Multivariate Econometrics

Master's Econometrics

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Part 1.

Question 1.

Simulate $T = 200$, 3×1 multivariate time-series data by using the DGP:

$$\boldsymbol{x}_t = \boldsymbol{\delta} + \boldsymbol{\Lambda}\boldsymbol{x}_{t-1} + \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\delta}_t$ is a 3×1 vector of time-invariant deterministic components, $\boldsymbol{x}_t$ is a 3×1 vector. The coefficient matrix $\boldsymbol{\Lambda}$ is a 3×3 matrix and $\boldsymbol{\varepsilon}_t$ is a 3×1 vector of innovations. Make your own assumptions about all model components to ensure ***wide-sense stationarity*** *of $\{\boldsymbol{x}_t\}$. Write all your assumptions about all model components in the report.*

In order for $\boldsymbol{x}_t$ to be covariance stationary the mean, the variance, and the autocovariance of the sequence should all be time - invariant.
To fulfill this requirement, the characteristic polynomial $|\boldsymbol{\Lambda} - zI|$ should be equal to 0, with eigenvalues < 1 (in order to be inside the unit circle).

For the Vector Autoregressive Model, the following components are being used:
- $\boldsymbol{x}_t$ is a (3×1) vector of the dependent variable;
- $\boldsymbol{\delta}$ is a (3×1) vector of time - invariant deterministic components;
- $\boldsymbol{\Lambda}$ is a (3×3) coefficient matrix;
- $\boldsymbol{\varepsilon}$ is a (3×1) vector of innovations, randomly drawn from a standard normal distribution;

For the model to be wide - sense stationary, three conditions must hold:
1. $\boldsymbol{\delta}_t$ is time-invariant (there should be no mean shifts) $\boldsymbol{\delta}_t = \boldsymbol{\delta}$;
2. $\{\boldsymbol{\varepsilon}_t\}$ should be i.i.d. for all t;
3. Stability condition, $\lim_{n\to\infty} \boldsymbol{\Lambda}^n = 0$;

$\boldsymbol{\delta}$ is set equal to $[1, 1, 1]$, fulfilling the first condition;
The innovations are drawn i.i.d. from a standard normal distribution function, fulfilling condition two;
$\boldsymbol{\Lambda}$ was chosen such that its eigenvalues satisfy the stability condition, condition 3.

Question 2.

Show that $\{\boldsymbol{x}_t\}$ is a wide-sense stationary multivariate time-series process. Explain how you conducted this analysis and elaborate on your findings in the report. how that $\{\boldsymbol{x}_t\}$ is a wide-sense stationary multivariate time-series process. Explain how you conducted this analysis and elaborate on your findings in the report.

The eigenvalues obtained are given by 0.421, 0.032 and 0.146.

$$E[\boldsymbol{x}_t] = \sum_{j=0}^{n-1} \boldsymbol{\Lambda}^j \boldsymbol{\delta}_{t-j} + \boldsymbol{\Lambda}^n E[\boldsymbol{x}_{t-n}],$$

For the stability condition to hold, $\lim_{n\to\infty} \boldsymbol{\Lambda}^n = 0$. This leaves the equation equal to $\boldsymbol{\Lambda}^j \boldsymbol{\delta}_{t-j}$. This occurs because $\boldsymbol{\Lambda}$ is diagonalizable and matrix decomposition should be applied: $\boldsymbol{\Lambda} = \boldsymbol{Q}\boldsymbol{M}\boldsymbol{Q}^{-1}$, with $\boldsymbol{Q}(3 \times 3)$ being the matrix with the eigenvectors on the columns, and $\boldsymbol{M}(3 \times 3)$ - the diagonal matrix, with the eigenvalues of $\boldsymbol{\Lambda}$ on the diagonal.

Question 3.

Plot the data set you generated. Include the plot in the report.

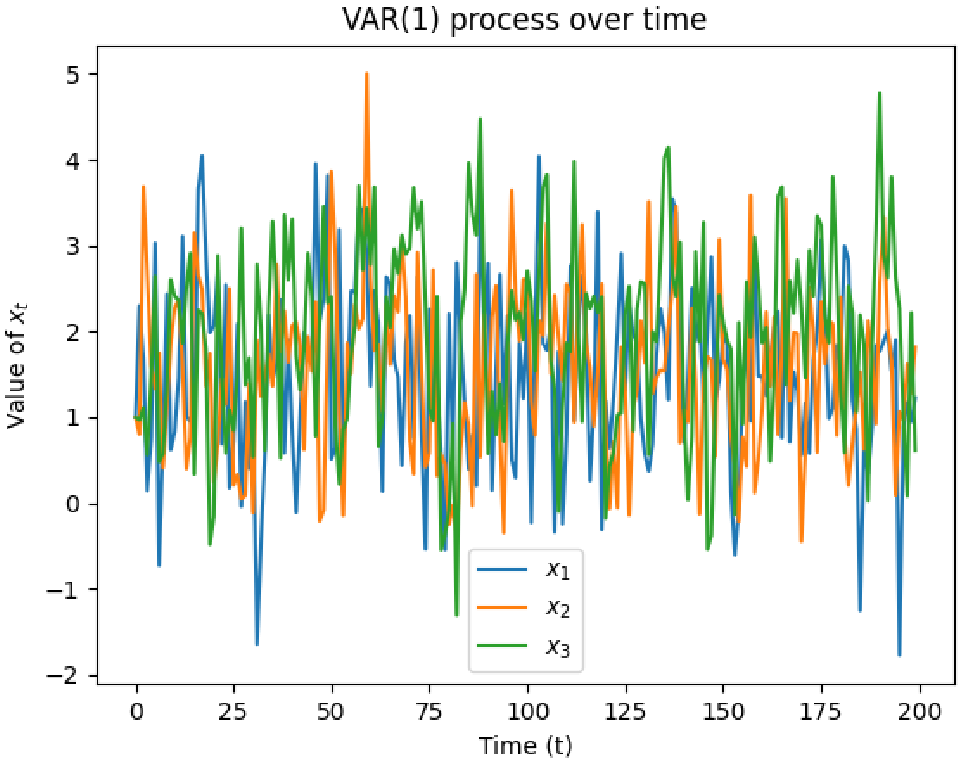

Figure 1: Stationary VAR(1) process.

Question 4.

Obtain the average for each 3 time series data you simulated ($\bar{\boldsymbol{x}}_t$). Obtain the means of the 3 time series processes ($E[\boldsymbol{x}_t]$). Compare the averages with the means. Report your findings. Briefly comment on your findings.

	$\boldsymbol{x}^{(1)}$	$\boldsymbol{x}^{(2)}$	$\boldsymbol{x}^{(3)}$
Average	1.506	1.530	1.945
Uncond.Mean	1.506	1.674	1.883

Table 1: averages and unconditional means of $\boldsymbol{x}^{(1)}$, $\boldsymbol{x}^{(2)}$ and $\boldsymbol{x}^{(3)}$

Stationarity has already been proven, so the concept of ergodicity could also be applied. The idea behind ergodicity is that the ensemble mean (the expected value of $\boldsymbol{x}_t$ for the fixed time period t) and the time average (the expected value of each realization of the sequence) are equal. This could be achieved by applying the law of large numbers. The results obtained from Python are given in table 1. These results confirm the rule only when n is high enough. Therefore, with only 200 = n realizations of the time - series, a perfect match of the ensemble and time mean cannot be established while with 2000 realizations or more the desired result can be obtained approximately.

