FIRST LECTURE

This series of lectures will be devoted to a subject which is becoming more and more popular and useful -- probability theory. I will not take the textbook line of development -- I will rather illustrate how some of the notions and methods have originated in important, simple problems in physical science. I will start with things which look very simple and progress towards more and more complicated things towards the end.

I would like to introduce one possible innovation in colloquia. To speak for an hour and a half is reasonably easy, especially if you are a college professor and are used to it. But to listen for an hour and a half is difficult. In fact, I think the span of attention of a human being is somewhat around fifty minutes. After that you may speak in the most flowing terms, tell the most beautiful jokes, and it is simply impossible for the listener to absorb it. Consequently, after about fifty minutes we will have a short break, so we can stretch ourselves for a few minutes. I would like also to devote some of the time, perhaps twenty minutes, to questions which might arise in the audience. The material will, perhaps, be somewhat strange to you, and it would be much better to settle some of the difficulties right off the bat. Do not be afraid to ask a question. Everybody has the fear that he may appear terribly ignorant. We all are ignorant -- the only difference is the degree of ignorance. Some of your questions I assure you I will not be able to answer; sometimes I may be able to give you a reference; sometimes I will go to the library and look it up and give you the answer tomorrow; and sometimes I will give you the answer right off the bat. The difference between you and me is that I have spent fifteen years on this subject, and you are going to spend probably only one week. Of course, I hope to convert you to the subject, so that you will be spending more and more time from now on.

I would like to start, after these preliminaries, with a little history. As you probably know, the theory originated in the very lowly problems of gambling. But that's not really how it entered science. It entered serious scientific thought in the latter part of the nineteenth century -- although some indications were already present early in the nineteenth century -- in connection with the kinetic theory of matter. Since the kinetic theory of matter is now so well known, it is interesting to tell you that it was by no means easy to convince people that atoms are something which exist. In fact, Boltzmann was severely attacked for empty speculation by the reviewers of his book. There are actually many reasons for that. One of the serious reasons was that the subject was plagued by certain paradoxes. It was in trying to resolve these paradoxes that the probability theory -- we might even say the statistical way of thinking -- came into being, at least in physics.

Now where did these paradoxes originate? That is a very interesting story, very educational, which I will tell you by way of an introduction to the subject. You probably know that Boltzmann, following people like Clausius and Meyer, tried to explain the behavior of gases on the basis of a mechanistic model. A gas was regarded as a system of a large number of particles, and the laws of mechanics were used to derive the equation of state and other things. The crowning achievement of the work of Boltzmann and of Maxwell is the derivation of the so-called H theorem. This is one of the things Professor Dresden spoke about when he discussed the Boltzmann equation and how it is related to hydrodynamics. We will devote a certain amount of time to Boltzmann' equation from a somewhat different point of view. We assume to begin with that

we have a spatially homogeneous gas. This means that the spatial distribution of particles is uniform. This is not the case Professor Dresden treated, because for hydrodynamics the whole problem is how the spatial distribution changes.

Now let $f(\vec{v},t)d\vec{v}$ be the velocity distribution in velocity space. Boltzmann actually interpreted this as the number of particles having velocity $\vec{v}$ at time t within the volume element $d\vec{v}$. He wrote out a complicated integro-differential equation from which he derived the H theorem:

$$\frac{d}{dt}\int f(\vec{v},t)\log f(\vec{v},t)\,d\vec{v} \leqslant 0 \qquad (1)$$

The integral (it is actually a triple integral) is denoted by H. Consequently,

$$H = \int f(\vec{v},t)\log f(\vec{v},t)\,d\vec{v} \qquad (2)$$

decreases, or at least does not increase in time. This was a remarkable achievement which particularly pleased Boltzmann because H clearly was some kind of an analog of the negative entropy. It was well known from classical thermodynamics that this function of state has the remarkable property that it never decreases. Here Boltzmann had managed to construct a mechanistic quantity which had a similar behavior.

