to v and, for each $i > 1$, choosing v_{i+1} to be any vertex adjacent to v_i except v_{i-1}; the existence of such a vertex is guaranteed by our hypothesis. Since G has only finitely many vertices, we must eventually choose a vertex that has been chosen before. If v_k is the first such vertex, then the part of the walk that lies between the two occurrences of v_k is the required cycle. ■

We come now to the main result of this section, which tells us that a given connected graph is Eulerian if and only if all of its vertex-degrees are even. In terms of the Königsberg bridges problem, this corresponds to a city map in which the number of bridges that emerge from each part of the city is even.

THEOREM 2.9 (Euler, 1735) *A connected graph G is Eulerian if and only if the degree of each vertex of G is even.*

Proof. $\Rightarrow$ Suppose that P is an Eulerian trail of G. Whenever P passes through a vertex, there is a contribution of 2 towards the degree of that vertex. Since each edge occurs exactly once in P, the degree of each vertex must be a sum of 2s, and is thus an even number.

$\Leftarrow$ The proof is by induction on the number of edges of G. Suppose that the degree of each vertex is even. Since G is connected, each vertex has degree at least 2 and so, by Lemma 2.8, G contains a cycle C.

If C contains every edge of G, the proof is complete. If not, we remove from G the edges of C to form a new (possibly disconnected) graph H with fewer edges than G, and in which each vertex still has even degree. By the induction hypothesis, each component of H has an Eulerian trail. But each component of H has at least one vertex in common with C, by connectedness. It follows that we can obtain the required Eulerian trail of G by tracing the edges of C until a non-isolated vertex of H is reached, tracing the Eulerian trail of the component of H that contains that vertex, and then continuing along the edges of C until we reach a vertex belonging to another component of H, and so on. The whole process terminates when we return to the initial vertex (see Fig. 2.17). ■

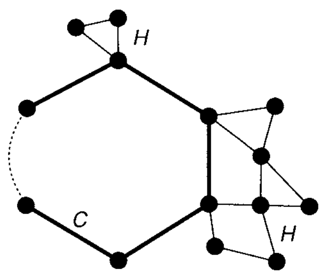


Figure 2.17

This proof can easily be modified to prove the following two results; we omit the details (see Exercise 2.18).

COROLLARY 2.10 *A connected graph is Eulerian if and only if its set of edges can be split up into edge-disjoint cycles.*

COROLLARY 2.11 *A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.*

Note that, in a semi-Eulerian graph, any semi-Eulerian trail must have one vertex of odd degree as its initial vertex and the other as its final vertex. Note also that, by the handshaking lemma, a graph cannot have exactly one vertex of odd degree.

We conclude our introduction to Eulerian graphs with an algorithm for constructing an Eulerian trail in a given Eulerian graph. The method is known as **Fleury's algorithm**.

THEOREM 2.12 *Let G be an Eulerian graph. Then the following construction is always possible, and produces an Eulerian trail of G.*

Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following rules:

(i) *erase the edges as they are traversed, and if any isolated vertices result, erase them too;*
(ii) *at each stage, use a bridge only if there is no alternative.*

Proof. We show first that the construction can be carried out at each stage.

Suppose that we have just reached a vertex v, erasing the edges as we go. If $v \neq u$, then the subgraph H that remains is connected and has only two vertices of odd degree, u and v. To show that the construction can be carried out, we must show that the removal of the next edge does not disconnect H – or, equivalently, that v is incident with at most one bridge. But if this is not the case, then there exists a bridge vw such that the component K of $H - vw$ containing w does not contain u (see Fig. 2.18). Since the vertex w has odd degree in K, some other vertex of K must also have odd degree, giving the required contradiction. If $v = u$, the proof is almost identical, as long as there are still edges incident with u.

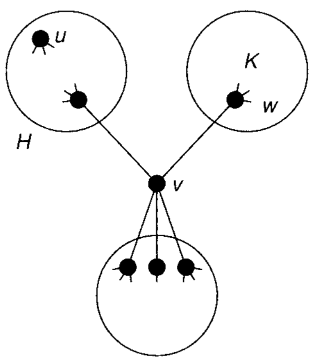


Figure 2.18

It remains only to show that this construction always yields a complete Eulerian trail. But this is clear, since no edges of G can remain untraversed when the last edge incident with u is used, since otherwise the removal of some *earlier* edge adjacent to one of these edges would have disconnected the graph, contradicting (ii). ■

Eulerian digraphs

We can also obtain digraph analogues of some of the above results.

A connected digraph D is **Eulerian** if there exists a closed directed trail that includes every arc of D; such a trail is an **Eulerian trail**. For example, the digraph in Fig. 2.19 is not Eulerian, even though its underlying graph is an Eulerian graph.

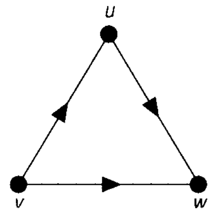


Figure 2.19

Note that the digraph must be strongly connected for an Eulerian trail to exist. Note also that any Eulerian graph is orientable, since we simply follow any Eulerian trail, directing the edges in the direction of the trail as we go.

Our first aim is to give a necessary and sufficient condition, analogous to the one in Theorem 2.9, for a connected digraph to be Eulerian.

THEOREM 2.13 *A strongly connected digraph is Eulerian if and only if, for each vertex v of D,*

$$\text{outdeg}(v) = \text{indeg}(v).$$

Proof. The proof is entirely analogous to that of Theorem 2.9, and is left as an exercise. ■

We leave it to you to define a semi-Eulerian digraph, and to obtain digraph analogues of Corollaries 2.10 and 2.11 (see Exercise 2.23).

Infinite Eulerian graphs

We conclude this section with a brief discussion of infinite Eulerian graphs. It seems natural to call a connected infinite graph G **Eulerian** if there exists a two-way infinite trail that includes every edge of G; such an infinite trail is a two-way **Eulerian trail**. Note that these definitions require G to be countable.

The following theorems give further necessary conditions for an infinite graph to be Eulerian.

THEOREM 2.14 *Let G be a countable connected graph which is Eulerian. Then*

(i) *G has no vertices of odd degree;*
(ii) *for each finite subgraph H of G, the infinite graph K obtained by deleting from G the edges of H has at most two infinite components;*
(iii) *if, in addition, each vertex of H has even degree, then K has exactly one infinite component.*

Proof.

(i) Suppose that P is an Eulerian trail. Then, by the argument given in the proof of Theorem 2.9, each vertex of G must have either even or infinite degree.

(ii) Let P be split up into three subtrails P_-, P_0 and P_+ in such a way that P_0 is a finite trail containing every edge of H (and possibly other edges as well), and P_- and P_+ are one-way infinite trails. Then the infinite graph formed by the edges of P_- and P_+, and the vertices incident with them, has at most two infinite components. Since K is obtained by adding only a finite set of edges to this graph, the result follows.

(iii) Let the initial and final vertices of P_0 be v and w. We wish to show that v and w are connected in K. If $v = w$, this is obvious. If not, then the result follows when we apply Corollary 2.11 to the graph obtained by removing the edges of H from P_0, this graph having exactly two vertices (v and w) of odd degree, by hypothesis. ■

The necessary conditions given in the previous theorem are also sufficient. We state this result formally in the following theorem. Its proof may be found in Ore [15].

THEOREM 2.15 *If G is a connected countable graph, then G is Eulerian if and only if conditions (i), (ii) and (iii) of Theorem 2.14 are satisfied.*

Exercises

2.15s Which of the following graphs are Eulerian or semi-Eulerian?
(i) the complete graph K_5;
(ii) the complete bipartite graph $K_{2,3}$;
(iii) the graph of the cube;
(iv) the graph of the octahedron;
(v) the Petersen graph.

2.16s In the table of Fig. 1.9, locate all the Eulerian and semi-Eulerian graphs.

2.17 (i) For which values of n is K_n Eulerian?
(ii) Which complete bipartite graphs are Eulerian?
(iii) Which Platonic graphs are Eulerian?
(iv) For which values of n is the wheel W_n Eulerian?
(v) For which values of k is the k-cube Q_k Eulerian?

2.18 Prove Corollaries 2.10 and 2.11.

2.19s Let G be a connected graph with k (> 0) vertices of odd degree.
(i) Show that the minimum number of trails, that together include every edge of G and that have no edges in common, is $\frac{1}{2}k$.
(ii) How many continuous pen-strokes are needed to draw the diagram in Fig. 2.20 without repeating any line?

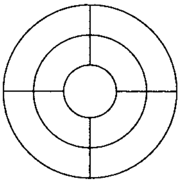


Figure 2.20

2.20^s Use Fleury's algorithm to produce an Eulerian trail for the graph in Fig. 2.21.

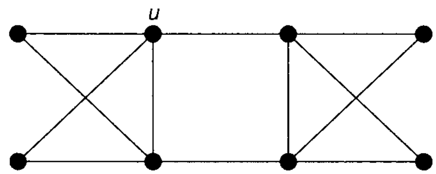


Figure 2.21

2.21 An Eulerian graph is **randomly traceable** from a vertex v if, whenever we start from v and traverse the graph in an arbitrary way never using any edge twice, we eventually obtain an Eulerian trail.

(i) Show that the graph in Fig. 2.22 is randomly traceable.

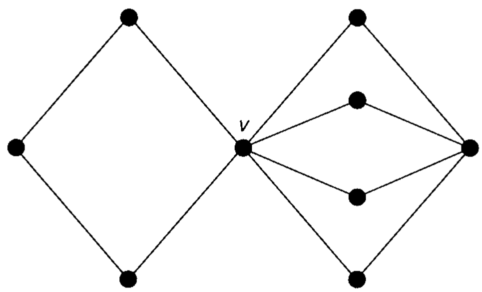


Figure 2.22

(ii) Give an example of an Eulerian graph that is not randomly traceable.

(iii) Why might a randomly traceable graph be suitable for the layout of an exhibition?

2.22^s Find an Eulerian trail in the tournament of Fig. 2.23.

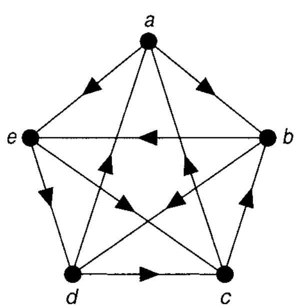


Figure 2.23

2.23 Prove Theorem 2.13 and the digraph analogues of Corollaries 2.10 and 2.11.

2.24 Let D be the digraph whose vertices are the pairs of integers 11, 12, 13, 21, 22, 23, 31, 32, 33, and whose arcs join ij to kl if and only if $j = k$. Find an Eulerian trail in D and use it to obtain a circular arrangement of nine 1s, nine 2s and nine 3s in which each of the 27 possible triples (111, 233, etc.) occurs exactly once. (Problems of this kind arise in communication theory.)

2.25 (i) Find an Eulerian trail in the infinite square lattice *S*.
(ii) Verify that *S* satisfies the conditions of Theorem 2.14.

2.26 Repeat Exercise 2.25 for the infinite triangular lattice.

2.3 Hamiltonian graphs and digraphs

In the previous section we discussed whether there exists a closed trail that includes every edge of a given connected graph *G*. A similar problem is to determine whether there exists a closed trail passing exactly once through each vertex of *G*: such a trail must be a cycle, except when *G* is the graph K_1. Such a cycle is a **Hamiltonian cycle** and a graph with a Hamiltonian cycle is a **Hamiltonian graph**. A non-Hamiltonian graph is **semi-Hamiltonian** if there exists a path through every vertex. Figures 2.24, 2.25 and 2.26 show graphs that are Hamiltonian, semi-Hamiltonian and non-Hamiltonian, respectively.

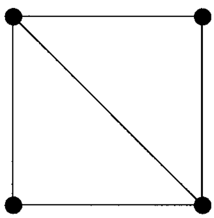


Figure 2.24

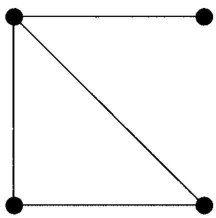


Figure 2.25

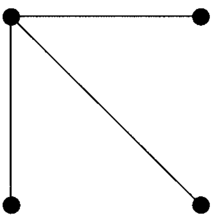


Figure 2.26

The name 'Hamiltonian cycle' arises from the fact that the Irish mathematician Sir William Hamilton investigated their existence in the dodecahedron graph in connection with a problem in algebra, although cycles on polyhedra in general had been studied earlier by the Revd. T. P. Kirkman. A cycle on a dodecahedron is shown with solid lines in Fig. 2.27.

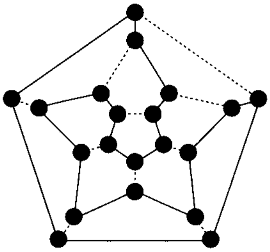


Figure 2.27

In Theorem 2.9 and Corollary 2.10 we obtained necessary and sufficient conditions for a connected graph to be Eulerian, and we may hope to obtain similar characterizations for Hamiltonian graphs – but no such characterization is known. In fact, little is known in general about Hamiltonian graphs, and most results have the form 'if G has enough edges, then G is Hamiltonian'. Probably the most celebrated of these is **Dirac's theorem**, due to G. A. Dirac (Corollary 2.17). We deduce it from the following more general result of O. Ore.

THEOREM 2.16 (Ore, 1960) *If G is a simple graph with n (≥ 3) vertices, and if*

$$\deg(v) + \deg(w) \geq n$$

for each pair of non-adjacent vertices v and w, then G is Hamiltonian.

Proof. We assume that the result is false, and derive a contradiction.

So let G be a *non-Hamiltonian* graph with n vertices, satisfying the given condition on the vertex degrees. By adding extra edges if necessary, we may assume that G is 'only just' non-Hamiltonian, in the sense that the addition of any further edge gives a Hamiltonian graph; note that adding an extra edge does not violate the condition on the vertex degrees. It follows that G contains a path

$$v_1 \to v_2 \to \cdots \to v_n$$

passing through every vertex. But since G is non-Hamiltonian, the vertices v_1 and v_n are not adjacent, and so $\deg(v_1) + \deg(v_n) \geq n$. It follows that there must be some vertex v_i adjacent to v_1 with the property that v_{i-1} is adjacent to v_n (see Fig. 2.28).

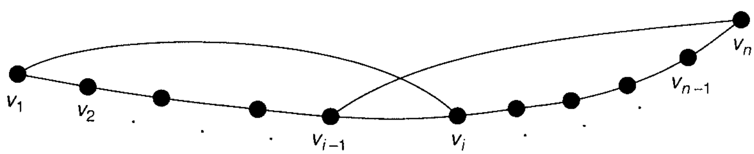


Figure 2.28

But this gives us the required contradiction since

$$v_1 \to v_2 \to \cdots \to v_{i-1} \to v_n \to v_{n-1} \to \cdots \to v_{i+1} \to v_i \to v_1$$

is then a Hamiltonian cycle. ■

COROLLARY 2.17 (Dirac, 1952) *If G is a simple graph with n (≥ 3) vertices, and if* $\deg(v) \geq \frac{1}{2}n$ *for each vertex v, then G is Hamiltonian.*

Proof. The result follows immediately from Theorem 2.16, since $\deg(v) + \deg(w) \geq n$ for each pair of vertices v and w, whether adjacent or not. ■

Hamiltonian digraphs

As we might expect, the study of Hamiltonian digraphs is less successful than that of Eulerian digraphs. A digraph D is **Hamiltonian** if there is a directed cycle that includes every vertex of D. A non-Hamiltonian digraph that contains a directed path through every vertex is **semi-Hamiltonian**. Little is known about Hamiltonian digraphs, and several theorems on Hamiltonian graphs do not generalize easily, if at all, to digraphs.

It is natural to ask whether there is a generalization to digraphs of Dirac's theorem (Corollary 2.17). One such generalization is due to M. A. Ghouila-Houri; its proof is more difficult than that of Dirac's theorem, and can be found in West [16].

THEOREM 2.18 *Let D be a strongly connected digraph with n vertices. If* $\mathrm{outdeg}(v) \geq \frac{1}{2}n$ *and* $\mathrm{indeg}(v) \geq \frac{1}{2}n$ *for each vertex v, then D is Hamiltonian.*

It seems that such results will not come easily, and so we consider instead which types of digraph are Hamiltonian. In this respect, the tournaments are particularly important, the results in this case taking a very simple form.

Because tournaments may have vertices with out-degree or in-degree 0, they are not in general Hamiltonian. However, the following theorem, due to L. Rédei and P. Camion, shows that every tournament is 'nearly Hamiltonian'.

THEOREM 2.19
(i) *Every non-Hamiltonian tournament is semi-Hamiltonian.*
(ii) *Every strongly connected tournament is Hamiltonian.*

Proof. (i) The statement is clearly true if the tournament has fewer than four vertices. We prove the result by induction on the number of vertices, and assume that every non-Hamiltonian tournament on n vertices is semi-Hamiltonian.

Let T be a non-Hamiltonian tournament on $n + 1$ vertices, and let T' be the tournament on n vertices obtained by removing from T a vertex v and its incident arcs. By the induction hypothesis, T' has a semi-Hamiltonian path $v_1 \to v_2 \to \cdots \to v_n$. There are now three cases to consider:

- if vv_1 is an arc in T, then the required path is

$$v \to v_1 \to v_2 \to \cdots \to v_n.$$

- if vv_1 is not an arc in T (which means that v_1v is), and if there exists an i such that vv_i is an arc in T, then choosing i to be the first such, we obtain the required path (see Fig. 2.29)

$$v_1 \to v_2 \to \cdots \to v_{i-1} \to v \to v_i \to \cdots \to v_n.$$

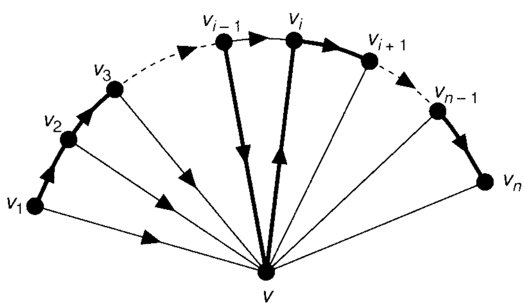


Figure 2.29

- if there is no arc in T of the form vv_i, then the required path is

$$v_1 \to v_2 \to \cdots \to v_n \to v.$$

(ii) We prove the stronger result that a strongly connected tournament T on n vertices contains cycles of length 3, 4, ..., n.

To show that T contains a cycle of length 3, let v be any vertex of T, and let W be the set of all vertices w such that vw is an arc in T, and Z be the set of all vertices z such that zv is an arc. Since T is strongly connected, W and Z must both be non-empty, and there must be an arc in T of the form $w'z'$, where w' is in W and z' is in Z (see Fig. 2.30). The required cycle of length 3 is then

$$v \to w' \to z' \to v.$$

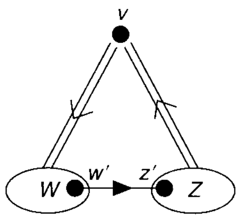


Figure 2.30

It remains only to show that, if there is a cycle of length k, where $k \leq n$, then there is one of length $k + 1$. Let

$$v_1 \to v_2 \to \cdots \to v_k \to v_1$$

be such a cycle. Suppose first that there exists a vertex v, not contained in this cycle, for which there exist arcs in T of the form vv_i and of the form v_jv. Then there must be a vertex v_i such that both $v_{i-1}v$ and vv_i are arcs in T. The required cycle is then (see Fig. 2.31)

$$v_1 \to v_2 \to \cdots \to v_{i-1} \to v \to v_i \to \cdots \to v_k \to v_1.$$

If no vertex exists with the above-mentioned property, then the set of vertices not contained in the cycle may be divided into two disjoint sets W and Z, where W is the set of vertices w such that v_iw is an arc for each i, and Z is the set of vertices z such

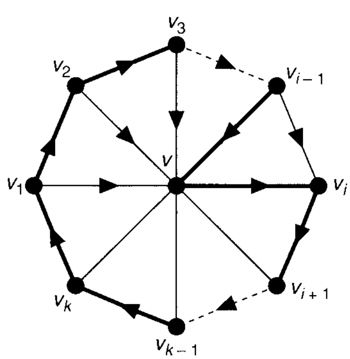


Figure 2.31

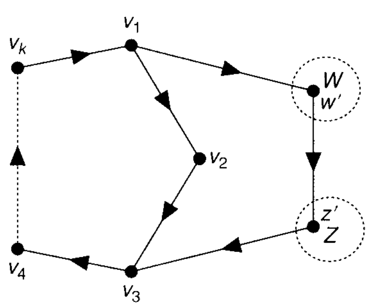


Figure 2.32

that zv_i is an arc for each i. Since T is strongly connected, W and Z must both be non-empty, and there must be an arc in T of the form $w'z'$, where w' is in W and z' is in Z. The required cycle is then (see Fig. 2.32)

$$v_1 \to w' \to z' \to v_3 \to \cdots \to v_k \to v_1. \qquad \blacksquare$$

Exercises

2.27ˢ Which of the following graphs are Hamiltonian or semi-Hamiltonian?
- (i) the complete graph K_5;
- (ii) the complete bipartite graph $K_{2,3}$;
- (iii) the graph of the octahedron;
- (iv) the wheel W_6;
- (v) the 4-cube Q_4.

2.28ˢ In the table of Fig. 1.9, locate all the Hamiltonian and semi-Hamiltonian graphs.

2.29
- (i) For which values of n is K_n Hamiltonian?
- (ii) Which complete bipartite graphs are Hamiltonian?
- (iii) Which Platonic graphs are Hamiltonian?
- (iv) For which values of n is the wheel W_n Hamiltonian?
- (v) For which values of k is the k-cube Q_k Hamiltonian?

2.30 Is the Grötzsch graph in Fig. 2.33 Hamiltonian?

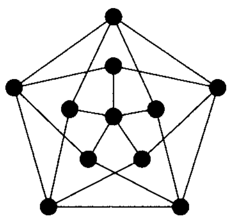


Figure 2.33

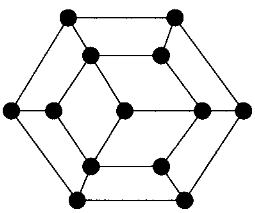


Figure 2.34

2.31
- (i) Prove that, if G is a bipartite graph with an odd number of vertices, then G is non-Hamiltonian.
- (ii) Deduce that the graph in Fig. 2.34 is non-Hamiltonian.

2.32 Use the result of Exercise 2.31(i) to show that a knight cannot visit all the squares of a 5×5 or 7×7 chessboard exactly once by knight's moves and return to its starting point. Can a knight visit all the squares of a 6×6 chessboard?

2.33ˢ Give an example to show that the condition '$\deg(v) \geq \frac{1}{2}n$', in the statement of Dirac's theorem, cannot be replaced by '$\deg(v) \geq \frac{1}{2}(n-1)$'.

2.34
- (i) Let G be a graph with n vertices and $\{\frac{1}{2}(n-1)(n-2)\} + 2$ edges. Use Ore's theorem to prove that G is Hamiltonian.
- (ii) Find a non-Hamiltonian graph with n vertices and $\{\frac{1}{2}(n-1)(n-2)\} + 1$ edges.

2.35ˢ Find a Hamiltonian cycle in the tournament of Fig. 2.23.

2.4 Applications

Many important advances in graph theory arose as a result of attempts to solve particular problems – Euler and the bridges of Königsberg (Section 2.2), Cayley and the enumeration of chemical molecules (Section 3.2), and Kirchhoff's work on electrical networks (Section 3.3), to name but three. Much present-day interest in the subject is due to the fact that, quite apart from being an elegant mathematical discipline in its own right, graph theory can be applied in a wide range of areas (see the Preface). In a book of this size we cannot discuss a large number of these applications, and you should consult Berge [9], Chartrand and Oellermann [20], Deo [21], Roberts [25] and Tucker [26] for a wide range of practical problems, often with algorithms for their solution.

In this section we outline four types of problem that involve paths and cycles: the *shortest path problem*, the *critical path problem*, the *Chinese postman problem* and the *travelling salesman problem*. The first two of these can be solved by efficient **algorithms** – finite step-by-step procedures that quickly yield the solutions. The third problem can also be solved by an efficient algorithm, but we consider only a special case here. For the fourth problem, no efficient algorithms are known; we must therefore choose between algorithms that take a long time to implement and heuristic algorithms that are quick to apply but give only an approximation to the solution.

The shortest path problem

Suppose that we have a diagram of the form shown in Fig. 2.35, in which the letters A–L refer to towns that are connected by roads. If the lengths of these roads are as marked, what is the length of a shortest path from A to L? (We write 'a shortest path' rather than 'the shortest path' since there may be more than one path with this shortest length.)

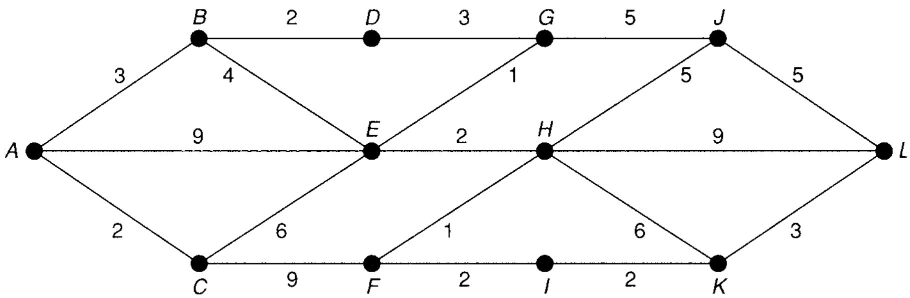


Figure 2.35

Note that the numbers in the diagram might also refer, not to the length of each road, but to the time taken to travel along it, or to the cost of doing so. Thus, if we have an algorithm for solving this problem in its original formulation, then this algorithm can also be used to find a quickest or a cheapest route.

Note also that we can easily obtain an upper bound for the answer by taking *any* path from A to L and calculating its length. For example, the path

$$A \to B \to D \to G \to J \to L$$

has total length 18, and so the length of a shortest path cannot exceed 18.

In solving such problems, we regard our diagram as a connected graph in which a non-negative number is assigned to each edge. Such a graph is called a **weighted graph**, and the number assigned to each edge e is the **weight** of e, denoted by $w(e)$. The problem is to find a path from A to L with minimum total weight. Note that, if we have a weighted graph in which each edge has weight 1, then the problem reduces to that of finding the number of edges in a shortest path from A to L.

There are several methods that we can use to solve this problem. One way is to make a model of the map by knotting together pieces of string whose lengths are proportional to the lengths of the roads. To find a shortest path, take hold of the knots corresponding to A and L – and pull tight!

However, there is a more mathematical way of solving it. The idea is to move across the map from left to right, labelling each vertex V with a number $l(V)$ that indicates the shortest distance from A to V. This means that, when we reach a vertex such as K in Fig. 2.35, then $l(K)$ is labelled either $l(H) + 6$ or $l(I) + 2$, whichever is the smaller. Our aim is to find $l(L)$.

To apply the algorithm, we first assign A the label 0 and give B, E and C the temporary labels $l(A) + 3$, $l(A) + 9$ and $l(A) + 2$ – that is, 3, 9 and 2. We take the *smallest* of these, and write $l(C) = 2$. *C is now permanently labelled 2.*

We next look at the vertices adjacent to this labelled vertex C. We assign F the temporary label $l(C) + 9 = 11$, and we can lower the temporary label at E to $l(C) + 6 = 8$. The smallest temporary label is now 3 (at B), so we write $l(B) = 3$. *B is now permanently labelled 3.*

Now we look at the vertices adjacent to B. We assign D the temporary label $l(B) + 2 = 5$, and we can lower the temporary label at E to $l(B) + 4 = 7$. The smallest temporary label is now 5 (at D), so we write $l(D) = 5$. *D is now permanently labelled 5.*

Now we look at the vertices adjacent to D. The only one is G, and we assign it the temporary label $l(D) + 3 = 8$. The smallest temporary label is now 7 (at E), so we write $l(E) = 7$. *E is now permanently labelled 7.*

Continuing in this way, we successively obtain the permanent labels

$$l(G) = 8,\ l(H) = 9,\ l(F) = 10,\ l(I) = 12,\ l(J) = 13,\ l(K) = 14,\ l(L) = 17;$$

these are shown on Fig. 2.36. It follows that the shortest path from A to L has length 17.

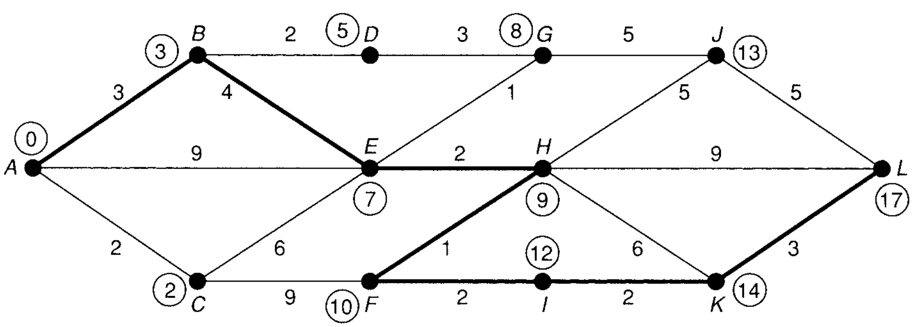


Figure 2.36

To find such a shortest path, we can restrict our attention to those edges whose length is the difference of the labels at its ends, such as KL and IK (since $l(L) - l(K) = 17 - 14 = 3$ and $l(K) - l(I) = 2$). Using the labels to help us, we then trace back from L, via K and I, obtaining

$$L \leftarrow K \leftarrow I \leftarrow F \leftarrow H \leftarrow E \leftarrow B \leftarrow A.$$

Thus the shortest path (which in this case is unique) from A to L is

$$A \rightarrow B \rightarrow E \rightarrow H \rightarrow F \rightarrow I \rightarrow K \rightarrow L.$$

The critical path problem

We now see how this algorithm can be adapted to yield the *longest* path in a digraph, and we illustrate its use in a 'critical path' problem relating to the scheduling of a series of operations.

Suppose that we have a job to perform, such as the building of a house, and that this job can be divided into a number of activities, such as laying the foundations, putting on the roof, doing the wiring, etc. Some of these activities can be performed simultaneously, whereas some may need to be completed before others can be started. Can we find an efficient method for determining how to schedule the activities so that the entire job is completed in minimum time?

In order to solve this problem, we construct a 'weighted digraph', or **activity network**, in which each arc represents the length of time taken for an activity. Such a network is given in Fig. 2.37. The vertex A represents the beginning of the job, and the vertex L represents its completion. Since the entire job cannot be completed until each path from A to L has been traversed, the problem reduces to that of finding the longest path from A to L.

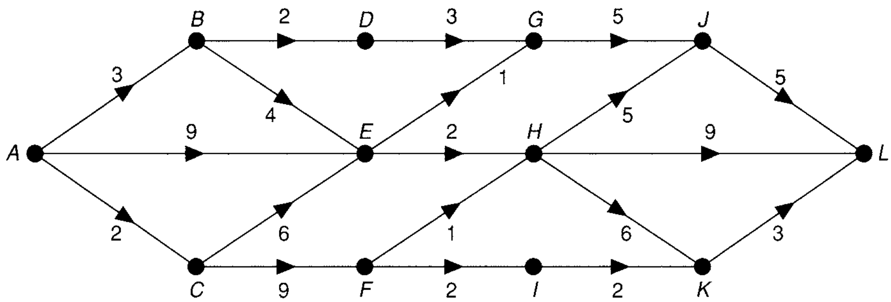


Figure 2.37

This is accomplished by using a technique known as programme evaluation and review technique (PERT), which is similar to the method we used to solve the shortest path problem above. This time, as we move across the digraph from left to right, we associate with each vertex V a label $l(V)$ indicating the length of the *longest* path from A to V. As in the shortest path problem, we keep track of these labels by writing them next to the vertices they represent. However, unlike the problem that we considered above, there is no 'zigzagging', since all arcs are directed from left to right, so we can assign the permanent labels as we proceed. For example, for the digraph of Fig. 2.37, we assign:

to vertex A, the number 0;
to vertex B, the number $l(A) + 3$ – that is, 3;
to vertex C, the number $l(A) + 2$ – that is, 2;
to vertex D, the number $l(B) + 2$ – that is, 5;
to vertex E, the largest of the numbers $l(A) + 9$, $l(B) + 4$ and $l(C) + 6$ – that is, 9;
to vertex F, the number $l(C) + 9$ – that is, 11;
to vertex G, the larger of the numbers $l(D) + 3$ and $l(E) + 1$ – that is, 10;
to vertex H, the larger of the numbers $l(E) + 2$ and $l(F) + 1$ – that is, 12;
to vertex I, the number $l(F) + 2$ – that is, 13;
to vertex J, the larger of the numbers $l(G) + 5$ and $l(H) + 5$ – that is, 17;
to vertex K, the larger of the numbers $l(H) + 6$ and $l(I) + 2$ – that is, 18;
to vertex L, the largest of the numbers $l(H) + 9$, $l(J) + 5$ and $l(K) + 33$ – that is, 22.

These labels are shown in Fig. 2.38. It follows that a longest path has length 22, and the job cannot therefore be completed until time 22.

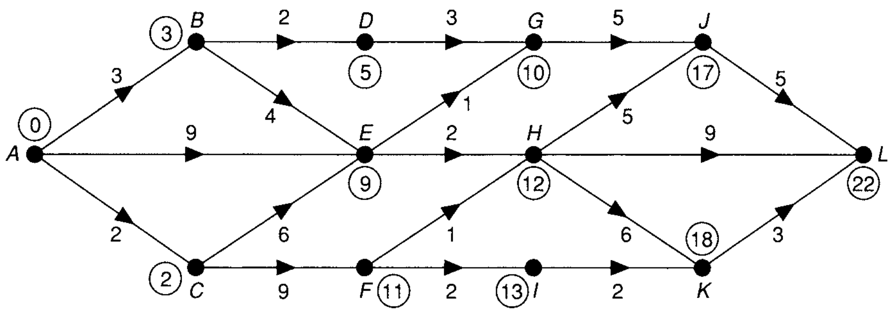


Figure 2.38

To find the longest path, we trace back from L, as we did for the shortest path problem. In this case, the unique longest path from A to L is

$$A \to C \to F \to H \to J \to L.$$

It is often called a **critical path**, since any delay in an activity on this path creates a delay in the whole job. In scheduling a job, we therefore need to pay particular attention to the critical paths.

We can also calculate the latest time by which any given operation must be completed if the work is not to be delayed. Working back from L, we see that we must reach

K by time $22 - 3 = 19$,
J by time $22 - 5 = 17$,
H by time $\min\{17 - 5, 22 - 9 \text{ and } 19 - 6\} = 12$,

and so on.

The Chinese postman problem

In this problem, discussed by the Chinese mathematician Meigu Guan, a postman wishes to deliver his letters, covering the least possible total distance and returning to his starting point. He must obviously traverse each road in his route at least once, but should avoid covering too many roads more than once.

This problem can be reformulated in terms of a weighted graph, where the graph corresponds to the network of roads, and the weight of each edge is the length of the corresponding road. In this reformulation, the requirement is to find a closed walk of minimum total weight that includes each edge at least once. If the graph is Eulerian, then any Eulerian trail is a closed walk of the required type; such an Eulerian trail can be found by Fleury's algorithm (see Section 2.2). If the graph is not Eulerian, then the problem is much harder, although an efficient algorithm for its solution is known (see Gibbons [23]).

To illustrate the ideas involved, we look at a special case, in which exactly two vertices have odd degree (see Fig. 2.39).

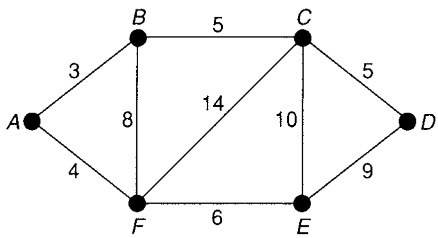


Figure 2.39

Since vertices B and E are the only vertices of odd degree, we can find a semi-Eulerian trail from B to E covering each edge exactly once. In order to return to the starting point, covering the least possible distance, we now find the shortest path from E to B using the algorithm described above. The solution of the Chinese postman problem is then obtained by combining this shortest path

$$E \to F \to A \to B$$

with the original semi-Eulerian trail, giving a total distance of $13 + 64 = 77$. Note that, when we combine the shortest path and the semi-Eulerian trail, we get an Eulerian graph (see Fig. 2.40).

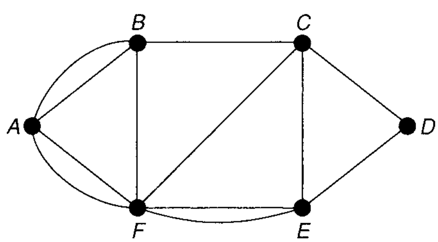


Figure 2.40

The travelling salesman problem

In this problem, a travelling salesman wishes to visit several given cities and return to his starting point, covering the least possible total distance. Such a route must be

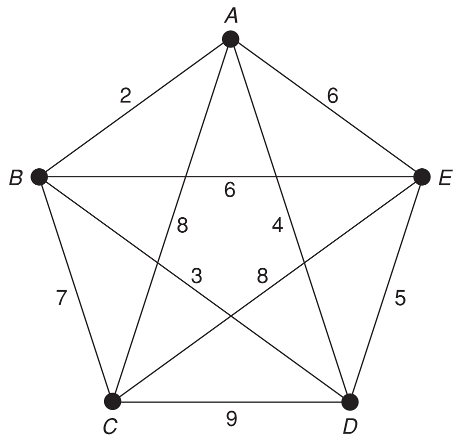


Figure 2.41

a cycle. For example, if there are five cities A, B, C, D and E, and if the distances are as given in Fig. 2.41, then two possible routes are the cycles

$$A \to B \to C \to D \to E \to A \quad \text{and} \quad A \to D \to B \to C \to E \to A,$$

with total distances of

$$2 + 7 + 9 + 5 + 6 = 29 \quad \text{and} \quad 4 + 3 + 7 + 8 + 6 = 28,$$

but neither is the shortest possible route. By inspection we find that the shortest route is

$$A \to B \to D \to E \to C \to A,$$

with a total distance of 26.

This problem can also be reformulated in terms of weighted graphs. In this case, the requirement is to find a Hamiltonian cycle of least possible total weight in a given weighted complete graph. Note that, as in the shortest path problem, the numbers can also refer to the times taken to travel between the cities, or the costs involved in doing so. Thus, if we could find an efficient algorithm for solving the travelling salesman problem in its original formulation, then we could apply the same algorithm to find the quickest or the cheapest route.

One possible algorithm is to calculate the total distance for *all* possible Hamiltonian cycles, but this is far too time consuming for more than about five cities. For example, if there are 20 cities, then the number of possible cycles is $\frac{1}{2}(19!)$, which is about 6×10^{16}. Various other algorithms have been proposed, but they take too long to apply – no efficient algorithm is known, and it is generally believed that none exists. On the other hand, there are several heuristic methods that quickly tell us *approximately* what the shortest distance is. One of these is described in Section 3.3.

Exercises

2.36^{s} Use the shortest path algorithm to find the shortest path from A to G in the weighted graph of Fig. 2.42.

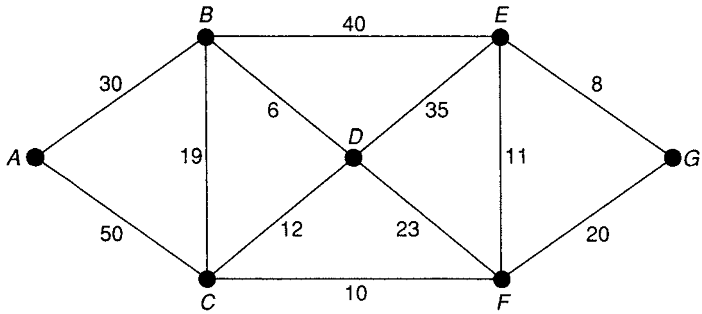


Figure 2.42

2.37 Use the shortest path algorithm to find the shortest path from L to A in Fig. 2.35, and check that your answer agrees with that given in Fig. 2.36.

2.38 Find the shortest path from S to each other vertex in the weighted graph of Fig. 2.43.

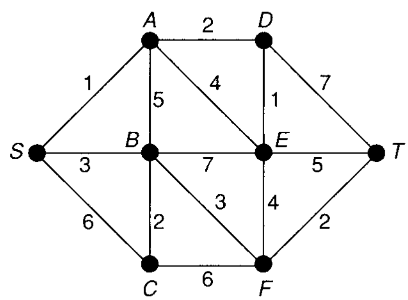


Figure 2.43

2.39[s] In the critical path problem in the text, calculate the latest times at which we can reach the vertices G, E and B.