Question 5.

Obtain the sample variance for each 3 time series data you simulated. Obtain the variance of the 3 time series processes. Compare the sample variance of the data with the variance of the processes. Report your findings. Briefly comment on your findings.

	$\boldsymbol{x}^{(1)}$	$\boldsymbol{x}^{(2)}$	$\boldsymbol{x}^{(3)}$
Cond. Var.	1.146	0.962	1.260
Uncond. Var.	1.036	1.071	1.132

Table 2: Conditional and unconditional variances of $\boldsymbol{x}^{(1)}$, $\boldsymbol{x}^{(2)}$ and $\boldsymbol{x}^{(3)}$

In order to obtain the variance of the time-series processes, the Kronecker product of the matrices should be acquired:

$$Vec(\boldsymbol{\Omega}_x) = Vec(\boldsymbol{\Lambda}\boldsymbol{\Omega}_x\boldsymbol{\Lambda'}) + Vec(\boldsymbol{\Omega}).$$

in order to obtain $vec(\Omega)$ that includes the variances. Consequently, $vec(\Omega)$ is acquired using the re-written formula:

$$Vec(\boldsymbol{\Omega}_x) = (\boldsymbol{I}_{m^2} - \boldsymbol{\Lambda} \otimes \boldsymbol{\Lambda})^{-1} Vec(\boldsymbol{\Omega}).$$

This approach derives $vec(\Omega)(9 \times 1)$, which includes the variances of the processes. Eventually, this vector is de-vectorized and turned into a (3×3) matrix. The results for each time series are given in table 2.
Again, since the realizations of the process are only 200, the values are not perfectly identical (as it should be expected). However, when applying the Central Limit Theorem, with realizations $n \to \infty$, the variances of the 3 time-series processes are the same as the sample variances of each simulated time-series.

Question 6.

Simulate $T = 200$, 3×1 multivariate time-series data by using the DGP:

$$\boldsymbol{x}_t = \boldsymbol{\Lambda}\boldsymbol{x}_{t-1} + \boldsymbol{\varepsilon}_t,$$

*where $\boldsymbol{x}_t$ is a 3×1 vector. The coefficient matrix $\boldsymbol{\Lambda}$ is a 3×3 matrix and $\boldsymbol{\varepsilon}_t$ is a 3×1 vector of innovations. Make your own assumptions about all model components to ensure **unit root non-stationarity** of $\{\boldsymbol{x}_t\}$. Write all your assumptions about all model components in the report.*

To form a non-stationary VAR model, it needs to be transformed it into a random walk.

$$\boldsymbol{x}_t = \boldsymbol{\Lambda}\boldsymbol{x}_{t-1} + \boldsymbol{U}_t,$$

where $\boldsymbol{\Lambda}$ is non-stationary, because the stability condition does not hold. For the process to be non-stationary the following $\boldsymbol{\Lambda}$ was chosen:

$$\begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0.1 & 0.4 \end{pmatrix}$$

This matrix has 1 real eigenvalue equal to 1, and 2 complex eigenvalues $-0.2-i/10$, $+0.2-i/10$ respectively (both below 1, using the modulus method for the complex roots). Although the first two conditions were met (there were no shifts in the model and the error term was an i.i.d. process), stability does not hold. The $\lim_{n\to\infty} \boldsymbol{\Lambda}^n$ will potentially diverge to infinity, producing an unstable system.

Question 7.

Show that $\{\boldsymbol{x}_t\}$ is a unit root non-stationary multivariate time-series process. Explain how you conducted this analysis and elaborate on your findings in the report.

Firstly, to prove that the process is non-stationary the eigenvalues are examined.
All eigenvalues are distinct. Two of them are below 1, but the third one is 1. This implies that when the power n in the stability condition goes to infinity, there is a possibility that $\boldsymbol{\Lambda}$ will diverge to infinity too. Therefore it would be non-stationary (tested, using Python code). Furthermore, a Dickey-Fuller test for unit root non-stationary test is performed. The hypotheses for this test are:

$$H_0 : |\lambda| = 1$$
$$H_1 : |\lambda| < 1$$

The respective p-values are 0.303, 0.430 and 0.319. Consequently, all of them are greater than 0.05 which means H_0 cannot be rejected, and the processes are non-stationary.
The graphical representation of the 3 processes confirms the tests applied.

Question 8.

Plot the data set you generated. Include the plot in the report. Comment on the behavior of the data.

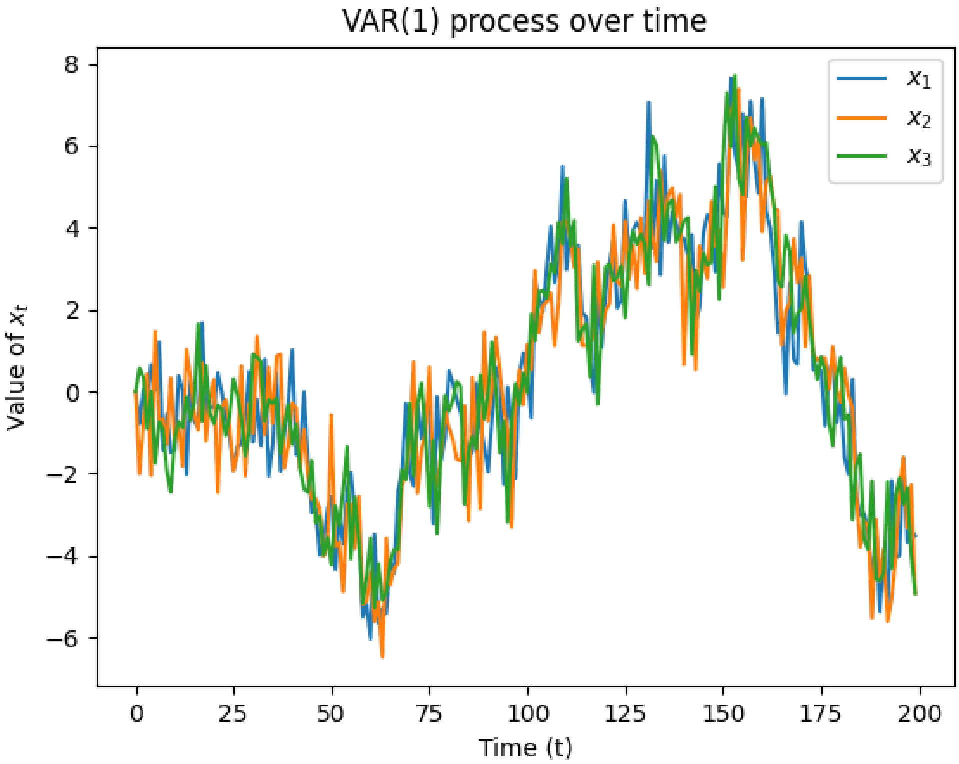

Figure 2: Non-stationary VAR(1) process.