All was well until it was pointed out that this was clearly untenable, since it was in contradiction with mechanics. The objections were crystallized in two paradoxes. One was the reversibility paradox of Loschmidt (around 1876) and the second was the recurrence paradox due to Zermelo and Poincare'. That

came a little later, after 1900, I think. The reversibility paradox is the simpler of the two and in a way more basic. It simply says the following: All equations of mechanics are time-reversible. This means that if you perform the transformation $t \to -t$ (that is, replace time by minus time) then the equations do not change. This is simply due to the fact that in mechanics all derivatives with respect to time are of second order. There is no way to distinguish the equations written with time going forward and time going backward. Using a more philosophically appealing terminology, there is no mechanical experiment that will tell you which way time is flowing. Consequently, said Loschmidt, something must be wrong, because this H presumably is a quantity which can be derived from a mechanistic description of the system. But now, if I change t to minus t, the quantity instead of decreasing, will increase. Thus from a purely reversible model, we draw an irreversible conclusion and something clearly is wrong.

It's interesting that in mathematics one contradiction is enough; it is even enough to think that something is wrong, to have doubts. But in physics, you must have several paradoxes before people will believe that something is wrong. So, to provide another one, Zermelo (who at the turn of the century was also interested in the logical foundations of this subject) recalled a theorem of Poincare'. It's a very famous and very beautiful theorem. It says that any conservative, closed, dynamical system is such that (I will first state it very loosely) if you start from somewhere, then unless you are extremely unlucky in choosing your starting point, you are bound to come back arbitrarily close to the starting point. To put it another way: conservative dynamical systems with finite energy are quasi-periodic. That is, states tend to recur. Consequently,

if this quantity H were a mechanical quantity then it would have to oscillate. Starting from a certain value, it would eventually have to come back arbitrarily near to that value. This would certainly contradict the result that H changes in only one direction.

Before I proceed to elucidate this paradox, I would like to prove the Poincare' theorem for you. It is very simple, and it illustrates a rather important pedagogical and fundamental point about general mathematical thought. I will emphasize this particular point many times during the conference. Suppose you have a system of mechanical particles with some strange arbitrary forces between them. Then everybody knows that the motion of the system can be described in terms of Hamilton's equations of motion. I don't even need to know the exact form of the Hamilton equations of motion. All I need to know is that there is a certain function H of the positions (generalized positions, if you wish) and momenta, and that in terms of this function certain differential equations can be written which describe the motion of the particles. Since the Hamiltonian depends on the positions and momenta, this suggests that it would be nice to look at a 6n-dimensional space, the coordinates being the q's and the p's. As most of you know, this space is known to us as the phase space.

The Hamiltonian equations of motion can be written in an extremely simple, concise way, under the name of the Liouville theorem. It simply says the following: suppose you take a little region or set A. (Refer to Figure 1 where the phase space is visualized as a plane).

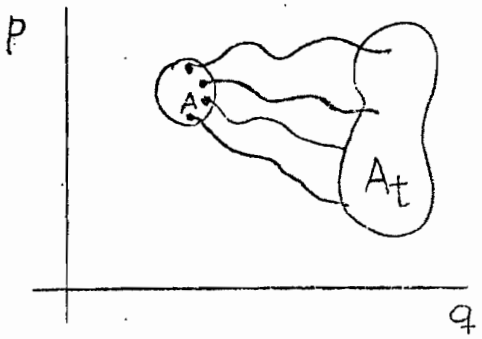

Figure 1

Now every point in this set A can be looked upon as a starting point for my system. And the trajectory will then describe the motion of the system according to the equations of motion. I now consider all the points in this little set A as the initial values for my system and look to see what happens after time t. After time t I collect them again into a set which I am going to call A_t. This is the set we get from A after time t due to the motion of the points. You might think of this set A_t as a very simply shaped one, something like a circle or a sphere. Actually, because of the complexity of the motion in the 6n-dimensional space, the shape of the thing would be tremendously complex. But the important thing is that its volume remains the same. And that's the Liouville theorem. It states that the volume of A is an invariant measure. This both implies and is implied by Hamilton's equations of motion. So, if you want to remember once and for all the simplest way to describe the motion of a system of particles, it is simply that the volume in phase space is conserved by the motion.