2.40 Find the longest path from A to G in the network of Fig. 2.44.

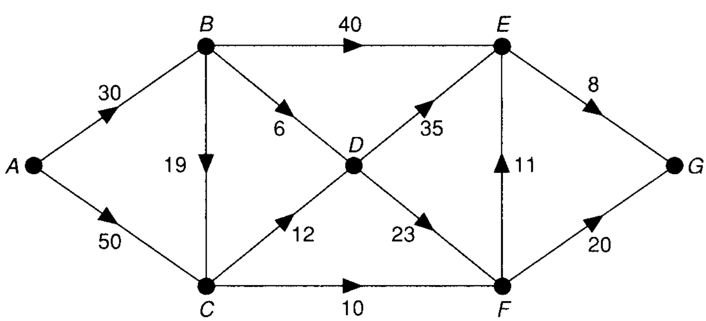


Figure 2.44

2.41[s] Solve the Chinese postman problem for the weighted graph of Fig. 2.45.

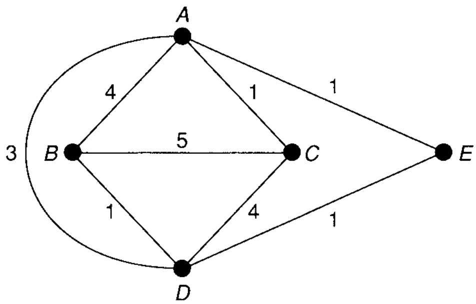


Figure 2.45

2.42[s] Solve the travelling salesman problem for the weighted graph of Fig. 2.46.

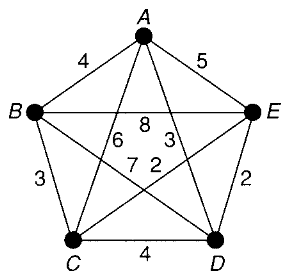


Figure 2.46

2.43 Find the Hamiltonian cycle of *greatest* weight in the graph of Fig. 2.41.

Challenge problems

2.44 Let G be a simple graph on $2k$ vertices containing no triangles. Prove, by induction on k, that G has at most k^2 edges, and give an example of a graph for which this upper bound is achieved. (This result is often called *Turán's extremal theorem*.)

2.45 Let G be a connected graph with vertex-set $\{v_1, v_2, \ldots, v_n\}$, m edges and t triangles.
(i) If $\mathbf{A}$ is the adjacency matrix of G, prove that the number of walks of length 2 from v_i to v_j is the ijth entry of the matrix $\mathbf{A}^2$. Deduce that $2m$ = the sum of the diagonal entries of $\mathbf{A}^2$.
(ii) Obtain a corresponding result for the number of walks of length 3 from v_i to v_j, and deduce that $6t$ = the sum of the diagonal entries of $\mathbf{A}^3$.

2.46 (i) Prove that, if two distinct cycles of a graph G each contain an edge e, then G has a cycle that does not contain e.
(ii) Prove a similar result with 'cycles' replaced by 'cutsets'.

2.47 (i) Prove that, if C is a cycle and C^* is a cutset of a connected graph G, then C and C^* have an even number of edges in common.
(ii) Prove that, if S is any set of edges of G with an even number of edges in common with each cutset of G, then S can be split into edge-disjoint cycles.

2.48 A set E of edges of a graph G is **independent** if E contains no cycle of G. Prove that:
(i) any subset of an independent set is independent;
(ii) if I and J are independent sets of edges with $|J| > |I|$, then there is an edge e that lies in J but not in I with the property that $I \cup \{e\}$ is independent.
Prove also that (i) and (ii) still hold if we replace the word 'cycle' by 'cutset'.

2.49 Let V be the vector space of a graph G (see Exercise 1.52).
(i) Use Corollary 2.10 to show that, if C and D are cycles of G, then their sum $C + D$ can be written as a union of edge-disjoint cycles.
(ii) Deduce that the set of such unions of cycles of C forms a subspace W of V (the **cycle subspace** of G), and find its dimension.
(iii) Show that the set of unions of edge-disjoint cutsets of G forms a subspace W^* of V (the **cutset subspace** of G), and find its dimension.

2.50 The line graph of a graph G was defined in Exercise 1.53.
(i) Show that the line graph of a simple Eulerian graph is Eulerian.
(ii) If the line graph of a simple graph G is Eulerian, must G be Eulerian?

2.51 Let T be a tournament. The **score** of a vertex of T is its out-degree, and the **score sequence** of T is the sequence formed by arranging the scores of its vertices in non-decreasing order. Prove that, if $(s_1, s_2, \ldots, s_n)$ is the score-sequence of a tournament T, then
(i) $s_1 + s_2 + \cdots + s_n = \frac{1}{2}n(n-1)$;
(ii) for each positive integer $k < n$, $s_1 + s_2 + \cdots + s_k \geq \frac{1}{2}k(k-1)$, with strict inequality for all k if and only if T is strongly connected;
(iii) T is transitive (see Exercise 2.12) if and only if $s_k = k - 1$ for each k.

2.52 Prove that the Petersen graph is non-Hamiltonian.

2.53 Let G be a Hamiltonian graph and let S be any set of k vertices in G. Prove that the graph $G - S$ has at most k components.

2.54 (i) Find four Hamiltonian cycles in K_9, no two of which have an edge in common.
(ii) What is the maximum number of edge-disjoint Hamiltonian cycles in K_{2k+1}?

2.55 Show that the infinite square lattice has both one-way and two-way infinite paths passing exactly once through each vertex.

Chapter 3

Trees

A fool sees not the same tree that a wise man sees.
William Blake

We are all familiar with the idea of a family tree. In this chapter, we study trees in general, with special reference to spanning trees in a connected graph and to Cayley's celebrated result on the enumeration of labelled trees. The chapter concludes with some further applications.

3.1 Properties of trees

A connected graph that has no cycles is a **tree**; Fig. 3.1 shows four trees.† Note that trees are simple graphs.

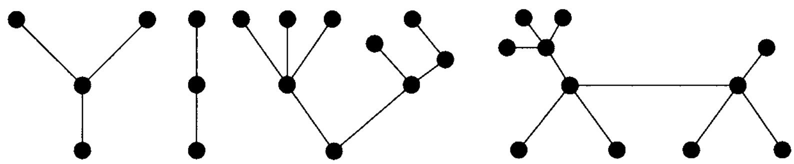


Figure 3.1

In many ways a tree is the simplest non-trivial type of graph. As we shall see, it has several 'nice' properties, such as the fact that any two vertices are connected by exactly one path. When trying to prove a general result for graphs, we sometimes find it useful to start by trying to prove it for trees; in fact, there are several conjectures that have not been proved for arbitrary graphs but are known to be true for trees.

† The last tree in Fig. 3.1 is particularly well known for its bark.

The following theorem lists some simple properties of trees.

THEOREM 3.1 *Let T be a graph with n vertices. Then the following statements are equivalent:*

(i) *T is a tree;*
(ii) *T contains no cycles, and has $n - 1$ edges;*
(iii) *T is connected, and has $n - 1$ edges;*
(iv) *T is connected, and each edge is a bridge;*
(v) *any two vertices of T are connected by exactly one path;*
(vi) *T contains no cycles, but the addition of any new edge creates exactly one cycle.*

Proof. If $n = 1$, all six results are trivial; we therefore assume that $n \geq 2$.

(i) $\Rightarrow$ (ii). Since T contains no cycles, the removal of any edge must disconnect T into two graphs, each of which is a tree. It follows by induction that the number of edges in each of these two trees is one fewer than the number of vertices. Replacing the removed edge, we deduce that the total number of edges of T is $n - 1$.

(ii) $\Rightarrow$ (iii). If T is disconnected, then each component of T is a connected graph with no cycles and hence, by the previous part, the number of vertices in each component exceeds the number of edges by 1. It follows that the total number of vertices of T exceeds the total number of edges by at least 2, contradicting the fact that T has $n - 1$ edges.

(iii) $\Rightarrow$ (iv). The removal of any edge results in a graph with n vertices and $n - 2$ edges, which must be disconnected by Theorem 2.2.

(iv) $\Rightarrow$ (v). Since T is connected, each pair of vertices is connected by at least one path. If a given pair of vertices is connected by two paths, then these paths contain a cycle, contradicting the fact that each edge is a bridge.

(v) $\Rightarrow$ (vi). If T contained a cycle, then any two vertices in the cycle would be connected by at least two paths, contradicting statement (v). If an edge e is added to T, then, since the vertices incident with e are already connected in T, a cycle is created. The fact that only one cycle is obtained follows from Exercise 2.46(i).

(vi) $\Rightarrow$ (i). Suppose that T is disconnected. If we add to T any edge joining a vertex of one component to a vertex in another, then no cycle is created. ■

Note that, by the handshaking lemma, the sum of the degrees of the n vertices of a tree is equal to twice the number of edges ($= 2n - 2$). It follows that:

If $n \geq 2$, any tree on n vertices has at least two end-vertices.

Given any connected graph G, we can choose a cycle (if there is one) and remove any one of its edges, and the resulting graph remains connected. We repeat this procedure with one of the remaining cycles, continuing until there are no cycles left. The graph that remains is a tree that connects all the vertices of G. It is called a **spanning tree** of G. An example of a graph and one of its spanning trees appears in Fig. 3.2.

If we carry out the above procedure, the total number of edges removed is the **cycle rank** of G, denoted by $\gamma(G)$. Note that $\gamma(G) = m - n + 1$, which is a non-negative integer by Theorem 2.2. It is convenient also to define the **cutset rank** of G to be

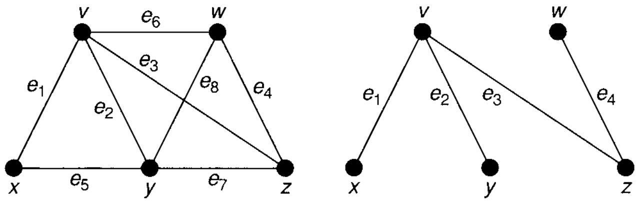


Figure 3.2

the number of edges in a spanning tree, denoted by $\xi(G)$; thus, $\xi(G) = n - 1$. Some properties of the cutset rank are given in Exercise 3.33.

Before proceeding, we prove a couple of simple results concerning spanning trees. In this theorem, the **complement** of a spanning tree T of a (not necessarily simple) connected graph G is the graph obtained from G by removing the edges of T.

THEOREM 3.2 *If T is any spanning tree of a connected graph G, then*

(i) *each cutset of G has an edge in common with T;*
(ii) *each cycle of G has an edge in common with the complement of T.*

Proof.

(i) Let C^* be a cutset of G, the removal of which splits a component of G into two subgraphs H and K. Since T is a spanning tree, T must contain an edge joining a vertex of H to a vertex of K, and this edge is the required edge.
(ii) Let C be a cycle of G having no edge in common with the complement of T. Then C must be contained in T, which is a contradiction. ■

Closely linked with the idea of a spanning tree T of a connected graph G is that of the fundamental set of cycles associated with T. This is formed as follows: if we add to T any edge of G not contained in T, then by Theorem 3.1(vi) we obtain a unique cycle. The set of all cycles formed in this way, by adding separately each edge of G not contained in T, is the **fundamental set of cycles associated with T**. Sometimes we are not interested in the particular spanning tree chosen, and refer simply to a **fundamental set of cycles of G**. Note that the number of cycles in any fundamental set must equal the cycle rank of G. Figure 3.3 shows the fundamental set of cycles of the graph shown in Fig. 3.2 associated with the given spanning tree.

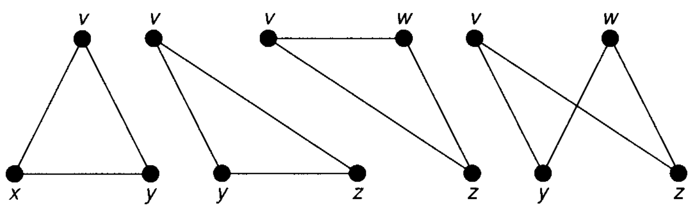


Figure 3.3

In the light of our remarks at the end of Section 2.1, we may hope to be able to define a fundamental set of cutsets of a connected graph G associated with a spanning tree T. This is indeed the case. By Theorem 3.1(iv), the removal of any edge of

T divides the vertex-set of T into two disjoint sets, V_1 and V_2. The set of all edges of G joining a vertex of V_1 to one of V_2 is a cutset of G, and the set of all cutsets obtained in this way, by removing separately each edge of T, is the **fundamental set of cutsets associated with** $\boldsymbol{T}$. Note that the number of cutsets in any fundamental set must equal the cutset rank of G. The fundamental set of cutsets of the graph in Fig. 3.2 associated with the given spanning tree is $\{e_1, e_5\}$, $\{e_2, e_5, e_7, e_8\}$, $\{e_3, e_6, e_7, e_8\}$ and $\{e_4, e_6, e_8\}$.

Exercises

3.1[s] In the table of Fig. 1.9, locate all the trees.

3.2[s] There are six trees on six vertices: draw them.

3.3 There are 11 trees on seven vertices: draw them.

3.4 (i) Prove that every tree is a bipartite graph.
(ii) Which trees are complete bipartite graphs?

3.5[s] Draw all the spanning trees in the graph of Fig. 3.4.

3.6 Draw all the spanning trees in the graph of Fig. 3.5.

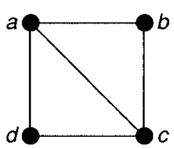


Figure 3.4

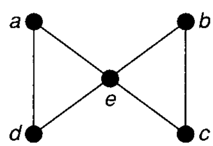


Figure 3.5

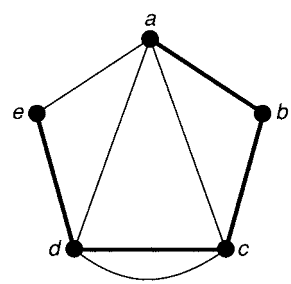


Figure 3.6

3.7[s] Find the fundamental sets of cycles and cutsets of the graph in Fig. 3.6 associated with the spanning tree shown.

3.8 Find the cycle and cutset ranks of
(i) K_5; (ii) $K_{3,3}$; (iii) W_5; (iv) N_5; (v) the Petersen graph.

3.9[s] Let G be a connected graph. What can you say about
(i) an edge of G that appears in every spanning tree?
(ii) an edge of G that appears in no spanning tree?

3.10 If G is a connected graph, a **centre** of G is a vertex v with the property that the maximum of the distances between v and the other vertices of G is as small as possible. By successively removing all the end-vertices, prove that every tree has either one centre or two adjacent centres. Give an example of a tree of each type with seven vertices.

3.11 Let T_1 and T_2 be spanning trees of a connected graph G.
(i) If e is any edge of T_1, show that there exists an edge f of T_2 such that the graph $(T_1 - \{e\}) \cup \{f\}$ (obtained from T_1 on replacing e by f) is also a spanning tree.
(ii) Deduce that T_1 can be 'transformed' into T_2 by replacing the edges of T_1 one at a time by edges of T_2 in such a way that a spanning tree is obtained at each stage.
(This result will be needed in Chapter 7.)

3.2 Counting trees

The subject of graph enumeration is concerned with the problem of determining how many non-isomorphic graphs satisfy a given property. The subject was initiated in the 1850s by Arthur Cayley, who later applied it to the problem of enumerating alkanes C_nH_{2n+2} with a given number of carbon atoms. As he realized, and as you will see below, this problem is that of counting the number of trees in which the degree of each vertex is either 4 or 1.

Many standard problems of graph enumeration have been solved. For example, it is possible to calculate the number of graphs, connected graphs, trees and Eulerian graphs with a given number of vertices and edges. Most of the known results can be obtained by using a fundamental enumeration theorem due to Polya (see Polya and Read [43]). Unfortunately, in almost every case it is impossible to express these results by means of simple formulas. For tables of some known results, see Appendix 2.

Counting chemical molecules

One of the earliest uses of trees was in the enumeration of chemical molecules. If we have a molecule consisting only of carbon atoms and hydrogen atoms, then we can represent it as a graph in which each carbon atom appears as a vertex of degree 4, and each hydrogen atom appears as a vertex of degree 1, as we saw in Exercise 0.3.

The graphs of *n*-butane and 2-methyl propane are shown in Fig. 3.7. Although they have the same chemical formula C_4H_{10}, they are different molecules because the carbon atoms are arranged differently within the molecule. These two molecules form part of a general class of molecules, known as *alkanes* or *paraffins*, with chemical formula C_nH_{2n+2}.

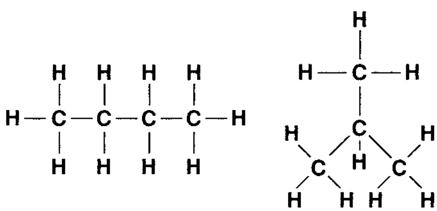


Figure 3.7

How many different molecules are there with this formula? To answer this, we note first that the graph of any molecule with formula C_nH_{2n+2} is a tree; this follows from Theorem 3.1(iii), since it is connected and has

$$n + (2n + 2) = 3n + 2 \text{ vertices} \quad \text{and} \quad {}^1\!/\!_2\{4n + (2n + 2)\} = 3n + 1 \text{ edges.}$$

We note also that the alkane is determined completely once the carbon atoms are arranged, since the hydrogen atoms can then be added in such a way as to bring the degree of each carbon vertex up to 4. We can thus discard the hydrogen atoms, as in Fig. 3.8, and the problem reduces to that of finding the number of trees with n vertices, each of degree 4 or less.

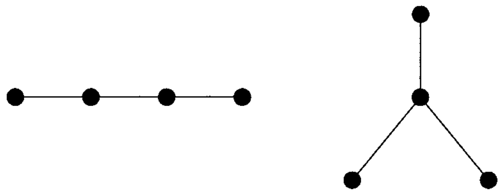


Figure 3.8

This problem was solved by Cayley in 1875, by counting the number of ways in which trees can be built up from their centre(s) (see Exercise 3.10); the details of this argument are too complicated to describe here, but may be found in Biggs, Lloyd and Wilson [17]. Much of Cayley's work was superseded by Polya and others, with the result that several chemical series have now been enumerated by graph-theoretic techniques.

Counting labelled trees

This section is devoted primarily to two proofs of a famous result, usually attributed to Cayley, on the number of *labelled* trees with a given number of vertices.

To see what is involved, consider Fig. 3.9, which shows three ways of labelling a path with four vertices. Since the second labelled tree is the reverse of the first one, we regard them as the same. On the other hand, neither is isomorphic to the third labelled tree, as you can see by comparing the degrees of the vertex labelled 3. Thus, since the reverse of any labelled path does not result in a new one, the number of ways of labelling the path is $(4!)/2 = 12$.

Figure 3.9

Similarly, the number of ways of labelling the 'star' in Fig. 3.10 is 4, since the central vertex can be labelled in four different ways, and each one determines the labelling.

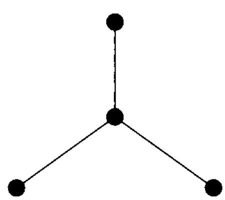


Figure 3.10

Since every tree with four vertices is either a path or a star, the total number of non-isomorphic labelled trees on four vertices is 12 + 4 = 16.

We now prove **Cayley's theorem**, which generalizes this result to labelled trees with n vertices. The following proofs are due to H. Prüfer and L. E. Clarke; for other proofs, see Moon [40].

THEOREM 3.3 (Cayley, 1889) *There are n^{n-2} distinct labelled trees with n vertices.*

First proof. We establish a one–one correspondence between the set of labelled trees with n vertices and the set of sequences $(a_1, a_2, \ldots, a_{n-2})$, where each a_i is an integer satisfying $1 \le a_i \le n$. Since there are precisely n^{n-2} such sequences, the result then follows immediately. We assume that $n \ge 3$, since the result is trivial if $n = 1$ or 2.

In order to establish the required correspondence, we first let T be a labelled tree of order n, and show how the sequence can be determined. If b_1 is the smallest label assigned to an end-vertex, we let a_1 be the label of the vertex adjacent to the vertex b_1; for example, if T is the labelled tree in Fig. 3.11, then $b_1 = 2$ and $a_1 = 6$. We then remove the vertex b_1 and its incident edge, leaving a labelled tree of order $n - 1$. We next let b_2 be the smallest label assigned to an end-vertex of our new tree, and let a_2 be the label of the vertex adjacent to the vertex b_2; in this example, $b_2 = 3$ and $a_2 = 5$. We then remove the vertex b_2 and its incident edge, as before. We proceed in this way until there are only two vertices left; the required sequence is then $(a_1, a_2, \ldots, a_{n-2})$; in this example, $b_3 = 4$ and $a_3 = 6$, $b_4 = 6$ and $a_4 = 5$, $b_5 = 5$ and $a_5 = 1$, and the required sequence is (6, 5, 6, 5, 1).

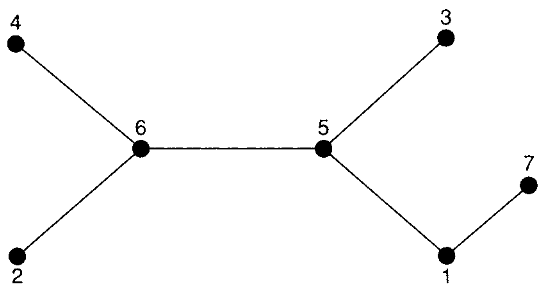


Figure 3.11

To obtain the reverse correspondence, we take a sequence $(a_1, a_2, \ldots, a_{n-2})$, let b_1 be the smallest number from 1 to n that does *not* appear in it, and join the vertices a_1 and b_1; for example, if the sequence is (6, 5, 6, 5, 1), then $a_1 = 6$, $b_1 = 2$ and the first edge is 62. We then remove a_1 from the sequence and remove the number b_1 from consideration, and proceed as before, building up the tree, edge by edge; in this example, $a_2 = 5$, $b_2 = 3$ and the second edge is 53, $a_3 = 6$, $b_3 = 4$ and the third edge is 64, $a_4 = 5$, $b_4 = 6$ and the fourth edge is 56, and $a_5 = 1$, $b_5 = 5$ and the fifth edge is 15. We conclude by joining the last two vertices not yet crossed out – in this case, 1 and 7.

It can be checked that if we start with any labelled tree, find the corresponding sequence, and then find the labelled tree corresponding to that sequence, we obtain the tree we started from. We have therefore established the required correspondence, and the result follows. ■

Second proof. Let $T(n, k)$ be the number of labelled trees on n vertices in which a given vertex v has degree k. We shall derive an expression for $T(n, k)$, and the result follows on summing from $k = 1$ to $k = n - 1$.

Let A be any labelled tree in which $\deg(v) = k - 1$. The removal from A of any edge wz that is not incident with v leaves two subtrees, one containing v and either w or z (w, say), and the other containing z. If we now join the vertices v and z, we obtain a labelled tree B in which $\deg(v) = k$ (see Fig. 3.12). We call a pair (A, B) of labelled trees a **linkage** if B can be obtained from A by the above construction. Our aim is to count the possible linkages (A, B).

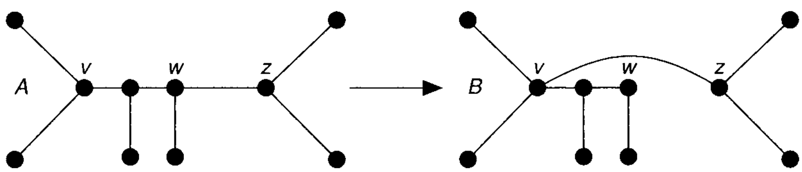


Figure 3.12

Since A may be chosen in $T(n, k - 1)$ ways, and since B is uniquely defined by the edge wz, which may be chosen in $(n - 1) - (k - 1) = n - k$ ways, the total number of linkages (A, B) is $(n - k)\,T(n, k - 1)$.

On the other hand, let B be a labelled tree in which $\deg(v) = k$, and let $T_1, T_2, \ldots, T_k$ be the subtrees obtained from B by removing the vertex v and each edge incident with v. Then we obtain a labelled tree A with $\deg(v) = k - 1$ by removing from B just one of these edges (vw_i, say, where w_i lies in T_i), and joining w_i to any vertex u of any other subtree T (see Fig. 3.13). Note that the corresponding pair (A, B) of labelled trees is a linkage, and that all linkages may be obtained in this way.

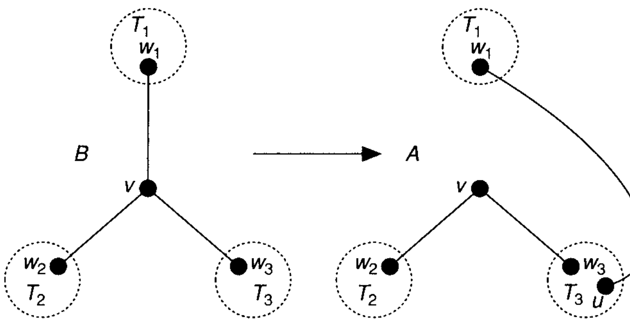


Figure 3.13

Since B can be chosen in $T(n, k)$ ways, and the number of ways of joining w_i to vertices in any other T_j is $(n - 1) - n_i$, where n_i is the number of vertices of T_i, the total number of linkages (A, B) is

$$T(n, k)\,\{(n - 1 - n_1) + (n - 1 - n_2) + \cdots + (n - 1 - n_k)\},$$

which equals $(n - 1)(k - 1)\,T(n, k)$, since $n_1 + n_2 + \cdots + n_k = n - 1$.

We have thus shown that

$$(n-k)\,T(n,k-1) = (n-1)(k-1)\,T(n,k).$$

Iterating this result, and using the obvious fact that $T(n, n-1) = 1$, we find that

$$T(n,k) = \binom{n-2}{k-1}(n-1)^{n-k-1},$$

and on summing this over all possible values of k, we deduce that the number $T(n)$ of labelled trees on n vertices is given by

$$T(n) = \sum_{k=1}^{n-1} T(n,k) = \sum_{k=1}^{n-1}\binom{n-2}{k-1}(n-1)^{n-k-1} = \{(n-1)+1\}^{n-2} = n^{n-2}. \qquad \blacksquare$$

The above result can also be interpreted in terms of spanning trees of the complete graph K_n.

COROLLARY 3.4 *The number of spanning trees of K_n is n^{n-2}.*

Proof. To each labelled tree on n vertices there corresponds a unique spanning tree of K_n. Conversely, each spanning tree of K_n gives rise to a unique labelled tree on n vertices. ■

We conclude this section by stating a result that can be used to calculate the number of spanning trees in any connected simple graph. It is called the **matrix-tree theorem** and a proof may be found in Harary [14].

THEOREM 3.5 *Let G be a connected simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let $\mathbf{M} = (m_{ij})$ be the $n \times n$ matrix in which $m_{ii} = \deg(v_i)$, $m_{ij} = -1$ if v_i and v_j are adjacent, and $m_{ij} = 0$ otherwise. Then the number of spanning trees of G is equal to the cofactor of any element of $\mathbf{M}$.*

Exercises

3.12^s Verify directly that there are exactly 125 labelled trees on five vertices.

3.13 Draw all the trees corresponding to alkanes with formulae C_5H_{12} and C_6H_{14}.

3.14^s Show that, for each value of n, the graph associated with the alcohol $C_nH_{2n+1}OH$ is a tree (the oxygen vertex has degree 2). Draw the tree corresponding to the molecule C_2H_5OH.

3.15^s In the first proof of Cayley's theorem, find:
(i) the labelled trees that correspond to the sequences (1, 2, 3, 4) and (3, 3, 3, 3);
(ii) the sequences that correspond to the labelled trees in Fig. 3.14.

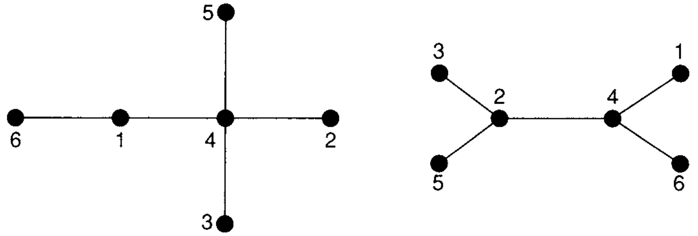


Figure 3.14

3.16 In the first proof of Cayley's theorem, find the labelled tree that corresponds to the sequence (7, 6, 5, 4, 3, 2, 1).

3.17 (i) Find the number of trees on n vertices in which a given vertex is an end-vertex.
(ii) Deduce that, if n is large, then the probability that a given vertex of a tree with n vertices is an end-vertex is approximately e^{-1}.

3.18[s] How many spanning trees has $K_{2,s}$?

3.19 Let $\tau(G)$ be the number of spanning trees in a connected graph G.
(i) Prove that, for any edge e, $\tau(G) = \tau(G - e) + \tau(G \setminus e)$.
(ii) Use this result to calculate $\tau(K_{2,3})$.

3.3 More applications

In Section 2.4 we considered four problems that arise in operational research – the shortest path problem, the critical path problem, the Chinese postman problem and the travelling salesman problem. In this section we consider four applications that involve the use of trees, taken from operational research, computer science, structural engineering and electrical network theory.

The minimum connector problem

Suppose that we wish to build a railway network connecting n given cities in such a way that a passenger can travel from any city to any other. If, for economic reasons, the total amount of track must be a minimum, then the graph formed by taking the n cities as vertices and the connecting rails as edges must be a tree. The problem is to find an efficient algorithm for deciding which of the n^{n-2} possible trees connecting these cities uses the least amount of track, assuming that all the distances between pairs of cities are known. As before, we can reformulate the problem in terms of weighted graphs. We denote the weight of the edge e by $w(e)$, and our aim is to find the spanning tree T with least possible total weight $W(T)$.

Unlike some of the problems we considered earlier, there is a simple algorithm that provides the solution. It is known as a **greedy algorithm**, and involves being 'greedy' at each stage, by choosing an edge of minimum weight in such a way that no cycle is created. For example, if there are five cities, as shown in Fig. 3.15, then we start by choosing the shortest edges AB (of weight 2) and BD (of weight 3). We cannot then choose the edge AD (of weight 4), since it would create the cycle ABD, so instead

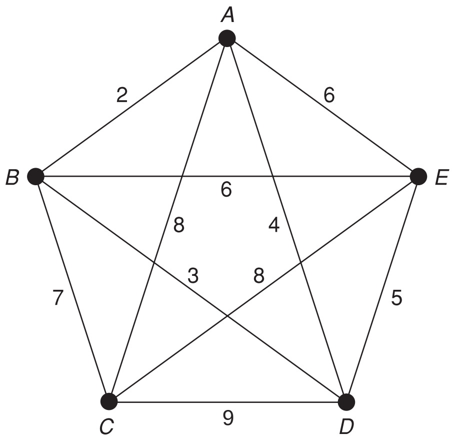


Figure 3.15

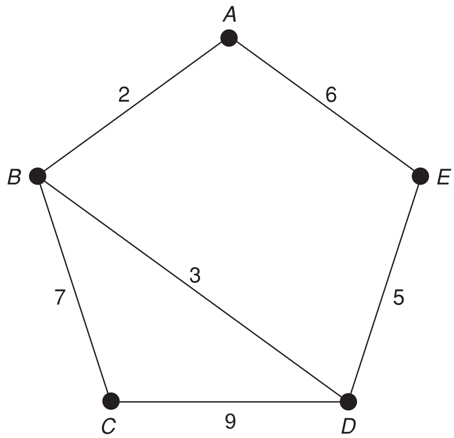


Figure 3.16

we choose the edge DE (of weight 5). We cannot then choose the edges AE or BE (of weight 6), since each would create a cycle, so instead we choose the edge BC (of weight 7). This completes a tree (see Fig. 3.16), which turns out to be of minimum total weight.

The algorithm is described in general in the following theorem.

THEOREM 3.6 *Let G be a connected graph with n vertices. Then the following construction gives a solution of the minimum connector problem:*

(i) *let e_1 be an edge of G of smallest weight;*
(ii) *define $e_2, e_3, \ldots, e_{n-1}$ by choosing at each stage a new edge of smallest possible weight that forms no cycle with the previous edges e_i.*

The required spanning tree is the subgraph T of G whose edges are $e_1, e_2, \ldots, e_{n-1}$.

Proof. The fact that T is a spanning tree of G follows immediately from Theorem 3.1(ii). We must show that the total weight of T is a minimum.

In order to do so, suppose that S is a spanning tree of G for which $W(S) < W(T)$. If e_k is the first edge in the above sequence that does not lie in S, then the subgraph of G formed by adding e_k to S contains a unique cycle C that includes the edge e_k. Since C contains an edge e lying in S but not in T, the subgraph obtained from S on replacing e by e_k is a spanning tree S'. But by the construction, $w(e_k) \leq w(e)$, so $W(S') \leq W(S)$, and S' has one more edge in common with T than S.

It follows on repeating this procedure that we can change S into T, one step at a time, with the total weight decreasing at each stage. Hence $W(T) \leq W(S)$, giving the required contradiction. ∎

We can also use the above algorithm to obtain a lower bound for the solution of the travelling salesman problem (see Section 2.4). This is useful, since the greedy algorithm is an efficient algorithm, whereas no efficient general algorithms are known for the travelling salesman problem.

If we take any Hamiltonian cycle in a complete graph and remove any vertex v, then we obtain a semi-Hamiltonian path through the remaining vertices, and such a path must be a spanning tree. So any solution of the travelling salesman problem must consist of a spanning tree of this type, together with two edges incident with v. It follows that if we take the weight of a *minimum weight* spanning tree through the remaining vertices and add the two *smallest* weights of the edges incident with v, then we get a *lower bound* for the solution of the travelling salesman problem.

For example, if we take the weighted graph of Fig. 3.15 and remove the vertex C, then the remaining graph has the four vertices A, B, D and E. The minimum weight spanning tree joining these four vertices is the tree whose edges are AB, BD and DE, with total weight 10, and the two edges of minimum weight incident with C are CB and CA (or CE) with total weight 15 (see Fig. 3.17). This gives 10 + 15 = 25 as the resulting lower bound for the travelling salesman problem. Since the correct answer is 26, we see that this approach to the travelling salesman problem can yield surprisingly good results.

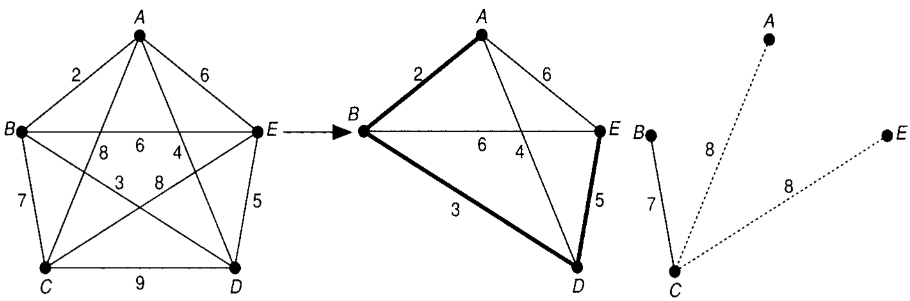


Figure 3.17

Searching trees

In many applications, the trees that we consider have a hierarchical structure, with one vertex at the top (inappropriately called the **root**), and the other vertices branching down from it, as in Fig. 3.18. For example, a computer file or a library classification system is often organized in this way, with information stored at the vertices.

If a particular piece of information is required, we need to be able to search the tree in a systematic way. This often involves examining every part of the tree until the

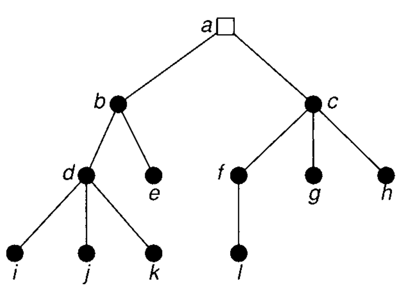


Figure 3.18

desired vertex is found. We wish to find a search technique that eventually visits all parts of the tree, but without visiting any vertex too often.

There are two well-known search procedures: **depth-first search** and **breadth-first search**. Both methods visit all the vertices, but in a different order. No rule can be given for which method should be used for a particular problem – each has its advantages. For example, a breadth-first search is used in the shortest path algorithm (Section 2.4), whereas a depth-first search is used for finding network flows (Section 6.3).

In a breadth-first search, we fan out to as many vertices as possible, before penetrating deeper into the tree. This means that we visit all the vertices adjacent to the current vertex before proceeding to another vertex. For example, consider the tree in Fig. 3.18. In order to perform a breadth-first search, we start at vertex *a* and visit the vertices *b* and *c* that are adjacent to *a*. We then visit the vertices *d* and *e* adjacent to *b*, and the vertices *f*, *g* and *h* adjacent to *c*. Finally we visit the vertices *i*, *j* and *k* adjacent to *d*, and *l* adjacent to *f*. This gives us the labelled tree in Fig. 3.19, where the labels correspond to the order in which the vertices are visited.

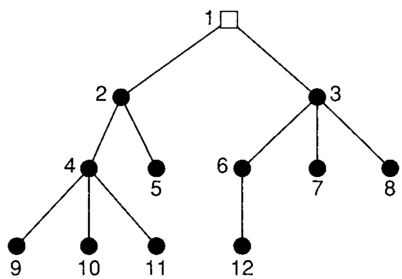


Figure 3.19

In a depth-first search, we penetrate as deeply as possible into a tree before fanning out to other vertices. For example, consider again the tree in Fig. 3.18. In order to perform a depth-first search, we start at vertex *a* and move down to *b*, *d* and *i*. Since we cannot penetrate further, we backtrack to *d* and then go down to *j*. We must then backtrack again, and go to *k*. We next backtrack via *d* to *b*, from which we can go down to *e*. Backtracking to *a* then takes us to *c*, *f* and *l*, and eventually to *g* and *h* before returning to *a*. This gives us the labelled tree in Fig. 3.20, where the labels correspond to the order in which the vertices are visited.

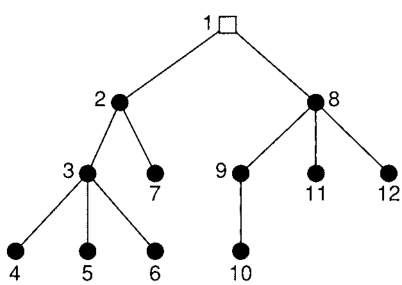


Figure 3.20

Bracing rectangular frameworks

Many buildings and other rigid structures are supported by rectangular steel frameworks, and such frameworks must remain rigid under heavy loads. For theoretical purposes, however, such structures can often be treated as planar structures with pin-joints (rather than rigid welds) holding the beams together.

Suppose that we have a rectangular framework consisting of rods linked by pin-joints, forming a pattern of square cells, so that the whole structure can move in the plane (see Fig. 3.21). We now brace three of the square cells with diagonal rods so that no distortion is possible and the entire framework becomes rigid. (The diagonal braces can go in either direction.)

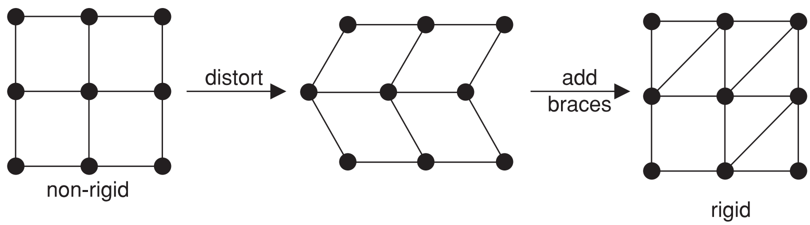


Figure 3.21

Further examples appear in Fig. 3.22. Framework (i) is not rigid, since it can be distorted as shown. Framework (ii) is rigid, but is over-braced since several of the diagonal braces can be removed without affecting the rigidity of the framework. But is framework (iii) rigid? If so, can any braces be removed without affecting the rigidity?