Looking at the graph, all sequences tend to follow a random walk process. This could be explained by the fact that the processes are non-stationary and would never converge to their means, based on the fact that we have 2 eigenvalues lower than 1 and one eigenvalue equal to 1.

Question 10.

Plot the data, include the plot in your report.

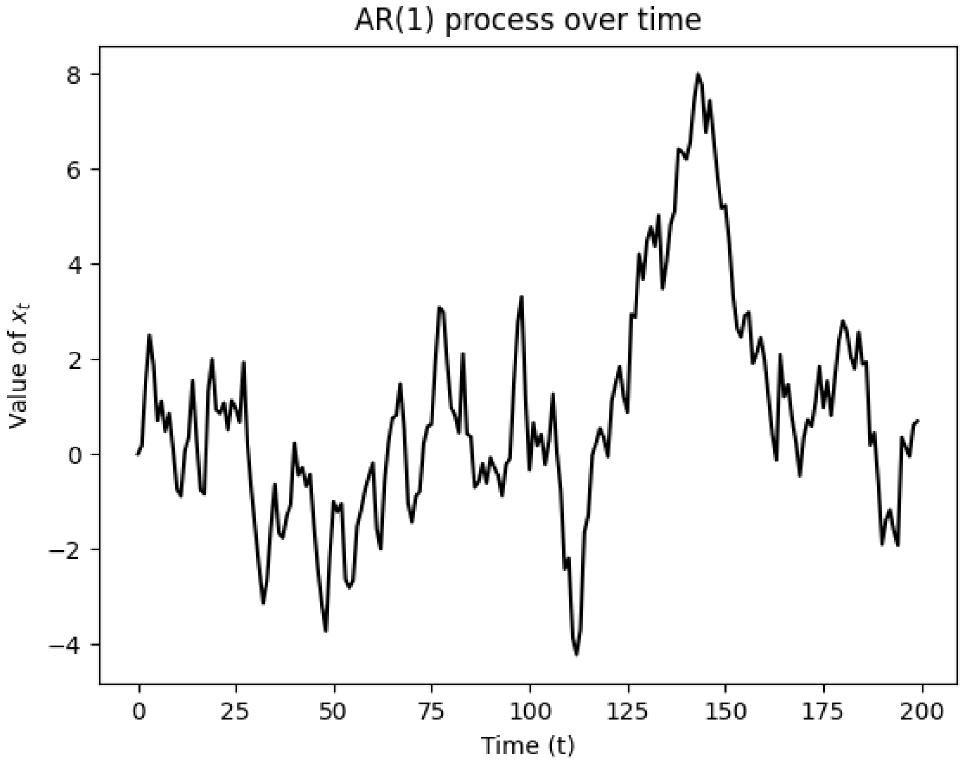

Figure 3: Generated data set, Random Walk Process.

Question 11.

Test for a unit root in the simulated data by using the Dickey Fuller test.

$$\boldsymbol{x}_t = \boldsymbol{x}_{t-1} + \boldsymbol{\varepsilon}_t,$$

The model above is initialized at $x_0 = 0$ and the innovations were drawn from a standard normal distribution. This suggests that the sequence is following a random walk process. The graphical representation confirms that the process is diverging from the mean (0), which implies that the process is non-stationary. After applying the Dickey-Fuller test a p-value of 0.089 is obtained. Since it is higher than 0.05, there is not enough evidence to reject the null hypothesis. Therefore, we conclude that the process contains a unit-root ($I(1)$).

Part 2.

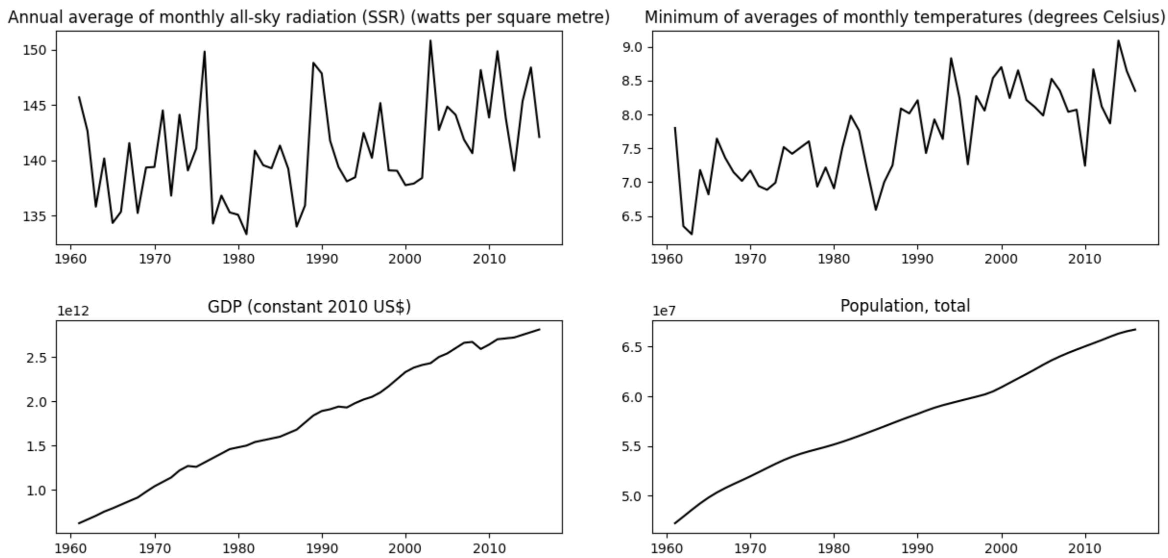

Figure 4: Levels of selected variables.