Since I mentioned that the system is conservative, we must take one further step. Not all phase space is being used, since the energy is being conserved. The Hamiltonian plays the role of the energy, so you are really not just anywhere in phase space, you are confined to the surface $H(\vec{q}, \vec{p}) = E$. This surface has to be assumed to be bounded. It is part of the assumptions of Poincare's theorem that this so-called energy surface is bounded. However, it is extremely easy to use Liouville's theorem (I will skip the details because that's unimportant for our purposes) to see what happens on the surface itself. All you have to do is to take a nearby surface and look at a little cylindrical

volume bounded by these two surfaces. Liouville's theorem tells us that this volume is invariant under the motion. We need only transform this knowledge to what happens on the surface by letting the two surfaces approach each other.

When you do that, you discover the following: If you take a surface element $d\sigma$, that is, a small element on the energy surface, and watch it move, then after time t the surface element is unfortunately not the same. What does remain the same is $d\sigma$ divided by the square root of the absolute value of the gradient of H. If you now take a set A which lies on the energy surface, then the set A_t also is on the energy surface -- this is because the energy is preserved. What we can say now is that

$$\int_A \frac{d\sigma}{|\text{grad}H|} = \int_{A_t} \frac{d\sigma}{|\text{grad}\,H|} \tag{3}$$

It is not a new fact. It is a simple, concise statement of the dynamics for closed, conservative systems.

Now the mathematician, if he is worth his name, extracts from this what is essential. What is essential here, for the whole picture, is not that we are dealing with a mechanical system pursuing the Hamilton equations of motion, but rather that we have the following very simple situation. Consider an abstract set, call it Ω, which will be the analog of the energy surface. I will call the points of this set ω. Suppose I also have a one-parameter family of transformations T_t of Ω -- this is an arbitrary family and depends only on time. The only condition these transformations satisfy is that $T_t \cdot T_s$ T_{t+s}. You can see this immediately: Where you are after time t + s is determined by where you were at the time t and, starting from there, the motion for

an additional time s. This property is called the semi-group property. That is just a name and it doesn't matter. The important thing is that we have such a one-parameter family of transformations which correspond to the motion. If ω is my starting point then $T_t(\omega)$ is where the starting point will be t seconds later.

Now finally comes the heart of the matter: I have a measure on this space. Now when I say measure I don't want to go into the matter in extreme detail; all I need is a class of sub-sets to which I know how to attach a number like volume. So for a large class of sub-sets there is a certain measure which I am going to call $\mu(A)$, or $|A|$. If you so wish, this corresponds simply to the integral:

$$\mu(A) = \int_A \frac{d\sigma}{|\text{grad } H|} \tag{4}$$

Finally, what is very, very important, is that the transformations are measure-preserving. Now, what does this mean? This means that if we take a set A and the transformed set A_t then the measure of A is the same as the measure of A_t.

Now I have abstracted all I know from mechanics: I abstracted the energy surface, I abstracted the motion, I abstracted the statement of the Liouville theorem, which simply is that the measure is preserved. With this I am going to prove to you in three lines the Poincare' theorem. Rather than to consider continuous time, I am simply going to discretize it and to select T_1, T_2, T_3, ..., which are the T_t's after one, two, three seconds, etc. We can write that T_1 is equal to T, T_2 equals T^2, T_4 equals T^4, etc. An important condition, which we have mentioned, is that the measure of the whole space is

finite. This is an abstraction of the assumption that we are dealing with a bounded energy surface.