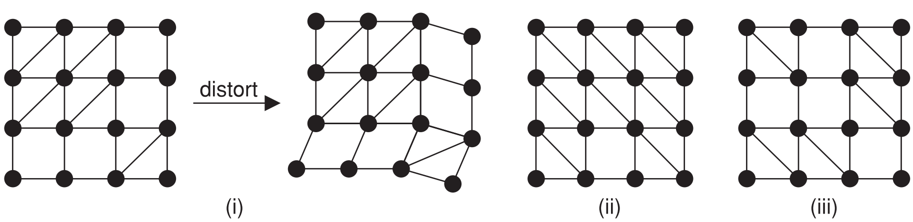


Figure 3.22

In general, the more braces we add to a framework, the more rigid the structure becomes, but it is often difficult to predict by looking at a given braced framework whether it is rigid or not, and whether any of the braces can be removed without affecting the rigidity of the entire framework.

The method we use is to draw a bipartite graph whose vertices correspond to the rows and the columns of the framework, and whose edges join a row-vertex and a column-vertex whenever there is a diagonal brace in the corresponding row and column. For example, the bipartite graphs corresponding to the frameworks of Fig. 3.22 are as shown in Fig. 3.23; in each case there are three row-vertices r_1, r_2 and r_3 and three column-vertices c_1, c_2 and c_3, and there is an edge from r_1 to c_1 since there is a diagonal brace in the r_1c_1 square.

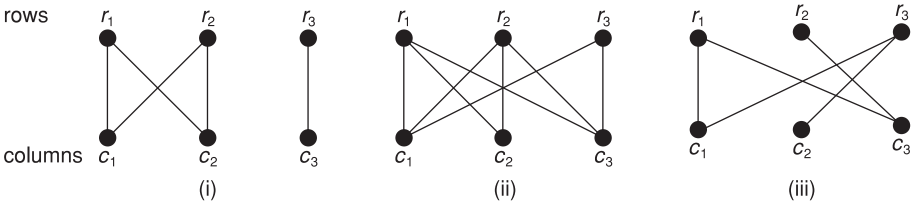


Figure 3.23

Note that the non-rigid framework (i) gives rise to a disconnected graph, whereas the rigid framework (ii) gives rise to a connected graph. This is because each diagonal brace forces the corresponding row and column to be at right angles. For example, in the bipartite graph of framework (ii), the edges from r_1 and r_2 to c_1, c_2 and c_3 show that rows 1 and 2 are at right angles to all three columns, and similarly row 3 is at right angles to columns 1 and 3; it follows that every row is at right angles to every column, and thus the framework cannot be distorted. However, in the bipartite graph of framework (i), the vertices r_3 and c_3 are not connected to the other vertices; thus, row 3 and column 3 need not remain at right angles to the other rows and columns, and the framework can be distorted.

These examples are instances of the following general rule, which shows that framework (iii) is also rigid, since its graph is connected.

THEOREM 3.7 *A braced rectangular framework is rigid if and only if the corresponding bipartite graph is connected.*

Note also that the graph of framework (ii) contains several cycles – for example, $r_1c_2r_2c_1r_1$. If we remove any edge from this cycle, the graph remains connected and so the framework remains rigid. We can continue removing edges from the cycles in the graph until there are no cycles left – for example, if we successively remove the edges r_1c_1, r_2c_2 and r_3c_1, the graph remains connected and the framework remains rigid. The resulting graph is a tree passing through all the vertices – that is, a spanning tree (see Fig. 3.24). The removal of any further edge would disconnect the tree, and the framework would then cease to be rigid.

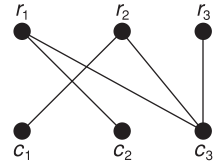


Figure 3.24

Thus we have the following rule which shows that the frameworks in Fig. 3.22(iii) and on the right of Fig. 3.21 have minimum bracings – that is, no further diagonal braces can be removed without affecting the rigidity of the framework:

If the bipartite graph is a spanning tree, then the bracing is a minimum bracing.

Electrical networks

Suppose that we are given the electrical network in Fig. 3.25, and that we wish to find the current in each wire.

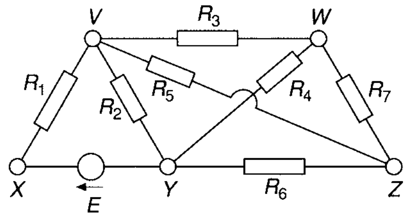


Figure 3.25

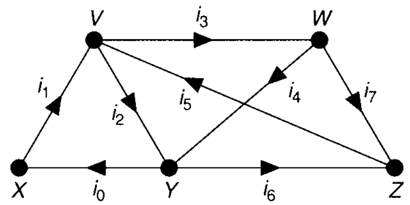


Figure 3.26

To do this, we assign an arbitrary direction to the current in each wire, as in Fig. 3.26, and apply **Kirchhoff's laws**:

(i) the algebraic sum of the currents at each vertex is 0;
(ii) the total voltage in each cycle is obtained by adding the products of the currents i_k and resistances R_k in that cycle.

Applying Kirchhoff's second law to the cycles *VYXV*, *VWYV* and *VWYXV*, we obtain the equations

$$i_1R_1 + i_2R_2 = E; \quad i_3R_3 + i_4R_4 - i_2R_2 = 0; \quad i_1R_1 + i_3R_3 + i_4R_4 = E.$$

But the last of these three equations is simply the sum of the first two, and gives us no further information. Similarly, if we have the Kirchhoff equations for the cycles *VWYV* and *WZYW*, then we can deduce the equation for the cycle *VWZYV*.

It will save us work if we can find a set of cycles that gives us the information we need without any redundancy, and this can be done by using a fundamental set of cycles (see Section 3.1). In this example, taking the fundamental system of cycles in Fig. 3.3, we obtain the following equations:

for the cycle *VYXV*,	$i_1R_1 + i_2R_2$	$= E;$
for the cycle *VYZV*,	$i_2R_2 + i_5R_5 + i_6R_6$	$= 0;$
for the cycle *VWZV*,	$i_3R_3 + i_5R_5 + i_7R_7$	$= 0;$
for the cycle *VYWZV*,	$i_2R_2 - i_4R_4 + i_5R_5 + i_7R_7$	$= 0.$

The equations arising from Kirchhoff's first law are:

for the vertex *X*,	$i_0 - i_1$	$= 0;$
for the vertex *V*,	$i_1 - i_2 - i_3 + i_5$	$= 0;$
for the vertex *W*,	$i_3 - i_4 - i_7$	$= 0;$
for the vertex *Z*,	$i_5 - i_6 - i_7$	$= 0.$

These eight equations can now be solved to give the eight currents $i_0, i_1, \ldots, i_7$. For example, if $E = 12$, and if each wire has unit resistance (that is, $R_i = 1$ for each i), then the solution is as given in Fig. 3.27.

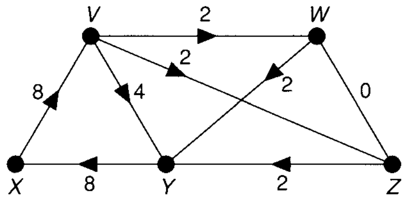


Figure 3.27

Exercises

3.20ˢ Use the greedy algorithm to find a minimum weight spanning tree in the graph in Fig. 3.28.

3.21 Find a minimum weight spanning tree in the graph in Fig. 3.29.

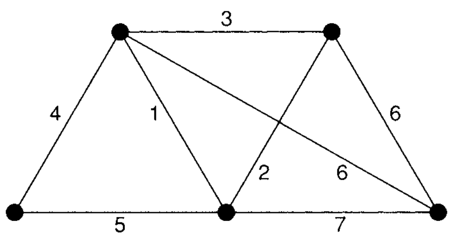


Figure 3.28

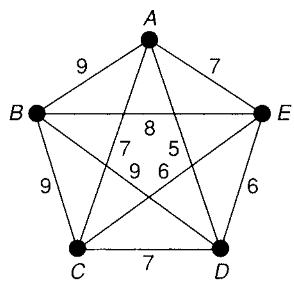


Figure 3.29

3.22 Show that if each edge of a connected weighted graph G has the same weight, then the greedy algorithm gives a method for constructing a spanning tree in G.

3.23 (i) How would you adapt the greedy algorithm to find a *maximum* weight spanning tree?
(ii) Find a maximum weight spanning tree for each of the weighted graphs in Figs 3.15 and 3.28.

3.24ˢ When applying the greedy algorithm to the travelling salesman problem in the text, what lower bounds do you get when you remove each of the vertices A, B, D and E in turn, instead of C?

3.25ˢ Perform a breadth-first search and a depth-first search on the tree in Fig. 3.30.

3.26 Perform a breadth-first search and a depth-first search on the tree in Fig. 3.31.

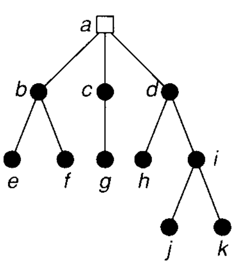


Figure 3.30

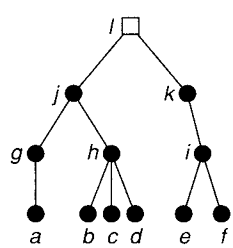


Figure 3.31

3.27[s] By constructing the corresponding bipartite graph, determine whether the braced framework in Fig. 3.32 is rigid. Is the bracing a minimum bracing?

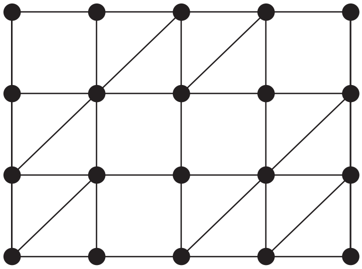


Figure 3.32

3.28 Determine whether the braced framework in Fig. 3.33 is rigid, and whether the bracing is a minimum bracing.

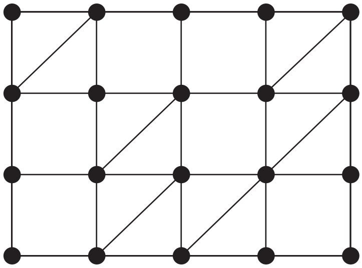


Figure 3.33

3.29 Determine whether each of the braced frameworks in Fig. 3.34 is rigid, and whether the bracing is a minimum bracing.

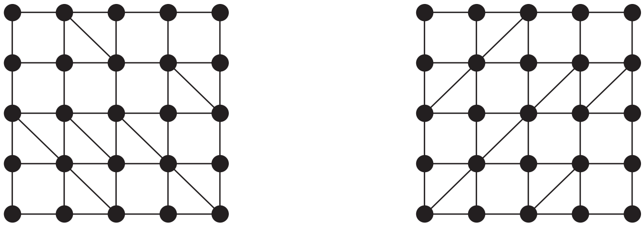


Figure 3.34

3.30[s] Verify the currents in Fig. 3.27, by applying Kirchhoff's laws to the fundamental cycles associated with the spanning tree with edges *VX*, *VW*, *WZ* and *YZ*.

3.31 Write down the Kirchhoff equations for the network of Fig. 3.35, in which the numbers are resistances; use the spanning tree with edges *VW*, *WZ*, *XV* and *WZ*.

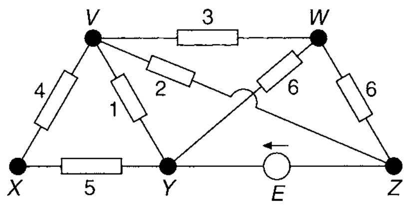


Figure 3.35

Challenge problems

3.32 (i) Let C^* be a set of edges of a connected graph G. Show that, if C^* has an edge in common with each spanning tree of G, then C^* contains a cutset.
(ii) Obtain a corresponding result for cycles.

3.33 Show that if H and K are subgraphs of a connected graph G, and if $H \cup K$ and $H \cap K$ are defined in the obvious way, then the cutset rank ξ satisfies:
(i) $0 \le \xi(H) \le |E(H)|$ (the number of edges of H);
(ii) if H is a subgraph of K, then $\xi(H) \le \xi(K)$;
(iii) $\xi(H \cup K) + \xi(H \cap K) \le \xi(H) + \xi(K)$.

3.34 Let V be the vector space associated with a simple connected graph G, and let T be a spanning tree of G.
(i) Show that the fundamental set of cycles associated with T forms a basis for the cycle subspace W.
(ii) Obtain a corresponding result for the cutset subspace W^*.
(iii) Deduce that the dimensions of W and W^* are $\gamma(G)$ and $\xi(G)$, respectively.

3.35 Use the matrix-tree theorem to prove Cayley's theorem.

3.36 Let $T(n)$ be the number of labelled trees on n vertices.
(i) By counting the number of ways of joining a labelled tree on k vertices and one on $n - k$ vertices, prove that

$$2(n-1)T(n) = \sum_{k=1}^{n-1} \binom{n}{k} k(n-k)T(k)T(n-k).$$

(ii) Deduce the identity

$$\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1)\, n^{n-2}.$$

3.37 Using the results of Exercise 3.31, find the current in each wire of the network in Fig. 3.35.

Chapter 4

Planarity

Flattery will get you nowhere.

Popular saying

We now embark upon a study of topological graph theory, in which graphs become tied up with topological notions such as planarity, genus, etc. In particular, we investigate when a graph can be drawn in the plane and on other surfaces. In Section 4.1, we discuss planar graphs, exhibit some graphs that are not planar, and state Kuratowski's characterization of planar graphs. In Section 4.2, we prove Euler's formula relating the numbers of vertices, edges and faces of a graph drawn in the plane. In Section 4.3 we study the important topic of duality. In Section 4.4 we generalize these ideas to graphs drawn on surfaces other than the plane.

4.1 Planar graphs

A **planar graph** is a graph that can be drawn in the plane without crossings – that is, so that no two edges intersect geometrically except at a vertex with which both are incident. Any such drawing is a **plane drawing**. For convenience, we often use the abbreviation **plane graph** for a plane drawing of a planar graph. For example, Fig. 4.1 shows three drawings of the planar graph K_4, but only the second and third drawings are plane graphs.

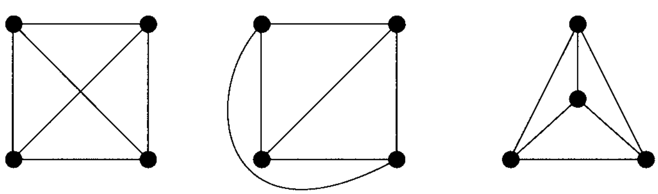


Figure 4.1

The right-hand drawing in Fig. 4.1 leads us to ask whether every planar graph can be drawn in the plane so that each edge is represented by a straight line. Although loops or multiple edges cannot be drawn as straight lines, it was proved independently by K. Wagner in 1936 and I. Fáry in 1948 that:

Every simple planar graph can be drawn with straight lines.

See Chartrand and Lesniak [11] or West [16] for details.

Not all graphs are planar, as the following theorem shows.

THEOREM 4.1 *$K_{3,3}$ and K_5 are non-planar.*

Remark. We give two proofs of this result. The first one, presented here, is constructive. The second proof, which we give in Section 4.2, appears as a corollary of Euler's formula.

Proof. Suppose first that $K_{3,3}$ is planar. Since $K_{3,3}$ has a cycle

$$u \to v \to w \to x \to y \to z \to u$$

of length 6, any plane drawing must contain this cycle drawn in the form of a hexagon, as in Fig. 4.2.

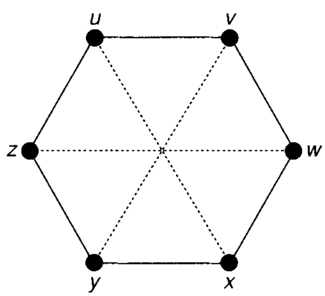


Figure 4.2

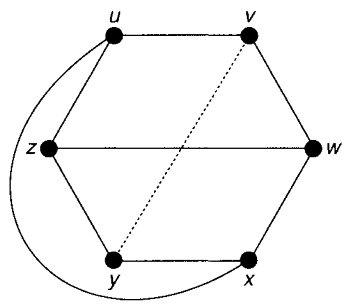


Figure 4.3

Now the edge wz must lie either wholly inside the hexagon or wholly outside it. We deal with the case in which wz lies inside the hexagon – the other case is similar. Since the edge ux must not cross the edge wz, it must lie outside the hexagon; the situation is now as in Fig. 4.3. It is then impossible to draw the edge vy, as it would cross either ux or wz. This gives the required contradiction.

Next, suppose that K_5 is planar. Since K_5 has a cycle

$$v \to w \to x \to y \to z \to v$$

of length 5, any plane drawing must contain this cycle drawn in the form of a pentagon, as in Fig. 4.4.

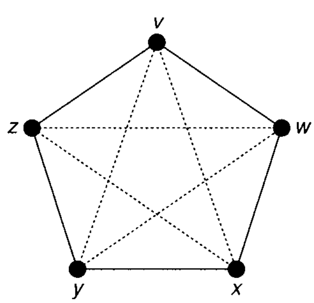


Figure 4.4

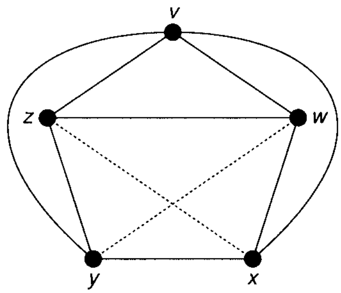


Figure 4.5

Now the edge wz must lie either wholly inside the pentagon or wholly outside it. We deal with the case in which wz lies inside the pentagon – the other case is similar. Since the edges vx and vy do not cross the edge wz, they must both lie outside the pentagon; the situation is now as in Fig. 4.5. But the edge xz cannot cross the edge vy and so must lie inside the pentagon; similarly the edge wy must lie inside the pentagon, and the edges wy and xz must then cross. This gives the required contradiction. ■

Note that, if we try to draw K_5 or $K_{3,3}$ on the plane, then there must be at least one crossing of two edges, since these graphs are not planar. However, we do not need more than one crossing (see Fig. 4.6), even when we require crossings to involve just two edges.

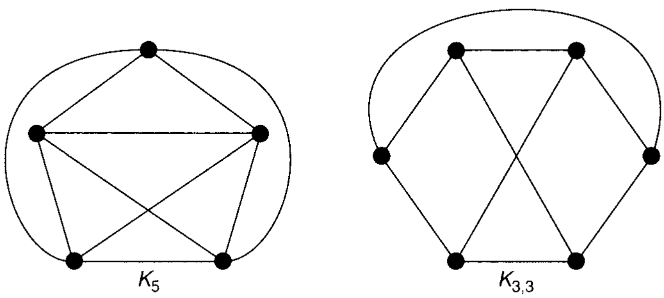


Figure 4.6

More generally, the **crossing number** cr(G) of a graph G is the smallest number of crossings (of two edges) that can occur when G is drawn in the plane. For example, the crossing number of any planar graph is 0, and $\mathrm{cr}(K_5) = \mathrm{cr}(K_{3,3}) = 1$. Unfortunately, little is known about the crossing number of graphs in general. Even the crossing numbers of most complete graphs and complete bipartite graphs are unknown.

Kuratowski's theorem

It is clear that every subgraph of a planar graph is planar, and that every graph with a non-planar subgraph must be non-planar. It follows that any graph with $K_{3,3}$ or K_5 as a subgraph is non-planar. In fact, as we shall see, these two graphs are the 'building blocks' for all non-planar graphs, in the sense that every non-planar graph must 'contain' at least one of them.

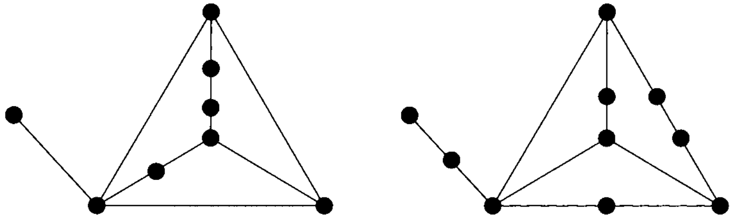


Figure 4.7

To make this statement more precise, we define two graphs to be **homeomorphic** if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges. For example, any two cycle graphs are homeomorphic, as are the two graphs of Fig. 4.7.

Note that the introduction of the term 'homeomorphic' is merely a technicality, as the insertion or deletion of vertices of degree 2 is irrelevant to considerations of planarity. However, it enables us to state the following important result, known as **Kuratowski's theorem**, which gives a necessary and sufficient condition for a graph to be planar.

THEOREM 4.2 (Kuratowski, 1930) *A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.*

The proof of Kuratowski's theorem is long and involved, and we omit it; see Chartrand and Lesniak [11] or West [16] for a proof. We shall, however, use Kuratowski's theorem to obtain another criterion for planarity. To do so, we first define a graph H to be **contractible** to K_5 or $K_{3,3}$ if we can obtain K_5 or $K_{3,3}$ by successively contracting edges of H. For example, the Petersen graph is contractible to K_5, as we can see by contracting the five 'spokes' joining the inner and outer 5-cycles (see Fig. 4.8).

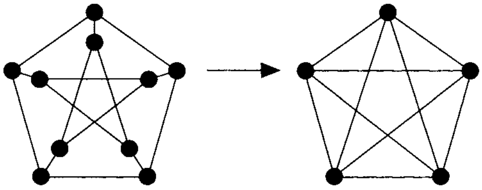


Figure 4.8

THEOREM 4.3 *A graph is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$.*

Sketch of proof. $\Leftarrow$ Assume first that the graph G is non-planar. Then, by Kuratowski's theorem, G contains a subgraph H homeomorphic to K_5 or $K_{3,3}$. On successively contracting edges of H that are incident to a vertex of degree 2, we see that H is contractible to K_5 or $K_{3,3}$.

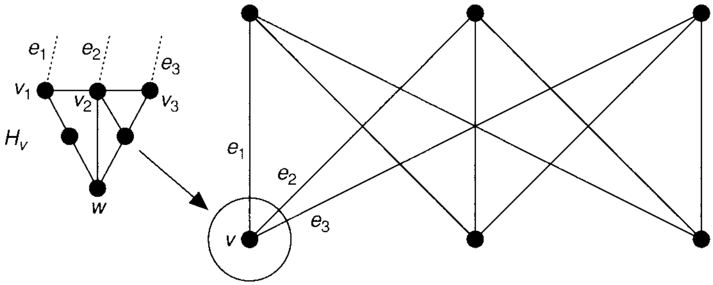


Figure 4.9

$\Rightarrow$ Now assume that G contains a subgraph H contractible to $K_{3,3}$, and let the vertex v of $K_{3,3}$ arise from contracting the subgraph H_v of H (see Fig. 4.9).

The vertex v is incident in $K_{3,3}$ with three edges e_1, e_2 and e_3. When regarded as edges of H, these edges are incident with three (not necessarily distinct) vertices v_1, v_2 and v_3 of H_v. If v_1, v_2 and v_3 are distinct, then we can find a vertex w of H_v and three paths from w to these vertices intersecting only at w. (There is a similar construction if the vertices are not distinct, the paths degenerating in this case to single vertices.) It follows that we can replace the subgraph H_v by a vertex w and three paths leading out of it. If this construction is carried out for each vertex of $K_{3,3}$, and the resulting paths joined up with the corresponding edges of $K_{3,3}$, then the resulting subgraph is homeomorphic to $K_{3,3}$. It follows from Kuratowski's theorem that G is non-planar (see Fig. 4.10).

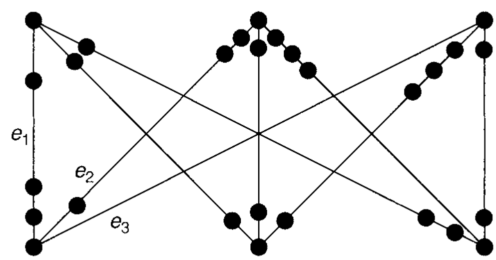


Figure 4.10

A similar argument can be carried out if G contains a subgraph contractible to K_5. Here the details are more complicated, as the subgraph we obtain by the above process can be homeomorphic to either K_5 or $K_{3,3}$; see Chartrand and Lesniak [11]. ∎

Infinite planar graphs

We conclude this section with a result on infinite planar graphs. Recall from Section 2.1 that König's lemma allows us to deduce results about infinite graphs from the corresponding results for finite graphs. The following theorem may be regarded as a typical example.

THEOREM 4.4 *If G is a countable graph, every finite subgraph of which is planar, then G is planar.*

Proof. Since G is countable, its vertices may be listed as $v_1, v_2, v_3, \ldots$. We now construct a strictly increasing sequence $G_1 \subset G_2 \subset G_3 \subset \ldots$ of subgraphs of G, by taking G_k to be the subgraph whose vertices are $v_1, v_2, \ldots, v_k$ and whose edges are those edges of G joining two of these vertices. Since each G_i can be drawn in the plane in only a finite number $m(i)$ of topologically different ways, we can construct another infinite graph H whose vertices w_{ij} (for $i \geq 1$ and $1 \leq j \leq m(i)$) correspond to the various drawings of the graphs G_i, and whose edges join the vertices w_{ij} and w_{kl} when $k = i + 1$ and the plane drawing corresponding to w_{kl} extends the drawing corresponding to w_{ij}. Since H is clearly connected and locally finite, it follows from König's lemma that H contains a one-way infinite path. Since G is countable, this infinite path gives the required plane drawing of G. ■

Exercises

4.1[s] Show how the graph of Fig. 4.11 can be drawn in the plane without crossings.

4.2 Show how the graph of Fig. 4.12 can be drawn in the plane without crossings.

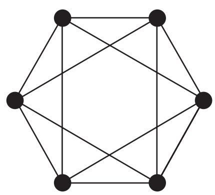


Figure 4.11

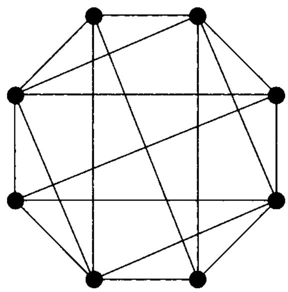


Figure 4.12

4.3[s] Three unfriendly neighbours use the same water, oil and treacle wells. In order to avoid meeting, they wish to build non-crossing paths from each of their houses to each of the three wells. Can this be done?

4.4[s] Which complete graphs and complete bipartite graphs are planar?

4.5 (i) For which values of k is the k-cube Q_k planar?
(ii) For which values of r, s and t is the complete tripartite graph $K_{r,s,t}$ planar?

4.6[s] Show that $K_{4,3}$ and the Petersen graph have crossing number 2.

4.7 Prove that the Petersen graph is non-planar
(i) by removing the two 'horizontal' edges and using Theorem 4.2;
(ii) by using Theorem 4.3.

4.8ˢ Give an example of
(i) a non-planar graph that is not homeomorphic to K_5 or $K_{3,3}$;
(ii) a non-planar graph that is not contractible to K_5 or $K_{3,3}$.
Why does the existence of these graphs not contradict Theorems 4.2 and 4.3?

4.9 If two homeomorphic graphs have n_i vertices and m_i edges ($i = 1, 2$), show that

$$m_1 - n_1 = m_2 - n_2.$$

4.10 A planar graph G is **outerplanar** if G can be drawn in the plane so that all of its vertices lie on the exterior boundary.
(i) Show that K_4 and $K_{2,3}$ are not outerplanar.
(ii) Deduce that, if G is an outerplanar graph, then G contains no subgraph homeomorphic or contractible to K_4 or $K_{2,3}$.
(The converse result also holds, yielding a Kuratowski-type criterion for a graph to be outerplanar.)

4.11 By placing its vertices at the points $(1, 1^2, 1^3)$, $(2, 2^2, 2^3)$, $(3, 3^2, 3^3)$, . . . , prove that any simple graph can be drawn without crossings in Euclidean three-dimensional space so that each edge is represented by a straight line.

4.12 Give an example of each of the following:
(i) a countable planar graph;
(ii) a countable non-planar graph.

4.2 Euler's formula

If G is a planar graph, then any plane drawing of G divides the set of points of the plane not lying on G into regions, called **faces**. For example, the plane graphs in Figs 4.13 and 4.14 have eight faces and four faces, respectively. Note, in each case, that the face f_4 is unbounded; it is called the **infinite face**.

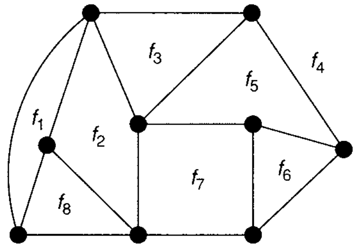


Figure 4.13

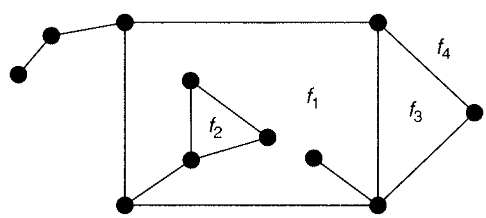


Figure 4.14

There is nothing special about the infinite face – in fact, any face can be selected as the infinite face. To see this, we map the graph onto the surface of a sphere by stereographic projection (see Fig. 4.15). We now rotate the sphere so that the point of projection (the north pole) lies inside our selected infinite face, and then project the graph down onto the plane tangent to the sphere at the south pole. The chosen face is now the infinite face.

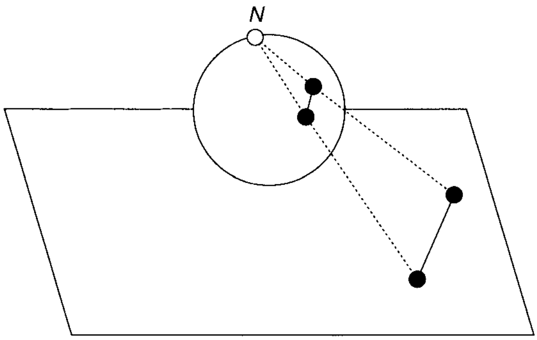


Figure 4.15

Figure 4.16 shows a representation of the graph of Fig. 4.14 in which the infinite face is f_3.

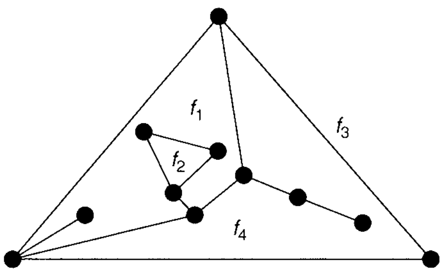


Figure 4.16

We now state and prove **Euler's formula** that tells us that whatever plane drawing of a planar graph we take, the number of faces is always the same and is given by a simple formula.

THEOREM 4.5 (Euler, 1750) *Let G be a plane drawing of a connected planar graph, and let n, m and f denote respectively the number of vertices, edges and faces of G. Then*

$$n - m + f = 2.$$

Remark. An example of this theorem is given by Fig. 4.14, where $n = 11$, $m = 13$, $f = 4$, and $n - m + f = 11 - 13 + 4 = 2$.

Proof. The proof is by induction on the number of edges of G.

If $m = 0$, then $n = 1$ (since G is connected) and $f = 1$ (the infinite face). The theorem is therefore true in this case.

Now suppose that the theorem holds for all plane graphs with at most $m - 1$ edges, and let G be a plane graph with m edges. If G is a tree, then $m = n - 1$ and $f = 1$, so that $n - m + f = 2$, as required. If G is not a tree, let e be an edge in some cycle of G. Then

$G - e$ is a connected plane graph with n vertices, $m - 1$ edges and $f - 1$ faces, so that $n - (m - 1) + (f - 1) = 2$, by the induction hypothesis. It follows that $n - m + f = 2$, as required. ■

This result is often called 'Euler's polyhedron formula', since it relates the numbers of vertices, edges and faces of a convex polyhedron. For example, for a cube we have $n = 8$, $m = 12$, $f = 6$, and $n - m + f = 8 - 12 + 6 = 2$ (see Fig. 4.17).

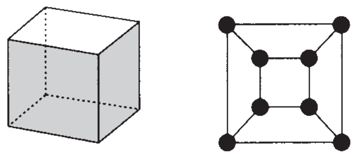


Figure 4.17

To see the connection in general, project the polyhedron out onto its circumsphere, and then use stereographic projection (as in Fig. 4.15) to project it down onto the plane. The resulting graph is a 3-connected plane graph in which each face is bounded by a polygon – such a graph is called a **polyhedral graph** (see Fig. 4.13). For convenience, we restate Euler's formula for such graphs; an alternative proof is outlined in Exercise 4.42.

COROLLARY 4.6 *Let G be a polyhedral graph. Then, with the above notation,*

$$n - m + f = 2.$$

Euler's formula can easily be extended to disconnected graphs:

COROLLARY 4.7 *Let G be a plane graph with n vertices, m edges, f faces and k components. Then*

$$n - m + f = k + 1.$$

Proof. We obtain the result by applying Euler's formula to each component separately, and remembering not to count the infinite face more than once. ■

All of the above results apply to arbitrary plane graphs. We now restrict ourselves to simple graphs.

COROLLARY 4.8

(i) *If G is a simple connected planar graph with n* (≥ 3) *vertices and m edges, then*

$$m \leq 3n - 6.$$

(ii) *If, in addition, G has no triangles, then* $m \leq 2n - 4$.

Proof.

(i) We can assume that we have a plane drawing of G. Since each face is bounded by at least three edges, it follows on counting up the edges around each face that $3f \le 2m$; the factor 2 appears since each edge bounds two faces (or the same face twice) and is therefore counted twice. We obtain the required result by combining this inequality with Euler's formula.

(ii) This part follows in a similar way, except that the above inequality $3f \le 2m$ is replaced by $4f \le 2m$. ■

Using this corollary, we can give an alternative proof of Theorem 4.1.

COROLLARY 4.9 *K_5 and $K_{3,3}$ are non-planar.*

Proof. If K_5 were planar then, applying Corollary 4.8(i), we would obtain $10 < 9$, which is a contradiction. If $K_{3,3}$ were planar then, applying Corollary 4.8(ii), we would obtain $9 < 8$, which is also a contradiction. ■

We can use a similar argument to prove the following theorem, which will be useful when we study the colouring of graphs in Chapter 5.

THEOREM 4.10 *Every simple planar graph G contains a vertex of degree at most* 5.

Proof. Without loss of generality we can assume that G is connected and has at least three vertices. If each vertex has degree at least 6, then, with the above notation, we have $6n \le 2m$, so $3n \le m$. It follows immediately from Corollary 4.8(i) that $3n \le 3n - 6$, which is a contradiction. ■

We conclude this section with a few remarks on the 'thickness' of a graph, which is another measure of how 'unplanar' a graph is. In electrical engineering, parts of networks are sometimes printed on one side of a non-conducting plate, and are called 'printed circuits'. Since the wires are not insulated, they must not cross and the corresponding graphs must be planar (see Fig. 4.18).

For a general network, we may need to know how many printed circuits are needed to complete the entire network. To this end, we define the **thickness** $t(G)$ of a graph

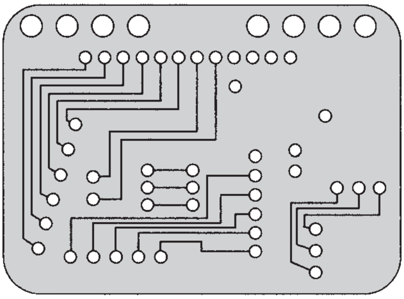


Figure 4.18

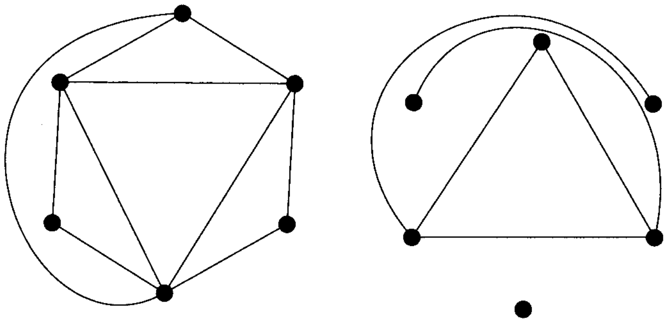


Figure 4.19

G to be the smallest number of planar graphs that can be superimposed to form G. For example, the thickness of a planar graph is 1, and of K_5 and $K_{3,3}$ is 2. Figure 4.19 shows that the thickness of K_6 is 2.

As we shall see, a lower bound for the thickness of a graph is easily obtained from Euler's formula. Surprisingly, this trivial lower bound frequently turns out to be the correct value, as we can verify by direct construction. In deriving this lower bound, we use the 'floor' and 'ceiling' symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote respectively the largest integer not greater than x and the smallest integer not less than x; for example,

$$\lfloor 3 \rfloor = \lceil 3 \rceil = 3; \quad \lfloor \pi \rfloor = 3; \quad \lceil \pi \rceil = 4.$$

THEOREM 4.11 *Let G be a simple graph with n (≥ 3) vertices and m edges. Then the thickness t(G) of G satisfies the inequalities*

$$t(G) \geq \lceil m/(3n-6) \rceil \quad \textit{and} \quad t(G) \geq \lfloor (m+3n-7)/(3n-6) \rfloor.$$

Proof. The first inequality is an immediate application of Corollary 4.4(i), with the brackets arising from the fact that the thickness must be an integer. The second inequality follows from the first by using the easily proved relation $\lceil a/b \rceil = \lfloor (a+b-1)/b \rfloor$, where a and b are positive integers. ∎

Exercises

4.13[s] Verify Euler's formula for
- (i) the wheel W_8;
- (ii) the graph of the octahedron;
- (iii) the graph of Fig. 4.13;
- (iv) the complete bipartite graph $K_{2,7}$.

4.14 Redraw the graph of Fig. 4.14 with
- (i) f_1 as the infinite face;
- (ii) f_2 as the infinite face.

4.15[s] (i) Use Euler's formula to prove that, if G is a connected planar graph of girth 5 with n vertices and m edges, then $m \le \frac{5}{3}(n-2)$. Deduce that the Petersen graph is non-planar.

(ii) Obtain an inequality, generalizing that in part (i), for connected planar graphs of girth r.

4.16 Let G be a polyhedron (or polyhedral graph), each of whose faces is bounded by a pentagon or a hexagon.

(i) Use Euler's formula to show that G must have at least 12 pentagonal faces.

(ii) Prove, in addition, that if G is such a polyhedron with exactly three faces meeting at each vertex (such as a football), then G has exactly 12 pentagonal faces.

4.17 Let G be a simple plane graph with fewer than 12 faces, in which each vertex has degree at least 3.

(i) Use Euler's formula to prove that G has a face bounded by at most four edges.

(ii) Give an example to show that the result of part (i) is false if G has 12 faces.

4.18[s] (i) Let G be a simple connected cubic plane graph, and let C_k be the number of k-sided faces. By counting the number of vertices and edges of G, prove that

$$3C_3 + 2C_4 + C_5 - C_7 - 2C_8 - 3C_9 - \cdots = 12.$$

(ii) Use this result to deduce the result of Exercise 4.16(ii).

(iii) Deduce also that G has at least one face bounded by at most five edges.

4.19 Let G be a simple graph with at least 11 vertices, and let $\bar{G}$ be its complement.

(i) Prove that G and $\bar{G}$ cannot both be planar.
(In fact, a similar result holds if 11 is replaced by 9, though this is difficult to prove.)

(ii) Find a graph G with eight vertices for which G and $\bar{G}$ are both planar.

4.20[s] Find the thickness of

(i) the Petersen graph;

(ii) the 4-cube Q_4.

4.21 (i) Show that the thickness of K_n satisfies $t(K_n) \ge \lfloor \frac{1}{6}(n+7) \rfloor$.

(ii) Use the results of Exercise 4.19 to prove that equality holds if $n = 8$, but not if $n = 9$ and 10.
(In fact, equality holds for all n other than 9 and 10.)

4.3 Dual graphs

In Theorems 4.2 and 4.3 we gave necessary and sufficient conditions for a graph to be planar – namely, that it contains no subgraph homeomorphic or contractible to K_5 or $K_{3,3}$. We now discuss conditions of a different kind, involving the concept of duality.

Given a plane drawing of a planar graph G, we construct another graph G^*, called the (**geometric**) **dual** of G. The construction is in two stages:

(i) inside each face f of G we choose a point v^* – these points are the vertices of G^*;

(ii) corresponding to each edge e of G we draw a line e^* that crosses e (but no other edge of G) and joins the vertices v^* in the faces f adjoining e – these lines are the edges of G^*.

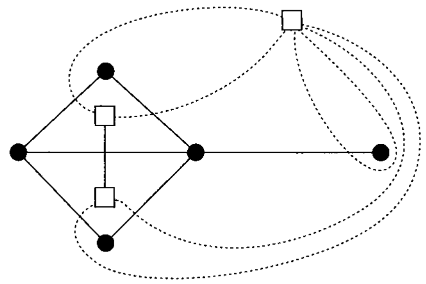


Figure 4.20

This procedure is illustrated in Fig. 4.20. The vertices v^* of G^* are represented by small squares, the edges e of G by solid lines and the edges e^* of G^* by dotted lines. Note that an end-vertex or a bridge of G gives rise to a loop of G^*, and that if two faces of G have more than one edge in common, then G^* has multiple edges.