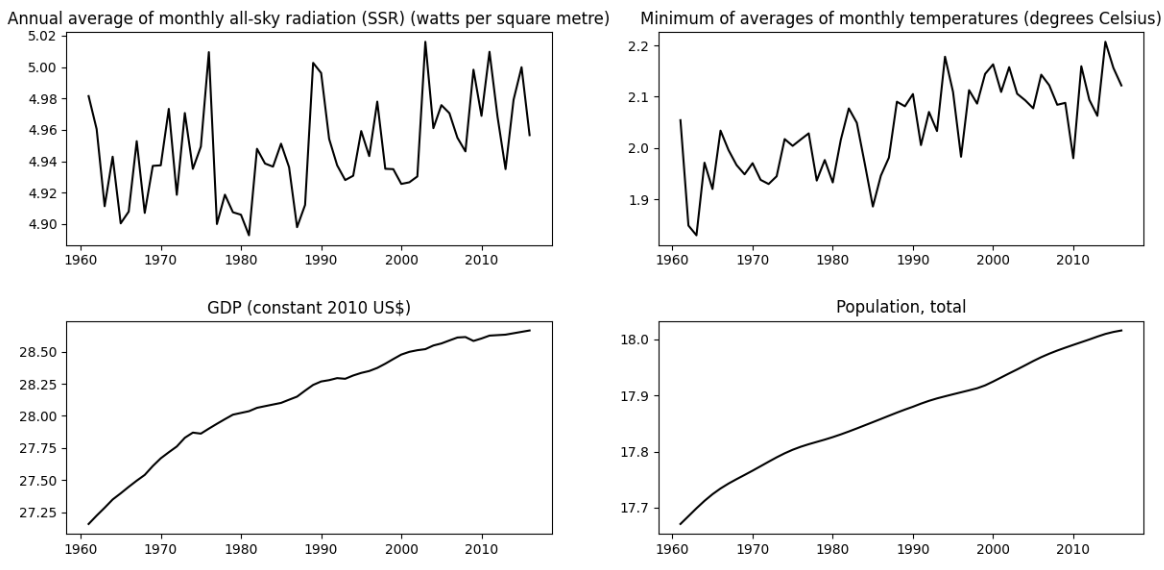

Figure 5: Logs of selected variables.

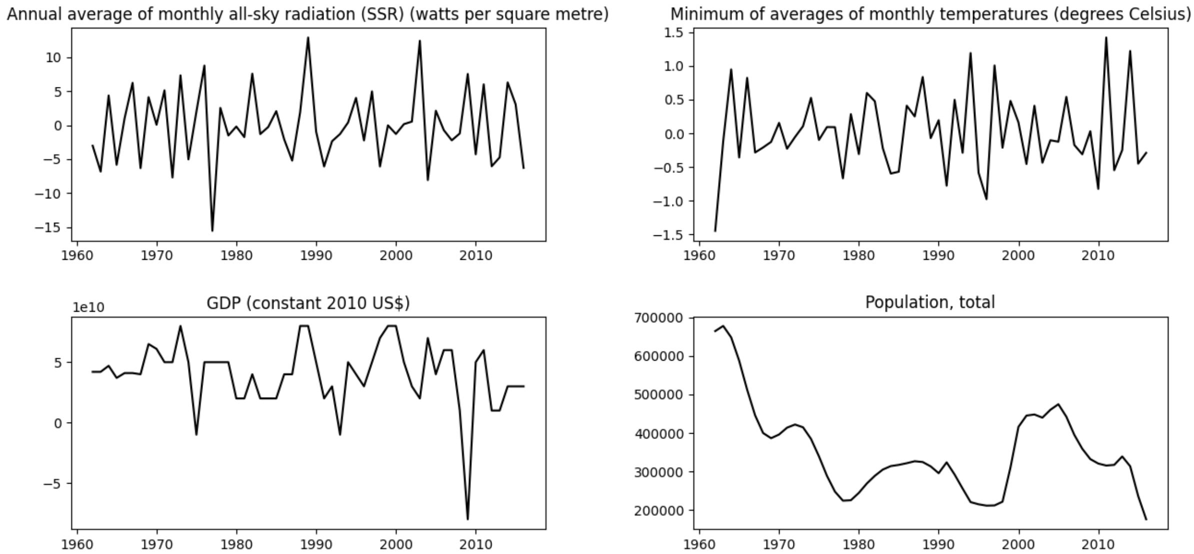

Figure 6: First differences of selected variables.

Question 1.

Do you see any evidence in favor or against the assumption of covariance stationarity? Do you suspect them to be I(2), I(1), I(0)?

By plotting the levels of the time-series for the selected variables, it can be inferred that the minimum temperature and average radiation variables could be stationary processes that always revert to their mean. Moreover, a positive trend could be observed for the two variables mentioned above.

From the diagrams it can be concluded that these trends are deterministic and not stochastic. An explanation could be the greenhouse effect or the fact that the more radiation the Earth's surface captures, the higher the minimum temperature will be over the years. This could suggest that there is a linear relationship between them that could be estimated and used to make predictions.

The plots of France's GDP and population imply that they are non-stationary processes, either $I(1)$ or $I(2)$. A trend line for these variables is obvious and could be a combination of a deterministic and a stochastic component since it never returns to the mean, but rather increases.

Question 2.

Do you see any evidence in favor or against for the presence of deterministic components such as a constant or a linear trend?

Before applying any tests and referring to external sources, the deterministic component that might be present in the time series is inferred only from the graphical representations.

All selected variables can have deterministic components, such as a constant or a trend. However, some of them may also have a stochastic trend. When analyzing these time-series, it is usually assumed that a constant and/or a deterministic trend are being used. Constants, are specifically recommended when researching economic or climate variables, as they are significant in most cases within the model.

Economic variables such as GDP and population that are used can potentially have a constant and a trend. The constant is the mean value of the dependent variable that is obtained when all independent variables are set equal to zero. The deterministic trend is the slope that a time series can have. This slope can be negative or positive. However, if this trend is deterministic, it can be predicted because it is not random, but rather follows a linear pattern that can be estimated.

For the climate variables (radiation and temperature) one could again assume a deterministic component, but in this particular case for the two observations (minimum temperature and radiation) this might not be so clear. According to the plots, both could have a potential upward trend, which could be considered normal when taking into account climate change and global warming.

Question 3.

Decide whether you want to use logarithms of any of the variables.

The logarithmic function is one of the most effective methods to reduce the large scaling of a given variable, and it is often used to transform a non-stationary time series into a stationary one.

In this case France's population and GDP appear to exhibit large scaling and do not follow a stationary process. For this reason, it would be useful to use the logarithmic function to adjust the scale of these specific variables and make them useful for conducting scientific analysis. If the scale is minimized for these variables, a more reasonable analysis and more accurate results could be obtained.

Part 3.

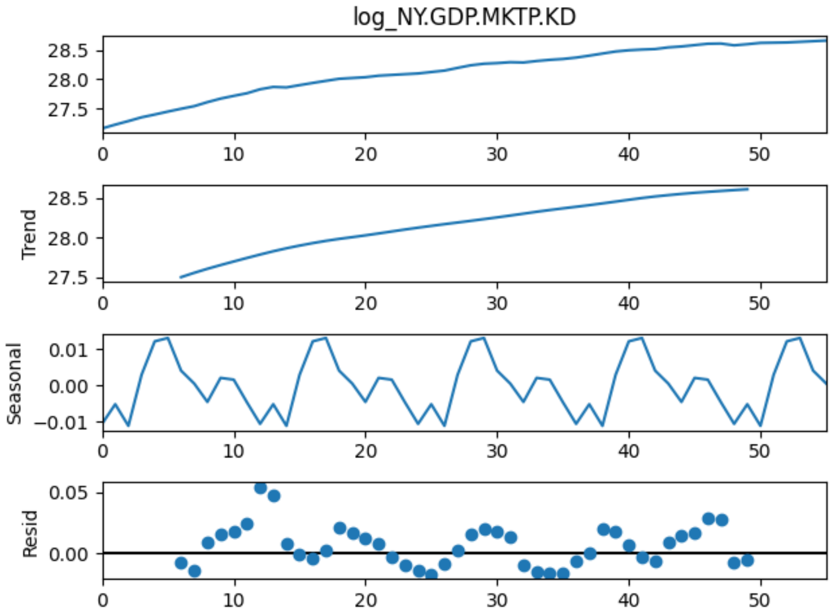

Figure 7: Seasonal decomposition of $log(GDP)$.