Now let me take a set A, and let me take a point ω from it and see what happens to this point. I start with ω ; this is transformed to $T(\omega)$, then to $T^2(\omega)$, and to $T^3(\omega)$, etc. This set of points is called an orbit. Now what I want to prove is that unless I was very unlucky in choosing my initial point ω , then eventually I will come back into A. That is, I want to prove that some $T^k(\omega)$ is again in the set A. This will help us decide also what it means to be unlucky.

Suppose that this is not so, that some points are such that they never return, or rather that their images never return. We are going to consider a set B, which is the set of points in A which never return. I must somehow show that set B must be very small: It must be an accident to choose a starting point from B. If this is so, then most of the starting points will be such that eventually I will return.

I now operate very simply. I take B, the image of B, the image of the image of B, etc. These are the sets B, $T(B)$, $T^2(B)$, etc. Now I claim that these sets do not overlap. Why not? Well, suppose for the sake of argument, that $T(B)$ and $T^3(B)$ overlapped; this means that there is a point which belongs to both of them. (I should have said that the transformation T is invertible, that it is uniquely invertible. This is certainly so in mechanics, since if you know where you are now, you can certainly determine where you were ten minutes ago.)

Now suppose that P is a point common to $T(B)$ and $T^3(B)$. Consider $T^{-1}(P)$: We are simply taking the point back one step. The result must be a

point of B since we started with a point in T(B). It must also be a point in $T^2(B)$ since the point P was in $T^3(B)$. This argument produces a point lying in both $T^2(B)$ and in B. But this can't be: B was a set, none of whose points returned to B. Thus, our assumption that T(B) and $T^3(B)$ had a point in common must be rejected. The same argument can be applied to show that none of our sets overlap.

Now we are through -- because now I have infinitely many sets which do not overlap. They all have the same measure, because the transformation preserves measure. Now all these sets lie on the energy surface, which is a set of finite measure. If the measure of each of these sets were positive, then we would have infinitely many non-overlapping chunks, each having positive measure, to be included in a set of finite measure. This is clearly impossible. That means that the measure of the set B must be zero. Otherwise these infinitely many sets could not be made to fit. When I say that I have to be terribly unlucky to choose a starting point which behaved badly, what I mean is that the measure of the set of such points is zero. And that's exactly the meaning of Poincaré's recurrence theorem.

I want to show you how very often mathematical abstraction--that is, throwing away what is non-essential, and keeping only what is pertinent to the problem--simplifies. Really, there is absolutely no problem at all in proof of the Poincaré theorem, once you know this. You don't have to think of a swarm of points moving in a complicated way in three-dimensional space. The whole thing is reduced to a very simple, very intuitive notion. That of the motion of a single point in phase space, the motion being described by means of a measure-preserving transformation.

Coming back to the paradoxes, we now have two. One was the reversibility paradox, and the other the recurrence paradox based on this theorem. Looking at these paradoxes, we are faced with a disturbing situation. On the one hand, it was extremely appealing to have the H theorem, which, so to speak, links thermodynamics with mechanics. On the other hand, it was extremely disquieting that the link was not entirely correct. Consequently, the problem was how to reconcile the proof of the H theorem -- which everybody believes contains some truth--with the difficulties exemplified by the paradoxes I have described. Boltzmann himself, as well as other people, has proposed that the solution of these difficulties would be found in a probabilistic treatment. We should say, well, it is not always so, but with overwhelming probability, it is so. To quote Gilbert and Sullivan, "What, never? No, never. What, never? Well, hardly ever." The intent was to express the immutable statement "in every case" into some kind of a probabilistic setting. Boltzmann himself and his followers of that time were not quite clear in their explanations.

