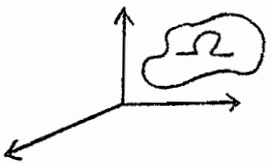
NINTH LECTURE

This lecture I will devote to giving you a little bit of an idea how the method of integration in function spaces can be applied to the classical potential theory. We will start just as before. Only now I will move myself in three-dimensional space. The reason why we go to three-dimensional space is because in two dimensions there are some very special features. Now let me have a region Ω and consider a three-dimensional Brownian motion.



I imagine that the particle starts from the origin $\vec{r} = 0$. Now remember what I did with $X(\tau)$, the Wiener process. I constructed certain basic sets of paths and assigned probabilities to them. Here it's a very similar thing that I do, in fact a pure analogy. $X(\tau)$ now becomes the vector $\vec{r}(\tau)$. Again you take n time points $t_1 < t_2 < \cdots < t_n$. And now, instead of using intervals, I will take n arbitrary open sets, $\Omega_1, \Omega_2, \ldots, \Omega_n$. What is the probability, the measure, of those paths for which $\vec{r}(t_1) \in \Omega_1$, $\vec{r}(t_2) \in \Omega_2$, . . . , and $\vec{r}(t_n) \in \Omega_n$? This again is going to be a multiple integral:

$$\int_{\Omega_1} d\vec{r}_1 \int_{\Omega_2} d\vec{r}_2 \cdots \int_{\Omega_n} d\vec{r}_n \, P(0|\vec{r}_1, t_1) P(\vec{r}_1|\vec{r}_2, t_2 - t_1) \cdots P(\vec{r}_{n-1}|\vec{r}_n, t_n - t_{n-1}) \quad (108)$$

Here each integral stands for a triple integral. $P(\vec{\rho}|\vec{r}, t)$ is now the Gaussian distribution in three-dimensions:

$$P(\vec{\rho}|\vec{r},t) = \frac{e^{-|\vec{r}-\vec{\rho}|^2/2t}}{(2\pi t)^{3/2}} \qquad (109)$$

Now you have a complete analogy. We can build up a measure theory in the same way as before, starting from these sets whose measure we know. As a matter of fact, I may as well mention that this is not really necessary. We don't have to go through this process a second time. We can take the measure already introduced and simply build up what's called product measure. Both ways get you there, so take your choice.

We now have a measure and hence also the possibility of integration in the space of all continuous paths. In this case, the paths emanate from the origin. With these tools we can build up very neatly the potential theory in three dimensions. The problem we are going to consider is the following: Take a region Ω and a point $\vec{y}$. In Figure 4 the point is drawn outside the region, but it can be anywhere.

Figure 4.

Now consider $\vec{y} + \vec{r}(\tau)$. This is a Brownian motion path which starts from $\vec{y}$ rather than from the origin. That is the only difference. The problem is to find out how much time is spent in the region Ω.

In order to get a nice symbolism going I will introduce the following function:

$$V(\vec{x}) = \begin{cases} 1 & \text{if } \vec{x} \in \Omega \\ 0 & \text{if } \vec{x} \notin \Omega \end{cases} \tag{110}$$

This is what is known as the characteristic function of the region Ω . Let me also consider the following functional:

$$T_\Omega(\vec{y}, \vec{r}(\tau)) = \int_0^\infty V\left(\vec{y} + \vec{r}(\tau)\right) d\tau \tag{111}$$

This integral may be infinite, but let me first of all see what its meaning is. Whenever $\vec{y} + \vec{r}(\tau)$ is in the region, I am integrating simply one. Otherwise I am integrating zero. So it clearly represents the total time the curve spends in the region Ω . It depends, of course, upon which curve $\vec{r}(\tau)$ the Brownian particle follows.