The geometrical idea of duality is very old. For example, the 'fifteenth book of Euclid', written about AD 500–600, observes that the dual of a cube is an octahedron, and that the dual of a dodecahedron is an icosahedron (see Exercise 4.24). Note that any two graphs formed from G in this way must be isomorphic; this is why we called G^* '*the* dual of G' instead of '*a* dual of G'. On the other hand, if G is isomorphic to H, it does not necessarily follow that G^* is isomorphic to H^*; an example that demonstrates this is given in Exercise 4.27.

If G is both plane and connected, then G^* is plane and connected, and there are simple relations between the numbers of vertices, edges and faces of G and G^*.

LEMMA 4.12 *Let G be a connected plane graph with n vertices, m edges and f faces, and let its geometric dual G^* have n^* vertices, m^* edges and f^* faces. Then*

$$n^* = f,\ m^* = m \text{ and } f^* = n.$$

Proof. The first two relations are direct consequences of the definition of G^*. The third relation follows on substituting these two relations into Euler's formula, applied to both G and G^*. ■

Since the dual G^* of a plane graph G is also a plane graph, we can repeat the above construction to form the dual G^{**} of G^*. If G is connected, then the relationship between G^{**} and G is particularly simple, as we now show.

THEOREM 4.13 *If G is a connected plane graph, then G^{**} is isomorphic to G.*

Proof. The result follows immediately, since the construction that gives rise to G^* from G can be reversed to give G from G^*; for example, in Fig. 4.20 the graph G is the dual of the graph G^*. We need to check only that a face of G^* cannot contain more than one vertex of G (it certainly contains at least one), and this follows immediately from the relations $n^{**} = f^* = n$, where n^{**} is the number of vertices of G^{**}. ■

If G is a planar graph, then a dual of G can be defined by taking any plane drawing and forming its geometric dual, but uniqueness does not always hold. Since duals have been defined only for planar graphs, it is trivially true to say that a graph is planar if and only if it has a dual. On the other hand, we cannot tell from the above whether a given graph is planar. It is obviously desirable to find a definition of duality that generalizes the geometric dual and tells us in principle whether a given graph is planar. One such definition exploits the relationship under duality between the cycles and cutsets of a planar graph G. We now describe this relationship and use it to obtain the definition we seek. An alternative definition is given in Exercise 4.43.

THEOREM 4.14 *Let G be a planar graph and let G* be a geometric dual of G. Then a set of edges in G forms a cycle in G if and only if the corresponding set of edges of G* forms a cutset in G*.*

Proof. We can assume that G is a connected plane graph. If C is a cycle in G, then C encloses one or more finite faces of G, and thus contains in its interior a non-empty set S of vertices of G^*. It follows immediately that those edges of G^* that cross the edges of C form a cutset of G^* whose removal disconnects G^* into two subgraphs, one with vertex-set S and the other containing those vertices that do not lie in S (see Fig. 4.21).

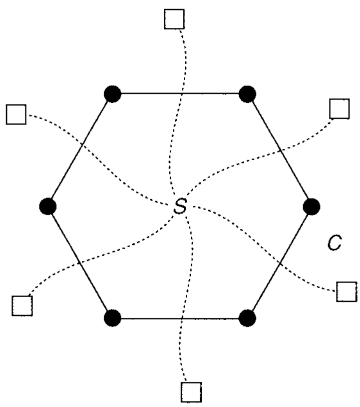


Figure 4.21

The converse implication is similar, and is omitted. ■

COROLLARY 4.15 *A set of edges of G forms a cutset in G if and only if the corresponding set of edges of G* forms a cycle in G*.*

Proof. The result follows on applying Theorem 4.14 to G^* and using Theorem 4.13. ■

Using Theorem 4.14 as motivation, we can now give an abstract definition of duality. Note that this definition does not invoke any geometrical properties of planar graphs, but concerns only the relationship between two graphs.

We say that a graph G^* is an **abstract dual** of a graph G if there is a one–one correspondence between the edges of G and those of G^*, with the property that a set of edges of G forms a cycle in G if and only if the corresponding set of edges of G^* forms a cutset in G^*. For example, Fig. 4.22 shows a graph and its abstract dual, with corresponding edges sharing the same letter.

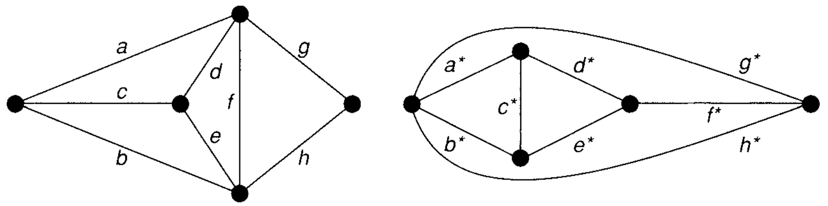


Figure 4.22

It follows from Theorem 4.14 that the concept of an abstract dual generalizes that of a geometric dual, in the sense that if G is a plane graph and G^* is a geometric dual of G, then G^* is an abstract dual of G. We should like to obtain analogues for abstract duals of our results on geometric duals. We present just one of these here – the analogue for abstract duals of Theorem 4.13. In this theorem we do not require that G should be connected.

THEOREM 4.16 *If G^* is an abstract dual of G, then G is an abstract dual of G^*.*

Proof. Let C be a cutset of G and let C^* denote the corresponding set of edges of G^*. We show that C^* is a cycle of G^*. By the first part of Exercise 2.47, C has an even number of edges in common with any cycle of G, and so C^* has an even number of edges in common with any cutset of G^*. It follows from the second part of Exercise 2.47 that C^* is either a cycle in G^* or an edge-disjoint union of at least two cycles. But the second possibility cannot occur, since we can show similarly that cycles in G^* correspond to edge-disjoint unions of cutsets in G, and so C would be an edge-disjoint union of at least two cutsets, rather than a single cutset. The result follows. ■

Although our definition of an abstract dual may seem strange, it turns out to have the properties required of it. We saw in Theorem 4.14 that a planar graph has an abstract dual (for example, any geometric dual). We now show that the converse result is true – that any graph with an abstract dual must be planar. This gives us an abstract definition of duality that generalizes the geometric dual and characterizes planar graphs. As we shall see in Section 7.3, the above definition of an abstract dual is a natural consequence of duality in matroids.

THEOREM 4.17 *A graph is planar if and only if it has an abstract dual.*

Remark. There are several proofs of this result. We outline a proof that uses Kuratowski's theorem.

Sketch of proof. As mentioned above, it is sufficient to prove that if G is a graph with an abstract dual G^*, then G is planar. The proof is in four steps.

(i) We note first that if an edge e is removed from G, then the abstract dual of the remaining graph is obtained from G^* by contracting the corresponding edge e^*. On repeating this procedure, we deduce that, if G has an abstract dual, then so does any subgraph of G.

(ii) We next observe that if G has an abstract dual, and if G' is a graph that is homeomorphic to G, then G' also has an abstract dual. This follows from the fact that the insertion or removal in G of a vertex of degree 2 results in the addition or deletion of a 'multiple edge' in G^*.

(iii) The third step is to show that neither K_5 nor $K_{3,3}$ has an abstract dual. But $K_{3,3}$ contains only cycles of length 4 or 6 and no cutsets with two edges. Thus, if G^* were a dual of $K_{3,3}$, then G^* would contain no multiple edges and each vertex of G^* would have degree at least 4. Hence G^* would have at least five vertices, and thus at least $\frac{1}{2}(5 \times 4) = 10$ edges, which is a contradiction. The argument for K_5 is similar.

(iv) Suppose, now, that G is a non-planar graph with an abstract dual G^*. Then, by Kuratowski's theorem, G has a subgraph H homeomorphic to K_5 or $K_{3,3}$. It follows from (i) and (ii) that H, and hence also K_5 or $K_{3,3}$, must have an abstract dual, contradicting (iii). ■

Exercises

4.22[s] Find the dual of the graph in Fig. 4.23 and verify Lemma 4.12.

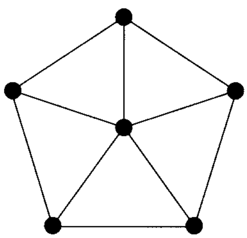


Figure 4.23

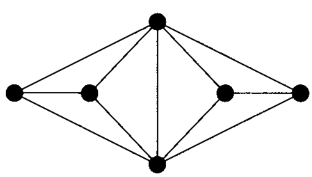


Figure 4.24

4.23 Find the dual of the graph in Fig. 4.24 and verify Lemma 4.12.

4.24 Show that the dual of the cube graph is the octahedron graph, and that the dual of the dodecahedron graph is the icosahedron graph. What is the dual of the tetrahedron graph?

4.25 Show that the dual of a wheel is a wheel.

4.26[s] Use duality to prove that there exists no plane graph with five faces, each pair of which shares an edge in common.

4.27[s] Show that the graphs in Fig. 4.25 are isomorphic, but that their geometric duals are not isomorphic.

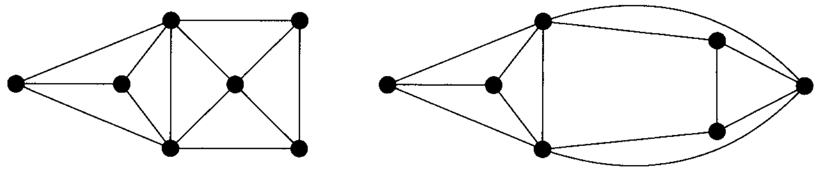


Figure 4.25

4.28 (i) Give an example to show that, if G is a disconnected plane graph, then G^{**} is not isomorphic to G.
(ii) Prove the result of part (i) in general.

4.29[s] Dualize the results of Exercise 4.16.

4.30[s] Prove that, if G is a 3-connected plane graph, then its geometric dual is a simple graph.

4.31[s] Let G be a connected plane graph. Using Theorem 2.1 and Corollary 2.10, prove that G is bipartite if and only if its dual G^* is Eulerian.
(This result will be needed in Chapters 5 and 7.)

4.32 (i) Give an example to show that, if G is a connected plane graph, then any spanning tree in G corresponds to the complement of a spanning tree in G^*.
(ii) Prove the result of part (i) in general.
(This result will also be needed in Chapter 7.)

4.4 Graphs on other surfaces

In the previous three sections we considered graphs drawn in the plane or (equivalently) on the surface of a sphere. We now consider drawing graphs on other surfaces – for example, the torus (see Fig. 4.26). It is easy to check that K_5 and $K_{3,3}$ can be drawn without crossings on the surface of a torus (see Exercise 4.33), and it is natural to ask whether there are analogues of Euler's formula and Kuratowski's theorem for graphs drawn on such surfaces.

The torus can be thought of as a sphere with one 'handle'. More generally, a surface is of **genus *g*** if it is topologically homeomorphic to a sphere with g handles. If you are unfamiliar with these terms, just think of graphs drawn on the surface of a doughnut with g holes in it. Thus the genus of a sphere is 0, and that of a torus is 1.

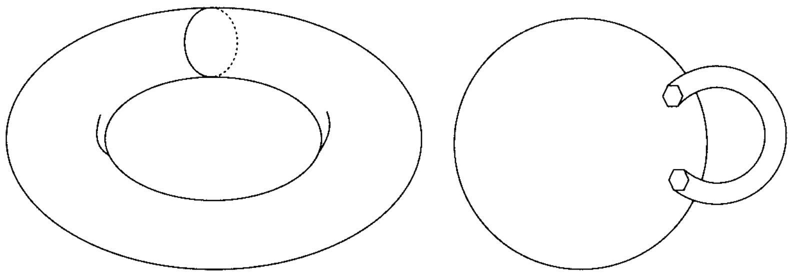


Figure 4.26

A graph that can be drawn without crossings on a surface of genus g, but not on one of genus $g - 1$, is a **graph of genus *g***. Thus, K_5 and $K_{3,3}$ are graphs of genus 1 (also called **toroidal graphs**).

The following result gives us an upper bound for the genus of a graph.

THEOREM 4.18 *The genus of a graph does not exceed the crossing number.*

Proof. We draw the graph on the surface of a sphere so that the number of crossings is as small as possible, and is therefore equal to the crossing number c. At each crossing, we construct a 'bridge' (as in Fig. 0.1 of the Introduction) and run one edge over the bridge and the other under it. Since each such bridge can be thought of as a handle, we have drawn the graph on the surface of a sphere with c handles. It follows that the genus does not exceed c. ■

There is currently no complete analogue of Kuratowski's theorem for surfaces of genus g, although Neil Robertson and Paul Seymour have proved that there exists a finite collection of 'forbidden' subgraphs of genus g, for each value of g, corresponding to the forbidden subgraphs K_5 and $K_{3,3}$ for graphs of genus 0. However, even for the torus, there seem to be many hundreds of forbidden subgraphs.

In the case of Euler's formula we are more fortunate, since there is a natural generalization for graphs of genus g. In this generalization, a **face** of a graph of genus g is defined in the obvious way.

THEOREM 4.19 *Let G be a connected graph of genus g with n vertices, m edges and f faces. Then*

$$n - m + f = 2 - 2g.$$

Sketch of proof. We outline the main steps in the proof, omitting the details.

Without loss of generality, we may assume that G is drawn on the surface of a sphere with g handles. We can also assume that the curves A at which the handles meet the sphere are cycles of G, by shrinking those cycles that contain these curves in their interior.

We next disconnect each handle at one end, in such a way that the handle has a free end E and the sphere has a corresponding hole H. We may assume that the cycle corresponding to the end of the handle appears at both the free end E and at the other end, since the additional vertices and edges required for this exactly balance each other, leaving $n - m + f$ unchanged (see Fig. 4.27).

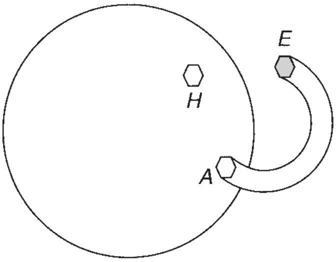


Figure 4.27

We complete the proof by telescoping each of these handles, leaving a sphere with $2g$ holes in it. This telescoping process does not change the value of $n - m + f$. But for a sphere, $n - m + f = 2$, and hence for a sphere with $2g$ holes in it,

$$n - m + f = 2 - 2g.$$

The result follows immediately. ■

COROLLARY 4.20 *The genus $g(G)$ of a simple graph G with n (≥ 4) vertices and m edges satisfies the inequality*

$$g(G) \geq \lceil \tfrac{1}{6}(m - 3n) + 1 \rceil.$$

Proof. Since each face is bounded by at least three edges, we have $3f \leq 2m$ (as in the proof of Corollary 4.8(i)). The result follows on substituting this inequality into Theorem 4.19, and using the fact that the genus of a graph is an integer. ■

As with the thickness, little is known about the genus of an arbitrary graph. The usual approach is to use Corollary 4.20 to obtain a lower bound for the genus, and then to try to obtain the required drawing by direct construction.

One case of particular historical importance is that of the genus of the complete graphs. Corollary 4.20 tells us that the genus of K_n satisfies

$$g(K_n) \geq \lceil \tfrac{1}{6}\{\tfrac{1}{2}n(n - 1) - 3n\} + 1 \rceil$$

or, after a little algebraic manipulation,

$$g(K_n) \geq \lceil \tfrac{1}{12}(n - 3)(n - 4) \rceil.$$

Percy Heawood asserted in 1890 that the inequality just obtained is an equality, and this was proved in 1968 by Gerhard Ringel and Ted Youngs after a long and difficult struggle; see Ringel [44] for a discussion and proof of this theorem.

THEOREM 4.21 (Ringel and Youngs, 1968)

$$g(K_n) = \lceil \tfrac{1}{12}(n - 3)(n - 4) \rceil.$$

Further results concerning the drawing of graphs on these surfaces, as well as on 'non-orientable' surfaces (such as the projective plane and the Möbius strip), can be found in Beineke and Wilson [34] or Gross and Tucker [37].

Exercises

4.33^s The surface of a torus can be regarded as a rectangle in which opposite edges are identified (see Fig. 4.28). Using this representation, find drawings of K_5 and $K_{3,3}$ on the torus.

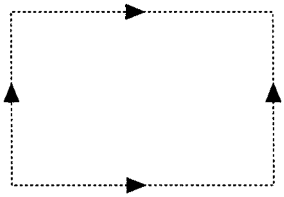


Figure 4.28

4.34 Using the representation in Exercise 4.33, show that the Petersen graph has genus 1.

4.35[s] (i) Calculate $g(K_7)$ and $g(K_{11})$.
(ii) Give an example of a complete graph of genus 2.

4.36 (i) Use Theorem 4.21 to prove that there is no value of n for which $g(K_n) = 7$.
(ii) What is the next integer that is not the genus of any complete graph?

4.37[s] (i) Give an example of a plane graph that is regular of degree 4 and in which each face is a triangle.
(ii) Show that there is no graph of genus $g \geq 1$ with these properties.

4.38 (i) Obtain a lower bound, analogous to that of Corollary 4.20, for a graph containing no triangles.
(ii) Deduce that $g(K_{r,s}) \geq \lceil \frac{1}{4}(r-2)(s-2) \rceil$.
(Ringel has shown that this is an equality.)

Challenge problems

4.39 Let G be a planar graph with vertex-set $\{v_1, v_2, \ldots, v_n\}$, and let $p_1, p_2, \ldots, p_n$ be any n distinct points in the plane. Give a heuristic argument to show that G can be drawn in the plane in such a way that the point p_i represents the vertex v_i, for each i.

4.40 If r and s are both even, prove that

$$\mathrm{cr}(K_{r,s}) \leq \tfrac{1}{16} rs(r-2)(s-2),$$

and obtain corresponding results when r and/or s is odd.
(Hint: place the r vertices along the x-axis with $\frac{1}{2}r$ vertices on each side of the origin, and the s vertices along the y-axis in a similar way; then join up the vertices by straight-line segments and count the crossings.)

4.41 (i) Use Corollary 4.8(ii) to prove that

$$t(K_{r,s}) \geq \lceil rs/(2r + 2s - 4) \rceil,$$

and verify that equality holds for $t(K_{3,3})$.
(ii) By splitting $K_{r,s}$ into a number of copies of $K_{2,s}$, prove that, if r is even, then $t(K_{r,s}) \leq r$, and deduce from part (i) that

$$t(K_{r,s}) = \tfrac{1}{2}r \text{ if } s > \tfrac{1}{2}(r-2)^2.$$

4.42 Let G be a polyhedral graph and let W be the cycle subspace of G.
(i) Show that the polygons bounding the finite faces of G form a basis for W.
(ii) Deduce Corollary 4.6.

4.43 A graph G^* is a **Whitney dual** of G if there is a one–one correspondence between $E(G)$ and $E(G^*)$ such that, for each subgraph H of G with $V(H) = V(G)$, the corresponding subgraph H^* of G^* satisfies

$$\gamma(H) + \xi(\bar{H}^*) = \xi(G^*),$$

where $\bar{H}^*$ is obtained from G^* by deleting the edges of H^*, and γ and ξ are as defined in Section 3.1.

(i) Show that this generalizes the idea of a geometric dual.

(ii) Prove that, if G^* is a Whitney dual of G, then G is a Whitney dual of G^*.

(In 1931, H. Whitney proved that a graph is planar if and only if it has such a dual.)

4.44 (i) Let G be a non-planar graph that can be drawn without crossings on a Möbius strip. Prove that, with the usual notation, $n - m + f = 1$.

(ii) Show how K_5 and $K_{3,3}$ can be drawn without crossings on the surface of a Möbius strip.

Chapter 5

Colouring graphs

With colours fairer painted their foul ends.
William Shakespeare (*The Tempest*)

In this chapter we investigate the colouring of graphs and maps, with special reference to the celebrated four-colour theorem and related topics. In Section 5.1, we colour the vertices of a graph so that each edge joins two vertices of different colours, and in Section 5.2, on chromatic polynomials, we discuss *in how many ways* such an assignment of colours can be made. Section 5.3 describes the connection between these vertex colourings and the colouring of maps, and in Section 5.4 we discuss the four-colour theorem. We conclude, in Section 5.5, by relating these ideas to the colouring of the edges of a graph.

5.1 Colouring vertices

If G is a graph without loops, then G is ***k*-colourable** if we can assign one of k colours to each vertex so that adjacent vertices have different colours. If G is k-colourable, but is not $(k-1)$-colourable, we say that G is ***k*-chromatic**, or that the **chromatic number** of G is k, and write $\chi(G) = k$. For example, Fig. 5.1 shows a graph G for which $\chi(G) = 4$; the colours are denoted by Greek letters: it is thus k-colourable if $k \geq 4$. We assume that all graphs here are simple, as multiple edges are irrelevant to our discussion. We also assume, when necessary, that they are connected.

It is clear that $\chi(K_n) = n$, and so there are graphs with arbitrarily high chromatic number. At the other end of the scale, $\chi(G) = 1$ if and only if G is a null graph, and $\chi(G) = 2$ if and only if G is a non-null bipartite graph. Note that every tree is 2-colourable, as is any cycle graph with an even number of vertices.

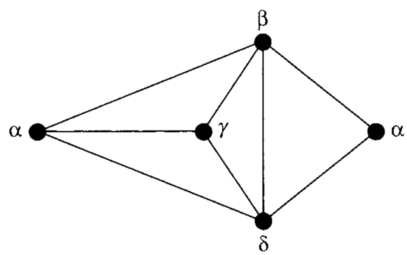


Figure 5.1

It is not known which graphs are 3-chromatic, although it is easy to give examples of such graphs. These examples include the cycle graphs or wheels with an odd number of vertices and the Petersen graph. The wheels with an even number of vertices are 4-chromatic.

There is little that we can say about the chromatic number of an arbitrary graph. If the graph has n vertices, then its chromatic number cannot exceed n, and if the graph contains K_r as a subgraph, then its chromatic number cannot be less than r, but such results do not usually take us very far. If, however, we know the degree of each vertex, then we can make good progress.

THEOREM 5.1 *If G is a simple graph with largest vertex-degree Δ, then G is $(\Delta + 1)$-colourable.*

Proof. The proof is by induction on the number of vertices of G.

Let G be a simple graph with n vertices. If we delete any vertex v and its incident edges, then the graph that remains is a simple graph with $n - 1$ vertices and largest vertex-degree at most Δ (see Fig. 5.2). By our induction hypothesis, this graph is $(\Delta + 1)$-colourable. A $(\Delta + 1)$-colouring for G is then obtained by colouring v with a different colour from the (at most Δ) vertices adjacent to v. ■

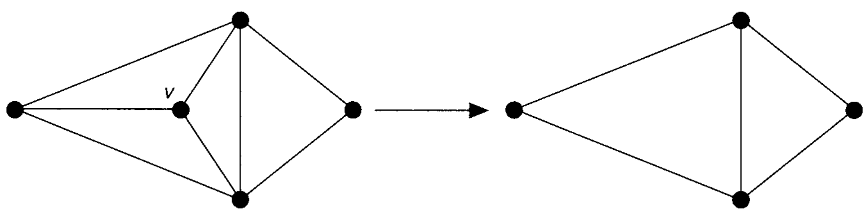


Figure 5.2

Brooks's theorem

By more careful treatment we can strengthen this theorem a little to give the following result, known as **Brooks's theorem**. Its proof is lengthy and can be omitted if desired.

THEOREM 5.2 (Brooks, 1941) *If G is a simple connected graph which is not a complete graph, and if the largest vertex-degree of G is Δ (≥ 3), then G is Δ-colourable.*

Proof. The proof is by induction on the number of vertices of G.

Suppose that G has n vertices. If any vertex of G has degree less than Δ, then we can complete the proof by imitating the proof of Theorem 5.1. We may thus suppose that G is regular of degree Δ.

If we delete a vertex v and its incident edges, then the graph that remains has $n - 1$ vertices and the largest vertex degree is still at most Δ. By our induction hypothesis, this graph is Δ-colourable. Our aim is now to colour v with one of the Δ colours. We can suppose that the vertices $v_1, v_2, \ldots, v_\Delta$ adjacent to v are arranged around v in clockwise order, and that they are coloured with distinct colours $c_1, c_2, \ldots, c_\Delta$, since otherwise there would be a spare colour that could be used to colour v.

We now define H_{ij} ($i \neq j$, $1 \leq i, j \leq \Delta$) to be the subgraph of G whose vertices are those coloured c_i or c_j and whose edges are those joining a vertex coloured c_i and a vertex coloured c_j. If the vertices v_i and v_j lie in different components of H_{ij}, then we can interchange the colours of all the vertices in the component of H_{ij} containing v_i (see Fig. 5.3). The result of this recolouring is that v_i and v_j both have colour c_j, enabling v to be coloured c_i. We may thus assume that, given any i and j, v_i and v_j are connected by a path that lies entirely in H_{ij}. We denote the component of H_{ij} containing v_i and v_j by C_{ij}.

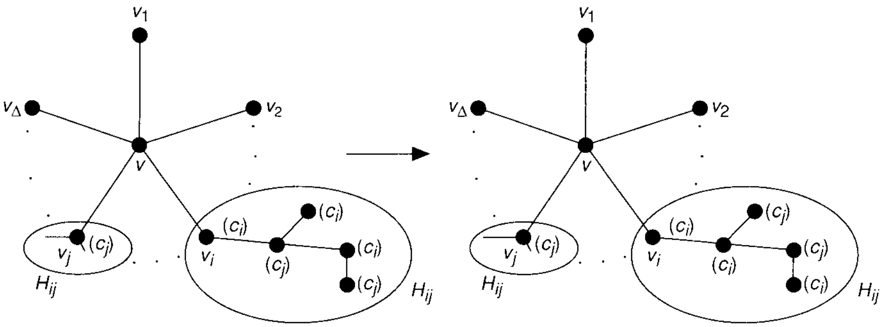


Figure 5.3

If v_i is adjacent to more than one vertex with colour c_j, then there is a colour (other than c_i) that is not assumed by any vertex adjacent to v_i. In this case, v_i can be recoloured using this colour, enabling v to be coloured with colour c_i. If this does not happen, then we can use a similar argument to show that every vertex of C_{ij} (other than v_i and v_j) must have degree 2. For, if w is the first vertex of the path from v_i to v_j with degree greater than 2, then w can be recoloured with a colour different from c_i or c_j, thereby destroying the property that v_i and v_j are connected by a path lying entirely in C_{ij} (see Fig. 5.4). We can thus assume that, for any i and j, the component C_{ij} consists only of a path from v_i to v_j.

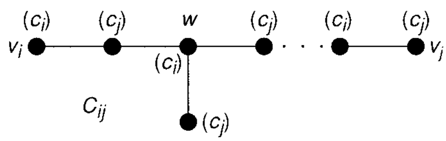


Figure 5.4

We can also assume that two paths of the form C_{ij} and C_{jl} (where $i \neq l$) intersect only at v_j, since any other point of intersection x can be recoloured with a colour different from c_i, c_j or c_l (see Fig. 5.5), contradicting the fact that v_i and v_j are connected by a path.

To complete the proof, we choose two vertices v_i and v_j that are not adjacent, and let y be the vertex with colour c_j that is adjacent to v_i. If C_{il} is a path (for some $l \neq j$),

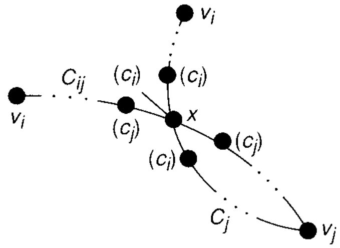


Figure 5.5

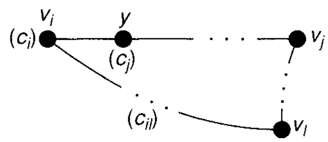


Figure 5.6

then we can interchange the colours of the vertices in this path without affecting the colouring of the rest of the graph (see Fig. 5.6). But if we carry out this interchange, then y would be a vertex common to the paths C_{ij} and C_{jl}, which is a contradiction. This contradiction establishes the theorem. ■

Both of these theorems are useful if all the vertex-degrees are approximately the same. For example, by Theorem 5.1, every cubic graph is 4-colourable, and by Theorem 5.2, every connected cubic graph, other than K_4, is 3-colourable. On the other hand, if the graph has a few vertices of large degree, then these theorems tell us very little. This is well illustrated by the graph $K_{1,s}$; Brooks's theorem tells us that it is s-colourable, but it is in fact 2-colourable for any s. There is no effective way of avoiding this situation, although there are techniques that help a little.

Colouring planar graphs

This rather depressing situation does not arise if we restrict our attention to planar graphs. In fact, we can easily prove the rather strong result that every simple planar graph is 6-colourable.

THEOREM 5.3 *Every simple planar graph is* 6-*colourable.*

Proof. The proof is similar to that of Theorem 5.1. We prove the theorem by induction on the number of vertices, the result being trivial for simple planar graphs with at most six vertices.

Suppose then that G is a simple planar graph with n vertices, and that all simple planar graphs with $n - 1$ vertices are 6-colourable. By Theorem 4.10, G contains a vertex v of degree at most 5. If we delete v and its incident edges, then the graph that remains has $n - 1$ vertices and is thus 6-colourable (see Fig. 5.7). A 6-colouring of G is then obtained by colouring v with a colour different from the (at most five) vertices adjacent to v. ■

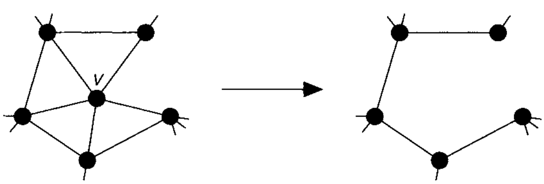


Figure 5.7

As with Theorem 5.1, this result can be strengthened by more careful treatment. The result is called the **five-colour theorem**.

THEOREM 5.4 *Every simple planar graph is* 5*-colourable.*

Proof. The proof is similar to that of Theorem 5.3, although the details are more complicated. We prove the theorem by induction on the number of vertices, the result being trivial for simple planar graphs with fewer than six vertices. Suppose then that G is a simple planar graph with n vertices, and that all simple planar graphs with $n - 1$ vertices are 5-colourable. By Theorem 4.10, G contains a vertex v of degree at most 5. As before, the deletion of v leaves a graph with $n - 1$ vertices, which is thus 5-colourable. Our aim is to colour v with one of the five colours, so completing the 5-colouring of G.

If $\deg(v) < 5$, then v can be coloured with any colour not assumed by the (at most four) vertices adjacent to v, completing the proof in this case. We thus suppose that $\deg(v) = 5$, and that the vertices v_1, v_2, v_3, v_4, v_5 adjacent to v are arranged around v in clockwise order as in Fig. 5.8. If these vertices v_i are all mutually adjacent, then G contains the non-planar graph K_5 as a subgraph, which is impossible. So at least two of them (say, v_1 and v_3) are not adjacent.

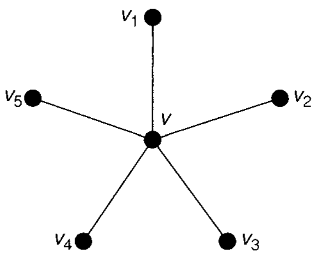


Figure 5.8

We now contract the two edges vv_1 and vv_3. The resulting graph is a planar graph with fewer than n vertices, and is thus 5-colourable. We now reinstate the two edges, giving both v_1 and v_3 the colour originally assigned to v. A 5-colouring of G is then obtained by colouring v with a colour different from the (at most four) colours assigned to the vertices v_i. ■

It is natural to ask whether this result can be strengthened further, and this leads to what was formerly one of the most famous unsolved problems in mathematics – the 'four-colour problem'. This problem, in its usual formulation for maps (see Sections 5.3 and 5.4), was first posed in 1852, and was eventually settled by K. Appel and W. Haken in 1976.

THEOREM 5.5 (Appel and Haken, 1976) *Every simple planar graph is 4-colourable.*

Their proof, which took them several years and a substantial amount of computer time, ultimately depends on a complicated extension of the ideas in the above proof of the five-colour theorem. We return to this topic in Section 5.4.

We conclude this section with a simple application of vertex colourings. Suppose that a chemist wishes to store five chemicals *a, b, c, d* and *e* in various areas of a warehouse. Some of these chemicals react violently when in contact, and so must be kept in separate areas. In the following table, an asterisk indicates those pairs of chemicals that must be separated. How many areas are needed?

	a	*b*	*c*	*d*	*e*
a	–	*	*	*	–
b	*	–	*	*	*
c	*	*	–	*	–
d	*	*	*	–	*
e	–	*	–	*	–

To answer this, we draw the graph whose vertices correspond to the five chemicals, with two vertices adjacent whenever the corresponding chemicals are to be kept apart (see Fig. 5.9). If we now colour the vertices, as shown by the Greek letters, then the colours correspond to the areas needed. In this case, the chromatic number is 4, and so four areas are needed. For example, chemicals *a* and *e* can be stored in area α, and chemicals *b*, *c* and *d* can be stored in areas β, γ and δ, respectively.

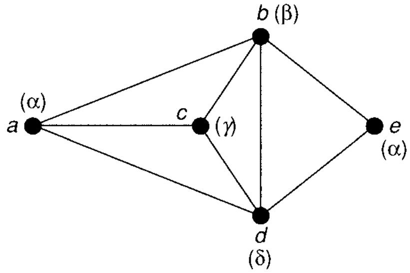


Figure 5.9

Exercises

5.1[s] Find the chromatic number of each graph in Fig. 5.10.

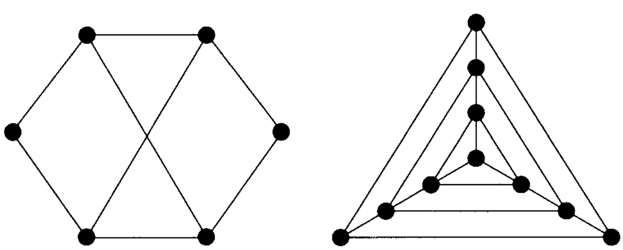


Figure 5.10

5.2 Find the chromatic number of each graph in Fig. 5.11.

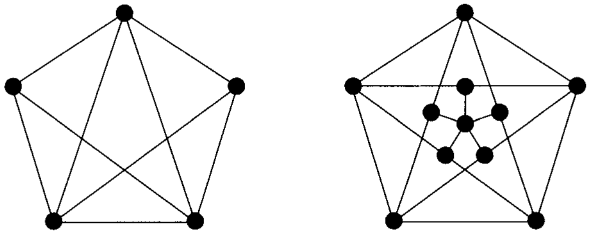


Figure 5.11

5.3[s] In the table of Fig. 1.9, locate all the 2-chromatic, 3-chromatic and 4-chromatic graphs.

5.4 What is the chromatic number of
(i) each of the Platonic graphs?
(ii) the complete tripartite graph $K_{r,s,t}$?
(iii) the k-cube Q_k?

5.5[s] Compare the upper bound for the chromatic number given by Brooks's theorem with the correct value, for
(i) the Petersen graph;
(ii) the k-cube Q_k.

5.6 A lecture timetable is to be drawn up. Since some students wish to attend several lectures, certain lectures must not coincide, as shown by asterisks in the following table. How many periods are needed to timetable all seven lectures?

	a	*b*	*c*	*d*	*e*	*f*	*g*
a	–	*	*	*	–	–	*
b	*	–	*	*	*	–	*
c	*	*	–	*	–	*	–
d	*	*	*	–	–	*	–
e	–	*	–	–	–	–	–
f	–	–	*	*	–	–	*
g	*	*	–	–	–	*	–

5.7[s] Let G be a simple graph with n vertices, which is regular of degree d. By considering the number of vertices that can be assigned the same colour, prove that $\chi(G) \geq n/(n-d)$.

5.8 Let G be a simple planar graph containing no triangles.
(i) Using Euler's formula, show that G contains a vertex of degree at most 3.
(ii) Use induction to deduce that G is 4-colourable.
(In fact, it can be proved that G is 3-colourable.)

5.2 Chromatic polynomials

In this section we continue our study of vertex colourings by associating with each graph a function that tells us, among other things, whether or not the graph is 4-colourable. By investigating this function, we may hope to gain useful information about the four-colour theorem. Without loss of generality, we restrict our attention to simple graphs.

Let G be a simple graph, and let $P_G(k)$ be the number of ways of colouring the vertices of G with k colours so that no two adjacent vertices have the same colour. P_G is called (for the time being) the **chromatic function** of G. For example, if G is the tree shown in Fig. 5.12, then $P_G(k) = k(k-1)^2$, since the middle vertex can be coloured in k ways, and then each end-vertex can be coloured in any of $k-1$ ways. This result can be extended to show that, if G is any tree with n vertices, then $P_G(k) = k(k-1)^{n-1}$. Similarly, if G is the complete graph K_3 in Fig. 5.13, then $P_G(k) = k(k-1)(k-2)$. This can be extended to $P_G(k) = k(k-1)(k-2) \cdots (k-n+1)$, if G is the complete graph K_n.

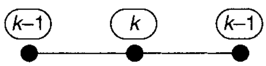


Figure 5.12

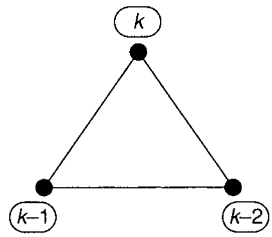


Figure 5.13

Clearly, $P_G(k) = 0$ *if* $k < \chi(G)$, and $P_G(k) > 0$ *if* $k \geq \chi(G)$. Note also that the four-colour theorem is equivalent to the statement:

If G is a simple planar graph, then $P_G(4) > 0$.

If we are given an arbitrary simple graph, it is usually difficult to obtain its chromatic function by inspection. The following theorem and corollary give us a systematic method for obtaining the chromatic function of a simple graph in terms of the chromatic functions of null graphs.

THEOREM 5.6 *Let G be a simple graph, and let G – e and G / e be the graphs obtained from G by deleting and contracting an edge e. Then* $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$.

For example, let G be the graph shown in Fig. 5.14. The corresponding graphs $G-e$ and G/e are shown in Fig. 5.15, and the theorem states that

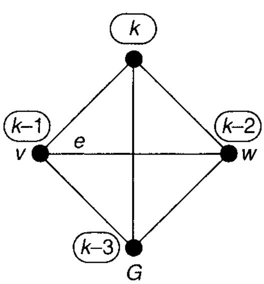


Figure 5.14

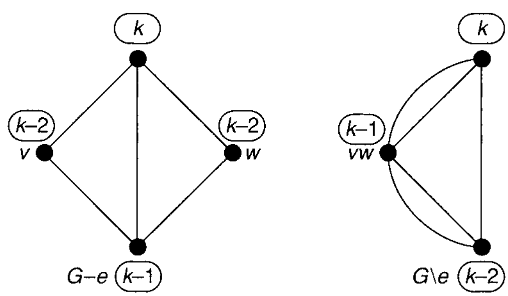


Figure 5.15

$$k(k-1)(k-2)(k-3) = \{k(k-1)(k-2)^2\} - \{k(k-1)(k-2)\}.$$

Proof. Let $e = vw$. The number of k-colourings of $G - e$ in which v and w have *different* colours is unchanged if the edge e is drawn joining v and w, and is therefore equal to $P_G(k)$. Similarly, the number of k-colourings of $G - e$ in which v and w have *the same* colour is unchanged if v and w are identified, and is therefore equal to $P_{G\backslash e}(k)$. The total number $P_{G-e}(k)$ of k-colourings of $G - e$ is therefore $P_G(k) + P_{G\backslash e}(k)$, as required. ■

Using this result, we can prove that the chromatic function of any simple graph is a polynomial.

COROLLARY 5.7 *The chromatic function of a simple graph is a polynomial.*

Proof. We continue the above procedure by choosing an edge in $G - e$ and an edge in $G \backslash e$ and deleting and contracting them. We then repeat the procedure for these four new graphs, and so on. The process terminates when no edges remain – in other words, when each remaining graph is a null graph. Since the chromatic function of a null graph is a polynomial ($= k^r$, where r is the number of vertices), it follows by repeated application of Theorem 5.6 that the chromatic function of the graph G must be a sum of polynomials, and so must itself be a polynomial. ■

A worked example that illustrates this procedure is given later in the section. In practice, we do not need to reduce each graph to a null graph. It is enough to reduce each graph to graphs whose chromatic polynomials we already know, such as trees.