As a matter of fact, even now we do not know the answers to all the questions. But at least we can give a consistent description without any contradictions. I will do so, at least with the case of the ideal gas, to show you how this can be done. By means of an extremely simple model I will now explain to you again both the illness and the cure. Actually, the model which I am going to describe to you is a simplification of another model which was proposed for similar purposes many years ago by Ehrenfest. I think I will skip the Ehrenfest model, because the one I am going to describe will have all the features. Moreover, it will be very easy to calculate. Now again, this is a

time-honored device in science: that you are trying to extract from an extremely complicated situation, a simple situation which has the difficulty of the old one, but not the morass of details. We want an example which will have similar difficulties, but from which we can compute, calculate, discuss, or do anything we want. I am going to give you such a simple model with which I can follow what Boltzmann did, and also what Gibbs did. I will show you all the difficulties which come up -- exactly the same two paradoxes -- and how finally one can resolve them, at least to some satisfaction. It will also give us the opportunity to go into the actual Boltzmann equation and to connect it with other things.

The model I am going to consider has nothing whatsoever to do with a gas or mechanical system. But don't let that worry you, because I will show you all the analogies. The model is the following: Take a circle, and on the circle take n equidistant points (vertices of a regular n-gon). Of these a certain number is marked, and I am going to say that m of them are marked. These m points form a set which I will call S, just for abbreviation. Now on each of the n points there sits a ball. This can be either black or white. In order to be completely definite, let us say that at time $t=0$, all the balls are black. Now comes the dynamics of this model. During each time unit (the model will be discrete in time, and the length of the elementary time interval will be unity) every ball moves one step counterclockwise. But with the following proviso: A ball moving from a point in the set S changes color. If the point isn't in S, the ball doesn't blush; it maintains its own color. At the next step some new balls will now be in S and again move counterclockwise. All those which are in S change color and those which aren't in S don't change color. So you go on.

Now the question is, what is going to happen in the long run? You can first of all argue quite intuitively that if the set S, where the blushing, the changing of color occurs, is sufficiently irregular, then eventually we are going to have roughly half white and half black ones. I can assure you that if you take a poll of public opinion, this is the answer you will always get. Let us now analyze this answer. Our analysis will be a direct imitation of what Boltzmann did, and you will see where one gets the paradox. Let me denote by $N_w(t)$ the number of white balls at time t, and by $N_b(t)$ the number of black balls at time t. At the beginning I have the condition that $N_b(0)$ is equal to n. That is my initial condition. Let $N_w(S,t)$ and $N_b(S,t)$ be the number of white and black balls in the set S at time t. We have the conservation law:

$$N_w(S,t) + N_b(S,t) = m \tag{5}$$

Now I am going to write the other conservation law for this model. I will follow Boltzmann, religiously copying what he did for the case of the gas. What can I say about $N_b(t+1)$, the number of black balls at time t+1? It is certainly the number of black balls at time t plus the ones I gain minus the ones I lose. Now what do I gain? I gain all the white ones which are in the set S. I lose all the black ones in the set S. Therefore:

$$N_b(t+1) = N_b(t) + N_w(S,t) - N_b(S,t) \tag{6}$$

$$N_w(t+1) = N_w(t) + N_b(S,t) - N_w(S,t) \tag{7}$$

To go further, we must make a special assumption. After some time has gone by, and if set S is pretty irregular, the balls should be fairly distributed. That is, the proportion of white balls in the set S ought to be in the same proportion as in the set of all the balls. That is, a reasonable approximation ought to be:

$$N_b(S,t) \cong \frac{m}{n} N_b(t) \tag{8}$$

$$N_w(S,t) \cong \frac{m}{n} N_w(t) \tag{9}$$

No sane person would argue with this. I assure you that if a situation like this came up in your daily work you would unflinchingly make this assumption. As a matter of fact, I wouldn't blame you, because if we didn't make such assumptions we would be working against the progress of science --- because it is clearly, obviously so.