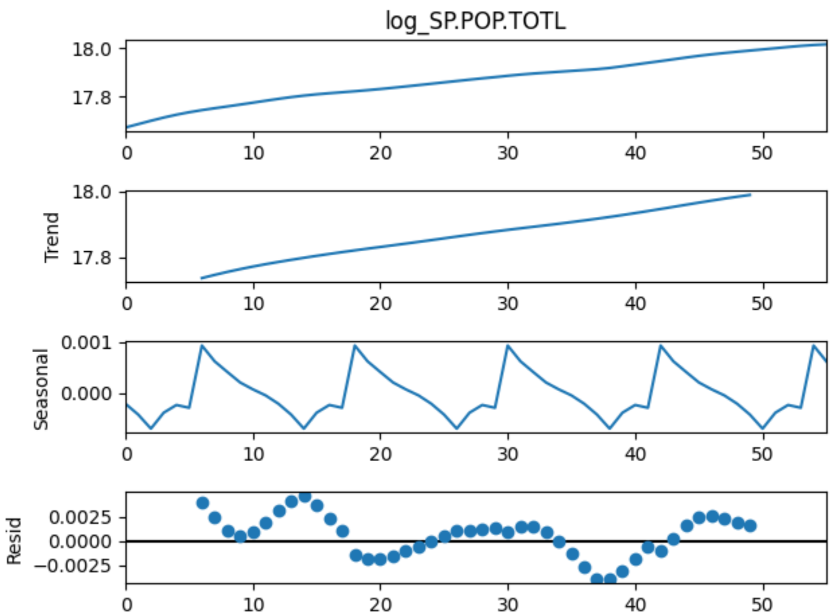

Figure 8: Seasonal decomposition of $log(population)$.

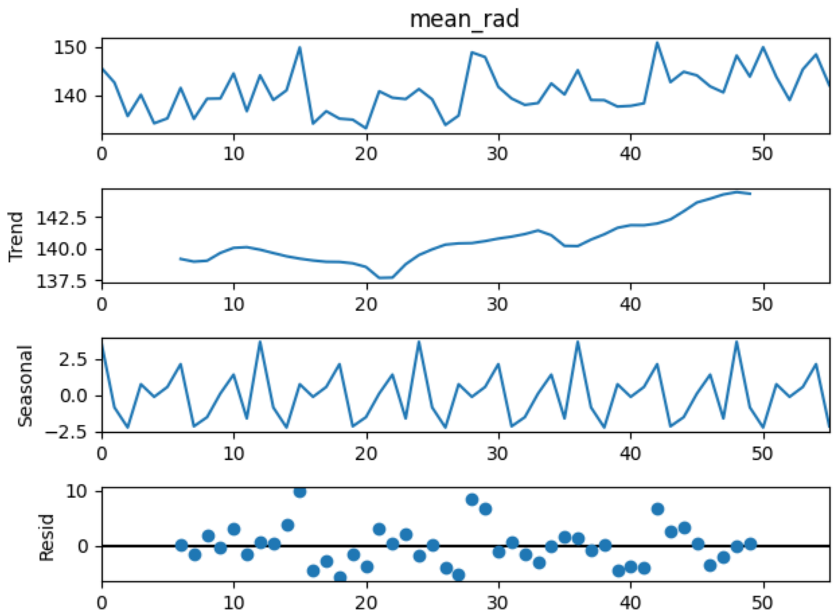

Figure 9: Seasonal decomposition of the radiation

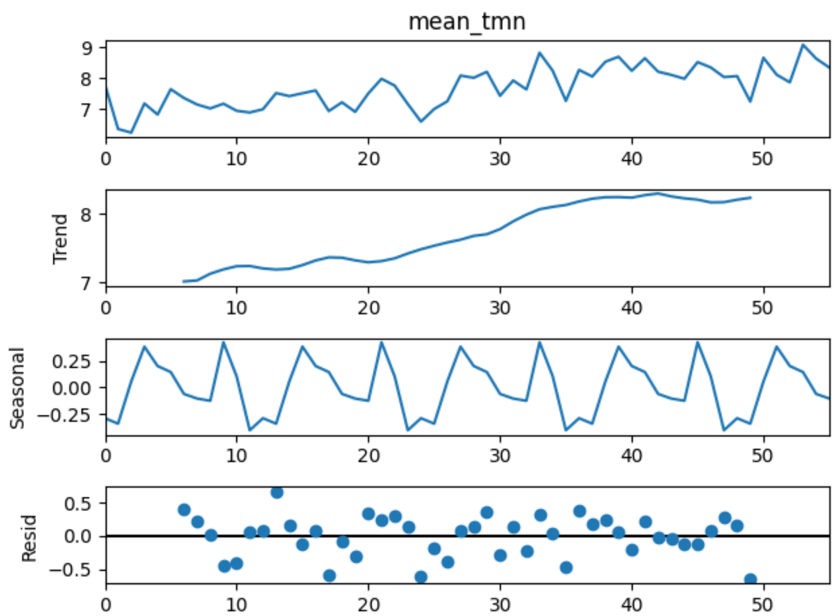

Figure 10: Seasonal decomposition of the temperature

Question 1.

Discuss carefully your choice of the deterministic components.

To decide which deterministic components to use in the Dickey-Fuller test, a technique called time-series decomposition is used. This method shows the seasonality, deterministic trend, and residuals and allows for a clearer representation and easier selection of the necessary components. There are two methods of time-series decomposition: additive and multiplicative. In this case, it is better to use the additive decomposition because when the time-series is plotted, a trend that is not linear cannot be seen (does not have a curve). The seasonal decompositions are shown in figures 7, 8, 9, and 10.

Moreover, it is useful to include constants in the Dickey-Fuller test, because in most cases the stationarity of an (economic or climatic) time series is examined with the "full" version of the DF test, which includes constant and trend. Additionally, it can be inferred from the plots that the deterministic trend always reverts to the general trend of the series in the long run, which is further confirmation that the observed models contain a deterministic component.

Question 2.

Discuss the possible evidence of serial correlation in the residuals of your Dickey-Fuller regression.

One of the problems in time-series analysis is the serial correlation between residuals after performing the Dickey-Fuller test. Serial correlation in time series can lead to the misspecification of a model, by resulting in biased coefficients. In order to remove the nuisance parameters (short and long-term variance) when performing the tests and to have a "clear" procedure, serial independence of the residuals needs to be assumed, i.e. they must be uncorrelated. In real data, it is not reasonable to assume no serial correlation in the residuals.

After accounting for autocorrelation in the residuals, a different approach to obtaining more reliable unit root tests can be taken, such as the Augmented DF test. This method includes lags of the differences of the dependent variable on the right-hand side of the test. Another alternative is the Phillips - Perron test.