In the light of Corollary 5.7, we can now call $P_G(k)$ the **chromatic polynomial** of G. Note from the above proof that, if G has n vertices, then $P_G(k)$ is of degree n, since no new vertices are introduced at any stage. Since the construction yields only one null graph on n vertices, the coefficient of k^n is 1. We can also prove by induction (see Exercise 5.15) that the coefficients alternate in sign, and that the coefficient of k^{n-1} is $-m$, where m is the number of edges of G. Since we cannot colour a graph if no colours are available, the constant term of any chromatic polynomial is 0.

We now give an example to illustrate the above ideas. We use Theorem 5.6 to find the chromatic polynomial of the graph G of Fig. 5.16 and then verify that this polynomial has the form

$$k^5 - 7k^4 + ak^3 - bk^2 + ck,$$

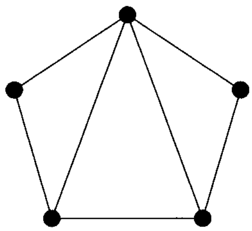


Figure 5.16

where a, b and c are positive constants, as predicted in the previous paragraph. It is convenient at each stage to draw the graph itself, rather than write down its chromatic polynomial. For example, instead of writing $P_G(k) = P_{G-e}(k) - P_{G\setminus e}(k)$, where G, $G - e$ and $G \setminus e$ are the graphs of Figs 5.14 and 5.15, we write down the 'equation' in Fig. 5.17.

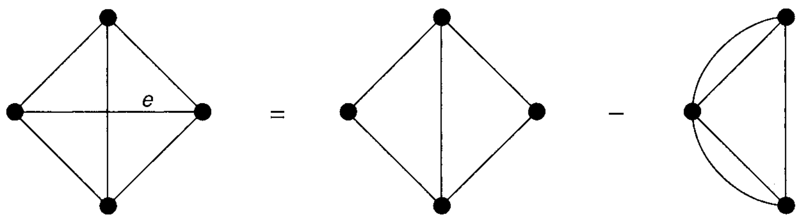
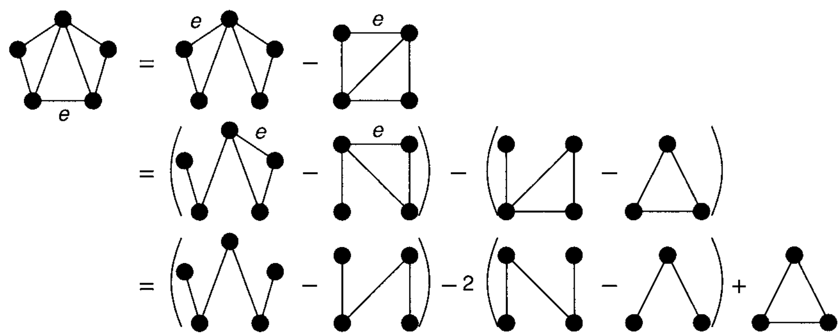


Figure 5.17

With this convention, and ignoring multiple edges as we proceed, we have



Thus

$$P_G(k) = k(k-1)^4 - 3k(k-1)^3 + 2k(k-1)^2 + k(k-1)(k-2)$$
$$= k^5 - 7k^4 + 18k^3 - 20k^2 + 8k.$$

Note that this result has the required form $k^5 - 7k^4 + ak^3 - bk^2 + ck$, where a, b and c are positive constants.

We conclude this section by recalling from Exercise 5.6 how vertex colourings can be applied to timetabling problems. Suppose that we wish to arrange the times at which certain lectures are to be given. Some pairs of lectures cannot be given at the same time, since there may be students who wish to attend both. In order to construct a timetable, we construct a graph whose vertices correspond to the lectures and whose edges join pairs of lectures that cannot be scheduled at the same time. If we associate a colour with each time available for lectures, then a colouring of the vertices corresponds to a timetabling of the lectures. The chromatic number of the graph tells us the number of lecture periods needed, and the chromatic polynomial tells us how many ways there are of timetabling the lectures.

Exercises

5.9ˢ Write down the chromatic polynomials of
(i) the complete graph K_6;
(ii) the complete bipartite graph $K_{1,5}$.
In how many ways can these graphs be coloured with seven colours?

5.10 Write down the chromatic polynomials of
(i) the complete graph K_7;
(ii) the complete bipartite graph $K_{1,6}$.
In how many ways can these graphs be coloured with ten colours?

5.11 (i) Find the chromatic polynomials of the six connected simple graphs on four vertices.
(ii) Verify that each of the polynomials in part (i) has the form

$$k^4 - mk^3 + ak^2 - bk,$$

where m is the number of edges, and a and b are positive constants.

5.12ˢ Find the chromatic polynomials of
(i) the complete bipartite graph $K_{2,5}$;
(ii) the cycle graph C_5.

5.13 (i) Prove that the chromatic polynomial of $K_{2,s}$ is

$$k(k-1)^s + k(k-1)(k-2)^s.$$

(ii) Prove that the chromatic polynomial of C_n is

$$(k-1)^n + (-1)^n(k-1).$$

5.14 Prove that, if G is a disconnected simple graph, then its chromatic polynomial $P_G(k)$ is the product of the chromatic polynomials of its components. What can you say about the degree of the lowest non-vanishing term?

5.15 Let G be a simple graph with n vertices and m edges. Use induction on m, together with Theorem 5.6, to prove that
(i) the coefficient of k^{n-1} is $-m$;
(ii) the coefficients of $P_G(k)$ alternate in sign.

5.16ˢ (i) Use the results of Exercises 5.14 and 5.15 to prove that, if

$$P_G(k) = k(k-1)^n,$$

then G is a tree on n vertices.
(ii) Find three graphs with chromatic polynomial

$$k^5 - 4k^4 + 6k^3 - 4k^2 + k.$$

5.3 Colouring maps

The four-colour problem arose historically in connection with the colouring of maps. Given a map with several countries, we may ask how many colours are needed to colour them so that no two countries with a common boundary line share the same

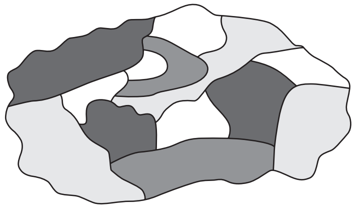


Figure 5.18

colour; for example, Fig. 5.18 shows a map that has been coloured with four colours. Although it may seem that the more complicated a map becomes, the more colours are needed to colour it, this is not the case. In fact:

Every map can be coloured with only four colours.

In order to make this statement precise, we must explain what we mean by a ‘map’. Since the two colours on either side of an edge must be different, we need to exclude maps containing a bridge (see Fig. 5.19). We may also exclude vertices of degree 2, as they can easily be eliminated (see Fig. 5.20). To cover these, and similar cases, we define a **map** to be a 3-connected plane graph; thus a map contains no cutsets with one or two edges, and in particular no vertices of degree 1 or 2. As we shall see, these exclusions correspond to our exclusion of loops and multiple edges in Section 5.1.

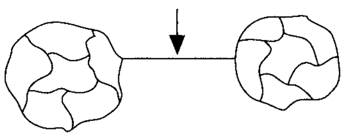


Figure 5.19

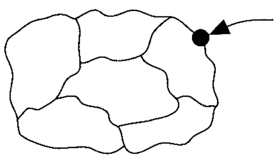


Figure 5.20

We now define a map to be ***k*-colourable-(f)** if its faces can be coloured with k colours so that no two faces with a boundary edge in common have the same colour. To avoid confusion, we use ***k*-colourable-(v)** to mean k-colourable in the usual sense. For example, the map in Fig. 5.21 is 3-colourable-(f) and 4-colourable-(v). The four-colour theorem for maps is thus the assertion that every map is 4-colourable-(f).

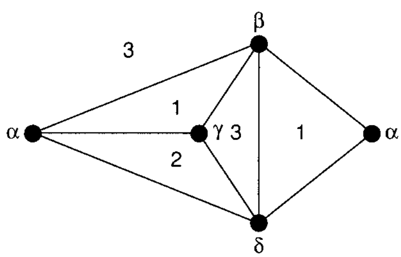


Figure 5.21

In Corollary 5.10 we prove the equivalence of the two forms of the four-colour theorem. But first we investigate the condition under which a map can be coloured with two colours. This condition takes a particularly simple form.

THEOREM 5.8 *A map G is 2-colourable-(f) if and only if G is an Eulerian graph.*

First proof. $\Rightarrow$ For each vertex v of G, the faces surrounding v must be even in number, since they can be coloured with two colours. It follows that each vertex has even degree and so, by Theorem 2.9, G is Eulerian.

$\Leftarrow$ If G is Eulerian, we colour its faces in two colours as follows. Choose any face F and colour it red. Draw a curve from a point x in F to a point in each other face, passing through no vertex of G. If such a curve crosses an even number of edges, colour the face red; otherwise, colour it blue (see Fig. 5.22).

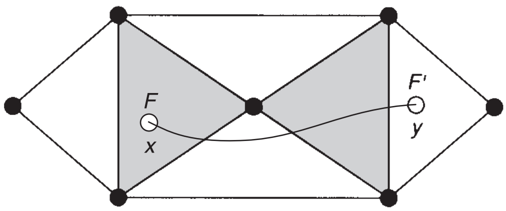


Figure 5.22

This colouring is well defined, as we can see by taking a ‘cycle’ of two such curves and proving that it crosses an even number of edges of G, using the fact that each vertex has an even number of edges incident with it. ■

A simpler proof of Theorem 5.8 involves translating the situation into one of colouring the vertices of the dual graph. We first justify this procedure, and then illustrate it by giving our alternative proof of Theorem 5.8 and by proving the equivalence of the two forms of the four-colour theorem.

THEOREM 5.9 *Let G be a plane graph without loops, and let G^* be a geometric dual of G. Then G is k-colourable-(v) if and only if G^* is k-colourable-(f).*

Proof. $\Rightarrow$ We can assume that G is simple and connected, so that G^* is a map. If we have a k-colouring of the vertices of G, then we can k-colour the faces of G^* so that each face inherits the colour of the unique vertex that it contains (see Fig. 5.23). No two adjacent faces of G^* can have the same colour because the vertices of G that they contain are adjacent in G and so must be differently coloured. Thus G^* is k-colourable-(f).

$\Leftarrow$ Suppose now that we have a k-colouring of the faces of G^*. Then we can k-colour the vertices of G so that each vertex inherits the colour of the face containing it. No two adjacent vertices of G have the same colour, by reasoning similar to the above. Thus G is k-colourable-(v). ■

It follows that we can dualize any theorem on the colouring of the vertices of a planar graph to give a theorem on the colouring of the faces of a map, and conversely. As an example of this, consider Theorem 5.8 again.

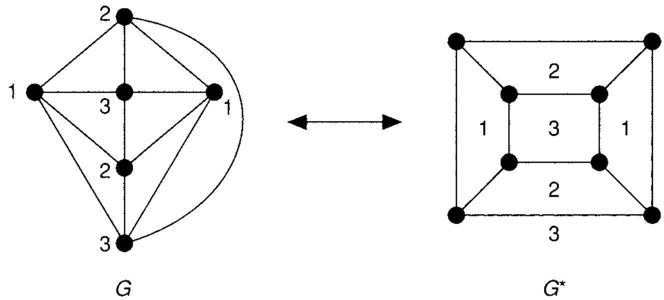


Figure 5.23

THEOREM 5.8 *A map G is 2-colourable-(f) if and only if G is an Eulerian graph.*

Second proof. By the result of Exercise 4.31, the dual of an Eulerian planar graph is a bipartite planar graph, and conversely. It is therefore sufficient to observe that a connected planar graph without loops is 2-colourable-(v) if and only if it is bipartite. ■

We can similarly prove the equivalence of the two forms of the four-colour theorem.

COROLLARY 5.10 *The four-colour theorem for maps is equivalent to the four-colour theorem for planar graphs.*

Proof. $\Rightarrow$ We may assume that G is a simple connected plane graph. Then its geometric dual G^* is a map and by assumption is 4-colourable-(f). It follows immediately from Theorem 5.9 that G is 4-colourable-(v).

$\Leftarrow$ Conversely, let G be a map and let G^* be its geometric dual. Then G^* is a simple planar graph and by assumption is 4-colourable-(v). It follows immediately from Theorem 5.9 that G is 4-colourable-(f). ■

Duality can also be used to prove the following theorem.

THEOREM 5.11 *Let G be a cubic map. Then G is 3-colourable-(f) if and only if each face is bounded by an even number of edges.*

Proof. $\Rightarrow$ Given any face F of G, the faces of G that surround F must alternate in colour. So there must be an even number of them, and so each face is bounded by an even number of edges (see Fig. 5.24).

$\Leftarrow$ We prove the dual result, that if G is a simple connected plane graph in which each face is a triangle and each vertex has even degree (that is, G is Eulerian), then G is 3-colourable-(v). We shall denote the three colours by α, β and γ.

Since G is Eulerian, it follows from Theorem 5.8 that its faces can be coloured with two colours, red and blue. The required 3-colouring of the vertices of G is then

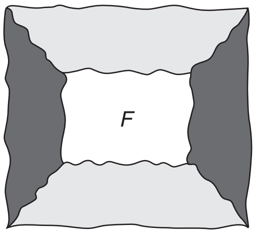


Figure 5.24

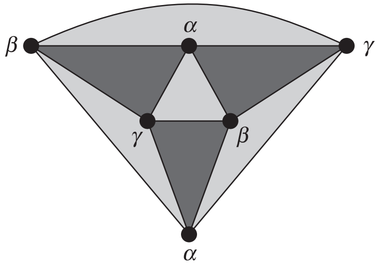


Figure 5.25

obtained by colouring the vertices of any red face so that the colours α, β and γ appear in clockwise order, and colouring the vertices of any blue face so that these colours appear in anti-clockwise order (see Fig. 5.25). This vertex colouring can be extended to the whole graph, thus proving the theorem. ■

In the above theorem, we assumed that the map is cubic. This need not be a severe restriction, as the following theorem shows.

THEOREM 5.12 *In order to prove the four-colour theorem, it is sufficient to prove that every cubic map is* 4*-colourable-*(*f*).

Proof. By Corollary 5.10, it is sufficient to prove that the 4-colourability of the faces of every cubic map implies the 4-colourability of the faces of any map.

Let *G* be any map. If *G* has any vertices of degree 2, then we can remove them without affecting the colouring. It remains only to eliminate vertices of degree 4 or more. But if *v* is such a vertex, then we can stick a ‘patch’ over *v*, as in Fig. 5.26. Repeating this for all such vertices, we obtain a cubic map that is 4-colourable-(f) by hypothesis. The required 4-colouring of the faces of *G* is then obtained by shrinking each patch to a single vertex and reinstating each vertex of degree 2.

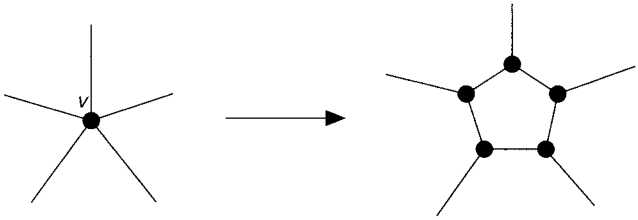


Figure 5.26

Exercises

5.17^s How many colours are needed to colour the countries of the maps in Fig. 5.27?

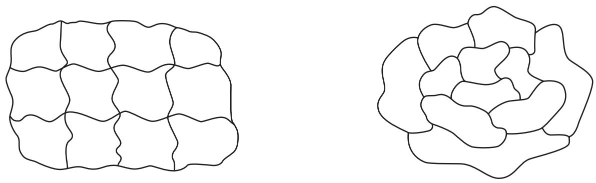


Figure 5.27

5.18 How many colours are needed to colour the countries of the maps in Fig. 5.28?

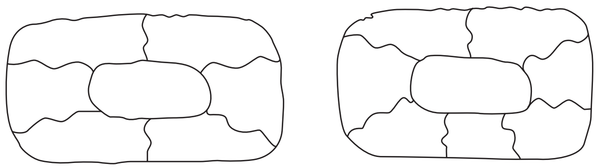


Figure 5.28

5.19 Consider the map in Fig. 5.29, in which the countries are to be coloured red, blue, green and yellow.

(i) Show that country A must be red.

(ii) What colour is country B?

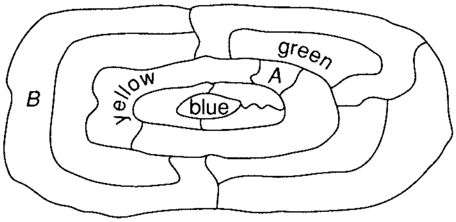


Figure 5.29

5.20^s Find the minimum number of colours needed to colour the faces of each of the Platonic graphs, so that adjacent faces are coloured differently.

5.21^s Give an example of a plane graph that is both 2-colourable-(f) and 2-colourable-(v).

5.22 The plane is divided into a finite number of regions by drawing infinite straight lines in an arbitrary manner. Show that these regions can be 2-coloured.

5.23^s By dualizing the proof of Theorem 5.3, prove the six-colour theorem for maps.

5.24 Let G be a simple plane graph with fewer than 12 faces, and suppose that each vertex of G has degree at least 3.

(i) Use Exercise 4.17 to prove that G is 4-colourable-(v).

(ii) Dualize the result of part (i).

5.4 The four-colour theorem

The four-colour problem was first posed in 1852 by a London student, Francis Guthrie, who thought of it while colouring a map of the counties of England. The problem remained dormant for several years until 1879, when Alfred Kempe, a London barrister, produced a 'proof' which for ten years was accepted as correct. But in 1890, Percy Heawood of Durham pointed out Kempe's error, salvaged enough of Kempe's argument to deduce the five-colour theorem, and generalized the problem to other surfaces. Not until 1976 was the four-colour theorem proved, by Kenneth Appel and Wolfgang Haken of the University of Illinois. Details of the proof can be found in Wilson [46].

Although Kempe's argument was faulty, it contained two crucial ideas that featured in the eventual proof: an **unavoidable set of configurations** and a **reducible configuration**.

Unavoidable sets

In Exercise 4.18(iii) we saw that *every* cubic map must contain at least one face bounded by three, four or five sides (see Fig. 5.30) – that is, a triangle, quadrilateral or pentagon. We call such a collection of faces an **unavoidable set of configurations**, because it cannot be avoided.

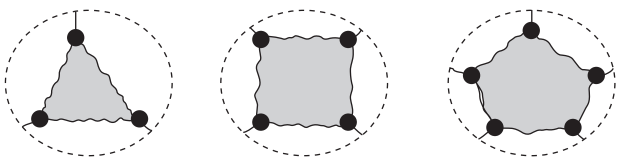


Figure 5.30

Another unavoidable set was obtained by Paul Wernicke in 1904, as shown in Fig. 5.31. He showed that every map that contains no triangle or quadrilateral must contain either two adjacent pentagons or a pentagon adjacent to a hexagon.

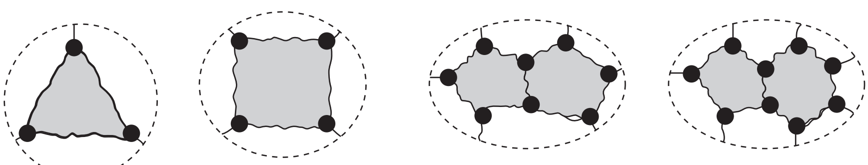


Figure 5.31

THEOREM 5.13 *The configurations in Fig. 5.31 form an unavoidable set.*

Proof. We assume that there exists a cubic map that contains none of these configurations, and derive a contradiction. Since no pentagon can be adjacent to a triangle or quadrilateral (because there are none), or to another pentagon or a hexagon, every pentagon can be adjacent only to faces bounded by seven edges or more.

We now assign to each face a number which we can think of as an 'electrical charge': to each face with k boundary edges, we assign a charge of $6 - k$, so that each pentagon receives a charge of 1, each hexagon receives zero charge, and each of the other faces receives negative charge. Note that if the map has C_5 pentagons, C_6 hexagons, C_7 heptagons, and so on, then the total charge on the map is

$$(1 \times C_5) + (0 \times C_6) + (-1 \times C_7) + (-2 \times C_8) - \ldots$$

But, by the result of Exercise 4.18(i) with $C_3 = C_4 = 0$,

$$C_5 - C_7 - 2C_8 - 3C_9 - \ldots = 12.$$

Thus, the total charge on the map is 12.

We now redistribute the charges around the map in such a way that no charge is created or destroyed. To do this, we transfer *one-fifth* of a unit of charge from each pentagon to each of its five negatively charged neighbours: recall that each such neighbour is bounded by seven edges or more. Then the total charge on the map remains 12, but each pentagon now has zero charge and each hexagon retains zero charge.

Note that if a heptagon (with initial charge -1) received enough contributions of $^1/_5$ to acquire positive charge, it would have at least six neighbouring pentagons – but two of these pentagons would then have to be adjacent, which is disallowed. Thus, after the redistribution, no heptagon can acquire positive charge. Similarly, if an octagon (with initial charge -2) received enough contributions of $^1/_5$ to acquire positive charge, it would have at least eleven neighbouring pentagons, which is impossible. Thus, after the redistribution, no octagon can acquire positive charge – and similarly, nor can each nonagon, decagon, ...

Thus, after the redistribution, no face can have positive charge, contradicting our assertion that the total charge on the map is 12. This contradiction proves that every cubic map must contain a triangle, a square, two adjacent pentagons, or a pentagon adjoining a hexagon – and so these four configurations form an unavoidable set. ■

Reducible configurations

If the four-colour theorem were false, then there would exist cubic maps that require five or more colours, and of these there would be at least one with a minimum number of faces. Such a map M (sometimes called a *minimal counter-example*) would require at least five colours, while any map with fewer faces than M could be face-coloured with four colours.

We note that such a map M cannot contain a triangle T. For if it did, we could remove a boundary edge from T and merge T with one of its former neighbours (see Fig. 5.32), giving a map with fewer faces. By our assumption, we can face-colour this new map with four colours.

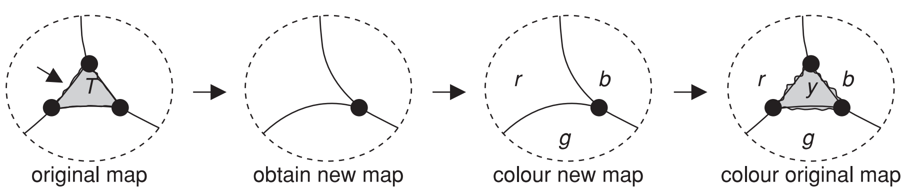


Figure 5.32

We now reinstate T by replacing the removed edge. Since the countries adjacent to T use only three of the four available colours, there must be a spare colour for T. Thus, we can colour M with four colours, which contradicts our assumption. This shows that M cannot contain a triangle.

We next show similarly that M cannot contain a quadrilateral Q. For if it did, we could remove a boundary edge and merge Q with one of its former neighbours (see Fig. 5.33), giving a map with fewer faces. By our assumption, we can colour this new map with four colours.

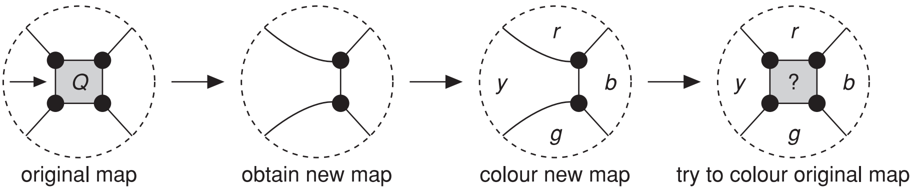


Figure 5.33

We now reinstate Q by replacing the removed line. But in this case the countries next to Q may already use all four colours, and so no spare colour is available for colouring Q. Thus, the proof cannot proceed as before.

To overcome this difficulty, Kempe introduced a method now known as the **method of Kempe chains**. In his method, we choose two of the colours surrounding Q that are not adjacent – say, red and green – and consider only those faces that are coloured with these colours. Each red or green face neighbouring Q is the starting point for a red-green part of the map – that is, a part of the map consisting entirely of countries coloured red or green. Two cases now arise, depending on whether these two red-green parts are separate from each other, or link up (see Fig. 5.34).

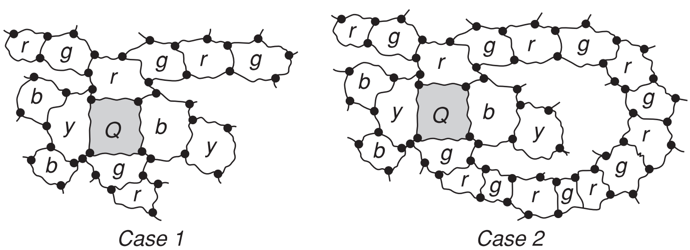


Figure 5.34

Case 1. In the first case, the red and green faces above Q that can be reached from the red neighbour of Q do not link up with the red and green faces below Q that can be reached from the green neighbour of Q. We can therefore interchange the colours of the red and green faces above Q, as shown in Fig. 5.35, without affecting the colouring of those below Q. The quadrilateral Q is then surrounded only by the colours green, blue and yellow, so that Q can be coloured red. This completes the colouring of M in this case.

Case 2. The second case, where the red-green part above Q does link up with the red-green part below Q, is a little more difficult, since nothing is gained by interchanging

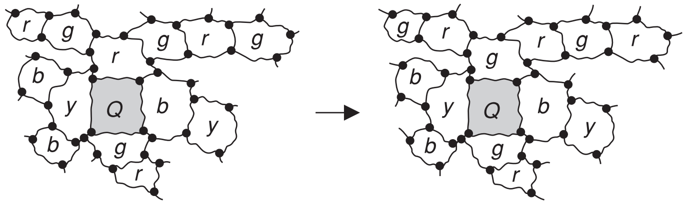


Figure 5.35

the colours red and green: the red neighbour of Q becomes green and the green neighbour of Q becomes red, and we are no better off than before.

In this case, we turn our attention to the blue-yellow parts of the map to the left and right of Q. Here, the blue-yellow part on the right of Q is cut off from the blue-yellow part on the left of Q, because the chain of red and green countries 'gets in the way', as illustrated in Fig. 5.36. Thus, we can interchange the colours of the blue and yellow faces on the right of Q without affecting the colouring of those on the left of Q. The quadrilateral Q is then surrounded only by the colours yellow, red and green, so that Q can be coloured blue. This completes the colouring of M in this case.

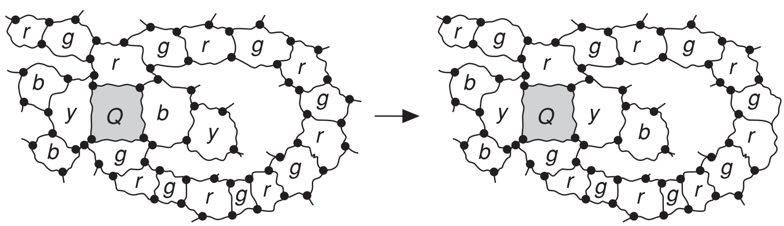


Figure 5.36

Thus in each case we can colour M with four colours, which contradicts our assumption. This shows that M cannot contain a quadrilateral.

We say that a configuration of faces is **reducible** if a 4-colouring of all the other faces can be extended (directly or after interchanging colours) to include the configuration; for example, as we have seen, a triangle and a quadrilateral are reducible configurations. Note that no reducible configuration can appear in a minimal counterexample to the four-colour theorem.

Another reducible configuration is the arrangement of four pentagons in Fig. 5.37. It is known as the **Birkhoff diamond**, after the American mathematician George Birkhoff. The following proof can be omitted if desired.

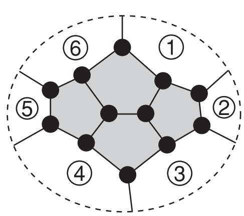


Figure 5.37

THEOREM 5.14 *The Birkhoff diamond is reducible.*

Sketch of proof. Suppose that we have a cubic map M that contains the Birkhoff diamond D. Removing D yields a new map with fewer faces. We assume that this new map can be face-coloured with four colours, and try to extend the colouring to the pentagons in D.

If the faces in the outer ring surrounding D are numbered 1, 2, 3, 4, 5 and 6, as shown in Fig. 5.37, then there are essentially 31 different ways in which they can be coloured with the four colours *red*, *green*, *blue* and *yellow*. These colourings of the six faces are as follows (the reason for the asterisks is given below):

rgrgrg	*rgrbrg**	*rgrbgy**	*rgbrgy*	*rgbryb*	*rgbgbg**	*rgbyrg*	*rgbygy**
*rgrgrb**	*rgrbrb*	*rgrbyg**	*rgbrbg**	*rgbgrg**	*rgbgby*	*rgbyrb*	*rgbybg**
rgrgbg	*rgrbry*	*rgrbyb**	*rgbrby*	*rgbgrb**	*rgbgyg*	*rgbyry**	*rgbyby**
*rgrgby**	*rgrbgb**	*rgbrgb*	*rgbryg*	*rgbgry**	*rgbgyb*	*rgbygb*	

Notice that we cannot include colourings such as *rgygbr* in which the two faces coloured *red* appear together, and we have also omitted colourings such as *rgrgry*, since it is essentially the same colouring as *rgrgrb* (on recolouring the final *yellow* as *blue*).

Consider the colouring *rgrgrb*. This can be extended directly to D, as shown in Fig. 5.38, and is called a 'good colouring'. In the same way, all the asterisked colourings above are good colourings.

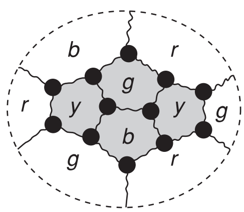


Figure 5.38

The colouring *rgrbrb* is not a good colouring, but by using Kempe-changes of colour that interchange the colours *red* and *yellow*, or *green* and *blue*, we can convert it into one of the good colourings *rgrgrb*, *rgrbrg* or *rgrbyb*. For example, if there is a *red-yellow* chain connecting countries 3 and 5, then we can interchange the colours in the *blue-green* chain from country 4 so as to recolour country 4 *green*. Similarly, if there is a *red-yellow* chain connecting countries 1 and 5, then we can interchange the colours in the *blue-green* chain from country 6 so as to recolour country 6 *green*. But if there is no *red-yellow* chain connecting countries 3 and 5, or 1 and 5, then we can interchange the colours in the *red-yellow* chain from country 5 so as to recolour country 5 *yellow*. (These three situations are illustrated in Fig. 5.39.) Thus, the colouring *rgrbrb* can be converted into a good colouring.

By using similar arguments, we can show that each of the 31 possible colourings of the outer ring either is a good colouring, or can be converted into a good colouring

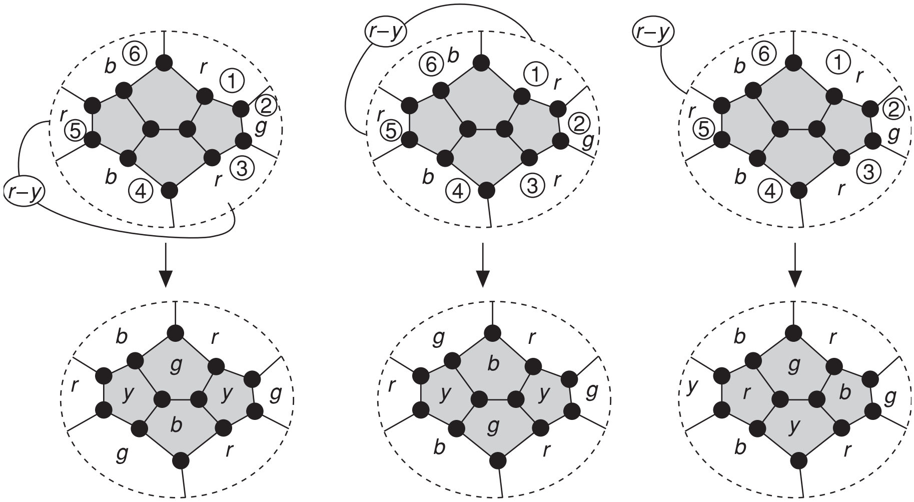


Figure 5.39

by suitable interchanges of colour. Thus, all 31 colourings of the ring can be extended to the diamond, and so the Birkhoff diamond is reducible. ■

We now note that:

In order to prove the four-colour theorem, it is sufficient to find an unavoidable set of reducible configurations.

This is because every map must contain at least one of these configurations (since the set is unavoidable), and whichever one it is, any colouring of the rest of the map can be extended to the configuration (since the configuration is reducible). Expressed in another way, every map must contain at least one of these configurations, and yet none of them can appear in a minimal counter-example to the four-colour theorem. Thus, no such counter-example can exist and so the four-colour theorem is true.

During the twentieth century many thousands of reducible configurations were discovered, and graph theorists spent a great deal of time trying to package them into an unavoidable set whose existence would prove the four-colour theorem. Eventually, Appel and Haken succeeded, but their approach was different: they started by constructing unavoidable sets of configurations that seemed likely to be reducible. They then tested each configuration for reducibility by programming a computer to check all possible colourings of the surrounding ring of faces, and replacing configurations that proved not to be reducible. This strategy was eventually successful and yielded an unavoidable set of 1482 configurations, thereby proving the four-colour theorem.

5.5 Colouring edges

In this section we colour the edges of a graph. As we shall see, the four-colour theorem for planar graphs is equivalent to a theorem concerning edge colourings of cubic maps.

A graph G is ***k*-colourable-(e)** (or k-edge colourable) if its edges can be coloured with k colours so that no two adjacent edges have the same colour. If G is k-colourable-(e) but not $(k-1)$-colourable-(e), we say that the **chromatic index** of G is k, and write $\chi'(G) = k$. For example, Fig. 5.40 shows a graph G for which $\chi'(G) = 4$.

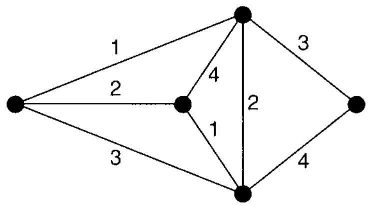


Figure 5.40

Note that, if Δ is the largest vertex degree of G, then $\chi'(G) \geq \Delta$. The following result, known as **Vizing's theorem**, gives very sharp bounds for the chromatic index of a simple graph G. Its proof can be found in West [16] or Fiorini and Wilson [35].

THEOREM 5.15 (Vizing, 1964) *If G is a simple graph with largest vertex-degree Δ, then*

$$\Delta \leq \chi'(G) \leq \Delta + 1.$$

It is not known which graphs have chromatic index Δ and which have chromatic index $\Delta + 1$. However, the results for particular types of graphs can easily be found. For example, $\chi'(C_n) = 2$ or 3, depending on whether n is even or odd, and $\chi'(W_n) = n - 1$ if $n \geq 4$.

We now determine the corresponding results for complete graphs.

THEOREM 5.16 $\chi'(K_n) = n$ *if n is odd* $(n \geq 3)$, *and* $\chi'(K_n) = n - 1$ *if n is even.*

Proof. The result is trivial if $n = 2$. We therefore assume that $n \geq 3$.

If n is odd, then we can n-colour the edges of K_n by placing the vertices of K_n in the form of a regular n-gon, colouring the edges around the boundary with a different colour for each edge, and then colouring each remaining edge with the colour used for the boundary edge parallel to it (see Fig. 5.41). The fact that K_n is not $(n-1)$-

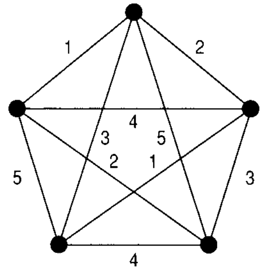


Figure 5.41

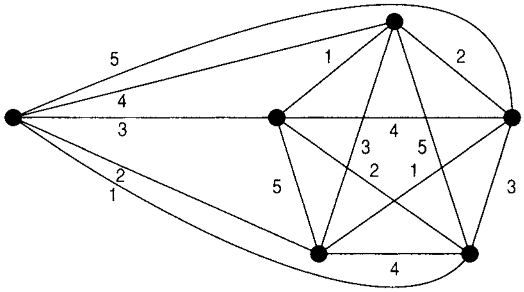


Figure 5.42

colourable-(e) follows by observing that the largest possible number of edges of the same colour is $\frac{1}{2}(n-1)$, and so K_n has at most $\frac{1}{2}(n-1) \times \chi'(K_n)$ edges.

If n is even, then we first obtain K_n by joining the complete graph K_{n-1} to a single vertex. If we now colour the edges of K_{n-1} as above, then there is one colour missing at each vertex, and these missing colours are all different. We complete the edge colouring of K_n by colouring the remaining edges with these missing colours (see Fig. 5.42). ■

We now show the connection between the four-colour theorem and the colouring of the edges of a graph. This connection accounts for much of the interest in edge colourings.

THEOREM 5.17 *The four-colour theorem is equivalent to the statement that* $\chi'(G) = 3$ *for each cubic map G.*

Proof. ⇒ Suppose that we have a 4-colouring of the faces of G, where the colours are denoted by $\alpha = (1, 0)$, $\beta = (0, 1)$, $\gamma = (1, 1)$ and $\delta = (0, 0)$. We can then construct a 3-colouring of the edges of G by colouring each edge e with the colour obtained by adding together (modulo 2) the colours of the two faces adjoining e; for example, if e adjoins two faces coloured α and γ, then e is coloured β, since $(1, 0) + (1, 1) = (0, 1)$. Note that the colour δ cannot occur in this edge colouring, since the two faces adjoining each edge must be distinct. Moreover, no two adjacent edges can share the same colour. We thus have the required edge colouring with three colours (see Fig. 5.43).

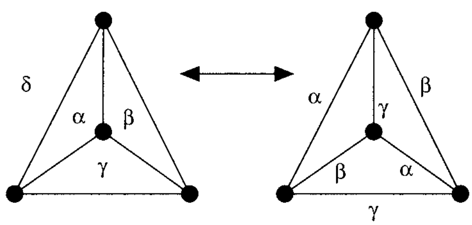


Figure 5.43

⇐ Suppose now that we have a 3-colouring of the edges of G. Then there is an edge of each colour at each vertex. The subgraph determined by those edges coloured α or β is regular of degree 2, and so, by an obvious extension of Theorem 5.8 to disconnected graphs, we can colour its faces with two colours, 0 and 1. In a similar way, we can colour the faces of the subgraph determined by those edges coloured α or γ with the colours 0 and 1. Thus, we can assign to each face of G two coordinates (x, y), where each of x and y is 0 or 1. Since the coordinates assigned to two adjacent faces of G must differ in at least one place, these coordinates (1, 0), (0, 1), (1, 1), (0, 0) give the required 4-colouring of the faces of G. ■

König's theorem

We conclude this section with a theorem of Dénes König on the chromatic index of a bipartite graph. The method of proof is similar to that of Brooks's theorem in Section 5.1 – we consider a two-coloured subgraph H_{ij} and interchange the colours.

THEOREM 5.18 (König, 1916) *If G is a bipartite graph with largest vertex-degree Δ, then $\chi'(G) = \Delta$.*

Proof. We use induction on the number of edges of G, and prove that if all but one of the edges have been coloured with at most Δ colours, then there is a Δ-colouring of the edges of G.

So suppose that each edge of G has been coloured, except for the edge vw. Then there is at least one colour missing at the vertex v, and at least one colour missing at the vertex w. If some colour is missing from both v and w, then we colour the edge vw with this colour. If this is not the case, then let α be a colour missing at v, and β be a colour missing at w, and let $H_{\alpha\beta}$ be the connected subgraph of G consisting of the vertex v and those edges and vertices of G that can be reached from v by a path consisting entirely of edges coloured α or β (see Fig. 5.44).

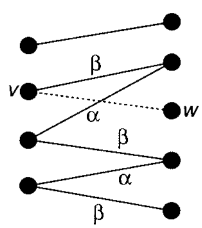


Figure 5.44

Since G is bipartite, the subgraph $H_{\alpha\beta}$ cannot contain the vertex w, and so we can interchange the colours α and β in this subgraph without affecting w or the rest of the colouring. The edge vw can now be coloured β, thereby completing the colouring of the edges of G. ■

COROLLARY 5.19

$$\chi'(K_{r,s}) = \max(r, s).$$

Exercises

5.25[s] Find the chromatic index of the graph in Fig. 5.45.