Now let me find the average of this quantity $T_\Omega(\vec{y}, \vec{r}(\tau))$ The average means the integral with respect to the measure we introduced. This is sometimes called the mathematical expectation and we have used the symbol E for it. What we want then is

$$E\left\{T_\Omega(\vec{y}, \vec{r}(\tau))\right\} = E\left\{ \int_0^\infty V\left(\vec{y} + \vec{r}(\tau)\right) d\tau \right\} \tag{112}$$

Now one does the usual trick. One interchanges these two operations (remember that this operator E is also an integration, an integration in function space) so you get:

$$E\left\{T_\Omega(\vec{y},\vec{r}(\tau))\right\} = \int_0^\infty d\tau\, E\left\{V(\vec{y}+\vec{r}(\tau))d\tau\right\} \tag{113}$$

Now this thing has a very vivid significance, because the function V is either zero or one depending on whether $\vec{y}+\vec{r}(\tau)$ *is in* Ω *or not.* A moment of thought will convince you that this just gives:

$$E\left\{T_\Omega(\vec{y},\vec{r}(\tau))\right\} = \int_0^\infty d\tau \left[\text{Prob}\left\{\vec{y}+\vec{r}(\tau)\in\Omega\right\}\right] \tag{114}$$

But this probability, that $\vec{y}+\vec{r}(\tau)\in\Omega$, I can immediately decipher from my basic assumption. It is the simplest case, where we have only one time involved. This gives us:

$$E\left\{T_\Omega(\vec{y},\vec{r}(\tau))\right\} = \int_0^\infty d\tau \int_\Omega d\vec{r}\, \frac{e^{-(\vec{r}-\vec{y})^2/2\tau}}{(2\pi\tau)^{3/2}} \tag{115}$$

Now we interchange the order of integration again:

$$E\left\{T_\Omega(\vec{y},\vec{r}(\tau))\right\} = \int_\Omega d\vec{r} \int_0^\infty d\tau\, \frac{e^{-(\vec{r}-\vec{y})^2/2\tau}}{(2\pi\tau)^{3/2}} \tag{116}$$

This last integral is well-known. You will find it in almost all texts on Laplace transform. So I will just put it in and obtain the final answer.

$$E\left\{T_\Omega(\vec{y},\vec{r}(\tau))\right\} = \frac{1}{2\pi}\int_\Omega \frac{d\vec{r}}{|\vec{r}-\vec{y}|} \tag{117}$$

This is finite in three or more dimensions. And this is true even if $\vec{y}$ would happen to be inside the region. It would still be finite, as you can see, by introducing polar coordinates. The integrand is essentially $1/r$ and the surface element will be of order r^2 . So there is plenty to spare to make it convergent.

Now what is the conclusion? So far, it is that the average of $T_\Omega(\vec{y},\vec{r}(\tau))$ is finite. But the quantity itself is non-negative. And if the average of a non-negative quantity is finite then with probability one the quantity itself must be finite. If it were infinite with a probability bigger than zero then of course the average would also have to be infinite. So you discover a very interesting, very simple fact: Almost every Brownian motion curve spends a finite amount of time in a region of finite volume. (I use "almost every" in a technical sense -- it means except for a set of curves of measure zero.) That is not so in the plane. For this reason the plane has to be treated separately. If you were to go through the whole argument you would come out with

$$\frac{e^{-\frac{|\vec{r}-\vec{\rho}|^2}{2\tau}}}{2\pi\tau} \tag{118}$$

as the integrand in (116). The t in the denominator means that this is not integrable at infinity. You end up with an infinite integral. This doesn't necessarily imply that almost every path spends an infinite time in the region. If the integral of a function is infinite that doesn't mean that the function is infinite. However, it can be shown by a somewhat more refined argument that in two dimensions almost every curve indeed spends an infinite amount of time in any region.

This distinction between the plane and space is very interesting. In fact, as you will see a little later, this is the beginning of a probabilistic explanation of the great difference between the two- and three-dimensional spaces in potential theory. Everybody knows there is a tremendous difference. In one case the basic Green's function is $1/r$ and in the other is $\log r$. That causes the great difference. From our point of view, you discover that the real difference is that in three dimensions almost every Brownian motion curve spends a finite amount of time in a given region. Whereas in two dimensions there is not enough room. The poor curve wanders around and comes back and back to the same region. In fact, one can refine the argument and prove the following: Almost every curve in two-dimensional space is everywhere dense. That means it comes arbitrarily close to every point. In three dimensions almost every curve is nowhere dense. Then you have the remarkable theorem that while every three-dimensional curve is nowhere dense the projections, the plane projections, are almost everywhere dense. Of course, one should not take this literally. This is not a statement about curves but a statement about the measures. It's not quite easy, actually, to imagine even one such curve. A curve which occupies hardly any space, which is kind of sparse,

but one whose projections on any plane go arbitrarily near to every point in that plane. Our particular argument shows that there are such curves.