Question 3.

Taking into account the presence of possible serial correlation, consider various extensions such as Augmented Dickey-Fuller test, and Phillips-Perron tests. In order to obtain more

robust results, Augmented Dickey-Fuller, Phillips-Perron, KPSS, and DFGLS tests are performed as they are the most commonly used ones and provide the best results.

At this point, it is good to mention that the ADF test treats serial correlation parametri-

cally, while the Phillips-Perron method - non-parametric. The DFGLS test is also considered the one that produces the most trustworthy results and has the greatest value when the AR coefficient is close to 1 (the random walk case). This test outperforms both ADF and Phillips-Perron and was proposed by (Elliot et al., 1996).

The Phillips-Perron method allows for the existence of serial correlation, but it can be calculated and excluded from the model. This technique corrects the estimator by subtracting the spurious parameters computed by the Newey and West method.

On the other hand, the ADF test adds lags of the first differences on the right side of the unit root test. These added lags 'remove' the autocorrelation parametrically. What should be considered here is that the researcher must be careful not to over- or under-parameterize the unit root test and select the ideal number of lags. This can be determined from the information criteria (AIC, BIC, etc.) or by constructing the correlation plot of each time series to check for autocorrelation.

The last test that will be performed to test for a unit root in this research is the KPSS test. It sets the case of stationarity at the null hypothesis and indicates that a series can be non-stationary, contain no unit root but still to be trend stationary (stationary around a deterministic trend).

Question 4.

Use some other tests that are robust too, for example, structural breaks. You need to find the test from the literature yourself.

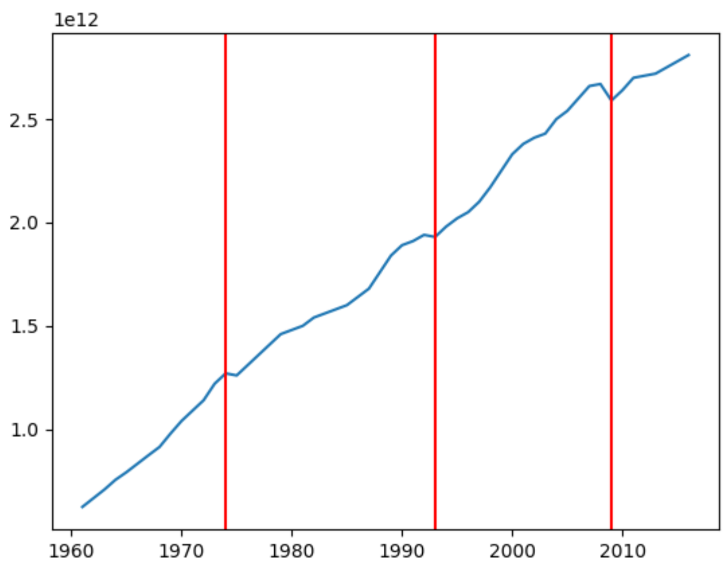

Figure 11: Potential structural break for $log(GDP)$.

A structural break is a sudden change in the time series, or more precisely in the trend. The most common approach to this type of problem is observing the graphical representation of the time series and detecting the potential structural break. The main objective is to divide the entire non-stationary time series into stationary sub-periods. For this purpose, some external literature on important economic, social and economic events could be used as an additional source. After selecting specific points in time, these can be further examined using Chow's test, which is a standard procedure in the econometrics literature for testing structural breaks.

The selected variables, the radiation and the minimum of the monthly average temperature, are stationary, so there is no reason to apply structural breaks. Although the population has an obvious non-stationary sequence, there is no historical reason to consider a significant difference in their values. Therefore, the only variable that would be examined for a structural break is the GDP, since there were significant historical events in the observed time frame that have affected the GDP of France.

According to figure 11, three potential structural breaks can be distinguished - in 1976 (due to the oil crisis which caused a substantial shock in the economies of most countries), in 1993, and in 2008 (due to the global economic crisis). The chow's test has:

$$H_0 : No - break$$
$$H_1 : Structural - break$$

The result is compared to an F distribution with a critical value at the 99.9% confidence interval being equal to 7.7. The respective results for 1976 and 1993 were t-stat equal to 79 and 18 which confirms that there were structural breaks. Although it was expected that the economic crisis in 2008 would affect the GDP, Chow's test gives a test statistic of 2.181 which does not confirm this result.

Question 5.

Present the results of the various tests and compare these results.

Unit-root test ($a = 5\%$, 'ct')	ADF	Phillips-Perron	DFGLS	KPSS
mean_rad	$0.00(-6.71)$	$0.00(-6.90)$	$0.00(-5.20)$	$0.10(0.10)$
mean_tmn	$0.00(-6.69)$	$0.00(-6.66)$	$0.00(-5.27)$	$0.10(0.07)$
$log(GDP)$	$0.37(-2.41)$	$0.08(-3.21)$	$0.94(-0.62)$	$0.01(0.26)$
$log(population)$	$0.46(-2.24)$	$0.00(-5.59)$	$0.84(-0.97)$	$0.02(0.18)$

By examining the table above it can be concluded that there are 2 variables that are trend stationary ($I(0)$) - the mean temperature and the mean radiation. In most of the conducted tests, these variables have a p-value equal to 0, therefore rejecting the null-hypothesis of the existence of a unit root and accepting the alternative that states that the series are stationary. Respectively, the other two variables (GDP and Population) can be confirmed to be $I(1)$ processes since the p-values were above the threshold of 0.05, which means that the Ho is rejected and a unit root is present. For the population the Phillips-Perron test returned a p-value equal to 0, implying that the series are stationary, and contradicting the result of the ADF test that was equal to $0.46 (> 0.05)$. This outcome could be attributed to the sample size or to the fact that the Phillips-Perron treats the autocorrelation non-parametrically.

To confirm whether the population is $I(0)$ or $I(1)$, a DFGLS test was conducted and returned a p-value equal to 0.84, ensuring that the time-series is non-stationary. By taking the first logarithmic differences of the population and running the proper unit root tests it can be found that $log(population)$ is not $I(1)$, but rather an $I(2)$ process, so it contains 2 unit roots.

For the KPSS test that has as a null hypothesis the absence of a unit root, the results comply with the ADF and the DFGLS.

For the temperature and the radiation we have p-values greater than 0.05 (meaning - accepting the null hypothesis) while for the $log(GDP)$ and the $log(population)$ the p-values of 0.01 and 0.02 respectively (rejecting the null of having no unit root).

Here it should be mentioned that during the tests (4 in total), both constant and deterministic trends were used to secure robust procedures and entirely isolate the stochastic trend that is required for the cointegration.

Final conclusion from the performed tests is:
$I(0)$ variables: mean_tmn and mean_rad
$I(1)$ variables: $log(GDP)$
$I(2)$ variables: $log(population)$

Part 4.

Question 1.

First briefly discuss the type of cointegrating relationship(s) you might expect by using some previous literature findings.