5.26 Find the chromatic index of the graph in Fig. 5.46.

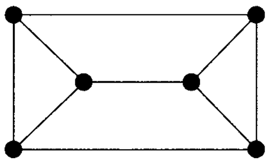


Figure 5.45

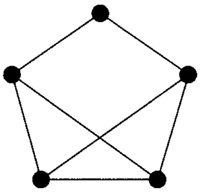


Figure 5.46

5.27[s] In the table of Fig. 1.9, locate all the graphs with chromatic index 2 and 3.

5.28[s] Compare the lower and upper bounds for the chromatic index given by Vizing's theorem with the correct value, for
(i) the cycle graph C_7;
(ii) the complete graph K_8;
(iii) the complete bipartite graph $K_{4,6}$.

5.29 What is the chromatic index of each of the Platonic graphs?

5.30[s] By exhibiting an explicit colouring for the edges of $K_{r,s}$, give an alternative proof of Corollary 5.19.

5.31[s] Prove that if G is a cubic Hamiltonian graph, then $\chi'(G) = 3$.

5.32 (i) By considering the possible 3-colourings of the outer 5-cycle, prove that the Petersen graph has chromatic index 4.
(ii) Using part (i) and Exercise 5.31, deduce that the Petersen graph is non-Hamiltonian.

Challenge problems

5.33 A graph G is ***k*-critical** if $\chi(G) = k$ and if the deletion of any vertex yields a graph with smaller chromatic number.
(i) Find all 2-critical and 3-critical graphs.
(ii) Give an example of a 4-critical graph.
(iii) Prove that, if G is k-critical, then
(a) every vertex of G has degree at least $k - 1$;
(b) G has no cut-vertices.

5.34 Generalize the results of Exercise 5.8 to the cases where
(i) G has girth r;
(ii) G has thickness t.

5.35 Try to prove the four-colour theorem by adapting the proof of the five-colour theorem in Section 5.1. At what point does the proof fail?

5.36 Let G be a countable graph, each finite subgraph of which is k-colourable.
(i) Use König's lemma (Theorem 2.7) to prove that G is k-colourable.
(ii) Deduce that every countable planar graph is 4-colourable.

5.37 (i) Let G be a simple graph which is not a null graph. Prove that $\chi'(G) = \chi(L(G))$, where $L(G)$ is the line graph of G.
(ii) By combining part (i) with Brooks's theorem, prove Vizing's theorem in the case $\Delta = 3$.

5.38 By dualizing the proof of Theorem 5.4, prove the five-colour theorem for maps.

5.39 (i) Prove that, if a toroidal graph is embedded on the surface of a torus, then its faces can be coloured with seven colours.
(ii) Find a toroidal graph whose faces cannot be coloured with six colours.

5.40 The map in Fig. 5.47 shown 16 countries numbered from 0 to 15. Each country other than country 0 has been coloured red, blue, green or yellow.
(i) Write down the numbers of the countries in the blue-green and blue-yellow parts of the map neighbouring country 0, and explain why interchanging the colours in just one of these parts does not yield a 4-colouring of the map.
(ii) By interchanging the colours of countries 1 and 2 and recolouring countries 3 and 5, find a 4-colouring of the map.

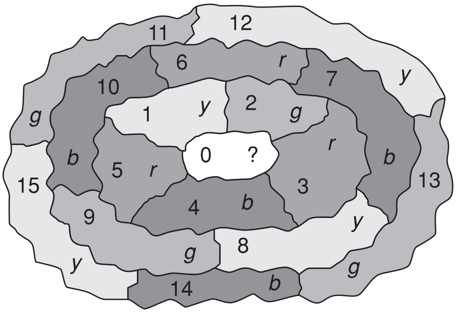


Figure 5.47

5.41 Let G be a simple graph with an odd number of vertices. Prove that if G is regular of degree Δ, then $\chi'(G) = \Delta + 1$.

Chapter 6

Matching, marriage and Menger's theorem

They drew all manner of things—
everything that begins with an M—.
Lewis Carroll

The results of this chapter are more combinatorial than those of the preceding chapters, although we shall see that they are closely related to graph theory. We begin with a discussion of Hall's 'marriage' theorem in various different contexts. In Section 6.2 we prove Menger's theorem (introduced in Section 2.1) on the number of disjoint paths connecting a given pair of vertices in a graph or digraph, and in Section 6.3 we give an alternative formulation of Menger's theorem, known as the *max-flow min-cut theorem*, which is of fundamental importance in connection with network flow problems. Further details can be found in Lovász and Plummer [39] and Dolan and Aldous [22].

6.1 Hall's 'marriage' theorem

The marriage theorem, proved in 1935 by Philip Hall, answers the following question, known as the **marriage problem**:

> *If there is a finite set of girls, each of whom knows several boys, under what conditions can all the girls marry boys in such a way that each girl marries a boy that she knows?*

For example, if there are four girls $\{g_1, g_2, g_3, g_4\}$ and five boys $\{b_1, b_2, b_3, b_4, b_5\}$, and the friendships are as shown below, then a possible solution is for g_1 to marry b_4, g_2 to marry b_1, g_3 to marry b_3, and g_4 to marry b_2:

girl	*boys known by girl*		
g_1	b_1	b_4	b_5
g_2	b_1		
g_3	b_2	b_3	b_4
g_4	b_2	b_4	

This problem can be represented graphically by taking G to be the bipartite graph in which the vertex-set is divided into two disjoint sets V_1 and V_2, corresponding to the girls and the boys, with each edge joining a girl to a boy she knows. Figure 6.1 shows the graph G corresponding to the above situation.

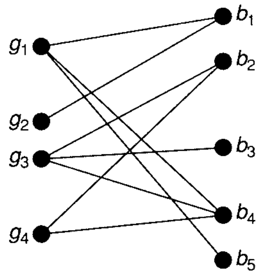


Figure 6.1

More generally, a **complete matching** from V_1 to V_2 in a bipartite graph $G(V_1, V_2)$ is a one–one correspondence between the vertices in V_1 and some of the vertices in V_2, such that corresponding vertices are joined. The marriage problem can then be expressed in graph-theoretic terms in the form:

If $G = G(V_1, V_2)$ is a bipartite graph, when does there exist a complete matching from V_1 to V_2 in G?

Returning to matrimonial language, we note that, in any solution of the marriage problem:

Each set of k girls must know collectively at least k boys, for all integers k satisfying $1 \le k \le m$, where m is the total number of girls.

This condition is necessary because, if it failed for a given set of k girls, then we could not marry the girls in that set, let alone the others. We refer to this condition as the **marriage condition**.

Surprisingly, the marriage condition also turns out to be sufficient: this is the content of Hall's 'marriage' theorem. Although this theorem is couched in the frivolous terms of the marriage problem, it also applies to more serious problems. For example, it gives a necessary and sufficient condition for the solution of a job assignment problem in which applicants must be assigned to jobs for which they are variously qualified: an example of such a problem is given in Exercise 6.2. Because of the importance of this theorem, we give three proofs of it; the first is due to P. Halmos and H. E. Vaughan.

THEOREM 6.1 (Hall, 1935) *A necessary and sufficient condition for a solution of the marriage problem is that each set of k girls collectively knows at least k boys, for $1 \le k \le m$.*

Proof. As noted above, the condition is necessary.

To prove that it is sufficient, we use induction on m. The theorem is clearly true when $m = 1$.

Suppose now that there are m girls and assume that the theorem is true if the number of girls is *less than m*. There are two cases to consider:

(i) If every k girls (where $k < m$) collectively know *at least* $k + 1$ boys, so that the condition is always true 'with one boy to spare', then we take any girl and marry her to any boy she knows. The original condition then remains true for the other $m - 1$ girls, who can be married by induction, completing the proof in this case.

(ii) If now there is a set of k girls ($k < m$) who collectively know *exactly* k boys, then these k girls can be married by induction to the k boys, leaving $m - k$ girls still to be married. But any collection of h of these $m - k$ girls, for $h \leq m - k$, must know at least h of the remaining boys, since otherwise these h girls, together with the above collection of k girls, would collectively know fewer than $h + k$ boys, contrary to our assumption. It follows that the original condition applies to the $m - k$ girls. They can therefore be married by induction in such a way that everyone is happy and the proof is complete. ∎

We can also state Hall's theorem in the language of complete matchings in a bipartite graph. Recall that the number of elements in a set S is denoted by $|S|$.

COROLLARY 6.2 *Let $G = G(V_1, V_2)$ be a bipartite graph, and for each subset A of V_1, let $\varphi(A)$ be the set of vertices of V_2 that are adjacent to at least one vertex of A. Then a complete matching from V_1 to V_2 exists if and only if $|A| \leq |\varphi(A)|$, for each subset A of V_1.*

Proof. The proof of this corollary is a translation into graph terminology of the above proof. ∎

Transversal theory

We now present an alternative proof of Hall's theorem, given in the language of transversal theory. We leave as an exercise the translation of this proof into matching or marriage terminology.

Recall from our above example (see Fig. 6.1) that the sets of boys known by the four girls are $\{b_1, b_4, b_5\}$, $\{b_1\}$, $\{b_2, b_3, b_4\}$ and $\{b_2, b_4\}$, and that a solution of the marriage problem is obtained by finding four distinct bs, one from each of these sets of boys; such a solution is given in Fig. 6.2.

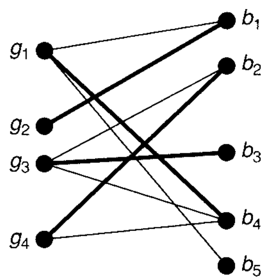


Figure 6.2

In general, if E is a non-empty finite set, and if $\mathcal{F} = (S_1, S_2, \ldots, S_m)$ is a family of (not necessarily distinct) non-empty subsets of E, then a **transversal** of $\mathcal{F}$ is a set of m distinct elements of E, one chosen from each set S_i; thus, $\{b_4, b_1, b_3, b_2\}$ is a transversal of the family

$$\mathcal{F} = (\{b_1, b_4, b_5\}, \{b_1\}, \{b_2, b_3, b_4\}, \{b_2, b_4\})$$

Now suppose that $E = \{1, 2, 3, 4, 5, 6\}$, and let

$$S_1 = S_2 = \{1, 2\}, S_3 = S_4 = \{2, 3\}, S_5 = \{1, 4, 5, 6\}.$$

Then it is impossible to find five distinct elements of E, one from each subset S_i; in other words, the family $\mathcal{F} = (S_1, S_2, \ldots, S_5)$ has no transversal. However, the subfamily $\mathcal{F}' = (S_1, S_2, S_3, S_5)$ has a transversal – for example, $\{1, 2, 3, 4\}$. We call a transversal of a subfamily of $\mathcal{F}$ a **partial transversal** of $\mathcal{F}$. In this example, $\mathcal{F}$ has several partial transversals, such as $\{1, 2, 3, 6\}$, $\{2, 3, 6\}$, $\{1, 5\}$ and $\varnothing$. Note that any subset of a partial transversal is a partial transversal.

It is natural to ask under what conditions a given family of subsets of a set has a transversal. The connection between this problem and the marriage problem is easily seen by taking E to be the set of boys, and S_i to be the set of boys known by girl g_i, for $1 \le i \le m$. A transversal in this case is then simply a set of m boys, one corresponding to, and known by, each girl. It follows that Theorem 6.1 gives a necessary and sufficient condition for a given family of sets to have a transversal.

We now restate Hall's theorem in this form, and give an alternative proof due to R. Rado. The beauty of his proof lies in the fact that essentially only one step is involved, in contrast to the Halmos–Vaughan proof which involves two separate cases, but it is more awkward to express it in the intuitive and appealing language of matrimony!

THEOREM 6.3 *Let E be a non-empty finite set, and let $\mathcal{F} = (S_1, S_2, \ldots, S_m)$ be a family of non-empty subsets of E. Then $\mathcal{F}$ has a transversal if and only if the union of any k of the subsets S_i contains at least k elements, for $1 \le k \le m$.*

Proof. The necessity of the condition is clear.

To prove the sufficiency, we show that if one of the subsets (S_1, say) contains more than one element, then we can remove an element from S_1 without altering the condition. By repeating this procedure, we eventually reduce the problem to the case in which each subset contains only one element, and the proof is then trivial.

It remains only to show the validity of this 'reduction procedure'. So, suppose that S_1 contains elements x and y, the removal of either of which invalidates the condition. Then there are subsets A and B of $\{2, 3, \ldots, m\}$ with the property that $|P| \le |A|$ and $|Q| \le |B|$, where

$$P = \bigcup_{j \in A} S_j \cup (S_1 - \{x\}) \quad \text{and} \quad Q = \bigcup_{j \in B} S_j \cup (S_1 - \{y\}).$$

Then

$$|P \cup Q| = \left| \bigcup_{j \in A \cup B} S_j \cup S_1 \right| \quad \text{and} \quad |P \cap Q| \ge \left| \bigcup_{j \in A \cap B} S_j \right|.$$

The required contradiction now follows, since

$$\begin{aligned}|A| + |B| &\geq |P| + |Q| = |P \cup Q| + |P \cap Q| \\ &\geq \left|\bigcup_{j \in A \cup B} S_j \cup S_1\right| + \left|\bigcup_{j \in A \cap B} S_j\right| \\ &\geq (|A \cup B| + 1) + |A \cap B| \quad \text{(by Hall's condition)} \\ &= |A| + |B| + 1. \end{aligned}$$

■

Before proceeding to some applications of Hall's theorem, we state a corollary that gives us a condition under which *at least t* girls can marry boys known to them.

COROLLARY 6.4 *If E and $\mathcal{F}$ are as before, then $\mathcal{F}$ has a partial transversal of size t if and only if the union of any k of the subsets S_i contains at least $k + t - m$ elements.*

Sketch of proof. The result follows on applying Theorem 6.1 to the family

$$\mathcal{F}' = (S_1 \cup D, S_2 \cup D, \ldots, S_m \cup D),$$

where D is any set disjoint from E and containing $m - t$ elements. Note that $\mathcal{F}$ has a partial transversal of size t if and only if $\mathcal{F}'$ has a transversal. ■

Exercises

6.1[s] Suppose that three boys a, b, c know four girls w, x, y, z as below:

boy	*girls known by boy*		
a	w	y	z
b	x	z	
c	x	y	

(i) Draw the bipartite graph corresponding to this table of relationships.
(ii) Find five different solutions of the corresponding marriage problem.
(iii) Check the marriage condition for this problem.

6.2 A building contractor advertises for a bricklayer, a carpenter, a plumber and a tool maker, and receives five applicants – one for the job of bricklayer, one for carpenter, one for bricklayer and plumber, and two for plumber and toolmaker.
(i) Draw the corresponding bipartite graph.
(ii) Check whether the marriage condition holds for this problem.
Can all of the jobs be filled by qualified people?

6.3[s] Explain why the graph in Fig. 6.3 has no complete matching from V_1 to V_2. Find a set of vertices in V_1 for which the marriage condition fails.

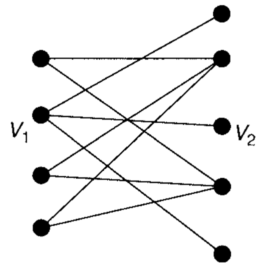


Figure 6.3

6.4 (The 'harem problem') Let B be a set of boys, and suppose that each boy in B wishes to marry more than one of his girl friends. Find a necessary and sufficient condition for the harem problem to have a solution.
(Hint: replace each boy by several identical copies of himself, and then use Hall's theorem.)

6.5 Prove that, if $G = G(V_1, V_2)$ is a bipartite graph in which the degree of each vertex in V_1 is not less than the degree of each vertex in V_2, then G has a complete matching.

6.6ˢ Decide which of the following families of subsets of $E = \{1, 2, 3, 4, 5\}$ have transversals, find a transversal for those that have them, and list all the partial transversals of those that have no transversal:
(i) $(\{1\}, \{2, 3\}, \{1, 2\}, \{1, 3\}, \{1, 4, 5\})$;
(ii) $(\{1, 2\}, \{2, 3\}, \{4, 5\}, \{4, 5\})$;
(iii) $(\{1, 3\}, \{2, 3\}, \{1, 2\}, \{3\})$;
(iv) $(\{1, 3, 4), \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\})$.

6.7 Repeat Exercise 6.6 for the set $\{G, R, A, P, H, S\}$:
(i) $(\{R\}, \{R, G\}, \{A, P\}, \{A, H\}, \{R, A\})$;
(ii) $(\{\mathrm{R}\}, \{R, G\}, \{A, G\}, \{A, R\})$;
(iii) $(\{G, R\}, \{R, P, H\}, \{G, S\}, \{R, H\})$;
(iv) $(\{R, P\}, \{R, P\}, \{R, G\}, \{R\})$.

6.8ˢ Let E be the set of letters in the word *MATROIDS*. Show that the family

$$(STAR, ROAD, MOAT, RIOT, RIDS, DAMS, MIST)$$

of subsets of E has exactly eight transversals.

6.9ˢ Let E be the set $\{1, 2, \ldots, 50\}$. How many different transversals has the family $(\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{50, 1\})$?

6.10 Verify the statement of Corollary 6.4 when

$$E = \{a, b, c, d, e\} \quad \text{and} \quad \mathcal{F} = (\{a, c, e\}, \{b, d\}, \{b, d\}, \{b, d\}).$$

6.11ˢ Let $E = \{\blacklozenge, \heartsuit, \spadesuit, \clubsuit, \bigstar\}$ and $\mathcal{F} = (\{\blacklozenge, \heartsuit, \spadesuit\}, \{\spadesuit, \clubsuit), \{\clubsuit\}, \{\clubsuit), \{\blacklozenge, \heartsuit, \bigstar\})$.
(i) List all the subfamilies of $\mathcal{F}$ for which the marriage condition is not satisfied.
(ii) Verify the statement of Corollary 6.4.

6.12 Rewrite
(i) the statement of Corollary 6.4 in marriage terminology;
(ii) the Halmos–Vaughan proof of Hall's theorem in the language of transversal theory.

6.2 Menger's theorem

We now revisit a theorem that is closely related to Hall's theorem and has far-reaching practical applications. This theorem, due to K. Menger, concerns the number of paths connecting two given vertices v and w in a graph G. We may ask for the maximum number of paths from v to w, no two of which have an *edge* in common – such paths are called **edge-disjoint paths**. Alternatively, we may ask for the maximum number of paths from v to w, no two of which have a *vertex* in common (apart from v and w) – these are called **vertex-disjoint paths**. For example, in the graph of Fig. 6.4, there are four edge-disjoint paths and two vertex-disjoint ones from v to w.

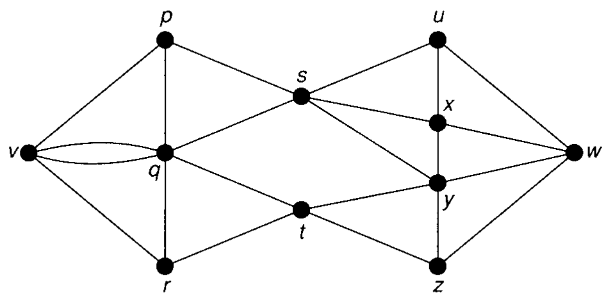


Figure 6.4

In order to investigate these problems, we need some further definitions. We shall assume that G is a connected graph and that v and w are distinct vertices of G. A ***vw*-disconnecting set** of G is a set E of edges of G such that each path from v to w includes an edge of E; note that a vw-disconnecting set is a disconnecting set of G. Similarly, a ***vw*-separating set** of G is a set S of vertices, other than v or w, such that each path from v to w passes through a vertex of S. In Fig. 6.4, the sets

$$E_1 = \{ps, qs, ty, tz\} \quad \text{and} \quad E_2 = \{uw, xw, yw, zw\}$$

are vw-disconnecting sets, and

$$V_1 = \{s, t\} \quad \text{and} \quad V_2 = \{p, q, y, z\}$$

are vw-separating sets.

In order to count the edge-disjoint paths from v to w, we note first that, if E is a vw-disconnecting set with k edges, then the number of edge-disjoint paths cannot exceed k, since otherwise some edge in E would be included in more than one path. In fact, if E is a vw-disconnecting set of minimum possible size, then the number of edge-disjoint paths is equal to k, and there is exactly one edge of E in each such path. This result is known as the edge form of **Menger's theorem**; it was first proved in this form by L. R. Ford, Jr. and D. R. Fulkerson in 1955.

THEOREM 6.5 *The maximum number of edge-disjoint paths connecting two distinct vertices v and w of a connected graph is equal to the minimum number of edges in a vw-disconnecting set.*

Remark. Our proof is non-constructive, in that it does not provide us with a systematic way of obtaining k edge-disjoint paths, or even of finding the value of k. An algorithm that can do so is given in the next section.

Proof. As we have just pointed out, the maximum number of edge-disjoint paths connecting v and w cannot exceed the minimum number of edges in a vw-disconnecting set. We use induction on the number of edges of the graph G to prove that these numbers are equal. The result is trivial if $m = 1$.

So, suppose that the number of edges of G is m, and that the theorem is true for all graphs with fewer than m edges. There are two cases to consider:

(i) Suppose first that there exists a vw-disconnecting set E of minimum size k, such that not all of its edges are incident with v, and not all are incident with w; for example, in Fig. 6.4, the above set E_1 is such a vw-disconnecting set. The removal from G of the edges in E leaves two disjoint subgraphs V and W containing v and w, respectively.

We now define two new graphs G_1 and G_2 as follows: G_1 is obtained from G by contracting every edge of V (that is, by shrinking V down to v), and G_2 is obtained by similarly contracting every edge of W; the graphs G_1 and G_2 obtained from Fig. 6.4 are shown in Fig. 6.5, with dotted lines denoting the edges of E_1. Since each of G_1 and G_2 has fewer edges than G, and since E is a vw-disconnecting set of minimum size for both G_1 and G_2, the induction hypothesis gives us k edge-disjoint paths in G_1 from v to w, and similarly for G_2. The required k edge-disjoint paths in G are obtained by combining these paths in the obvious way.

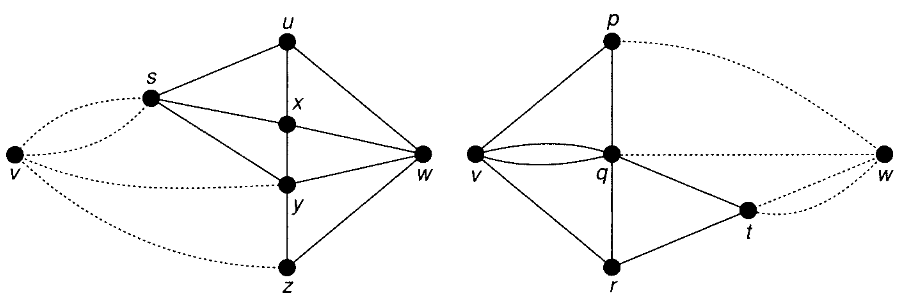


Figure 6.5

(ii) Now suppose that each vw-disconnecting set of minimum size k consists only of edges that are all incident to v or all incident to w; for example, in Fig. 6.4, the set E_2 is such a vw-disconnecting set. We can assume without loss of generality that each edge of G is contained in a vw-disconnecting set of size k, since otherwise its removal would not affect the value of k and we could use the induction hypothesis to obtain k edge-disjoint paths. If P is any path from v to w, then P must consist of either one or two edges, and can thus contain at most one edge of any vw-disconnecting set of size k. By removing from G the edges of P, we obtain a graph with at least $k - 1$ edge-disjoint paths, by the induction hypothesis. These paths, together with P, give the required k paths in G. ■

We turn now to the other problem mentioned at the beginning of the section – to find the number of vertex-disjoint paths from v to w. It was actually this problem that

Menger himself solved, although his name is usually given to both of Theorems 6.5 and 6.6. The proof of Theorem 6.5 goes through with only minor changes, mainly involving the replacement of such terms as 'edge-disjoint' and 'incident' by 'vertex-disjoint' and 'adjacent'. We now state the vertex form of Menger's theorem – its proof is omitted.

THEOREM 6.6 (Menger, 1927) *The maximum number of vertex-disjoint paths connecting two distinct non-adjacent vertices v and w of a graph is equal to the minimum number of vertices in a vw-separating set.*

Using Theorems 6.5 and 6.6, we can immediately deduce the conditions that we met in Section 2.1 for a graph to be *k*-connected and *k*-edge-connected:

COROLLARY 6.7 *A graph G is k-edge-connected if and only if any two distinct vertices of G are connected by at least k edge-disjoint paths.*

COROLLARY 6.8 *A graph G with at least* $k + 1$ *vertices is k-connected if and only if any two distinct vertices of G are connected by at least k vertex-disjoint paths.*

The above discussion can be modified to give the number of arc-disjoint paths from a vertex *v* to a vertex *w* in a digraph. The resulting theorem is similar to Theorem 6.5, and its proof goes through almost word for word. Note that, in a digraph, a *vw*-disconnecting set is a set *A* of arcs such that each path from *v* to *w* includes an arc in *A*.

THEOREM 6.9 *The maximum number of arc-disjoint paths from a vertex v to a vertex w in a digraph is equal to the minimum number of arcs in a vw-disconnecting set.*

For example, if the digraph is as shown in Fig. 6.6, then there are six arc-disjoint paths from *v* to *w*. A corresponding *vw*-disconnecting set consists of the arcs *vz*, *xz*, *yz* (twice) and *xw* (twice).

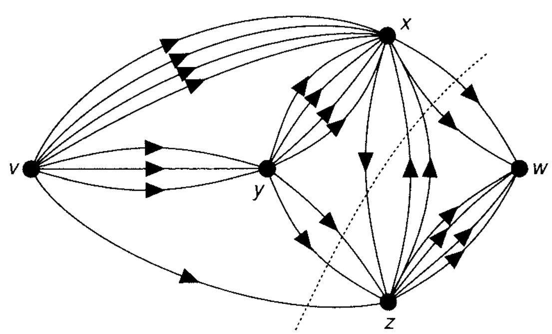


Figure 6.6

Clearly these diagrams will become very cumbersome as the number of arcs joining pairs of vertices increases. To overcome this, we draw just one arc and write next to it the number of arcs there should be (see Fig. 6.7). This seemingly innocent remark is fundamental to the study of network flows, which we discuss in the next section.

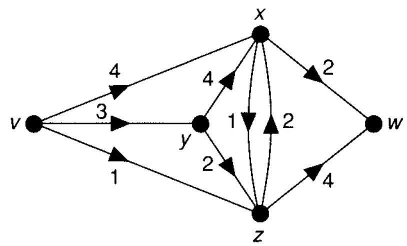


Figure 6.7

We end this section by deducing Hall's theorem from Menger's theorem. We prove the version of Hall's theorem that appears in Corollary 6.2.

THEOREM 6.10 *Menger's theorem implies Hall's theorem.*

***Proof*.** Let $G = G(V_1, V_2)$ be a bipartite graph. We must prove that, if $|A| \le |\varphi(A)|$ for each subset A of V_1, then there is a complete matching from V_1 to V_2.

To do this, we apply the vertex form of Menger's theorem (Theorem 6.6) to the graph obtained by adjoining to G a vertex v adjacent to every vertex in V_1 (the girls) and a vertex w adjacent to every vertex in V_2 (the boys) (see Fig. 6.8).

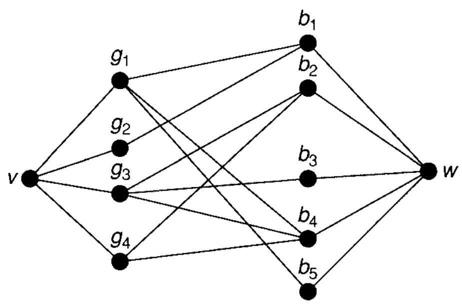


Figure 6.8

Since a complete matching from V_1 to V_2 exists if and only if the number of vertex-disjoint paths from v to w is equal to the number of vertices in V_1 (= k, say), it is enough to show that every vw-separating set has at least k vertices. So, let S be a vw-separating set consisting of a subset A of V_1 and a subset B of V_2. Since $A \cup B$ is a vw-separating set, no edge can join a vertex of $V_1 - A$ to a vertex of $V_2 - B$, and hence $\varphi(V_1 - \mathrm{A}) \subseteq B$. It follows that

$$|V_1 - A| \le |\varphi(V_1 - A)| \le |B|,$$

and so $|S| = |A| + |B| \ge |V_1| = k$, as required. ■

Exercises

6.13[s] Verify Theorems 6.5 and 6.6 for the graphs in Fig. 6.9.

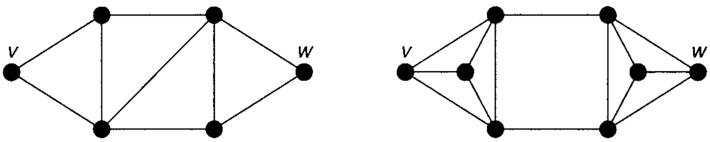


Figure 6.9

6.14 Verify Theorems 6.5 and 6.6 for the Petersen graph in the two cases:
(i) when v and w are adjacent vertices;
(ii) when v and w are not adjacent.

6.15[s] Verify Corollary 6.7 for each of the following graphs:
(i) W_5; (ii) $K_{3,4}$; (iii) Q_3.

6.16 Verify Corollary 6.8 for each of the following graphs:
(i) $K_{3,5}$; (ii) $K_{3,3,3}$; (iii) the graph of the octahedron.

6.17 Verify Theorem 6.9 for each digraph in Fig. 6.10.

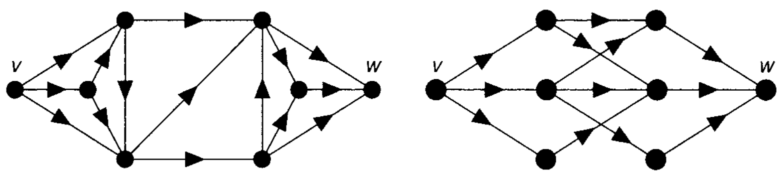


Figure 6.10

6.3 Network flows

Our society today is largely governed by networks – transportation, communication, etc. – and the mathematical analysis of such networks has become of fundamental importance. In this section we illustrate that network analysis is essentially the study of digraphs.

A computer manufacturer wishes to send several computers to a given market. There are various channels through which the boxes can be sent, as shown in Fig. 6.11, with v representing the manufacturer and w the market. The number next to each arc

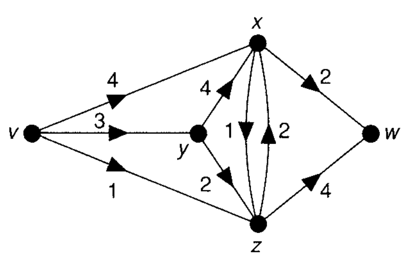


Figure 6.11

refers to the maximum load that can pass through the corresponding channel. The manufacturer wishes to find the maximum number of boxes that can be sent through the network without exceeding the permitted capacity of any channel.

Figure 6.11 can also describe other situations. For example, if each arc represents a one-way street and the number next to each arc is the maximum possible flow of traffic along that street, in vehicles per hour, then we may ask for the greatest possible number of vehicles that can travel from v to w in one hour. Alternatively, if the diagram depicts an electrical network, we may ask for the maximum current that can safely be passed through the network, given the currents at which individual wires burn out.

Using these examples as motivation, we define a **network** N to be a weighted digraph – that is, a digraph to each arc a of which is assigned a non-negative real number $c(a)$ called its **capacity**. The **out-degree** outdeg(x) of a vertex x is the sum of the capacities of the arcs of the form xz, and the **in-degree** indeg(x) is similarly defined. For example, in the network of Fig. 6.11, outdeg(v) = 8 and indeg(x) = 10. Note that the handshaking dilemma now takes the form:

The sum of the out-degrees of the vertices of a network is equal to the sum of the in-degrees.

A vertex with in-degree 0 is a **source**, and one with out-degree 0 is a **sink**; for example, in Fig. 6.11 v is the only source and w is the only sink. Usually we assume that any network has exactly one source v and one sink w. The general case of several sources and sinks, corresponding to more than one manufacturer and market, is easily reduced to this special case (see Exercise 6.22).

A **flow** in a network is a function φ that assigns to each arc a a non-negative real number $\varphi(a)$, called the **flow in *a***, in such a way that

(i) for each arc a, $\varphi(a) \le c(a)$;
(ii) the out-degree and in-degree of each vertex, other than v or w, are equal.

Thus, the flow in any arc cannot exceed its capacity and the 'total flow' into each vertex, other than v or w, is equal to the 'total flow' out of it. Figure 6.12 gives a possible flow for the network of Fig. 6.11. Another flow is the **zero flow** in which the flow in every arc is 0; any other flow is a **non-zero flow**. An arc a for which $\varphi(a) = c(a)$ is called **saturated**. In Fig. 6.12, the arcs vz, xz, yz, xw and zw are saturated, and the remaining arcs are **unsaturated**.

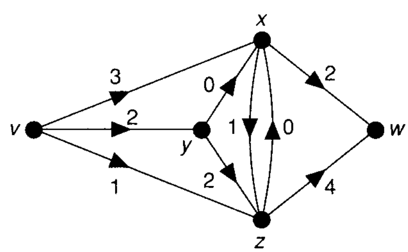


Figure 6.12

It follows from the handshaking dilemma that the sum of the flows in the arcs out of v is equal to the sum of the flows in the arcs into w; this sum is called the **value of the flow**. Prompted by our earlier examples, we are mainly interested in flows whose value is as large as possible – the **maximum flows**. You can easily check that the flow of Fig. 6.12 is a maximum flow for the network of Fig. 6.11, and that its value is 6. Although a network may have several different maximum flows, their values must all be equal.

The study of maximum flows in a network is closely tied up with the concept of a **cut**, which is a set A of arcs such that each path from v to w includes an arc in A. Thus, a cut in a network is a vw-disconnecting set in the corresponding digraph D. The **capacity of a cut** is the sum of the capacities of the arcs in the cut. We are concerned mainly with those cuts whose capacity is as small as possible, the so-called **minimum cuts**. In Fig. 6.13, a minimum cut consists of the arcs vz, xz, yz and xw, but not the arc zx; the capacity of this cut is $2 + 2 + 2 = 6$.

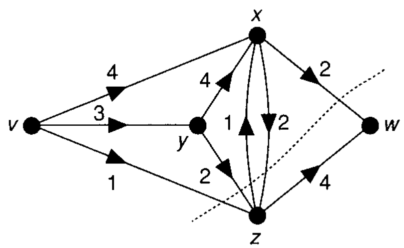


Figure 6.13

Note that the value of any flow cannot exceed the capacity of any cut, and so the value of any *maximum* flow cannot exceed the capacity of any *minimum* cut. It turns out that these last two numbers are always equal. This famous result, known as the **max-flow min-cut theorem**, was proved by L. R. Ford, Jr. and D. R. Fulkerson in 1955. We present two proofs. The first shows that the max-flow min-cut theorem is equivalent to Menger's theorem; the second is a direct proof.

THEOREM 6.11 (Max-flow min-cut theorem) *In any network, the value of any maximum flow is equal to the capacity of any minimum cut.*

Remark. When applying this theorem, it is often simplest to find a flow and a cut for which the value of the flow equals the capacity of the cut. It follows from the theorem that the flow must be a maximum flow and that the cut must be a minimum cut. If all the capacities are integers, then the value of a maximum flow is also an integer; this turns out to be useful in certain applications of network flows.

First proof. Suppose first that the capacity of each arc is an integer. Then the network can be regarded as a digraph D whose capacities represent the number of arcs connecting the various vertices (as in Figs 6.6 and 6.7). The value of a maximum flow is the total number of arc-disjoint paths from v to w in D, and the capacity of a minimum cut is the minimum number of arcs in a vw-disconnecting set of D. The result now follows from Theorem 6.9.

The extension of this result to networks in which the capacities are rational numbers is effected by multiplying these capacities by a suitable integer d to make them all integers. We then have the case described above, and the result follows on dividing by d.

Finally, if some capacities are irrational, then we approximate them as closely as we please by rationals and use the above result. By choosing these rationals carefully, we can ensure that the value of any maximum flow and the capacity of any minimum cut are altered by an amount that is as small as we wish. Note that, in practical examples, irrational capacities rarely occur, since the capacities are usually given in decimal form. ■

Second proof. Since the value of any maximum flow cannot exceed the capacity of any minimum cut, it is sufficient to prove the existence of a cut whose capacity is equal to the value of a given maximum flow.

Let φ be a maximum flow. We define two sets V and W of vertices in the network as follows. If G is the underlying graph of the network, then a vertex z is in V if and only if there exists in G a path

$$v = v_0 \to v_1 \to v_2 \to \ldots \to v_{m-1} \to v_m = z,$$

in which each edge v_iv_{i+1} corresponds either to an unsaturated arc v_iv_{i+1}, or to an arc $v_{i+1}v_i$ that carries a non-zero flow. The set W consists of all those vertices that do not lie in V. For example, in Fig. 6.12, the set V consists of the vertices v, x and y, and the set W consists of the vertices z and w.

Clearly, v is contained in V. We now show that W contains the vertex w. If this is not so, then w lies in V, and hence there exists in G a path

$$v \to v_1 \to v_2 \to \ldots \to v_{m-1} \to w$$

of the above type. We now choose a positive number ε that does not exceed the amount needed to saturate any unsaturated arc v_iv_{i+1}, and does not exceed the flow in any arc $v_{i+1}v_i$ that carries a non-zero flow. It is now easy to see that, if we increase by ε the flow in all arcs of the first type and decrease by ε the flow in all arcs of the second type, then we increase the value of the flow by ε, contradicting our assumption that φ is a maximum flow. It follows that w lies in W.

To complete the argument, we let E be the set of all arcs of the form xz, where x is in V and z is in W. Clearly, E is a cut. Moreover, each arc xz of E is saturated and each arc zx carries a zero flow, since otherwise z would also be an element of V. It follows that the capacity of E must equal the value of φ, and that E is the required minimum cut. ■

The max-flow min-cut theorem provides a useful check on the maximality or otherwise of a given flow, as long as the network is fairly simple. In practice, the networks we have to deal with are large and complicated, and it is usually difficult to find a maximum flow by inspection. Most methods for finding a maximum flow involve determining **flow-augmenting paths** from v to w. These are paths that consist entirely of unsaturated arcs xz and arcs zx carrying a non-zero flow. For example, consider the network of Fig. 6.14.

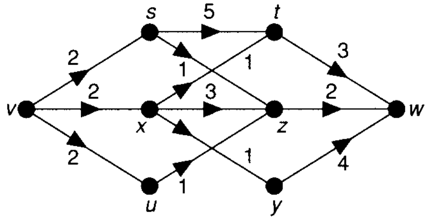


Figure 6.14

Starting with the zero flow, we can construct the flow-augmenting paths

$v \to s \to t \to w$, along which the value of the flow can be increased by 2,

$v \to x \to z \to w$, along which the value of the flow can be increased by 2,

and $v \to u \to z \to x \to y \to w$, along which the value of the flow can be increased by 1.

The resulting flow of value 5 is as shown in Fig. 6.15.

Since the network has a cut of capacity 5, this flow is a maximum flow and the cut is a minimum cut.

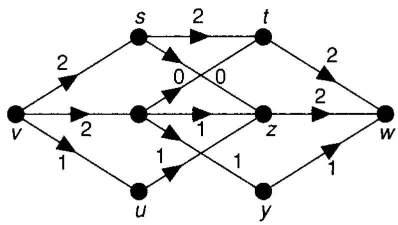


Figure 6.15

In this section we have been able only to scratch the surface of this very diverse and important subject. If you wish to pursue these topics, see Lawler [24] or Dolan and Aldous [22].

Exercises

6.18[s] Consider the network of Fig. 6.16.

(i) List all the cuts in this network, and find a minimum cut.

(ii) Find a maximum flow, and verify the max-flow min-cut theorem.

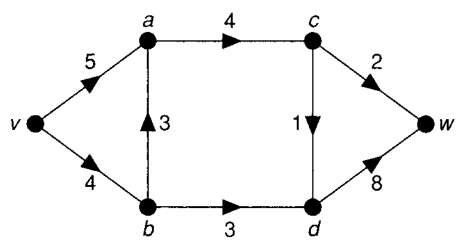


Figure 6.16

6.19 Repeat Exercise 6.18 for the network of Fig. 6.17.

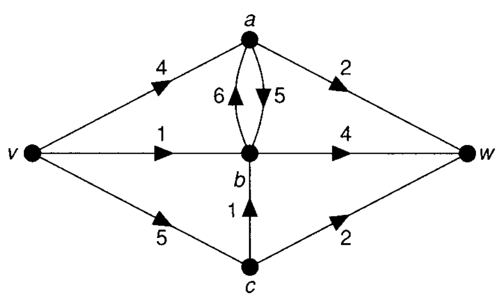


Figure 6.17

6.20[s] Verify the max-flow min-cut theorem for the network of Fig. 2.37.