Now this fact, this strange difference between two and three dimensions, was already noticed in a much simpler situation long, long ago by Polya. It was in 1923, I think, that Polya considered the following problem: Take a lattice, a square lattice, in the plane and consider a random walk, an ordinary discrete random walk. From every point you go to one of its four neighbors with probability one-fourth. Then, wherever you are, you go again to one of the neighbors with probability one-fourth. And so you continue. What is the probability that starting, say, from the origin you will eventually pass through a given point in the lattice? And the answer, in the plane, is one. That means that with probability one the curve will eventually pass through any pre-assigned point. You pick yourself a point and you are almost sure to hit it. In space, however, that isn't so. If you consider the corresponding problem in space you have six possible ways to go, six neighbors, and you make the probability one-sixth. If you now ask the same question for this cubic lattice, the answer is distinctly less than one. The probability that you will eventually hit a chosen point is no longer one. It can actually be calculated. The answer is expressible in terms of a certain integral.

It is really quite interesting, this distinction between the plane and space. In the plane the Brownian motion path tends to come arbitrarily close to every point and in space it does not. That's a very important point for people like Stan Natanson who don't know the city, and drives at random. Because, assuming that Dallas is a plane there is a chance of one that he will eventually be at the Melrose Hotel if at each intersection he will flip

a coin. It may be by way of Ithaca, of course, or Los Angeles. And, in fact, if you would calculate the average number of steps he takes, it is infinite. And indeed that was experimentally verified when he drove me to the Melrose.

These results can be extended to higher dimensions. The higher the dimension, the less is the probability that it will hit a given point. For every compact creature, like a sphere the probability will always be one, because then you really have no room. The only problem is for a non-compact thing, where you have definitely a possibility of escape. The phenomenon has not been investigated with any care on surfaces other than the plane, but the method would apply. As a matter of fact, in principle it is easy to tell you what the answer is on any surface, although nobody has ever proved this with complete rigor. You take the heat conduction equation on that surface and find its fundamental solution. Then you take the integral of this from zero to infinity in time as in equation (116). If the integral is finite then you know the probability is less than one. If the integral is infinite then you have the same case as in the plane. It's purely a property of the kind of singularity the heat conduction solution has.

This actually is a very superficial theorem. It is extremely elegant and very appealing but there is really no depth at all. There is no dynamical content to it. It is essentially a theorem which combines dimensionality of the space you are considering with some very, very rough idea of compactness. If the dimension is high enough and if it isn't compact then you will have the possibility of escaping. However, these facts form a beginning of an entirely new, pictorial way of looking at potential theory.

To go to potential theory I must do something more refined than simply calculating the average time (117). I must attempt to calculate the distribution of the times. This is really the crucial step. And this means that we must calculate all the moments, not simply the average (117). Now let me just perform for you the calculation of the second moment:

$$E\left\{\left[T_n(\vec{y},\vec{r}(\tau))\right]^2\right\} = E\left\{\left[\int_0^\infty V(\vec{y}+\vec{r}(\tau))\,d\tau\right]^2\right\} \tag{119}$$

Our procedure will be almost identical in spirit to the one we used with the telegrapher's equation. Let me again write the square of an integral as a double integral:

$$E\left\{\left[T_n(\vec{y},\vec{r}(\tau))\right]^2\right\} = E\left\{\int_0^\infty\int_0^\infty V(\vec{y}+\vec{r}(\tau_1))\,V(\vec{y}+\vec{r}(\tau_2))\,d\tau_1 d\tau_2\right\} \tag{119'}$$

Now this integral is over a square, an infinite square to be sure. Just as before I can integrate over only half of the square and then multiply by two:

$$2!\,E\left\{\int_0^\infty d\tau_2\int_0^{\tau_2} d\tau_1 V(\vec{y}+\vec{r}(\tau_1))\,V(\vec{y}+\vec{r}(\tau_2))\right\} \tag{120}$$

