TENTH LECTURE

Now I would like to make a few remarks about this method of looking at things from a probabilistic point of view. If we look at the problem of the capacitory potential classically then what one wants is the following. One wants a harmonic function which vanishes at infinity and approaches one on the boundary. Now you might take a very naive point of view. I'll try to find a mass distribution $\psi(\vec{\rho})$ over the region Ω so that the corresponding potential:

$$U(\vec{y}) = \frac{1}{2\pi} \int_{\Omega} \frac{\psi(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|} \tag{139}$$

has the desired properties. Well, certainly it's a potential all right. It's a harmonic function. It is also zero at infinity. But what is the best way to make it one on the boundary? Well, you say, I'll try to make it one on the boundary by making it one inside; everywhere inside. In fact, you know from electrostatics that that's how it's going to be. The potential inside is going to be uniform. This gives the integral equation:

$$\frac{1}{2\pi} \int_{\Omega} \frac{\psi(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho}-\vec{y}|} = 1 \quad , \quad \vec{y} \in \Omega \tag{140}$$

This is what people did when they first tried to solve problems of this sort. They tried to solve this integral equation. Now how do we solve such an integral equation? One of the simplest ways is to suppose that the function $\psi(\vec{\rho})$ can be expanded in the eigenfunctions φ_j of the integral equation (128). Lo and behold, you get

$$\psi(\vec{\rho}) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \int_{\Omega} \varphi_j(\vec{r})\, d\vec{r}\; \varphi_j(\vec{\rho}) \tag{141}$$

This is simply the expression I would get formally if I put $1/u$ equal to zero in (138). Then, if I substitute this in to find the potential, it comes out

$$U(\vec{y}) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \int_{\Omega} \varphi_j(\vec{r})\, d\vec{r}\; \frac{1}{2\pi} \int_{\Omega} \frac{\varphi_j(\vec{\rho})\, d\vec{\rho}}{|\vec{\rho}-\vec{y}|} \tag{142}$$

Again, this compares with my formula (136) if I put $1/u$ equal to zero.

Now, why is this bad? This is, to be sure, the most natural way to solve the problem. But it is bad because the series (142) makes no sense. That is already obvious on physical grounds. Because you know that there is no mass distribution inside Ω which will give such a potential. All the charge is concentrated on the boundary. So you could not expect (142) to give you anything sensible. Since everybody knew that there was no mass distribution inside which would give a reasonable result this approach was abandoned.

The interesting thing is that the simple change made in (136) makes possible a perfectly sensible interpretation. And, moreover, this interpretation is so closely related to the probability viewpoint. Now I might add, in this connection, the following observation: One can very easily prove that the series

$$\sum_{j=1}^{\infty} \frac{1}{\frac{1}{u} + \lambda_j} \int_{\Omega} \varphi_j(\vec{\rho})\,d\vec{\rho}\ \frac{1}{2\pi} \int_{\Omega} \frac{\varphi_j(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho} - \vec{y}|} \tag{143}$$

for a finite u , is the potential of a mass distribution. It can be written

$$\frac{1}{2\pi} \int_{\Omega} \frac{\psi_u(\vec{\rho})\,d\vec{\rho}}{|\vec{\rho} - \vec{y}|} \tag{143'}$$

Moreover, $\psi(\vec{\rho})$, the mass density, is non-negative. For every finite u you get a perfectly good mass distribution. But as u goes to infinity what happens to this function is that it gets smaller and smaller everywhere inside. The mass gets more and more concentrated near the boundary. So the process $u \to \infty$ is a sweeping-out process. The mass gets swept out from the inside and in the limit you collect it all on the boundary.

Now let's take another look at relation (136). From it one can very easily see what the capacity is. There are very many definitions of capacity, but one of them is:

$$U(\vec{y}) \sim \frac{C}{|\vec{y}|} \quad , \quad |\vec{y}| \to \infty \tag{144}$$

This says that at infinity the capacitory potential behaves like a certain constant divided by the distance of the point from the origin. The constant is known as the capacity. Now we can find this constant very easily. Of course, there is a matter of interchanging various limiting processes. But this can be easily justified. Now for large $\vec{y}$ (136) becomes:

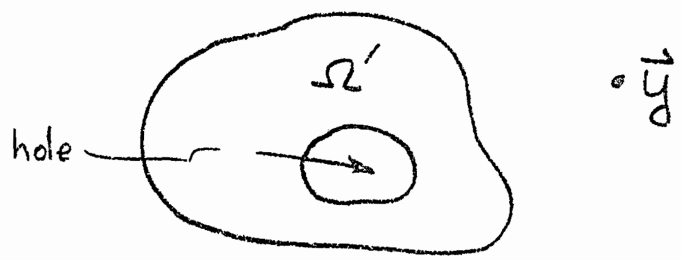
$$U(\vec{y}) \sim \lim_{u \uparrow \infty} \sum_{j=1}^{\infty} \frac{1}{\frac{1}{u} + \lambda_j} \frac{\left[\int_{\Omega} \varphi_j(\vec{\rho})\, d\vec{\rho}\right]^2}{2\pi |\vec{y}|} \tag{145}$$

But now all the terms in the series are positive so you ought to be able to let u go to infinity inside the sum. Then you get the following formula for the capacity:

$$C = \frac{1}{2\pi} \sum_{j=1}^{\infty} \frac{\left[\int_{\Omega} \varphi_j(\vec{\rho})\, d\vec{\rho}\right]^2}{\lambda_j} \tag{146}$$

This gives you an expression for the capacity in terms of the eigenvalues and eigenfunctions of the integral equation (128). It is equivalent to the formula obtained from a variational principle.