6.21 Find a flow with value 20 in the network of Fig. 6.18. Is it a maximum flow?

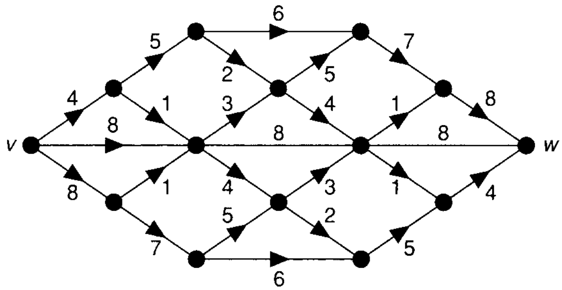


Figure 6.18

6.22[s] (i) Consider a network with several sources and sinks. Show how the analysis of the flows in this network can be reduced to the standard case by the addition of a new 'source vertex' and 'sink vertex'.

(ii) Illustrate your answer to part (i) with reference to the network in Fig. 6.19.

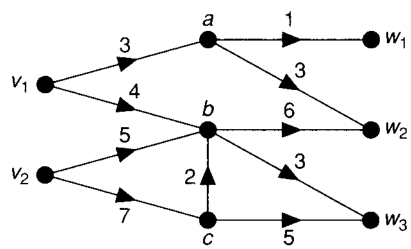


Figure 6.19

Challenge problems

6.23 (i) Use the marriage condition to show that if each girl has r (≥ 1) boy friends and each boy has r girl friends, then the marriage problem has a solution.

(ii) Use the result of part (i) to prove that, if G is a bipartite graph which is regular of degree r, then G has a complete matching. Deduce that the chromatic index of G is r.

(This is a special case of Theorem 5.18.)

6.24 Suppose that the marriage condition is satisfied, and that each of the m girls knows at least t boys. Show, by induction on m, that the marriages can be arranged in at least $t!$ ways if $t \leq m$, and in at least $t!/(t - m)!$ ways if $t > m$.

6.25[s] Let E and $\mathcal{F}$ have their usual meanings, let T_1 and T_2 be transversals of $\mathcal{F}$, and let x be an element of T_1. Show that there exists an element y of T_2 such that $(T_1 - \{x\}) \cup \{y\}$ (the set obtained from T_1 on replacing x by y) is also a transversal of $\mathcal{F}$. Compare this result with Exercise 3.11.

(This result will be needed in Chapter 7.)

6.26 Let $\mathcal{F}$ be a family consisting of m non-empty subsets of E, and let A be a subset of E. By applying Hall's theorem to the family consisting of $\mathcal{F}$, together with $|E| - m$ copies of $E - A$, prove that there is a transversal of $\mathcal{F}$ containing A if and only if

(i) $\mathcal{F}$ has a transversal;

(ii) A is a partial transversal of $\mathcal{F}$.

(A simpler proof, using matroids, is given in Chapter 7.)

6.27 The **rank** $r(A)$ of a subset A of E is the number of elements in the largest partial transversal of $\mathcal{F}$ contained in A. Show that

(i) $0 \le r(A) \le |A|$;

(ii) if $A \subseteq B \subseteq E$, then $r(A) \le r(B)$;

(iii) if $A, B \subseteq E$, then $r(A \cup B) + r(A \cap B) \le r(A) + r(B)$.

Compare these results with Exercise 3.33.

(This result will also be needed in Chapter 7.)

6.28 Let E be a countable set, and let $\mathcal{F} = (S_1, S_2, \ldots)$ be a countable family of non-empty *finite* subsets of E.

(i) Defining a transversal of $\mathcal{F}$ in the natural way, show, by König's lemma (Theorem 2.7), that $\mathcal{F}$ has a transversal if and only if the union of any k subsets S_i contains at least k elements, for all finite k.

(ii) By considering

$$E = \{1, 2, 3, \ldots\},\ S_1 = E,\ S_2 = \{1\},\ S_3 = \{2\},\ S_4 = \{3\}, \ldots,$$

show that the result of part (i) is false if not all of the S_i are finite.

6.29 Prove Theorem 6.6 in detail.

6.30 Show how the max-flow min-cut theorem can be used to prove Hall's theorem.

Chapter 7

Matroids

The first of earthly blessings, independence.

Edward Gibbon

In this chapter we investigate an unexpected similarity between certain results in graph theory and their analogues in transversal theory, as illustrated by Exercises 3.11 and 6.25, or Exercises 3.33 and 6.27. To do this, we introduce the idea of a matroid, first studied in 1935 in a pioneering paper by Hassler Whitney. As we shall see, a matroid is a set with an 'independence structure' defined on it, where the notion of independence generalizes that of independence in graphs, as defined in Exercise 2.48, and of linear independence in vector spaces. The link with transversal theory is provided by Exercise 6.25.

In Section 7.3 we define duality in matroids in such a way as to explain the similarity between the properties of cycles and cutsets in a graph. The unintuitive definition of an abstract dual of a graph in Section 4.3 then arises as a natural consequence of matroid duality. We also indicate how matroids can be used to give 'easy' proofs of results in transversal theory, and we conclude with a matroid analogue of Kuratowski's theorem. Throughout this chapter we state results without proof, where convenient. The omitted proofs may be found in Oxley [42] or Welsh [45].

7.1 Introduction to matroids

In Section 3.1 we defined a spanning tree in a connected graph G to be a connected subgraph of G that contains no cycles and includes every vertex of G. Clearly a spanning tree cannot contain another spanning tree as a proper subgraph. We also saw in Exercise 3.11 that if B_1 and B_2 are spanning trees of G and e is an edge of B_1, then there is an edge f in B_2 such that $(B_1 - \{e\}) \cup \{f\}$ (the graph obtained from B_1 on replacing e by f) is also a spanning tree of G.

Analogous results hold in the theory of vector spaces and in transversal theory. If V is a vector space and if B_1 and B_2 are bases of V and e is an element of B_1, then we can find an element f of B_2 such that $(B_1 - \{e\}) \cup \{f\}$ is also a basis of V. The corresponding result in transversal theory appeared in Exercise 6.25.

Using these examples as motivation, we now give our first definition of a matroid:

A **matroid** *M consists of a non-empty finite set E and a non-empty collection $\mathcal{B}$ of subsets of E, called* **bases**, *satisfying the following properties:*

$\mathcal{B}$(i) *no base properly contains another base;*

$\mathcal{B}$(ii) *if B_1 and B_2 are bases and if e is any element of B_1, then there is an element f of B_2 such that $(B_1 - \{e\}) \cup \{f\}$ is also a base.*

By repeatedly using property $\mathcal{B}$(ii), we can easily show that any two bases of a matroid M have the same number of elements (see Exercise 7.5). This number is called the **rank** of M.

As we indicated above, a matroid can be associated with any connected graph G by letting E be the set of edges of G and taking as bases the edges of the spanning trees of G. For reasons that will appear later, this matroid is called the **cycle matroid** of G, and is denoted by $M(G)$. Similarly, if E is a finite set of vectors in a vector space V (such as $\mathbf{R}^n$), then we can define a matroid on E by taking as bases all linearly independent subsets of E that span the same subspace as E. A matroid obtained in this way is called a **vector matroid**.

A subset of E is **independent** if it is contained in some base of the matroid M. For a vector matroid, a subset of E is independent whenever its elements are linearly independent as vectors in the vector space. For the cycle matroid $M(G)$ of a graph G, the independent sets are those sets of edges of G that contain no cycle.

Since the bases of M are the maximal independent sets (that is, those independent sets that are contained in no larger independent set), a matroid is uniquely defined by specifying its independent sets. It is natural to ask whether there is a simple definition of a matroid in terms of its independent sets. One such definition is as follows. A proof of the equivalence of this definition and the one above is given in Welsh [45]:

A **matroid** *M consists of a non-empty finite set E and a non-empty collection $\mathcal{I}$ of subsets of E* (*called* **independent sets**), *satisfying the following properties:*

$\mathcal{I}$(i) *any subset of an independent set is independent;*

$\mathcal{I}$(ii) *if I and J are independent sets with $|J| > |I|$, then there is an element e, contained in J but not in I, such that $I \cup \{e\}$ is independent.*

With this definition, a **base** is defined to be a maximal independent set. Property $\mathcal{I}$(ii) can then be used repeatedly to extend any independent set to a base.

If M is a matroid defined in terms of its independent sets, then a subset of E is **dependent** if it is not independent, and a minimal dependent set is called a **cycle**. If $M(G)$ is the cycle matroid of a connected graph G, then the cycles of $M(G)$ are precisely the cycles of G. Since a subset of E is independent if and only if it contains no cycles, a matroid can be defined in terms of its cycles. One such definition, generalizing to matroids the result of Exercise 2.46(i), is as follows:

A **matroid** *consists of a non-empty finite set E and a collection $\mathcal{C}$ of non-empty subsets of E* (*called* **cycles**) *satisfying the following properties:*

$\mathcal{C}$(i) *no cycle properly contains another cycle;*

$\mathcal{C}$(ii) *if C_1 and C_2 are two distinct cycles each containing an element e, then there exists a cycle in $C_1 \cup C_2$ that does not contain e.*

Before proceeding to some examples, we give one further definition of a matroid. This definition, in terms of a rank function r, is essentially the one given by Whitney.

If M is a matroid defined in terms of its independent sets, and if A is a subset of E, then its **rank** $r(A)$ is the size of the largest independent set contained in A; note that the previously defined rank of M is equal to $r(E)$.

Since a subset A of E is independent if and only if $r(A) = |A|$, we can define a matroid in terms of its rank function. The following theorem extends to matroids the results of Exercises 3.33 and 6.27.

THEOREM 7.1 *A matroid consists of a non-empty finite set E and an integer-valued function r defined on the set of subsets of E, satisfying:*

r(i) $0 \le r(A) \le |A|$, *for each subset A of E;*
r(ii) *if $A \subseteq B \subseteq E$, then $r(A) \le r(B)$;*
r(iii) *for any $A, B \subseteq E$, $r(A \cup B) + r(A \cap B) \le r(A) + r(B)$.*

Proof. We assume first that M is a matroid defined in terms of its independent sets, and prove properties r(i), r(ii) and r(iii).

Properties r(i) and r(ii) are trivial. To prove r(iii), we let X be a base (a maximal independent subset) of $A \cap B$. Since X is an independent subset of A, X can be extended to a base Y of A, and then to a base Z of $A \cup B$. Since $X \cup (Z - Y)$ is an independent subset of B, we have

$$r(B) \ge r(X \cup (Z - Y)) = |X| + |Z| - |Y| = r(A \cap B) + r(A \cup B) - r(A),$$

as required.

Conversely, let M be a matroid defined in terms of a rank function r, and define a subset A of E to be independent if and only if $r(A) = |A|$. It is then simple to prove property $\mathcal{I}$(i). To prove $\mathcal{I}$(ii), let I and J be independent sets with $|J| > |I|$, and suppose that $r(I \cup \{e\}) = k$ for each element e that lies in J but not in I. If e and f are two such elements, then

$$r(I \cup \{e\} \cup \{f\}) \le r(I \cup \{e\}) + r(I \cup \{f\}) - r(I) = k,$$

and so $r(I \cup \{e\} \cup \{f\}) = k$. We now continue this procedure, adding one new element of J at a time. Since the rank is k at each stage, we conclude that $r(I \cup J) = k$, and hence (by r(ii)) that $r(J) \le k$, which is a contradiction. It follows that there exists an element f that lies in J but not in I, such that $r(I \cup \{f\}) = k + 1$. ■

We conclude this section with some further definitions. A **loop** of a matroid M is an element e of E satisfying $r(\{e\}) = 0$, and a pair of **parallel elements** of M are a pair $\{e, f\}$ of elements of E that satisfy $r(\{e, f\}) = 1$ and are not loops. Note that, if M is the cycle matroid of a graph G, then the loops and parallel elements of M correspond to loops and multiple edges of G.

We also need the concept of isomorphism. We call two matroids M_1 and M_2 **isomorphic** if there is a one–one correspondence between their underlying sets E_1 and E_2 that preserves independence: thus, a set of elements of E_1 is independent in M_1 if and only if the corresponding set of elements of E_2 is independent in M_2. For example, the cycle matroids of the three graphs in Fig. 7.1 are all isomorphic. Note that, although matroid isomorphism preserves cycles, cutsets and the number of edges in a graph, it does not necessarily preserve connectedness, the number of vertices or their degrees.

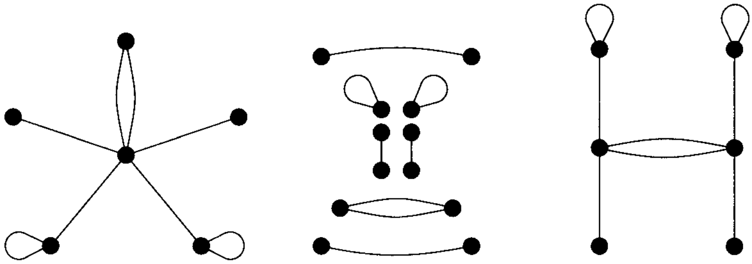


Figure 7.1

Exercises

7.1[s] Let $E = \{a, b, c, d, e\}$. Find matroids on E for which

(i) E is the only base;
(ii) the empty set Ø is the only base;
(iii) the bases are those subsets of E containing exactly three elements.

For each matroid, write down the independent sets, the cycles (if there are any) and the rank function.
(This question is answered in the next section.)

7.2[s] Let G_1 and G_2 be the graphs shown in Fig. 7.2. Write down the bases, cycles and independent sets of the cycle matroids $M(G_1)$ and $M(G_2)$.

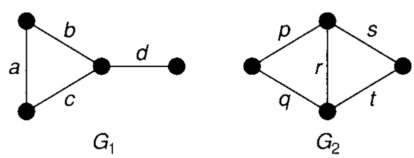


Figure 7.2

7.3 Let M be the matroid on the set $E = \{a, b, c, d\}$ whose bases are

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\} \text{ and } \{c, d\}.$$

Write down the cycles of M, and deduce that there is no graph G with M as its cycle matroid.

7.4[s] Let $E = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{F} = (S_1, S_2, S_3, S_4, S_5)$, where

$$S_1 = S_2 = \{1, 2\}, S_3 = S_4 = \{2, 3\}, S_5 = \{1, 4, 5, 6\}.$$

(i) Write down the partial transversals of $\mathcal{F}$ and check that they form the independent sets of a matroid on E.
(ii) Write down the bases and cycles of this matroid.

7.5 Use properties $\mathcal{B}$(i) and $\mathcal{B}$(ii) to prove that any two bases of a matroid on a set E have the same number of elements.

7.6^s (i) Use the result of Exercise 2.46(ii) to show that the cutsets of a graph satisfy conditions $\mathcal{C}$(i) and $\mathcal{C}$(ii).
(ii) Write down the bases of the corresponding matroids for the graphs of Fig. 7.2.

7.7 Show how the definition of a fundamental system of cycles in a graph can be extended to matroids.

7.2 Examples of matroids

In this section we examine several important types of matroid.

Trivial matroids

Given any non-empty finite set E, we can define on it a matroid whose only independent set is the empty set Ø. This matroid is the **trivial matroid** on E, and has rank 0.

Discrete matroids

At the other extreme is the **discrete matroid** on E, in which every subset of E is independent. Note that the discrete matroid on E has only one base, E itself, and that the rank of any subset A is the number of elements in A.

Uniform matroids

The previous examples are special cases of the ***k*-uniform matroid** on E, whose bases are those subsets of E with exactly k elements; the trivial matroid on E is 0-uniform and the discrete matroid is $|E|$-uniform. Note that the independent sets are those subsets of E with at most k elements, and that the rank of any subset A is either $|A|$ or k, whichever is smaller.

Graphic matroids

As we saw in the previous section, we can define a matroid $M(G)$ on the set of edges of a connected graph G by taking the cycles of G as the cycles of the matroid. $M(G)$ is the **cycle matroid** of G and its rank function is the cutset rank ξ.

It is natural to ask whether a given matroid M is the cycle matroid of some graph; in other words, does there exist a graph G such that M is isomorphic to $M(G)$? Such matroids are called **graphic matroids**, and a characterization of them is given in the next section. For example, the matroid M on the set {1, 2, 3} whose bases are {1, 2} and {1, 3} is a graphic matroid isomorphic to the cycle matroid of the graph in Fig. 7.3. An example of a non-graphic matroid is the 2-uniform matroid on a set of four elements (see Exercise 7.3).

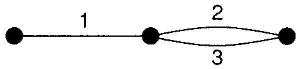


Figure 7.3

Cographic matroids

Given a connected graph G, the cycle matroid $M(G)$ is not the only matroid that can be defined on the set of edges of G. Because of the similarity between the properties of cycles and cutsets in a graph, we can construct a matroid by taking the *cutsets* of G as cycles of the matroid. We saw in Exercise 7.6 that this construction defines a matroid, and we call it the **cutset matroid** of G, denoted by $M^*(G)$. Note that a set of edges of G is independent if and only if it contains no cutset of G.

We call a matroid M **cographic** if there exists a graph G such that M is isomorphic to $M^*(G)$. The reason for the name 'cographic' is given in the next section.

Planar matroids

A matroid that is both graphic and cographic is a **planar matroid**. In the next section, we investigate the connection between planar matroids and planar graphs.

Bipartite and Eulerian matroids

We can also define bipartite and Eulerian matroids. Since the usual definitions of bipartite and Eulerian graphs, as given in Sections 1.2 and 2.2, are unsuitable for matroid generalization, we must use alternative characterizations of these graphs. In the case of bipartite graphs, we use Theorem 2.1 and define a **bipartite matroid** to be a matroid in which each cycle has an even number of elements. For Eulerian graphs we use Corollary 2.10 and define a matroid on a set E to be an **Eulerian matroid** if E can be written as a union of disjoint cycles. In the next section we see that Eulerian matroids and bipartite matroids are dual concepts (in a sense to be made precise), as we might expect from Exercise 4.31.

Representable matroids

Since the definition of a matroid is partly motivated by the idea of linear independence in a vector space (such as $\mathbf{R}^n$), it is of interest to investigate those matroids that arise as vector matroids associated with some set of vectors in a vector space. Given a matroid M on a set E, we say that M is **representable over a field *F*** if there exist a vector space V over F and a map φ from E to V, such that a subset A of E is independent in M if and only if φ is one–one on A and $\varphi(A)$ is linearly independent in V. This amounts to saying that, if we ignore loops and parallel elements, then M is isomorphic to a vector matroid defined in some vector space over F. We also say that M is a **representable matroid** if there exists some field F such that M is representable over F.

It turns out that some matroids are representable over every field (the **regular matroids**), some are representable over no fields, and some are representable only over a restricted class of fields. Of particular importance are the **binary matroids**, which are representable over the field of integers {0, 1} modulo 2. For example, if G is any graph, then its cycle matroid $M(G)$ is a binary matroid. To see this, we associate with each edge of G the corresponding row of the incidence matrix of G, regarded as a vector with components 0 or 1. Note that, if a set of edges of G forms a cycle, then the sum (modulo 2) of the corresponding vectors is 0.

An example of a binary matroid that is neither graphic nor cographic is the Fano matroid, described below.

Transversal matroids

Our next example provides the link between matroids and transversal theory. Recall from Exercises 6.25, 6.27 and 7.4 that, if E is a non-empty finite set and if $\mathcal{F} = (S_1, S_2, \ldots, S_m)$ is a family of non-empty subsets of E, then the partial transversals of $\mathcal{F}$ can be taken as the independent sets of a matroid on E, denoted by $M(\mathcal{F})$ or $M(S_1, S_2, \ldots, S_m)$. Any matroid obtained in this way (for suitable choices of E and $\mathcal{F}$) is a **transversal matroid**. For example, the above graphic matroid M is a transversal matroid on the set $\{1, 2, 3\}$, since its independent sets are the partial transversals of the family $\mathcal{F} = (S_1, S_2)$, where $S_1 = \{1\}$ and $S_2 = \{2, 3\}$. Note that the rank of a subset A of E is the size of the largest partial transversal contained in A. An example of a non-transversal matroid is given in Exercise 7.11.

As an example of the use of transversal matroids, recall from Exercise 6.26 that a family $\mathcal{F}$ of subsets of E has a transversal containing a given subset A if and only if $\mathcal{F}$ has a transversal and A is a partial transversal of $\mathcal{F}$. Clearly these conditions are necessary. To prove that they are sufficient, we observe that, since A is a partial transversal of $\mathcal{F}$, A is an independent set in the transversal matroid M determined by $\mathcal{F}$, and so can be extended to a base of M. Since $\mathcal{F}$ has a transversal, every base of M must be a transversal of $\mathcal{F}$, and the result follows immediately. If you have worked through Exercise 6.26, you will realize how much simpler this argument is.

It can be proved that every transversal matroid is representable over some field, but is binary if and only if it is graphic.

The Fano matroid

The **Fano matroid** F is the matroid defined on the set $E = \{1, 2, 3, 4, 5, 6, 7\}$, whose bases are all the subsets of E with three elements, *except*

$$\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\} \text{ and } \{7, 1, 3\}.$$

This matroid can be represented geometrically by Fig. 7.4; the bases are precisely those sets of three elements that do not lie on a line. It can be shown that F is binary and Eulerian, but is not graphic, cographic, transversal or regular.

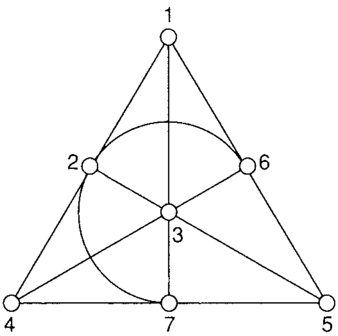


Figure 7.4

Restrictions and contractions

In graph theory we often investigate the properties of a graph by looking at its subgraphs, or by considering the graph obtained by contracting some of its edges. We now define the corresponding notions for matroids.

If M is a matroid defined on a set E, and if A is a subset of E, then the **restriction** of M to A, denoted by $M \times A$, is the matroid whose cycles are precisely those cycles of M that are contained in A. Similarly, the **contraction** of M to A, denoted by $M \cdot A$, is the matroid whose cycles are the minimal members of the collection $\{C_i \cap A\}$, where C_i is a cycle of M. (A simpler definition is given in Exercise 7.22.) You can verify that these are indeed matroids, and that they correspond to the deletion and contraction of edges in a graph.

A matroid obtained from M by restrictions and/or contractions is called a **minor** of M. It can be proved that if M is graphic, cographic, binary and/or regular, then so is any minor of M (see Exercise 7.26).

Exercises

7.8[s] Let $E = \{a, b\}$. Show that there are (up to isomorphism) exactly four matroids on E, and list their bases, independent sets and cycles.

7.9 Let $E = \{a, b, c\}$. Show that there are (up to isomorphism) exactly eight matroids on E, and list their bases, independent sets and cycles.

7.10[s] Let G_1 and G_2 be the graphs of Fig. 7.2.
(i) Are $M(G_1)$ and $M(G_2)$ transversal matroids?
(ii) Are $M^*(G_1)$ and $M^*(G_2)$ transversal matroids?

7.11 Show that $M(K_4)$ is not a transversal matroid.

7.12[s] Show that every uniform matroid is a transversal matroid.

7.13 Show that the graphic matroids $M(K_5)$ and $M(K_{3,3})$ are not cographic.

7.14[s] Describe the cycles of the Fano matroid.

7.15 Let M be a matroid on a set E, and let $A \subseteq B \subseteq E$. Prove that

$$(M \times B) \times A = M \times A \quad \text{and} \quad (M \cdot B) \cdot A = M \cdot A.$$

7.3 Matroids and graphs

We now study duality in matroids. Our aim is to show how several of our earlier results become more natural when looked at in this light. We shall see, for example, that the rather artificial definition of an abstract dual of a planar graph in Section 4.3 arises as a direct consequence of the corresponding definition of a matroid dual. Indeed, not only do concepts in matroid theory generalize their counterparts in graph theory, but they sometimes simplify them as well.

The dual of a matroid

Recall that we can form a matroid $M^*(G)$ on the set of edges of a connected graph G by taking as cycles of $M^*(G)$ the cutsets of G. In view of Theorem 4.14, it is sensible to define a matroid dual so as to make this matroid the dual of the cycle matroid $M(G)$.

This is achieved as follows. If M is a matroid on a set E, defined in terms of its rank function, we define the **dual matroid** M^* of M to be the matroid on E whose rank function r^* is given by the expression

$$r^*(A) = |A| + r(E - A) - r(E), \text{ for } A \subseteq E.$$

We must first verify that r^* is the rank function of a matroid on E.

THEOREM 7.2 *M^* is a matroid on E.*

Proof. We verify the properties r(i) and r(iii) of Section 7.1 for the function r^*: the proof of r(ii) is equally straightforward (see Exercise 7.18).

To prove r(i), note that $r(E - A) \le r(E)$, and hence that $r^*(A) \le |A|$. Also, by r(iii) applied to the function r, we have

$$r(E) + r(\varnothing) \le r(A) + r(E - A)$$

and hence

$$r(E) - r(E - A) \le r(A) \le |A|.$$

It follows immediately that $r^*(A) \ge 0$.

To prove r(iii), we have, for any $A, B \subseteq E$,

$$\begin{aligned}
&r^*(A \cup B) + r^*(A \cap B) \\
&= |A \cup B| + |A \cap B| + r(E - (A \cup B)) + r(E - (A \cap B)) - 2r(E) \\
&= |A| + |B| + r((E - A) \cap (E - B)) + r((E - A) \cup (E - B)) - 2r(E) \\
&\le |A| + |B| + r(E - A) + r(E - B) - 2r(E) \quad (\text{by } r(\text{iii}), \text{ applied to } r) \\
&= r^*(A) + r^*(B),
\end{aligned}$$

as required. ■

Although the above definition seems contrived, it turns out that the bases of M^* can be described very simply in terms of those of M, as we now show; this result is often used to define M^*.

THEOREM 7.3 *The bases of M^* are precisely the complements of the bases of M.*

Proof. We show that, if B^* is a base of M^*, then $E - B^*$ is a base of M; the converse result is obtained by reversing the argument.

Since B^* is independent in M^*, $|B^*| = r^*(B^*)$ and hence $r(E - B^*) = r(E)$. It remains only to prove that $E - B^*$ is independent in M. But this follows immediately from the fact that $r^*(B^*) = r^*(E)$, on using the above expression for r^*. ■

It follows from the above definition that, unlike the duality of planar graphs:

Every matroid has a dual and this dual is unique.

It also follows immediately from Theorem 7.3 that the double-dual M^{**} is equal to M. As we shall see, this trivial result is the natural generalization to matroids of the non-trivial Theorems 4.13 and 4.16.

Cycles and cutsets

We next show that the cutset matroid $M^*(G)$ of a connected graph G is the dual of the cycle matroid $M(G)$.

THEOREM 7.4 *If G is a connected graph, then $M^*(G) = (M(G))^*$.*

Proof. Since the cycles of $M^*(G)$ are the cutsets of G, we must check that C^* is a cycle of $(M(G))^*$ if and only if C^* is a cutset of G.

Suppose first that C^* is a cutset of G. If C^* is independent in $(M(G))^*$, then C^* can be extended to a base B^* of $(M(G))^*$, and so $C^* \cap (E - B^*)$ is empty. This contradicts the result of Theorem 3.2(i), since $E - B^*$ is a spanning tree of G. Thus, C^* is a dependent set in $(M(G))^*$, and therefore contains a cycle of $(M(G))^*$.

If, on the other hand, D^* is a cycle of $(M(G))^*$, then D^* is not contained in any base of $(M(G))^*$. It follows that D^* intersects every base of $M(G)$ – that is, every spanning tree of G. Thus, by the result of Exercise 3.32(i), D^* contains a cutset. The result follows. ■

Before proceeding further, we introduce some more terminology. We say that elements of a matroid M form a **cocycle** of M if they form a cycle of M^*. Note that, in view of Theorem 7.4, the cocycles of the cycle matroid of a graph G are precisely the cutsets of G. We similarly define a **cobase** of M to be a base of M^*, with corresponding definitions for **corank**, **co-independent set**, etc. We also say that a matroid M is **cographic** if and only if its dual M^* is graphic. In view of Theorem 7.4, this definition agrees with the one given in the previous section.

The reason for introducing this 'co-notation' is that we can now restrict ourselves to a single matroid M, without having to bring in M^*. To illustrate this, we prove the analogue for matroids of Theorem 3.2(i).

THEOREM 7.5 *Every cocycle of a matroid intersects every base.*

Proof. Let C^* be a cocycle of a matroid M, and suppose that there exists a base B of M with the property that $C^* \cap B$ is empty. Then C^* is contained in $E - B$, and so C^* is a cycle of M^* which is contained in a base of M^*. This contradiction establishes the result. ■

COROLLARY 7.6 *Every cycle of a matroid intersects every cobase.*

Proof. Apply the result of Theorem 7.5 to the matroid M^*. ■

From a matroid point of view, the two results in Theorem 3.2 are dual forms of a single result. Thus, instead of proving two results for graphs, as we had to in Chapter 3, it is sufficient to prove a single result for matroids and then use duality. This represents a saving of time and effort, and also gives us greater insight into several results that we met earlier.

As a further example of these simplifications, let us return to Exercise 2.46. A straightforward justification of these results involved two separate proofs – one for cycles and a different one for cutsets. If, however, we prove the matroid analogue of the result for cycles, as stated in Section 7.1, then we simply apply it to the matroid $M^*(G)$ to deduce the corresponding result for cutsets. Conversely, we can use duality to deduce the result for cycles from the result for cutsets.

Planar graphs

We now turn our attention to planar graphs, and show how the definitions of a geometric dual and an abstract dual of a graph arise as consequences of matroid duality. The Whitney dual of a graph, introduced in Exercise 4.43, is also a consequence of matroid duality, since the equation given in that exercise is simply a restatement of the expression for r^* at the beginning of this section.

We start with the abstract dual.

THEOREM 7.7 *If G^* is an abstract dual of a graph G, then $M(G^*)$ is isomorphic to $(M(G))^*$.*

Proof. Since G^* is an abstract dual of G, there is a one–one correspondence between the edges of G and those of G^* such that cycles in G correspond to cutsets in G^* and conversely. It follows immediately that the cycles of $M(G)$ correspond to the cocycles of $M(G^*)$. Thus, by Theorem 7.4, $M(G^*)$ is isomorphic to $M^*(G)$. ■

COROLLARY 7.8 *If G^* is a geometric dual of a connected plane graph G, then $M(G^*)$ is isomorphic to $(M(G))^*$.*

Proof. This follows immediately from Theorems 7.7 and 4.14. ■

As remarked earlier, a planar graph G can have several different duals, whereas a matroid can have only one. The reason for this is that, if we have two (possibly non-isomorphic) duals of G, then their cycle matroids are isomorphic matroids.

Graphic matroids

We conclude by answering the question ‘under what conditions is a given matroid M graphic?’ It is not difficult to find necessary conditions. For example, it follows from our discussion of representable matroids in Section 7.2 that such a matroid must be binary. Furthermore, by our discussions of the Fano matroid F and of minors, M cannot contain as a minor any of the matroids $M^*(K_5)$, $M^*(K_{3,3})$, F or F^*. It was shown by W. T. Tutte that these necessary conditions are sufficient. The proof of this result is too difficult to be given here (see Welsh [45]).

THEOREM 7.9 (Tutte, 1958) *A matroid M is graphic if and only if it is binary and contains no minor isomorphic to $M^*(K_5)$, $M^*(K_{3,3})$, F or F^*.*

On applying Theorem 7.9 to M^*, and noting that the dual of a binary matroid is binary, we obtain necessary and sufficient conditions for a matroid to be cographic.

COROLLARY 7.10 *A matroid M is cographic if and only if it is binary and contains no minor isomorphic to $M(K_5)$, $M(K_{3,3})$, F or F^*.*

Tutte also proved that:

A binary matroid is regular if and only if it contains no minor isomorphic to F or F^*.

By combining this result with those of Theorem 7.9 and Corollary 7.10, we immediately deduce the following matroid analogue of Kuratowski's theorem (Theorem 4.2).

THEOREM 7.11 *A matroid is planar if and only if it is regular and contains no minor isomorphic to $M(K_5)$, $M(K_{3,3})$ or their duals.*

Exercises

7.16[s] (i) Show that the dual of a discrete matroid is a trivial matroid.
(ii) What is the dual of the k-uniform matroid on a set of n elements?

7.17 Find the duals of the eight matroids on the set $E = \{a, b, c\}$, obtained in Exercise 7.9.

7.18 Verify property r(ii) of Section 7.1 for the function r^*.

7.19[s] Verify the result of Theorem 7.4 for the graph K_3.

7.20[s] What are the cocycles and cobases of
(i) the 3-uniform matroid on a set of nine elements?
(ii) the cycle matroids of the graphs in Fig. 7.2?
(iii) the cycle matroid of the graph in Fig. 7.3?
(iv) the Fano matroid?

7.21 Give an example to show that the dual of a transversal matroid need not be a transversal matroid.

7.22 Show that the contraction matroid $M \cdot A$ is the matroid whose cocycles are precisely those cocycles of M that are contained in A.

Challenge problems

7.23 Consider a matroid on a set E. Use properties $\mathcal{B}$(i) and $\mathcal{B}$(ii) to prove that if $A \subseteq E$, then any two maximal independent subsets of A have the same number of elements.

7.24 State and prove a matroid analogue of the greedy algorithm (Theorem 3.6).

7.25 Let E be a set of n elements. Prove that, up to isomorphism,
(i) the number of matroids on E is at most 2^{2^n};
(ii) the number of transversal matroids on E is at most 2^{n^2}.

7.26 Prove that, if M is graphic, cographic, binary and/or regular, then so is any minor of M.

7.27 Let M be a binary matroid on a set E.
(i) Prove that, if M is an Eulerian matroid, then M^* is bipartite.
(ii) Use induction on $|E|$ to prove the converse result.
(iii) By considering the 5-uniform matroid on a set of 11 elements, show that the word ‘binary’ cannot be omitted.
(This result generalizes Exercise 4.31.)

7.28 If C is a cycle and C^* is a cocycle in a matroid, prove that $|C \cap C^*| \neq 1$.
(This result generalizes the result of Exercise 2.47.)

Appendix 1: Algorithms

In this book we have encountered a number of problems, and have presented systematic methods for solving them: these problems include the *shortest path problem*, the *critical path problem*, the *Chinese postman problem*, the *travelling salesman problem*, the *minimum connector problem* and the *network flow problem*. Some of the algorithms we presented were efficient, while for others we claimed that no efficient algorithm is known. But what do we mean by the efficiency of an algorithm?

Algorithms

An **algorithm** is a finite step-by-step procedure for solving a problem. You can think of it as a black box into which you input some data, turn the handle, and produce an output that solves the problem. The word 'algorithm' derives from the name of the ninth-century Persian mathematician al-Khwarizmi, who wrote an influential arithmetic book on the Hindu-Arabic numerals. When this book was translated into Latin, the opening words became *Dixit algorismus* and the word *algorism* came to be used in the Middle Ages to mean 'arithmetic'.

In his famous 1798 *Essay on population*, Thomas Malthus contrasted the steady linear growth of food supplies with the exponential growth in population. He concluded that however well one may cope in the short term, the exponential growth would win in the long term, and there would be severe food shortages – a conclusion that has been borne out in practice.

A similar distinction can be made for algorithms used to solve problems. Each problem has an *input size n*, such as the number of cities in a network, and the algorithm has a *running time*, which may be the time that a computer needs to carry out all the necessary calculations, or the actual number of such calculations. Usually, the running time depends on the size n of the input.

Particularly important, because they are the most efficient, are the *polynomial-time algorithms*, in which the maximum running time is proportional to a power of the input size – say, to n^2 or n^7. The collection of all problems that can be solved by a polynomial-time algorithm is called *P*.

In contrast, there are algorithms that are known not to take polynomial time, such as the *exponential-time algorithms* in which the running time may grow like 2^n or faster. The collection of all problems that are known to have no polynomial algorithm is called *not-P*.

To see the difference, consider the following table that compares the running times for input sizes $n = 10$ and $n = 50$, for various types of polynomial-time and exponential-time algorithms on a computer performing one million operations per second; note how rapidly the time taken for exponential-time algorithms increases with n:

	$n = 10$	$n = 50$
n	0.00001 seconds	0.00005 seconds
n^2	0.0001 seconds	0.0025 seconds
n^3	0.001 seconds	0.125 seconds
n^5	0.1 seconds	5.2 minutes
2^n	0.001 seconds	35.7 years
3^n	0.059 seconds	2.3×10^{10} years

There are many polynomial-time algorithms around: their running time is at worst some power of the input – usually the square or the cube. We say that the running time of a graph algorithm is $O(n^k)$ if it is at most a constant multiple of n^k, where n is the input size (usually the number of vertices of the graph). Examples include algorithms for:

- the *shortest path problem*, with running time $O(n^2)$;
- the *Chinese postman problem*, with running time $O(n^3)$;
- *finding an Eulerian trail* (Fleury's algorithm), with running time $O(m)$, where m is the number of edges;
- the *minimal connector problem*, with running time $O(n^2)$;
- the *planar graph problem* (is a given graph planar?), with running time $O(n)$.

So P is the set of 'easy' problems that can be solved with a polynomial-time algorithm. On the other hand, there are several problems for which no polynomial algorithm has ever been found: these include the *travelling salesman problem*. But if you suggest a possible route for the salesman, then you can check in polynomial time whether it is a cyclical route and whether its length exceeds the minimum known. In general, the collection of 'non-deterministic polynomial-time problems' – those for which a solution, when given, can be checked in polynomial time – is called NP. So the travelling salesman problem is in NP.

The $P = NP$? question

Clearly P is contained in NP, since if a problem can be solved in polynomial time, then the solution can certainly be checked in polynomial time – checking a solution is far easier than finding it in the first place. But are they actually the same? Is $P = NP$? It seems very unlikely – indeed, few people believe that they are the same – but it's never been proved, and is one of the major unsolved problems in mathematics today with a $1 million prize for its solution.

In 1971 Stephen Cook, of the University of Toronto, wrote a short but fundamental paper, 'The complexity of theorem-proving procedures', in which he considered a particular problem in NP called the *satisfiability problem*, and proved the amazing result that any other problem in NP can be transformed into it in polynomial time. This means that, going via the satisfiability problem:

> *Any problem in NP can be transformed to any other problem in NP in polynomial time.*

So if the satisfiability problem is in P, then so is everything in NP, and $P = NP$. But if the satisfiability problem is not in P, then $P \neq NP$. Thus the whole $P = NP$? question depends on whether we can find a polynomial algorithm for just one problem. But not just one – there are thousands of such problems to choose from!

We say that a problem is *NP-complete* if its solution in polynomial time means that *every* problem in *NP* can be solved in polynomial time. These include the satisfiability problem, the travelling salesman problem and thousands of others. If you could find a polynomial algorithm for just one of them, then polynomial algorithms would exist for the whole lot, and *P* would be the same as *NP*. On the other hand, if you could prove that *just one* of them has no polynomial algorithm, then none of the others could have a polynomial algorithm either, and *P* would be different from *NP*.

Examples of *NP*-complete problems include the following:

- the *travelling salesman problem*;
- the *isomorphism problem* (are two given graphs isomorphic?);
- the *Hamiltonian cycle problem* (does a given graph have a Hamiltonian cycle?);
- the *3-colourability problem* (is a given graph 3-colourable?).

Appendix 2: Table of numbers

This table lists the number of graphs and digraphs of various types with n vertices, for $n \leq 8$. Numbers greater than 10^8 are given to two significant figures.

Types of graphs $n =$	1	2	3	4	5	6	7	8
Simple graphs	1	2	4	11	34	156	1044	12,346
Connected simple graphs	1	1	2	6	21	112	853	11,117
Eulerian simple graphs	1	0	1	1	4	8	37	184
Hamiltonian simple graphs	1	0	1	3	8	48	383	6020
Trees	1	1	1	2	3	6	11	23
Labelled trees	1	1	3	16	125	1296	16,807	262,144
Simple digraphs	1	3	16	218	9608	1,540,944	8.8×10^8	1.8×10^{12}
Connected simple digraphs	1	2	13	199	9364	1,530,843	8.8×10^8	1.8×10^{12}
Strongly connected simple digraphs	1	1	5	83	5048	1,047,008	7.1×10^8	1.6×10^{12}
Tournaments	1	1	2	4	12	56	456	6880

List of symbols

I've got a little list.