Now you interchange expectation and integration and write it so:

$$2!\int_0^\infty d\tau_2 \int_0^{\tau_2} d\tau_1\, E\left\{V\left(\vec{y}+\vec{r}(\tau_1)\right)V\left(\vec{y}+\vec{r}(\tau_2)\right)\right\} \tag{121}$$

This is perfectly justifiable. Now $V\left(\vec{y}+\vec{r}(\tau_1)\right)$ is either zero or one depending on whether $\vec{y}+\vec{r}(\tau_1)$ is in Ω at time τ_1 or not. And the same goes also for $V\left(\vec{y}+\vec{r}(\tau_2)\right)$. So the expectation of the product of these two is again included in formula (108), except that I start from $\vec{y}$ rather than from zero. So I simply apply formula (108) with Ω_1 and Ω_2 both equal to Ω:

$$2!\int_0^\infty d\tau_2 \int_0^{\tau_2} d\tau_1 \int_\Omega\int_\Omega P\left(\vec{y}\,|\vec{r}_1,\tau_1\right)P\left(\vec{r}_1\,|\vec{r}_2,\tau_2-\tau_1\right)d\vec{r}_1\,d\vec{r}_2 \tag{122}$$

You can see this easily. I must simply make a transition from $\vec{y}$ to some point $\vec{r}_1$ in Ω and then from that point I must again make a transition to a point $\vec{r}_2$ in Ω. Now you can see that it would be much more convenient to take the integration over the spatial variables lastly:

$$2!\int_\Omega\int_\Omega d\vec{r}_1\,d\vec{r}_2 \int_0^\infty d\tau_2 \int_0^{\tau_2} d\tau_1\, P\left(\vec{y}\,|\vec{r}_1,\tau_1\right)P\left(\vec{r}_1\,|\vec{r}_2,\tau_2-\tau_1\right) \tag{123}$$

The integrations over time are again related to convolution. That is simply the Markovian property making an appearance, together with the temporal homogeneity. Temporal homogeneity means that everything depends on the difference of times. You see $\tau_2 - \tau_1$ rather than τ_2, τ_1 separately. Now you can easily see what happens with the time integrals. Because suppose I insert $e^{-s\tau_2}$:

$$\int_0^\infty d\tau_2\, e^{-s\tau_2} \int_0^{\tau_2} d\tau_1\, P(\vec{y}\,|\vec{r}_1,\tau_1)P(\vec{r}_1|\vec{r}_2,\tau_2-\tau_1) \tag{124}$$

With this factor there, (124) is simply the Laplace transform of the convolution. Of course, I don't really have the factor there. But this means I simply put $s=0$. Now the Laplace transform of a convolution is simply the product of Laplace transforms. But these Laplace transforms are known for $s=0$ (refer back to formulas (116) and (117)). It immediately follows from this argument that

$$E\left\{\left[T_\Omega(\vec{y},\vec{r}(\tau))\right]^2\right\} = \frac{2!}{2\pi}\int_\Omega\int_\Omega \frac{1}{|\vec{r}_1-\vec{y}|}\,\frac{1}{|\vec{r}_2-\vec{r}_1|}\,d\vec{r}_1\,d\vec{r}_2 \tag{125}$$

This is the second moment. Now it's perfectly clear how the moments will come out in general. You can use exactly the same argument, the only difference is that a little more writing will be required. You have the following general formula for the kth moment M_k:

$$\frac{\mu_k}{k!} = \frac{1}{k!} E\left\{T_\Omega^k(\vec{y}, \vec{r}(\tau))\right\} = \tag{126}$$

$$= \frac{1}{(2\pi)^k} \int_\Omega \cdots \int_\Omega d\vec{r}_1 \cdots d\vec{r}_k \frac{1}{|\vec{r}_1 - \vec{y}|} \cdot \frac{1}{|\vec{r}_2 - \vec{r}_1|} \cdots \frac{1}{|\vec{r}_k - \vec{r}_{k-1}|}$$