Cointegration is a common occurrence in economic events, as it represents that two or more variables have a common stochastic trend. Normally, it appears between two integrated processes of the same order different than zero and demonstrates that a linear combination of these variables can be stationary and in the long - run the variables reach equilibrium.

Since in this case two of the variables (radiation and minimum of average monthly temperature) are stationary processes, both integrated of order 0, no cointegrating relationship can be considered.However, cointegration can be inferred to the GDP and Population of France (but in this case they are not integrated of the same order, because Population is an I(2) process).

(Kim, 2016) discusses the relation between population and GDP growth for the time period between 1950 and 2016 in the OECD countries. The conclusion is that even though in the long - run after a certain point it is predicted that increasing the population of economically developed countries (like France) would have a negative effect on the GDP, for the same observed historical period as in this report, the relation between population and GDP is positive. This observation proves the idea that there is cointegration between the variables and that they could possibly share the same stochastic trend that moves the trend lines together.

Although cointegration is considered to occur between non-stationary variables, external resources, such as (Kahn et al., 2019) suggest that in the long run, there could be potential cointegration between economic and climatic variables, and more specifically between GDP and temperature. For this specific article, data from 1960-2014 and 174 different countries was used and the conclusion is that no matter if a country is rich or poor, cold or warm, in the long-run the rise of the temperature will have negative results on the GDP.

Question 2 & 3.

Assuming that your series are I(1), test for (no)-cointegration using some of the techniques discussed during the course, in Davidson 2000 and in the related literature. Compute and compare the outcomes of various tests for no-cointegration. You may use:
– The standard residual based cointegration tests using a static regression. (Engle and Granger approach, ADF type tests, Phillips-Ouliaris' test);
– The Maximum Likelihood based tests (Johansen Trace and Maximum Eigenvalue tests);

Is there evidence in favor or against cointegration? Is there any evidence in favor of more than one cointegrating vector? If yes, what identification scheme one should adopt and why?

Several tests for cointegration between two $I(1)$ variables can be conducted, such as the Engle and Granger test, the Phillips-Ouliaris test, and ADF type tests.
By taking into account previous studies and the results from part 3, a test for the 4 variables $log(GDP)$(dependent one), $log(population)$,$mean\_rad$ and $mean\_tmn$ (independent ones) can be run. These tests show any evidence of no cointegration among these variables, checking the resulting p-values for 5% confidence level. In the conducted test both trends and constants are used. Here it is important to mention that when we conducted the ADF type tests we did not use the simple Dickey-Fuller approach, because Dickey-Fuller needs raw data as input. So, to run a simple OLS regression between $log(GDP)$(dependent one), $log(population)$,$mean\_rad$ and $mean\_tmn$ (independent ones),get the residuals from this regression and run a simple AR, so the 'residuals = $\hat{u}$' of the AR model (dependent variable is the residuals from the OLS regression with the 4 variables) ($\hat{\varepsilon} = \rho\hat{\varepsilon}_{t-1} + \hat{u}$) can be obtained. The last step is to conduct stationarity tests on $\hat{u}$ (like the Dickey-Fuller test), using James-MacKinnon critical values.

Cointegration tests	p-val.(t-stat.)
Engle and Granger test	0.94(-1.32)
Phillips-Ouliaris test	0.95(-1.34)
ADF-type test	-1.33(t-stat)

Table 3: Tests for cointegration among the 4 time-series.

By looking at table 3 it can be seen that there is no evidence of cointegration among the variables because all of the conducted tests have p-values greater than 0.05, while the t-stats of both regressions are lower than -3.14, in other words, the residuals are not stationary, so the null-hypothesis, claiming there is no cointegration, cannot be rejected.

Johansen Trace and Maximum Eigenvalue Tests:

Johansen Trace test allows for checking if three or more time-series are cointegrated. The conduction of the test consists of a series of tests, whose purpose is to demonstrate if there is cointegration and if so, how many cointegrated vectors there are.

The first null hypothesis states that there is no cointegrating vector, and there is no cointegration between the inspected processes ($r = 0$). The alternative states that there is a cointegrating vector ($r > 0$). If the null hypothesis is rejected, and cointegration is detected, then the test is run again with a new null hypothesis, $H_0 : (r = 1)$, and alternative hypothesis, $H_1 : (r > 1)$; this process continues until the null hypothesis cannot be rejected. The result is the number of cointegrating vectors.

No. of coint. vectors	t-stat.	90% crit. val.	95% crit. val.	99% crit. val.
$r = 0$	86.04	51.65	55.25	62.52
$r = 1$	44.20	32.06	35.01	41.08
$r = 2$	16.14	16.16	18.40	23.15
$r = 3$	0.08	2.71	3.84	6.63

Table 4: Tests for cointegration among the 4 time-series using the Johansen trace test.

Table 4 tests for cointegration between all variables in this research, assuming that they are all $I(1)$ processes. After performing the test, it is noticeable that cointegration is present, as initially, using the first t-statistics (86.05) the null hypothesis (no cointegration) can be rejected, as it exceeds the critical value of (55.25) at the 95%-confidence level. Repeating the process leads to the following result: the rank of the matrix is 2 ($r = 2$), therefore there are two cointegrating vectors at the 95%-confidence level.

No. of coint. vectors	t-stat.	90% crit. val.	95% crit. val.	99% crit. val.
$r = 0$	41.84	28.23	30.81	36.16
$r = 1$	28.06	21.87	24.25	29.26
$r = 2$	16.06	15.00	17.14	21.74
$r = 3$	0.07	2.70	3.84	6.63

Table 5: Tests for cointegration among the 4 time-series using the max. eigenvalue test.

In table 5 the results of the maximum eigenvalue test are shown. The maximum eigenvalue test confirms the Johansen's Trace test as again the null hypothesis is rejected for ($r = 0$) and ($r = 1$), leading to the conclusion that there are ($r = 2$) cointegrating vectors. t-statistic for $H_0 : (r = 2)$ is 16.06, which is less than the critical value of (17.14) at the $95 - \%$ confidence interval, therefore it cannot be rejected).

Identification scheme: Since there are two cointegrating vectors ($r = 2 > 1$) the Johansen method approach can be adopted, so that the coefficients of the system can be computed. The Johansen approach, using the Maximum Likelihood method allows for a test of higher orders of cointegration such as $r = 2, 3, 4, etc.$

Question 4.

Assuming there is a single cointegrating vector, estimate the cointegrating regression by using various approaches discussed in the lectures and in (Davidson 2000), such as static least squares, DOLS, FMOLS and ECM.

Performing the Static, Dynamic, and Fully - Modified OLS is possible when there are at least two $I(1)$ processes present. These variations of the OLS estimator are used since some of the methods have flaws which could be improved by changing the approach. Therefore, all of them will be applied in the context of this research and the final results will be compared. For the purposes of this work, it is assumed that the Population, the Mean Radiation, and the Minimum Monthly Temperature are $I(1)$ processes, when in reality the observed variables are $I(2)$, $I(0)$ and $I(0)$ respectively. As a general rule to follow, the idea that obtaining a consistent estimator for the cointegrating vector is possible only when there is a unique cointegrating vector.