We can now begin to solve the whole thing. For subtracting equation (7) from equation (6) and making use of our assumptions, we get:

$$N_b(t+1) - N_w(t+1) \cong \left(1 - 2\frac{m}{n}\right)\left[N_b(t) - N_w(t)\right] \tag{8'}$$

This is the simplest possible difference equation; it simply relates what happens at time t+1 to the situation at time t. It will take you exactly a second to see from this that the excess of black balls over white balls at time t is given by:

$$\frac{N_b(t) - N_w(t)}{n} \cong \frac{N_b(0) - N_w(0)}{n}\left(1 - 2\frac{m}{n}\right)^t \tag{9'}$$

I will now make the assumption that $2m$ is less than n ; in fact, I am going to put μ equal to $\frac{m}{n}$, and assume that μ is less than 1/2. Now if we believe all this, then the excess of black over white balls will decay exponentially, and hence, after a long enough time, they will nearly equalize. And moreover, we have the feature of the H theorem: not only will the numbers equalize, but the equalization will be irreversible.

This corresponds so well to our intuition that we are apt to overlook the difficulties. So now we take a second look at the model. First of all, the model is completely reversible. Again, I have exactly the same thing as in dynamics, because suppose that it were an exact conclusion that the numbers equalize. I start with n black balls and take, say, a thousand steps. If I rotate the set S one unit counterclockwise and then calling every black ball white and every white ball black, I make the balls go around clockwise instead of counterclockwise. The model doesn't know the difference, so the excess would still keep decreasing exponentially. But after 999 steps I will have to come back to where I started. So clearly, the conclusion is untenable. That's the reversibility paradox applied to the model.

The second paradox is very simple here, because you see that after $2n$ steps you come exactly to the initial situation. The model is so designed that it is strictly periodic. Let us represent the color by either +1 or -1; let's say that black is +1 and white is -1. Then in n steps each ball, no matter where it started, will have passed through all the points in the set S. The color will, therefore, have changed m times, so that the final color will be $(-1)^m$. If m were odd, then the black ball would have come out white.

However, if you run over the course again you again change color m times so that you certainly come back to the initial color. So, the model is completely periodic with period $2m$, and you have exactly -- or certainly a very strong analog with -- the situation of an actual gas, with the exponential decay in the role of the H theorem.

Now the question is, where is the offender? This model is very nice, because the offender sticks out. The offender clearly sits in the assumptions (8) and (9), because there is no doubt that the conservation laws are valid, and that's all we've done. From there on, it was the simplest possible elementary algebra. This assumption, or the analog thereof, was already made by Boltzmann -- only he didn't know it was an assumption. It is now so recognized and is referred to usually by the German word "stosszahlansatz". There seems to be no good English translation, but we may call it the "collision number postulate". In the original derivation of Boltzmann, it seemed that this was not a new assumption, but that it was somehow inherent in the model, so to speak. However, we now see clearly that it is an assumption and must try to justify it in some way.

If we want to maintain our conclusion -- which we think is more or less reasonable -- then we must somehow re-formulate or re-interpret the model. Perhaps by calling things by different names we can save our result. This is also a time-honored scientific practice: if you can't beat them, join them. If there is something wrong with the derivation, but the result seems to be more or less reasonable, the problem is to re-label and reorganize things to make sense. And that is what I will show you, that we can get out of all these difficulties by simply thinking statistically.

Some of the objections can be answered almost immediately. After all, with the recurrence paradox, we only run into difficulties at extremely long times, of the order of n. If n is the Avogadro number, $\sim 10^{23}$, and if it takes a tenth, or hundredth, or a millionth of a second to perform an experiment, then it will be after all quite a long time before this happens. Maybe we can save our results by saying that they hold only for time small compared to n. If that is correct, then we will discover how this proviso has to be made. But the other objection, namely of going for awhile in one direction, then violently reversing the model and coming back, still holds. That's a short time business. It holds for one step or for two steps. There is nothing to prevent us from moving just a few steps counterclockwise, shifting the set S one unit counterclockwise, changing the colors of the balls, and going back. So clearly it cannot be tenable to say that the excess of black balls is strictly decreasing as our formula indicates. Now the question is, how to extricate ourselves in a natural way from this situation and still end up with a result which is at least reminiscent of this one. And that is where the probabilistic analysis comes in.