Now everybody who has ever dealt with integral equations will recognize that in (126) we have something which looks very much like an iterated kernel. It is like raising a matrix to a power. To see this simply write the integrand of (126) in the form:

$$a(\vec{y}, \vec{r}_1)\, a(\vec{r}_1, \vec{r}_2)\, a(\vec{r}_2, \vec{r}_3) \cdots a(\vec{r}_{k-1}, \vec{r}_k) \tag{127}$$

Now I will have to assume a little bit of knowledge of integral equations. If you have this knowledge, then it is very natural to associate with this problem the following integral equation:

$$\frac{1}{2\pi} \int_\Omega \frac{1}{|\vec{r} - \vec{\rho}|} \varphi(\vec{\rho})\, d\vec{\rho} = \lambda \varphi(\vec{r}), \quad \vec{r} \in \Omega \tag{128}$$

I want to call your attention to the fact that this is a three-dimensional integral equation. The integral is not taken over the surface of Ω. It is taken over the volume. It is an honest-to-goodness three-dimensional integral equation. Now this kernel $1/|\vec{r} - \vec{\rho}|$ which may look singular

to you, is not really very bad. In fact, the integral:

$$\int_{\Omega} \frac{d\vec{\rho}}{|\vec{r}-\vec{\rho}|^2} \tag{129}$$

is finite. It is what is known as a completely continuous kernel. So all the theory of integral equations is applicable -- in particular, the whole theory of Hilbert and Schmidt. You can then show that the integral equation (128) has eigenvalues. In fact, the eigenvalues are all positive. It also has a complete set of normalized eigenfunctions.

In terms of the eigenfunctions and eigenvalues of the integral equation one can very easily express the moments. That is really the only thing you have to know from the theory of integral equations. The only difficulty is that the equation (128) holds only if $\vec{r} \in \Omega$. Now the only variable which may refer to a point outside Ω is $\vec{y}$. So $\vec{y}$ has to be treated a little bit separately. You get the formula

$$\frac{\mu_k}{k!} = \frac{1}{2\pi} \sum_{j=1}^{\infty} \lambda_j^{k-1} \int_{\Omega} \varphi_j(\vec{\rho})\, d\vec{\rho} \int_{\Omega} \frac{\varphi_j(\vec{\rho})\, d\vec{\rho}}{|\vec{\rho}-\vec{y}|} \tag{130}$$

This is very straight-forward. It is a formula for the iterated kernel that you'll find in any text book on integral equations. It is completely analogous to expressing the kth power of a matrix in terms of the eigenvalues and eigenvectors of the original matrix. Now you might jump at me and ask why I don't use the integral equation to replace

$$\int_{\Omega} \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|} \tag{131}$$

by $2\pi\lambda_j \varphi_j(\vec{y})$. I can do this provided $\vec{y}$ is in the region Ω . But if $\vec{y}$ is outside then I cannot. So, since I don't know where $\vec{y}$ is I must leave it in this form. Now we will calculate the moment generating function:

$$E\left\{e^{-u\,T_\Omega(\vec{y},\,\vec{r}(\tau))}\right\} \tag{132}$$

It is easy to see how to do it. You simply expand in a power series in u . You need a little bit of a discussion of convergence and you finally get the following:

$$E\left\{e^{-u\,T_\Omega(\vec{y},\,\vec{r}(\tau))}\right\} = 1 - \sum_{j=1}^{\infty} \frac{u}{1+\lambda_j u} \int_{\Omega} \varphi_j(\vec{\rho})\,d\vec{\rho}\, \frac{1}{2\pi} \int_{\Omega} \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|} \tag{133}$$

Let me call your attention to the following feature: On the right hand side there is a perfectly classical expression. And on the left hand side we have an average, an integral, over a function space. We will play the game of deciphering the properties of the classical expression by looking at the average over the function space.

Notice that the right hand side, the whole thing, satisfies Laplace's equation for $\vec{y}$ outside the region. That is perfectly obvious. (131) is a sum of potentials of spatial mass distributions. The mass distributions are given by $\varphi_j(\vec{\rho})$. Of course, you have an infinite series. But it is easy to show that the series converges uniformly in every finite region.

