

I considered the infinite potential well with potential $V(x)$ defined as

$$V(x) = \begin{cases} 0 & , \quad |x| < L \\ \infty & , \quad L \leq |x| \end{cases} \quad (1)$$

Our initial wave function is

$$\psi(x, 0) = C \exp \left(-\frac{(x - x_0)^2}{4s^2} + ik_0 x \right) \quad (2)$$

where C is some normalization constant, and k_0 and x_0 are real numbers.

The eigenvalues in the interior region of infinite well are

$$\varphi_n(x, t) = A_n \cos(k_n x) e^{-i \frac{k_n^2}{2} t} + B_n \sin(k_n x) e^{-i \frac{k_n^2}{2} t} \quad \text{where} \quad k_n = n \frac{\pi}{2L}, \quad n \in \mathbb{Z} \quad (3)$$

$$= \left[A_n \frac{e^{ik_n x} - e^{-ik_n x}}{2i} + B_n \frac{e^{ik_n x} - e^{-ik_n x}}{2i} \right] e^{-i \frac{k_n^2}{2} t} \quad (4)$$

$$= \left[A_n \frac{e^{ik_n x} + e^{-ik_n x}}{2} + B_n \frac{e^{ik_n x} - e^{-ik_n x}}{2i} \right] e^{-i \frac{k_n^2}{2} t} \quad (5)$$

$$= \left[\underbrace{\left(\frac{A_n}{2} + \frac{B_n}{2i} \right)}_{\tilde{A}_n} \underbrace{e^{ik_n x}}_{\varphi_n^{(1)}(x)} + \underbrace{\left(\frac{A_n}{2} - \frac{B_n}{2i} \right)}_{\tilde{B}_n} \underbrace{e^{-ik_n x}}_{\varphi_n^{(2)}(x)} \right] e^{-i \frac{k_n^2}{2} t} \quad (6)$$

$$= \left[\tilde{A}_n \varphi_n^{(1)}(x) + \tilde{B}_n \varphi_n^{(2)}(x) \right] \exp \left(-i \frac{k_n^2}{2} t \right) \quad (7)$$

By applying boundary conditions, we can conclude that whenever n is even, we must take into account only the sine part of the solution. On the other hand, if it is odd, we should take cosine part.

$$\sin(k_n L) = \sin(-k_n L) = 0 \implies k_n := n \frac{\pi}{2L} = \frac{m\pi}{L} \quad \text{where} \quad n, m \in \mathbb{Z} \quad (8)$$

$$\cos(k_n L) = \cos(-k_n L) = 0 \implies k_n := n \frac{\pi}{2L} = (2m+1) \frac{\pi}{2L} \quad \text{where} \quad n, m \in \mathbb{Z} \quad (9)$$

Therefore, we can obtain our solution with

$$\psi(x, t) = \sum_{n=0}^{\infty} A_{2n+1} \cos(k_{2n+1} x) e^{-i \frac{k_{2n+1}^2}{2} t} + B_{2n} \sin(k_{2n} x) e^{-i \frac{k_{2n}^2}{2} t} \quad (10)$$

$$= \sum_{n=0}^{\infty} \left(\tilde{A}_{2n+1} + \tilde{B}_{2n+1} \right) \cos(k_{2n+1} x) e^{-i \frac{k_{2n+1}^2}{2} t} + i \left(\tilde{A}_{2n} - \tilde{B}_{2n} \right) \sin(k_{2n} x) e^{-i \frac{k_{2n}^2}{2} t} \quad (11)$$

I tried to find projection of our initial wave function to eigenstates by using Gaussian

integration

$$\begin{aligned}
\tilde{A}_n &= \langle \varphi_n^{(1)} | \psi \rangle = \int_{-L}^L dx \varphi_n^{(1)*} \psi(x, 0) = C \int_{-L}^L dx \exp \left(-\frac{(x-x_0)^2}{4s^2} + i(k_0 - k_n)x \right) \\
&= C \int_{-L-x_0}^{L-x_0} dx' \exp \left(-\frac{x'^2}{4s^2} + i(k_0 - k_n)(x' + x_0) \right) \\
&= C e^{i(k_0 - k_n)x_0} \int_{-L-x_0-i2s^2(k_0-k_n)}^{L-x_0-i2s^2(k_0-k_n)} dx'' \exp \left(-\frac{(x'' + i2s^2(k_0 - k_n))^2}{4s^2} + i(k_0 - k_n)(x'' + i2s^2(k_0 - k_n)) \right) \\
&= C e^{i(k_0 - k_n)x_0} \int_{-L-x_0-i2s^2(k_0-k_n)}^{L-x_0-i2s^2(k_0-k_n)} dx'' \exp \left(-\frac{x''^2 + i4s^2(k_0 - k_n)x'' - 4s^4(k_0 - k_n)^2}{4s^2} \right. \\
&\quad \left. + i(k_0 - k_n)(x'' + i2s^2(k_0 - k_n)) \right) \\
&= C e^{i(k_0 - k_n)x_0} \int_{-L-x_0+i2s^2(k_0-k_n)}^{L-x_0+i2s^2(k_0-k_n)} dx'' \exp \left(-\frac{x''^2}{4s^2} + s^2(k_0 - k_n)^2 - 2s^2(k_0 - k_n)^2 \right) \\
&= C e^{-s^2(k_n - k_0)**2 - i(k_n - k_0)x_0} \int_{-L-x_0+i2s^2(k_0-k_n)}^{L-x_0+i2s^2(k_0-k_n)} dx'' \exp \left(-\frac{x''^2}{4s^2} \right) \\
&\approx C e^{-s^2(k_n - k_0)**2 - i(k_n - k_0)x_0} \int_{-\infty}^{\infty} dx'' \exp \left(-\frac{x''^2}{4s^2} \right) = C \sqrt{4s^2\pi} e^{-s^2(k_n - k_0)**2 - i(k_n - k_0)x_0}
\end{aligned}$$

$$\implies \tilde{A}_n = C 2s\sqrt{\pi} e^{-s^2(k_n - k_0)**2 - i(k_n - k_0)x_0} \quad (12)$$

$$\tilde{B}_n = A_{(-n)} = C 2s\sqrt{\pi} e^{-s^2(k_n + k_0)**2 + i(k_n + k_0)x_0} \quad (13)$$