Now let us be slightly more sophisticated -- the sophistication will increase as we go along. I'm going to get a little formalism going, which I will start now and finish in the next lecture. Let me introduce the following notation: I'm going to number the points from 1 to n; p will be a number between 1 and n;

$$\varepsilon_p = \begin{cases} -1 & \text{if } p \text{ is in } S \\ +1 & \text{if } p \text{ is not in } S \end{cases} \qquad (10)$$

$$\eta_p(t) = \begin{cases} +1 & \text{if the ball at } p \text{ at time } t \text{ is black} \\ -1 & \text{if the ball at } p \text{ at time } t \text{ is white} \end{cases} \tag{11}$$

This last quantity, of course, changes from time to time -- at various times the balls will be black or white. Now let me write out what I know about the model.

$\eta_p(t+1)$ is the color of the ball at the point p and time t+1. The ball came there from p-1, and it changed color or it did not change color depending on whether p-1 was in the set S or wasn't in the set S. So it immediately follows that:

$$\eta_p(t+1) = \varepsilon_{p-1}\eta_{p-1}(t) \tag{12}$$

This is merely the symbolic way of writing the dynamical description of my model. It says that the color of the ball at p at time t+1, is the color of the ball at the previous point at the previous time multiplied by -1 or +1 depending on whether the color changed there or did not change there. If we continue this recursion we get, very easily, that

$$\eta_p(t) = \eta_{p-t}(0)\varepsilon_{p-1}\varepsilon_{p-2}\cdots\varepsilon_{p-t} \tag{13}$$

That's simply repeating this thing, p times. Since I assumed that at time zero all the balls are black then $\eta_{p-t}(0)$ is simply 1, because at time zero all the η's are equal to one. That was my simplifying assumption. Now take the sum of all the η_p :

$$\sum_p \eta_p(t) = \sum_p \varepsilon_{p-1}\varepsilon_{p-2}\cdots\varepsilon_{p-t} \tag{14}$$

What is the meaning of this sum? η_p is +1 for a black ball and -1 for a white ball, so this sum is simply the number of black balls minus the number of white balls. Now I will divide the whole thing by n and speak then of the proportional excess:

$$\frac{1}{n}\left[N_b(t)-N_w(t)\right]=\frac{1}{n}\sum_p \varepsilon_{p-1}\,\varepsilon_{p-2}\cdots\cdot\varepsilon_{p-t} \tag{15}$$

You see again that this is as far as I can go. But now I'm going to say to myself the following: Suppose that all I know about this set S is that it has exactly m elements But I have no idea where they are located. Then for each possible location of the set S I can follow through and find this number which I call the proportional excess. At the end, I can look at all these numbers. In other words, I will not study one individual set S, but I will rather study all possible sets S. I will look at the numbers

$$\frac{1}{n}\left[N_b(t)-N_w(t)\right]_S \tag{16}$$

one for each possible set S. And I hope that I will notice the remarkable phenomena that most of them cluster around $(1-2\mu)^t$. True, we are going to have a few of them away from this value, but most of them, in fact an overwhelming proportion of them, I hope to be very close to $(1-2\mu)^t$.

But first I will aim for the more modest result that the average of all these numbers -- that is, the average over all possible positions of the set S -- is precisely $(1-2\mu)^t$. So the problem now is to perform this average, and I will demonstrate with complete precision in the next lecture that the result is exactly the one we want. The operation is not completely specified,

because I must tell you how to average. And that's always an element of arbitrariness which, in my opinion, can never be entirely disposed of in any physical situation. Somewhere you are going to get to a point where you will have to decide how to perform the averaging. For the immediate purposes I am going to assume that all positions of the set S are equally probable. That means, there is no way to distinguish, to prefer, one position over another. Our results must then be qualified by saying that they are obtained under the assumption that the positions of the set S are equally probable.