Firstly, the Static OLS will be performed on the two variables. The deficiencies of this approach are related to the fact that the asymptotic limit produces results that are seriously affected by dependency and bias. More specifically, there are 2 nuisance parameters in the limit - one representing the autocorrelation effect, and one for the simultaneity effect. Moreover, there is a dependence between B1 and B2 (which are the Brownian motions of the processes, for the regressors and for the error term respectively), that is the third source of bias. The results from the Static OLS performed on the assumed $I(1)$ variables confirm some of the issues - the Adjusted R2 tends to unity (=1) as in this case it is equal to 0.974. Although the p-values for the constant and Population are approximately 0.00 (being smaller than 0.05 and therefore significant), this may not necessarily be the case. The high Adjusted R2 - value could suggest spurious regression. The p-values for the other 2 variables are higher than 0.05 which confirms that they are statistically insignificant.

A partial solution to the bias in the results from the Static OLS could normally be the application of the Dynamic OLS which includes $I(0)$ variables within the model, so that the asymptotic distribution of the parameter estimate can be more tractable. However, Dynamic OLS does not completely remove the bias as well as the simultaneity effect also remains intact, due to the join determination of the regressor variables and the error terms.

The result obtained from the Dynamic OLS demonstrates again the problem that R2 tends to unity (=1). Although in this case, the p-values are all significant, the estimated parameters and standard errors are huge, and therefore they can be considered ill-suited since the actual data contains two $I(0)$ and one $I(2)$ processes. The results are not reliable because for the purpose of the task no change has been done on the data.

As an extra test for the quality of the results, the Population variable $I(2)$ can be transformed into an $I(1)$ by taking the first difference of the logarithmic value. However, after applying Dynamic OLS again with two $I(1)$ variables and adding two $I(0)$ variables the results remain imprecise, which supports the argument that these methods are not well suited for the specific piece of data that is used. The problem of simultaneity, autocorrelation, and dependence of

B2 on the B1 can be solved by the application of Fully Modified Least Squares. This technique approaches the model by using non-parametric correction and the second difference of independent variables. Since this method is applied only for $I(1)$ variables, in this case, it is assumed that all the variables are $I(1)$, but the results are expected to be inconsistent. Indeed, the R2 goes to unity and p-values are significant for population and constant, but not for radiation and temperature.

The conclusion regarding the three applied methods above confirms that they are not performing well in this case (as expected), since the data is only theoretically assumed to contain $I(1)$ processes, but in reality that is not correct.

If the variables are all assumed to be $I(1)$ processes, then a Vector Error Correction Model can be performed as a logical next step to adjust the short-run changes in the variables. By doing this, it also restricts the long-run behavior of the endogenous variables. This way the model secures that in the long - run, the time - series are converging to the cointegrating relationship which does not allow for a huge deviation from the long - run equilibrium.
*(A remark about this part is that again the data is not suitable for this model, consequently it is not expected to produce meaningful results).

VECM is an alternative to the Dynamic OLS in a way that it subtracts the lagged term of the dependent variable from both sides, meaning that the first difference of the dependent variable is being estimated. Also, it takes into account the error correction term and lags of all of the other variables of the system .

Important part of the interpretation of the VECM model are the obtained coefficients. For instance, the short-run effects are captured through coefficients of the endogenous variables. The most pivotal coefficient is the one before the error correction term represents which identifies the speed with which the variables adjust to their long - run equilibrium.
In this case the error correction coefficient is $|-0.0182|$. However, the p-value is greater than 0.05, implying that it is insignificant. In this case insignificance implies that the past equilibrium errors do not determine the long-run impact.
It can be then confirmed that the application of the VECM is not relevant for the conducted research.

Last but not least, a good and useful approach in order to overcome the problem of the different order of integration of the variables would be to conduct an ARDL model, but in this specific case we have a variable that is I(2), and ARDL model consists only of I(0) and I(1) variables. This model enables us to analyze long run relationships at the same time among the variables of different order of integration.

Question 5.

Adopt a systems approach and use Johansen's analysis (Chapter 16 of Davidson 2000) to estimate the cointegrated system. Report your estimation and inference results (coefficient estimates, test statistics, hypothesis tests, test results etc.) Discuss the similarities and differences between the results of Johansen's systems approach and the results of the single equation approach. Discuss the differences between the assumptions of these methods.

Johansen's analysis examines the cointegrating system when there have more than one cointegrating vectors. In this work there are 2 cointergrating vectors r=2, so Johansen's system approach will be ideal for this case.
Here it should be mentioned that this approach holds under the assumption that the variables are I(1). When the Johansen trace test is conducted, the eigenvectors are also obtained, as they contain the parameters for each series. The eigenvectors are as follows:

Eigenvector 1: [1.12,7.04,7.45,1.10], Eigenvector 2: [−1.55,−8.63,−6.95,4.36].

The first coefficient in each eigenvector corresponds to each time-series.
Firstly, eigenvector corresponds to log of GDP, the second one to logarithm of population, the third to temperature, and the fourth to the radiation.
By plotting the series that are computed by using the respective coefficients of each variable in Python it can be inferred that the series look stationary (plot available in the Python code). These results hold for both eigenvectors that represent the linear combination of the 4 variables in the long-run. Compared to the other methods such as the single equation approach, Johansen' approach seems to be the strongest one and offers the most reliable results.

*Note (here it is assumed that all four variables were I(1). In this case there is r=2 because there are 2 variables that are I(0) (the mean temperature and the mean radiation) and these specific variables are the ones that secure the two cointegrating vectors, because the $\boldsymbol{\gamma}$ (prime) in the cointegration system has rank=2. In addition, the Johansen trace test to be sufficient has to return maximum m-1 cointegrating vectors, with m = to the number of I(1) variables in the system).

Question 6.

Are the results in accordance with your prior expectations? How should we interpret these results?

The overall conclusion, as stated in the previous sections, is that cointegration is not present, therefore long-term relationship between the observed variables is cannot be theoretically supported to exist. This is an expected result, since cointegration appears only when variables are $I(1)$ processes.

However, when the data is assumed to be $I(1)$, then proper tests can be performed. If all variables were $I(1)$ then the Johansen test could be applied and the actual number of cointegrating vectors shared between the variables could be obtained. Then, a proper and theoretically hold up result could be acquired and a final decision about the actual long-run relationship between the variables can be made.

Question 7.

Are the long run relations you discover apparently constant through time? If not, do you think whether the interventions of policymakers affect these relations?

The relationship that was examined is unstable in the long-run, because in most cases tests showed the lack of cointegration among the system of the four variables. This means that in the future these four variables would not share a common trend, consequently they are not reaching an equilibrium.

Policymakers would play an important role in this case, because by introducing specific interventions, they could certainly affect the way climate and economic variables develop over time. A good case in point would be the long-run relationship between temperature and solar radiation. By minimizing the amount of solar radiation reaching the Earth's surface through certain policies and measures, the pace of global warming could be potentially altered.

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