Now let me see what happens as $u \to \infty$. This will be very interesting. To see what happens, just look at $\exp -u T_\Omega(\vec{y}, \vec{r}(\tau))$. Now as u goes to infinity this can go to two limits. It goes to zero if $T_\Omega(\vec{y}, \vec{r}) > 0$. And it goes to one if $T_\Omega(\vec{y}, \vec{r}) = 0$. These are the only possibilities because the time $T_\Omega(\vec{y}, \vec{r})$ is certainly non-negative. Now let's try to understand what it means. Don't forget that $T_\Omega(\vec{y}, \vec{r})$ depends upon the path $\vec{r}(\tau)$. If this path penetrates the region then you get the limit zero. Because then the particle spends a finite time in the region. For every path which does not penetrate the region the limit is one. That is the limit of the function, but I still have to take the expectation. That is, I still have to integrate over all the paths. For those of you who studied measure theory in your youth, this is one of the few places one needs to use the more refined properties of measure, namely the complete additivity property. Because now you need the theorem that the limit of the integral is the integral of the limit. In this case it is very easy because you are dealing with a set of decreasing functions, as u goes to infinity. So dominated convergence is operable. So in the limit the left hand side of (133) is simply the integral of a function which is zero for those paths which penetrate the region and is one for those paths which do not. But what is the integral of

a function which is one on some set and zero outside that set? That is just the probability attached to that set. So in the limit we get:

$$\lim_{u\uparrow\infty} E\left\{e^{-u\,T_\Omega(\vec{y},\vec{F}(\tau))}\right\} = \text{Prob}\left\{T_\Omega(\vec{y},\vec{F}(\tau))=0\right\}$$

$$= 1 - \lim_{u\uparrow\infty} \sum_{j=1}^{\infty} \frac{1}{\frac{1}{u}+\lambda_j} \int_\Omega \varphi_j(\vec{\rho})\,d\vec{\rho}\,\frac{1}{2\pi}\int_\Omega \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|} \qquad (134)$$

Now it's actually more convenient to write this in the form

$$\text{Prob}\left\{T_\Omega(\vec{y},\vec{F}(\tau))>0\right\} = \lim_{u\uparrow\infty} \sum_{j=1}^{\infty} \frac{1}{\frac{1}{u}+\lambda_j} \int_\Omega \varphi_j(\vec{\rho})\,d\vec{\rho}\cdot\frac{1}{2\pi}\int_\Omega \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|} \qquad (135)$$

This says the same thing but it's much nicer. This gives the probability that you will penetrate the region in the form of a limit as u goes to infinity of a series. So far, this is true for every $\vec{y}$, on the boundary, inside or out.

I want to call your attention to one thing immediately. The limit (135) is also a harmonic function. To prove that I must appeal to a very well-known theorem of Harnack. It states that the limit of a bounded decreasing sequence of harmonic functions is harmonic. This is certainly the case we have. You can see this most easily from the left-hand side of (134). So if you believe this theorem, which you can find in any standard textbook on potential

theory, then (135) is also a harmonic function. I am going to call it $U(\vec{y})$. And I know it satisfies $\Delta U = 0$ when $\vec{y} \notin \Omega$.

Not only is $U(\vec{y})$ a harmonic function, it is nothing but the capacitory potential. That, by definition, is the harmonic function which vanishes at infinity and which assumes the boundary value one on the boundary of the region Ω. Incidentally, we produce as a by-product the formula:

$$U(\vec{y}) = \lim_{u \uparrow \infty} \sum_{j=1}^{\infty} \frac{1}{\frac{1}{u} + \lambda_j} \int_{\Omega} \varphi_j(\vec{\rho})\, d\vec{\rho} \cdot \frac{1}{2\pi} \int_{\Omega} \frac{\varphi_j(\vec{\rho})\, d\vec{\rho}}{|\vec{\rho} - \vec{y}|} \tag{136}$$

for the capacitory potential. It is a remarkable thing that despite the advanced age of potential theory no one had ever discovered this formula. It comes, you see, very naturally out of this argument, which I have originally made around 1949. The key to the answer is not hard to find. You might say, that if u goes to infinity, and I have $1/u$ in (136), then why bother with it? Simply cross away the limit and put zero for $1/u$. Unfortunately, then, the series makes no sense whatsoever. It diverges. That was the reason why the approach based on the integral equation (128) was originally abandoned. Because it led directly to the divergent series:

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} \int_{\Omega} \varphi_j(\vec{\rho})\, d\vec{\rho} \cdot \frac{1}{2\pi} \int_{\Omega} \frac{\varphi_j(\vec{\rho})\, d\vec{\rho}}{|\vec{\rho} - \vec{y}|} \tag{137}$$

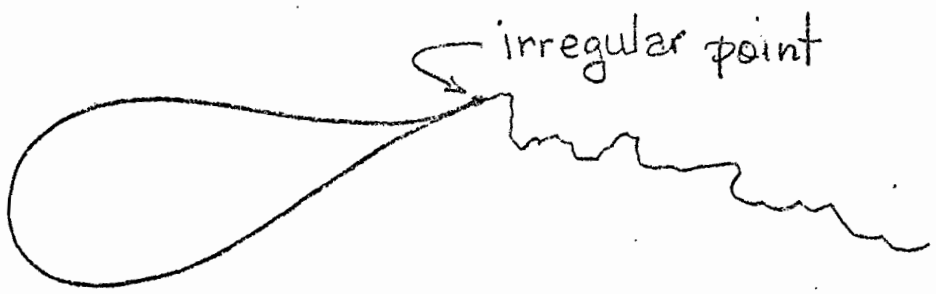
This was perhaps fortunate for the history of science, because it forced a different approach. The double layer idea was introduced which led to a very nice development of certain mathematical techniques and ideas. But (136) is a formula of a type that Karl Neumann, and others, tried to arrive at in 1870 but couldn't succeed. It simply introduces what we call in mathematics a summability method. The serious trouble with (137) is divergence. In (136) we have done a little repair work and then it makes perfect sense.

Let's have a look at what happens if $\vec{y}$ is inside the region. The integral equation then becomes operative and (135) can be simplified. The whole thing simplifies dramatically and becomes:

$$\text{Prob}\left\{T_\Omega(\vec{y},\vec{r}(\tau))>0\right\} = \lim_{u\to\infty} \sum_1^\infty \frac{\lambda_j}{\frac{1}{u}+\lambda_j} \cdot \varphi_j(\vec{y}) \int_\Omega \varphi_j(\vec{\rho})\, d\vec{\rho} \qquad (138)$$

It's very easy to tell what this is. If you start inside the region you've got to spend some positive time there. So the probability on the left hand side is simply one. The Brownian path, being continuous, can't simply jump out of the region. Notice how easy it was to see that (138) converged to one. It is almost trivial. You just use the fact that the particle can't jump out from inside the region.

Now what is really interesting is what happens on the boundary. Suppose that $\vec{y}$ is on the boundary. Then you can't say anymore that the probability is one that you spend sometime within the region Ω. In fact, there are points, called irregular points, for which this is not true. The boundary has to be very, very sharp at such points:



An example was first given by Lebesque in 1913, or so. It corresponds to a fact which is well-known in electrostatics. If you have an extremely sharp point on your conductor you cannot maintain a constant potential. The charge leaks off. This leak is simply interpretable. If you start the Brownian motion at such a point it is possible with non-zero probability to go out without ever coming into the region. However, the limit (135) still exists. It exists for all values of $\vec{y}$ -- there is no problem at all there. But it need not be equal to one. If a point is a regular point, if it is not a point where the boundary is very sharp, then the limit clearly is going to be one again. Intuitively it is perfectly obvious that if you have a round point then you have a pretty good chance of almost immediately entering the region. The point has got to be extremely sharp to prevent it.

Now let's take a look at this formula (136) and see what happens when $\vec{y}$ goes to infinity. It's perfectly clear that it goes to zero because each term goes to zero. This can be made precise without any trouble. Now what happens as $\vec{y}$ approaches the boundary is a little bit more complicated. It depends on what kind of point you have. If the point is an irregular point then the potential will not approach one. But if you have a reasonable

boundary then you can show that $U(\vec{y})$ approaches one as $\vec{y}$ approaches a boundary point. So, if you believe my intuitive argument, you see that $U(\vec{y})$ is indeed what is called the capacitory potential. It is that potential which is zero at infinity and which approaches one on the boundary wherever it can. If there are bad points, then of course it can't.