I would like to make one final observation in this connection. Throughout the whole theory here, we worked strictly with the volume integral equation (128). The surface, as such, never entered the consideration. From the purely mathematical point of view this was a great convenience. Surface considerations are always very tedious. You might wonder if we really got something for nothing. Well, we didn't, really. And again, this point of view makes it very clear. To see it, suppose I scoop a hole out of Ω . Let me call the region I have left Ω' .



Now, if I start a Brownian particle from the point $\vec{y}$ then what is the probability that it will spend a positive time in Ω' ? Why, it's clearly the same as the probability it will spend a positive time in Ω . Because when it enters Ω' it automatically enters Ω , also. On the other hand, the eigenfunctions and the eigenvalues for Ω' will be vastly different. But, nevertheless, the strange combination (135) must be the same. So you see that you can cut out an arbitrary chunk and still get the same answer. From this it is obvious that we are really dealing with a surface phenomenon, because you can badly butcher up the volume Ω and it makes no difference. For a reasonable region, one can say it as follows: The probability of spending positive time in the region Ω is the same as the probability of crossing the boundary. But if you try to build the theory from this viewpoint, that of crossing the boundary, then you're in trouble. Things would become very tedious and messy. So you simply say that this is the same as spending positive time in the interior. And then one is led to all these formulas in terms of volume integrals. It's much nicer.

Now what about two dimensions? There is no complication in going to higher dimensions. And that is, after all, only an academic exercise. But in two dimensions all these fundamental difficulties we mentioned come in. Almost every Brownian path spends an infinite amount of time in any region Ω. So one cannot carry through the same theory. It has to be modified. The modifications lead to more involved computations which I will spare you. I will just give you the results; at least, the probabilistic part of the results. These are the following: We again take a region Ω and a point $\vec{y}$. Let us start a Brownian particle from $\vec{y}$.

Now I know, with probability one, that the curve will eventually enter the region Ω. But I can ask for the probability that it will not enter the region up to time t. In other words, I can ask for:

$$\text{Prob}\left\{\vec{y}+\vec{r}(\tau)\notin\Omega \text{ for } 0\leqslant\tau\leqslant t\right\} \tag{147}$$

In the plane, as t approaches infinity, this probability must approach zero. But the remarkable thing is that it approaches zero, asymptotically, as

$$\frac{R(\vec{y})}{\log\sqrt{t}} \tag{148}$$

$R(\vec{y})$ is the two-dimensional analogue of the capacitory potential. It is, namely, that harmonic function which is zero on the boundary of the region and behaves like

$$C + \log|\vec{y}| \tag{149}$$

at infinity. In other words, it has a logarithmic singularity at infinity.

So even here you have a probabilitistic interpretation of the capacitory potential, a much more unpleasant one. It would be useless for Monte Carlo purposes, because you would really have to wait a long, long time. Convergence would be woefully slow.

To prove these things the methods have to be modified. The proofs become reasonably unpleasant. Some of it has been published. This theorem was conjectured by me and then proved by one of my colleagues. I have now, under somewhat stringent conditions, a reasonably simple proof. It will eventually appear in print.

Now, to finish up this thing I will make some remarks. You remember that all this measure theory, starting from a one-dimensional case, was built on this function:

$$\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-x_0)^2}{2t}} \tag{150}$$

The reason we needed such a function was that it was a solution of the Chapman-Kolmogoroff equation. It is of interest to see what would happen if we were to take other solutions of this equation. For instance, the one which I showed you:

$$\frac{t}{\pi}\frac{1}{t^2+(x-x_0)^2} \tag{151}$$

The second moment of this one is infinite. It has to be. If the second moment is finite then you can only get (150).

So suppose we try to build a measure theory based on (151). We introduce the windows (see page 136) and consider the set of paths passing through them, and so forth. But then you discover to your great surprise that with probability one the paths are discontinuous. This is one of the most interesting facts in this business. I think it was first discovered by Paul Levy. You cannot build a measure in the set of continuous paths based on this function (151). So the best you can do is to build the measure in a certain set of discontinuous functions.

One of the very interesting outcomes is the change this makes in our potential theory. The integral equation (128) has to be changed. But you can get a formula similar to (136). But now, as u goes to infinity, the mass becomes distributed all over the region. In the other case it got concentrated on the boundary. The difference is a direct consequence of the discontinuity of the paths. That is very interesting and very charming. Before, the only reason that I could use the argument that crossing the boundary was the same as entering the interior was that the paths were continuous. That's not true anymore, because the paths can jump. You might enter the interior without ever crossing the boundary. So you see, the fact that in ordinary potential theory the paths are continuous is equivalent to the statement that the charges giving rise to the capacitory potential are distributed on the boundary.

The measure in the space of all these discontinuous paths is a very pathological affair. In fact, considerable caution has to be exercised. Nevertheless, it is very intimately connected with purely analytic questions. You might think that such a measure would only be of interest in itself, to see how bad things can get. But one finds that it can be applied with reasonable success to analytical problems.

I may as well finish up with a famous statement due either to Russell or to Whitehead. I don't remember which. One of them gave a lecture and the other was presiding. Let us say that Russell was presiding and Whitehead gave the lecture. The lecture was on the foundations of quantum mechanics. It was apparently not only too much for the audience, but also too much for the presiding officer. The whole thing was extraordinarily difficult, abstruse, and unclear. Yet, when the lecture was over the chairman felt that he must make some comment. He made one which was both polite and true. He simply said, "We must be thankful to the speaker for not further darkening this vastly obscure subject."