W. S. Gilbert

$\mathbf{A}$	adjacency matrix
$A(D)$	arc-family of D
B	base of M
$\mathcal{B}$	set of bases of a matroid
C_n	cycle graph
$\mathcal{C}$	set of cycles of a matroid
$\mathrm{cr}(G)$	crossing number of G
D	digraph
$\deg(v)$	degree of v
E	non-empty finite set
$E(G)$	edge-set of G
f	number of faces
F	Fano matroid
$\mathcal{F}$	family of subsets
g	genus
G	graph
G	complement of G
G^*	dual of G
$G(V_1, V_2)$	bipartite graph
$G_1 \cup G_2$	union of graphs
I	independent set
$\mathcal{I}$	set of independent sets of a matroid
k	number of components
K_n	complete graph
$K_{r,s}$	complete bipartite graph
$K_{r,s,t}$	complete tripartite graph
$L(G)$	line graph of G
m	number of edges
M	matroid
M^*	dual matroid
$M \cdot A$	contraction matroid
$M \times A$	restriction matroid
$M(G)$	cycle matroid
$M(S_1, \ldots, S_m)$	transversal matroid
n	number of vertices
N_n	null graph
$P_G(k)$	chromatic polynomial of G
P_n	path graph
Q_k	k-cube
r	rank function of M
r^*	rank function of M^*
$t(G)$	thickness of G
T	tree
u, v, w, z	vertices of G
$v_0 \to \cdots \to v_m$	walk, trail or path
$V(D)$	vertex-set of digraph D
$V(G)$	vertex-set of graph G
W_n	wheel
$\alpha, \beta, \gamma, \delta$	colours
$\gamma(G)$	cycle rank of G
Δ	largest vertex degree of G
$\kappa(G)$	connectivity of G
$\lambda(G)$	edge-connectivity of G
$\xi(G)$	cutset rank of G
$\chi(G)$	chromatic number of G
$\chi'(G)$	chromatic index of G

Bibliography

Of making many books there is no end;
and much study is a weariness of the flesh.
Ecclesiastes

Although we have almost reached the end of this book, we have by no means reached the end of the subject. We hope that you will continue your study of graph theory, and for this reason we suggest possible directions for further reading.

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Solutions to selected exercises

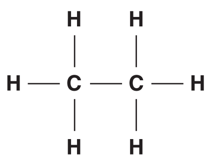
Introduction

0.1 (i) There are five vertices and eight edges; vertices P and T have degree 3, vertices Q and S have degree 4, and vertex R has degree 2.

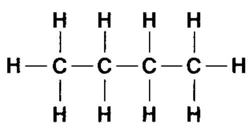
(ii) There are six vertices and five edges; vertices A, B, E and F have degree 1 and vertices C and D have degree 3.

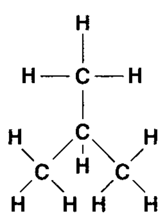
0.3 (i) Each carbon atom vertex has degree 4 and each hydrogen atom vertex has degree 1.

(ii) The graph is as follows:

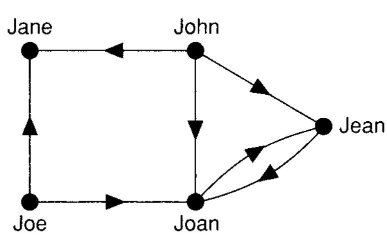


(iii) The graphs are as follows:





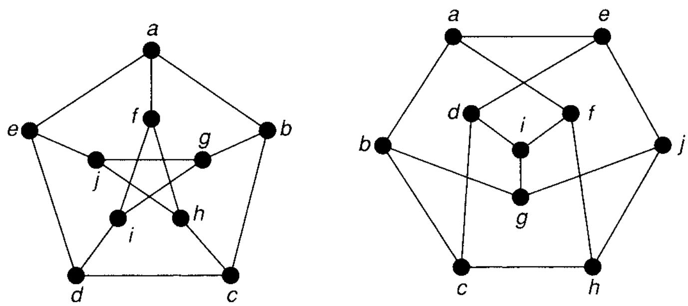
0.5 A suitable digraph is as follows:



Chapter 1

1.1 $V(G) = \{u, v, w, x, y, z\}$ and $E(G) = \{ux, uy, uz, vx, vy, vz, wx, wy, wz\}$;
$V(G) = \{l, m, n, p, q, r\}$ and $E(G) = \{lp, lq, lr, mp, mq, mr, np, nq, nr\}$.

1.4 We can label the vertices as follows:



1.5 In the first graph, no vertices of degree 2 are adjacent; in the second graph they are adjacent in pairs. Since isomorphism preserves adjacency of vertices, the graphs are not isomorphic.

1.8 (i) graph 12; (ii) graph 27; (iii) graph 30.

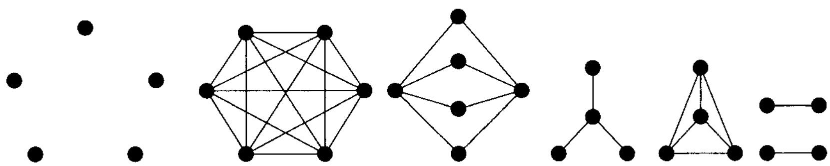
1.11 graph 5: degree sequence (1, 1, 1, 3); sum of degrees = 6, number of edges = 3.
graph 6: degree sequence (1, 1, 2, 2); sum of degrees = 6, number of edges = 3.
graph 7: degree sequence (1, 2, 2, 3); sum of degrees = 8, number of edges = 4.
graph 8: degree sequence (2, 2, 2, 2); sum of degrees = 8, number of edges = 4.
graph 9: degree sequence (2, 2, 3, 3); sum of degrees = 10, number of edges = 5.
graph 10: degree sequence (3, 3, 3, 3); sum of degrees = 12, number of edges = 6.

In each case, the sum of the degrees is twice the number of edges.

1.16 The cycles with five and six vertices.

1.20
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

1.25

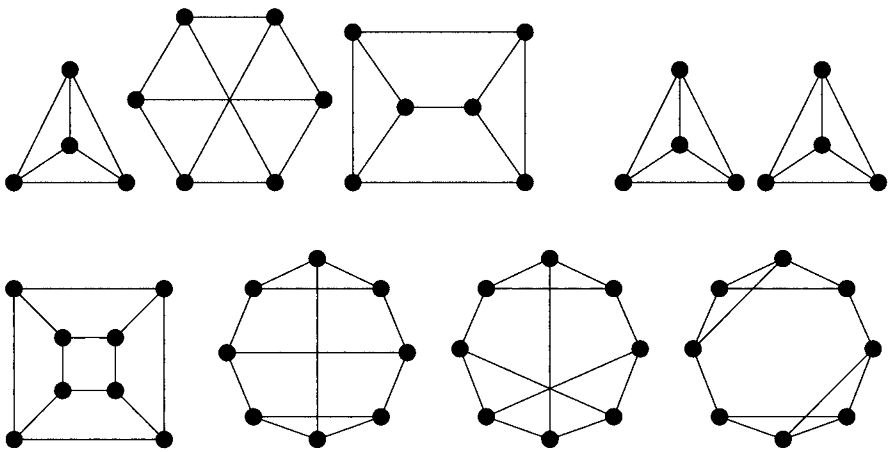


(i) (ii) (iii) (iv) (v)

1.26 Regular graphs: 1, 2, 4, 8, 10, 18, 31;
bipartite graphs: 2, 3, 5, 6, 8, 11, 12, 13, 17, 23.

1.27 (i) 45; (ii) 35; (iii) 32; (iv) 14; (v) 15.

1.32 There are eight such graphs:



1.35 $V(D) = \{u, v, w, z\}$ and $A(D) = \{uv, vw, wu, wv, zw\}$.

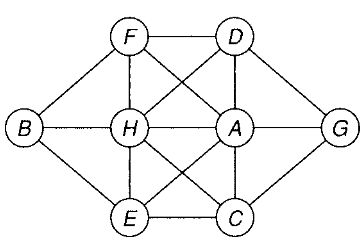
1.37 The first and last digraphs.

1.39 Sum of out-degrees = 1 + 3 + 2 + 1 = number of arcs = 7;
sum of in-degrees = 1 + 3 + 3 + 0 = number of arcs = 7.

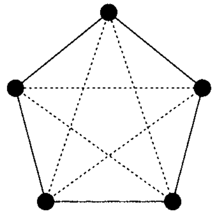
1.41 $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

1.45 (i) The infinite square lattice, or an 'infinite star' obtained by joining the origin to infinitely many points on the unit circle;
(ii) the infinite hexagonal lattice.

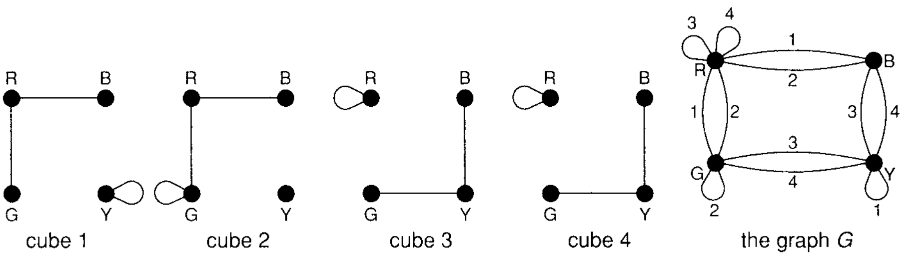
1.47 There are several possible solutions, all of them modifications of the solution in the text. One of these is as follows:



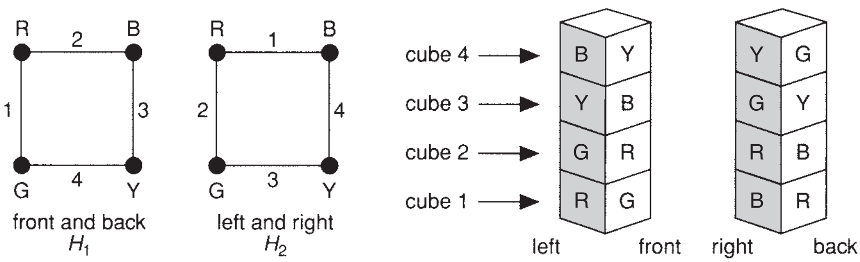
1.48 The following diagram illustrates such a gathering, with solid and dotted edges used as in the text.



1.49 Using the method described in the text, we obtain the following graphs:

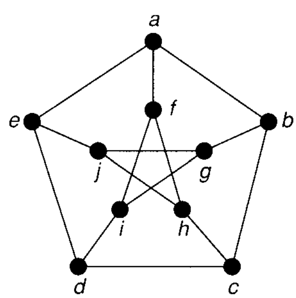


A pair of subgraphs H_1 and H_2 and a corresponding solution are as follows; there are several other solutions.



Chapter 2

2.1



There are many possibilities; for example:

(i) $a \to b \to c \to d \to e \to j$;
(ii) $a \to b \to c \to d \to e \to j \to h \to f \to i \to g$;
(iii) $a \to b \to c \to d \to e \to a$;
$a \to b \to c \to d \to i \to f \to a$;
$a \to b \to c \to d \to e \to j \to h \to f \to a$;
$a \to b \to c \to d \to e \to j \to g \to i \to f \to a$;
(iv) $\{ab, ae, af\}$, $\{ab, af, de, ej\}$, $\{ab, af, cd, di, ej\}$.

2.3 (i) 3; (ii) 4; (iii) 8; (iv) 3; (v) 4; (vi) 5; (vii) 5.

2.5 Let G be disconnected, and let v and w be vertices of G. If v and w lie in different components of G, then they are adjacent in $\bar{G}$. If v and w lie in the same component of G and z lies in another component, then $v \to z \to w$ is a path in $\bar{G}$. In either case, any two vertices can be connected by a path in $\bar{G}$, and hence $\bar{G}$ is connected.

2.6 (i) $\kappa = \lambda = 2$; (ii) $\kappa = \lambda = 3$; (iii) $\kappa = \lambda = 4$; (iv) $\kappa = \lambda = 4$.

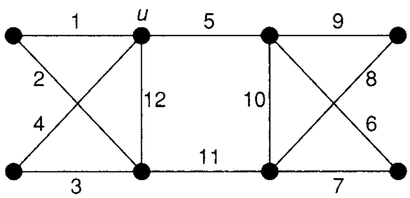
2.14 Consider the infinite star of Exercise 1.45(i).

2.15 (i) Eulerian; (ii) semi-Eulerian; (iii) neither; (iv) Eulerian; (v) neither.

2.16 Eulerian graphs: 1, 4, 8, 18, 21, 25, 31;
semi-Eulerian graphs: 2, 3, 6, 7, 9, 13, 14, 16, 17, 19, 22, 23, 26, 28, 30.

2.19 (i) At least $\frac{1}{2}k$ trails are needed, so as to 'use up' all k vertices of odd degree. If we now add $\frac{1}{2}k$ edges to G and join these vertices in pairs, then we obtain an Eulerian graph G'. We obtain the required $\frac{1}{2}k$ trails by writing down an Eulerian trail for G' and then omitting the added edges.
(ii) Four.

2.20 There are many possible solutions; for example, traverse the edges in the order indicated by the following diagram:



2.22 *aedabdcbeca*.

2.27 (i) Hamiltonian; (ii) semi-Hamiltonian; (iii) Hamiltonian; (iv) Hamiltonian; (v) Hamiltonian.

2.28 Hamiltonian: 1, 4, 8, 9, 10, 18, 22, 26, 27, 28, 29, 30, 31;
semi-Hamiltonian: 2, 3, 6, 7, 13, 15, 16, 17, 19, 20, 21, 23, 24, 25.

2.33 The complete bipartite graph $K_{(n/2)-1,(n/2)+1}$ if n is even, and $K_{(n-1)/2,(n+1)/2}$ if n is odd.

2.35 *aecbda*.

2.36 The permanent labels are, successively,

$$l(A) = 0,\ l(B) = 30,\ l(D) = 36,\ l(C) = 48,\ l(F) = 58,\ l(E) = 69,\ l(G) = 77.$$

The shortest path, of length 77, is

$$A \to B \to D \to C \to F \to E \to G.$$

2.39 G, 12; E, 10; B, 6.

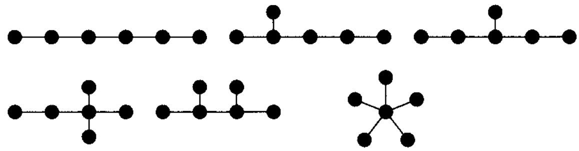
2.41 Doubling the edges along the path $B \to D \to E \to A \to C$ yields a solution with total weight 24.

2.42 The required Hamiltonian cycle is $A \to B \to C \to E \to D \to A$, with total weight 14.

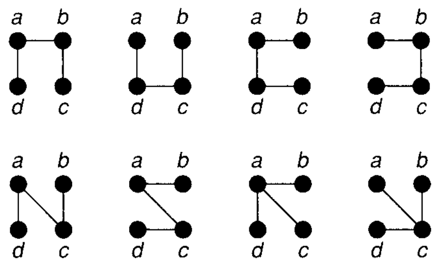
Chapter 3

3.1 Trees: 1, 2, 3, 5, 6, 11, 12, 13.

3.2



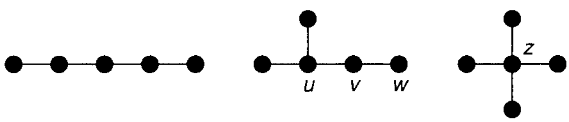
3.5



3.7 Cycles: *abcdea*, *abca*, *abcda*, *cdc*;
cutsets: {*ab*, *ac*, *ad*, *ae*}, {*ac*, *ad*, *ae*, *bc*}, {*ad*, *ae*, *cd*, *cd*}, {*ae*, *de*}.

3.9 (i) It is a bridge; (ii) it is a loop.

3.12 There are three unlabelled trees on five vertices.

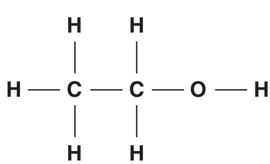


The first tree can be labelled in $(5!)/2 = 60$ ways; the second tree can be labelled in $5 \times 4 \times 3 = 60$ ways, corresponding to the 60 possible choices for u, v and w; the third tree can be labelled in 5 ways, corresponding to the 5 possible choices for z. The total number is therefore $60 + 60 + 5 = 125$.

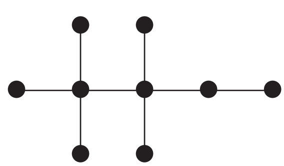
3.14 The graph is a connected graph with

$$n + (2n + 1) + 1 + 1 = 3n + 3 \text{ vertices and } \tfrac{1}{2}\{4n + (2n + 1) + 2 + 1\} \text{ edges,}$$

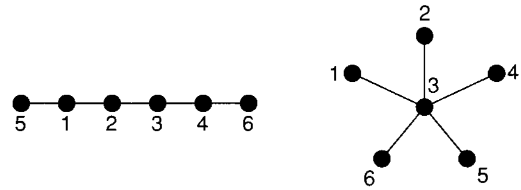
and is therefore a tree, by Theorem 3.1(iii).



or

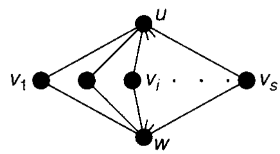


3.15 (i)



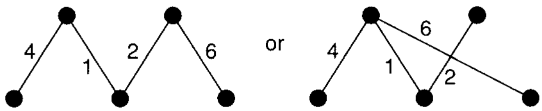
(ii) (4, 4, 4, 1) and (4, 2, 2, 4).

3.18



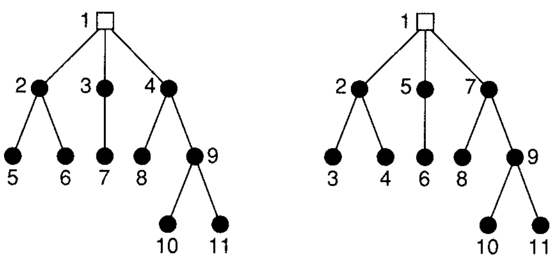
Each spanning tree in $K_{2,s}$ contains one of the two edges uv_i and v_iw, for each i, together with one extra edge. The number of spanning trees is therefore $2^s \times \frac{1}{2}s = s2^{s-1}$.

3.20 We obtain either of the following weighted trees with total weight 13:



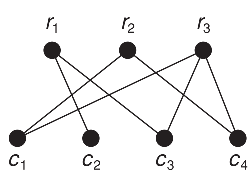
3.24 vertex A: 15 + (2 + 4) = 21; vertex B: 17 + (2 + 3) = 22;
vertex D: 15 + (3 + 4) = 22; vertex E: 12 + (5 + 6) = 23.

3.25 We obtain the following labelled trees, where the labels correspond to the order in which the vertices are visited:



breadth-first search depth-first search

3.27



Since the bipartite graph is connected, the framework is rigid.
The graph contains a cycle, so the bracing is not a minimum bracing.

3.30 The fundamental cycle equations are:

$$VWZYXV\colon\ i_1 + i_3 - i_6 + i_7 = 12; \quad VWZV\colon\ i_3 + i_5 + i_7 = 0;$$
$$VWZYV\colon\ -i_2 + i_3 - i_6 + i_7 = 0; \quad WZYW\colon\ -i_4 - i_6 + i_7 = 0.$$

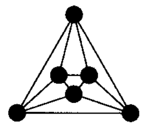
The vertex equations are:

$$V\colon\ i_1 + i_5 = i_2 + i_3;\ W\colon\ i_3 = i_4 + i_7;\ X\colon\ i_0 = i_1;\ Y\colon\ i_0 + i_6 = i_2 + i_4;\ Z\colon\ i_5 = i_6 + i_7.$$

These equations have the solution $i_0 = i_1 = 8$, $i_2 = 4$, $i_3 = i_4 = 2$, $i_5 = i_6 = -2$, $i_7 = 0$.

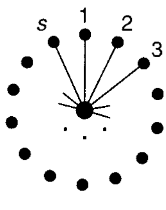
Chapter 4

4.1

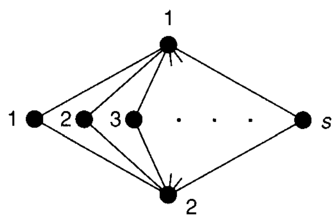


4.3 No, since $K_{3,3}$ is non-planar.

4.4 The complete graph K_n is planar if $n \le 4$.
The complete bipartite graph $K_{r,s}$ ($r \le s$) is planar if $r = 1$ or 2, as shown by the following plane drawings:

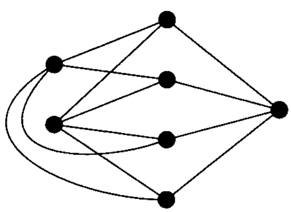


$K_{1,s}$

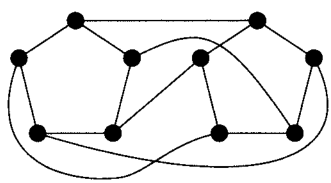


$K_{2,s}$

4.6 The following drawings show how the graphs can be drawn with two crossings. Some experimentation should convince you that no drawing with just one crossing is possible.

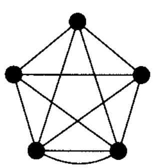


$K_{4,3}$

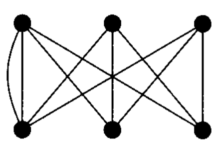


Petersen graph

4.8 (i) and (ii)



or



Although these graphs are not homeomorphic or contractible to K_5 or $K_{3,3}$, each contains a *subgraph* homeomorphic or contractible to K_5 or $K_{3,3}$.

4.13 (i) $n = 8$, $m = 14$, $f = 8$, and $8 - 14 + 8 = 2$;
(ii) $n = 6$, $m = 12$, $f = 8$, and $6 - 12 + 8 = 2$;
(iii) $n = 9$, $m = 15$, $f = 8$, and $9 - 15 + 8 = 2$;
(iv) $n = 9$, $m = 14$, $f = 7$, and $9 - 14 + 7 = 2$.

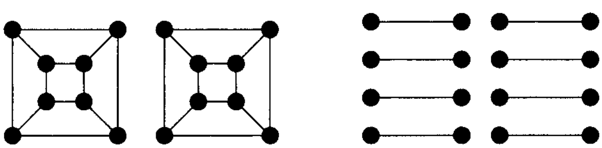
4.15 (i) Since G has girth 5, we have $5f \le 2m$.
Combining this with Euler's formula $n - m + f = 2$ gives the required inequality.
If the Petersen graph were planar, then this inequality would be $15 \le {}^{40}/_{3}$, which is false. Thus, the Petersen graph is non-planar.
(ii) If G has girth r, then $rf \le 2m$.
Combining this with Euler's formula gives the inequality $m \le r(n - 2)/(r - 2)$.

4.18 (i) If G has n vertices, m edges and f faces, then

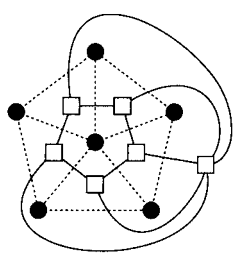
$$\begin{aligned} f &= C_3 + C_4 + C_5 + C_6 + \ldots, \\ 2m &= 3C_3 + 4C_4 + 5C_5 + 6C_6 + \ldots, \\ 3n &= 3C_3 + 4C_4 + 5C_5 + 6C_6 + \ldots. \end{aligned}$$

Substituting these expressions for f, m and n into Euler's formula yields the result.
(ii) Since $C_3 = C_4 = C_7 = C_8 = \cdots = 0$, we deduce that $C_5 = 12$.
(iii) If G has no face bounded by at most five edges, then $C_3 = C_4 = C_5 = 0$, and the left-hand side is negative; this is a contradiction.

4.20 (i) Since the Petersen graph is non-planar, its thickness is at least 2. But the Petersen graph can be obtained by superimposing two planar graphs, such as the outer pentagon and the 'spokes', and the inner pentagon. So the Petersen graph has thickness 2.
(ii) Q_4 is not planar, as can be seen by applying Corollary 4.8(ii). So its thickness is at least 2. But Q_4 can be obtained by superimposing two planar graphs as follows. So Q_4 has thickness 2.



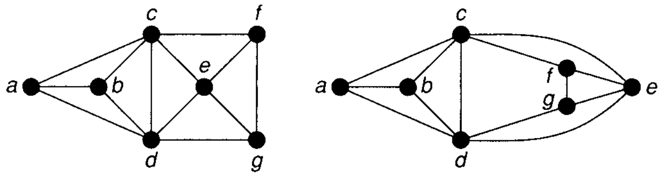
4.22



$$n^* = f = 6,\ m^* = m = 10,\ f^* = n = 6.$$

4.26 If such a plane graph existed, then its dual would be a plane graph with five mutually adjacent vertices. Since K_5 is non-planar, this is impossible.

4.27



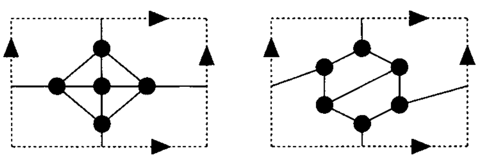
The above labelling shows that the given graphs are isomorphic. In the dual graphs the vertex-degrees are all 3 or 5 on the left and 3 or 4 on the right, so these duals are not isomorphic.

4.29 If G is a simple plane graph in which each vertex has degree 5 or 6, then G has at least 12 vertices of degree 5; if, in addition, each face is a triangle, then G has exactly 12 vertices of degree 5.

4.30 If G is 3-connected, then G has no vertices of degree 1 or 2, and hence G^* has no loops or multiple edges, and is therefore a simple graph.

4.31 If G is bipartite, then each cycle of G has even length, and thus each cutset of G^* has an even number of edges; in particular, each vertex of G^* has even degree, and thus G^* is Eulerian. The reverse implication is obtained by reversing the argument.

4.33



4.35 (i) $g(K_7) = \lceil {}^1\!/_{12}(7-3)(7-4) \rceil = 1$;
$g(K_{11}) = \lceil {}^1\!/_{12}(11-3)(11-4) \rceil = \lceil {}^{56}\!/_{12} \rceil = 5$.
(ii) K_8, since $g(K_8) = \lceil {}^1\!/_{12}(8-3)(8-4) \rceil = \lceil {}^{20}\!/_{12} \rceil = 2$.

4.37 (i) The octahedron graph.
(ii) For such a graph, $4n = 2m = 3f$. It follows from Theorem 4.19 that

$$ {}^1\!/_2 m - m + {}^2\!/_3 m = 2 - 2g, $$

and so $m = 12(1 - g)$, which is not positive.
This contradiction shows that no such graph can exist.

Chapter 5

5.1 2 and 4.

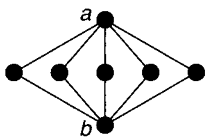
5.3 2-chromatic: 2, 3, 5, 6, 8, 11, 12, 13, 17, 23;
3-chromatic: 4, 7, 9, 14, 15, 16, 18, 19, 20, 21, 22, 25, 26, 27, 29;
4-chromatic: 10, 24, 28, 30.

5.5 (i) Upper bound = 3, chromatic number = 3;
(ii) upper bound = k, chromatic number = 2.

5.7 If c_i is the number of vertices coloured i, for $1 \le i \le \chi(G)$, then $c_i \le n - d$. Thus, $n \le c_1 + c_2 + \cdots + c_{\chi(G)} \le \chi(G) \times (n - d)$, and so $\chi(G) \ge n/(n - d)$.

5.9 (i) $k(k-1)(k-2)(k-3)(k-4)(k-5)$;
(ii) $k(k-1)^5$.
K_6 can be coloured with seven colours in $7 \times 6 \times 5 \times 4 \times 3 \times 2 = 5040$ ways;
$K_{1,5}$ can be coloured with seven colours in $7 \times 6^5 = 54{,}432$ ways.

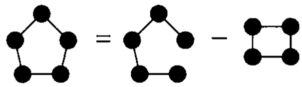
5.12 (i)



If vertices a and b have the same colour, then there are $k(k-1)^5$ colourings; if they have different colours, then there are $k(k-1)(k-2)^5$ colourings. Thus,

$$P_G(k) = k(k-1)^5 + k(k-1)(k-2)^5.$$

(ii) We have, by Theorem 5.6,

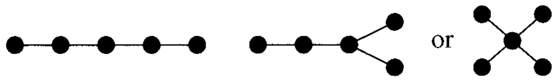


$$= k(k-1)^4 - k(k-1)(k^2 - 3k + 3) = k(k-1)(k^3 - 4k^2 + 6k - 4).$$

5.16 (i) Since

$$k(k-1)^{n-1} = k^n - (n-1)k^{n-1} + \cdots + (-1)^{n-1}k,$$

G has n vertices, $n - 1$ edges and one component.
It follows from Theorem 3.1(iii) that G is a tree on n vertices.
(ii) Since $P_G(k) = k(k-1)^4$, G must be a tree on five vertices – that is,



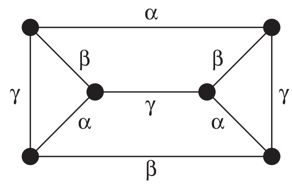
5.17 2 and 4.

5.20 Tetrahedron, 4; octahedron, 2; cube, 3; icosahedron, 3; dodecahedron, 4.

5.21 Any cycle graph with an even number of vertices; for example, C_4.

5.23 We prove the result by induction on the number of countries, the result being trivial for maps with at most six countries. Let G be a map with n countries, and assume that all maps with $n - 1$ countries are 6-colourable-(f). By Euler's theorem, G contains a country F bounded by at most five edges. If we shrink F to a point, then the remaining graph has $n - 1$ countries, and is thus 6-colourable-(f). A 6-colouring of the countries of G is then obtained by colouring F with a different colour from the (at most five) faces surrounding F. Thus, G is 6-colourable-(f).

5.25 Since the graph G contains vertices of degree 3, we have $\chi'(G) \geq 3$.
The following diagram illustrates a 3-edge-colouring of G, so $\chi'(G) = 3$.

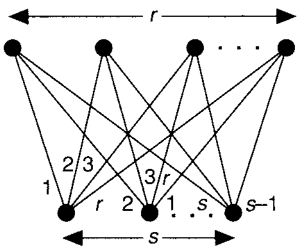


5.27 Chromatic index 2: 3, 6, 8, 13;
chromatic index 3: 4, 5, 7, 9, 10, 12, 15, 16, 17, 18, 20, 22, 23.

5.28 (i) Lower bound 2, upper bound 3, actual value 3;
(ii) lower bound 7, upper bound 8, actual value 7;
(iii) lower bound 6, upper bound 7, actual value 6.

5.30 Assume that $r \geq s$ and that $K_{r,s}$ is drawn as shown below, with the s vertices below the r vertices. Now successively colour the edges using the colours

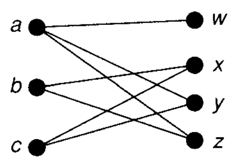
$$\{1, 2, \ldots, r\}, \{2, 3, \ldots, r, 1\}, \ldots, \{s, s+1, \ldots, r, 1, 2, \ldots, s-1\}.$$



5.31 Since G is regular of degree 3, we have $\chi'(G) \geq 3$. To obtain a 3-colouring of the edges of G, we colour the edges of a Hamiltonian cycle alternately red and blue, and then colour the remaining edges green.

Chapter 6

6.1 (i)



(ii) *aw, bx, cy*; *aw, bz, cx*; *aw, bz, cy*; *ay, bz, cx*; *az, bx, cy*.

(iii)

A	Ø	a	b	c	ab	ac	bc	abc
$\lvert A \rvert$	0	1	1	1	2	2	2	3
$\lvert \varphi(A) \rvert$	0	3	2	2	4	4	3	4

Thus $|A| \leq |\varphi(A)|$ for each subset A of $\{a, b, c\}$.

6.3 The first, third and fourth vertices in V_1 are collectively joined to only two vertices of V_2, and hence the marriage condition fails.

6.6 (i) No transversal.
The partial transversals are

Ø, 1, 2, 3, 4, 5, 12, 13, 14, 15, 23, 24, 25, 35, 123, 124, 125, 134, 135, 234, 235, 1234 and 1235.

(ii) Transversal – for example, {1, 2, 4, 5}.
(iii) No transversal.
The partial transversals are Ø, 1, 2, 3, 12, 13, 23 and 123.
(iv) Transversal – for example, {1, 4, 2, 5}.

6.8 By inspection, we can find eight transversals, each omitting just one of the eight letters in the word *MATROIDS*. For example, omitting *M*, we successively choose the letters *S*, *R*, *O*, *I*, *D*, *A*, *T*.

6.9 There is only one transversal – namely, {1, 2, . . . , 50}.

6.11 (i) Let $\mathcal{F} = (S_1, \ldots, S_5)$. Then the marriage condition fails for $\{S_3, S_4\}$ and $\{S_2, S_3, S_4\}$.
(ii) The union of any k of the subsets contains at least one element if $k = 1$ or 2, at least two elements if $k = 3$, at least four elements if $k = 4$, and five elements if $k = 5$, and so contains at least $k - 1$ elements for any value of k.
But $t = 4$, $m = 5$, so $k + t - m = k - 1$, as required.

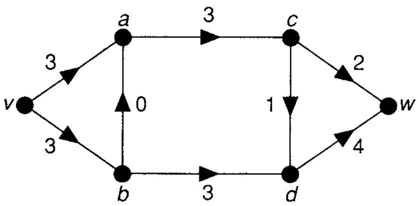
6.13 Edge form: the result is true with $k = 2$ for both graphs.
Vertex form: the result is true with $k = 2$ for both graphs.

6.15 The appropriate value of k is 3 for each graph.

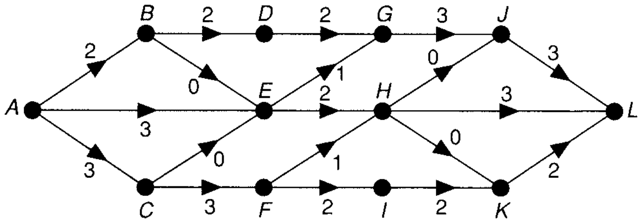
6.18 (i) The cuts are

{*va*, *vb*}, {*va*, *ba*, *bd*}, {*va*, *ba*, *cd*, *dw*}, {*vb*, *ba*, *ac*), {*vb*, *ba*, *cd*, *cw*}, {*ac*, *bd*), {*ac*, *cd*, *dw*}, {*bd*, *cd*, *cw*} and {*cw*, *dw*}.

The unique minimum cut is {*bd*, *cd*, *cw*} with capacity 6.
(ii) A corresponding maximum flow with value 6 is as follows:

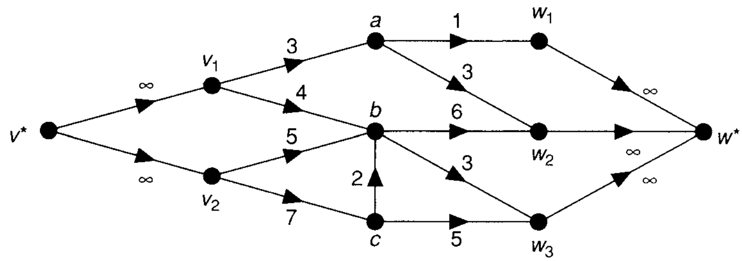


6.20 A cut with capacity 8 is {*BD*, *EG*, *EH*, *FH*, *FI*}.
A flow with value 8 is as follows:



6.22 (i) Add a new source vertex v^* joined to all the sources v_i with arcs v^*v_i of infinite capacity, and a new sink vertex w^* joined to all the sinks w_i with arcs w_iw^* of infinite capacity.

(ii)



Chapter 7

7.1 (i) Take the matroid in which each subset of E is independent, there are no cycles, and $r(A) = |A|$ for each subset A of E; this is the *discrete matroid*.

(ii) Take the matroid in which the only independent set is the empty set. The cycles are $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$, and the rank function is identically 0; this is the *trivial matroid*.

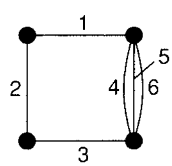
(iii) Take the matroid in which the independent sets are those subsets of E with zero, one, two or three elements, the cycles are those subsets with four elements, and if A is a subset of E, then $r(A) = \min\{|A|, 3\}$; this is the 3-*uniform matroid*.

7.2 $M(G_1)$ has bases *abd, acd* and *bcd,*
just one cycle *abc,*
and independent sets Ø, *a, b, c, d, ab, ac, ad, bc, bd, cd, abd, acd* and *bcd.*
$M(G_2)$ has bases *pqs, pqt, prt, pst, qrs, qrt, qst* and *prs,*
cycles *pqr, rst* and *pqst,*
and independent sets Ø, *p, q, r, s, t, pq, pr, ps, pt, qr, qs, qt, rs, rt, st, pqs, pqt, prt, pst, qrs, qrt, qst* and *prs.*

7.4 (i) The partial transversals are

Ø, 1, 2, 3, 4, 5, 6, 12, 13, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 123, 124, 125, 126, 134, 135, 136, 234, 235 and 236.

These partial transversals are the independent sets of the matroid $M(G)$, where G is the following graph:



(ii) The bases are 123, 124, 125, 126, 134, 135, 136, 234, 235 and 236;
the cycles are 1234, 1235, 1236, 45, 46 and 56.

7.6 (i) This follows immediately from the result of Exercise 2.46(ii) and the definition of a cutset.

(ii) graph G_1: *a*, *b* and *c*;
graph G_2: *pr*, *ps*, *pt*, *qr*, *qs*, *qt*, *rs* and *rt*.

7.8 Up to isomorphism, the four matroids on {*a*, *b*) are:

bases	*independent sets*	*Ecycles*
Ø	Ø	*a*, *b*
a	Ø, *a*	*b*
a, *b*	Ø, *a*, *b*	*ab*
ab	Ø, *a*, *b*, *ab*	–

7.10 (i) Yes: with the notation of Exercise 7.2,

$$M(G_1) = M(ab, bc, d) \quad \text{and} \quad M(G_2) = M(pq, qrs, st).$$

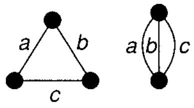
(ii) Yes: $M^*(G_1) = M(abc)$ and $M^*(G_2) = M(pqr, rst)$.

7.12 If M is a k-uniform matroid on E, then $M = M(\mathcal{F})$, where $\mathcal{F}$ consists of k copies of E.

7.14 The cycles of the Fano matroid are the lines, such as {1, 2, 4}, and the complements of the lines, such as {1, 2, 3, 6}.

7.16 (i) The only base of a discrete matroid on E is E itself, so the only base of its dual is Ø; the dual matroid is thus the trivial matroid on E.
(ii) The $(n - k)$-uniform matroid on the same set of n elements.

7.19 The bases of $M(K_3)$ are *ab*, *ac* and *bc*, so the bases of $(M(K_3))^*$ are *c*, *b* and *a*. Thus, $M(K_3^*)$ is isomorphic to $(M(K_3))^*$.



7.20 (i) The cocycles are the subsets of cardinality 7; the cobases are the subsets of cardinality 6.
(ii) $M(G_1)$ has cocycles *ab*, *ac*, *bc*, *d*, and cobases *a*, *b*, *c*.
$M(G_2)$ has cocycles *pq*, *prs*, *prt*, *qrs*, *qrt*, *st*, and cobases *pr*, *ps*, *pt*, *qr*, *qs*, *qt*, *rs*, *rt*.
(iii) The cocycles are 1 and 23, and the cobases are 2 and 3.
(iv) The cocycles are 1236, 1257, 1467, 1345, 2347, 2456 and 3567, and the cobases are all subsets of four elements containing a line, such as 1247.

Index

Only a little more
I have to write,
Then I'll give o'er
And bid the world Goodnight.
Robert Herrick

Here is my journey's end.

William Shakespeare (Macbeth)

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Robin Wilson is Emeritus Professor of Pure Mathematics at the Open University, and Emeritus Professor of Geometry at Gresham College, London. He is also a former Fellow in Mathematics at Keble College, Oxford University, and now teaches at Pembroke College. He has written and edited almost 40 books on graph theory, combinatorics, the history of mathematics, and music, and is very involved with the communication and popularisation of mathematics.

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