1. Solve the following LPs using the simplex method **(10 marks)**

(a)

$$\begin{array}{llllllll}
\text{Maximize} & x_1 & - & 2x_2 & + & x_3 & & \\
\text{Subject to} & x_1 & + & 2x_2 & + & x_3 & \leq 12 \\
& 2x_1 & + & x_2 & - & x_3 & \leq 6 \\
& -x_1 & + & 3x_2 & & & \leq 9 \\
& x_1, & & x_2, & & x_3, & \geq 0
\end{array}$$

(b)

$$\begin{array}{llllll}
\text{Maximize} & 3x_1 & + & 5x_2 & \\
\text{Subject to} & x_1 & - & 2x_2 & \leq 6 \\
& x_1 & & & \leq 10 \\
& & & x_2 & \geq 1 \\
& x_1, & & x_2, & \geq 0
\end{array}$$

2. Provide an algorithm based on the simplex method to check if a given system of inequalities is feasible. **(5 marks)**

3. Given a bipartite graph with vertex set $V = X \cup Y$ with $|X| = |Y|$ and edge set E. Each edge e has a weight w_e. A matching is a subset $M \subseteq E$ of edges such that each vertex of both X and Y is incident to exactly one edge of M.

   (a) Write an ILP for finding a maximum weight matching.

   (b) Show that in the relaxed LP, every non-integral feasible solution can be expressed as a convex combination of two distinct feasible solutions. **(10 marks)**

4. Prove that the simplex method using Bland's pivoting rule terminates. **(5 marks)**

5. An LP in *general form* is given as: maximize $c^T x$ subject to $Ax \leq b$ where A is an $m \times n$ matrix, c and x are $n \times 1$, and b is $m \times 1$. Each constraint $A_i x \leq b_i$ is a half-space defined using the *hyperplane* $A_i x = b_i$. The general form LP is said to be *degenerate* if the feasible region contains a point obtained as an intersection of more than n hyperplanes.

   An LP in *equational form* is given as: maximize $c^T x$ subject to $Ax = b$, $x \geq 0$. Equational form LP is degenerate if there are several feasible bases corresponding to a single basic feasible solution.

   (a) Convert the following general form LP to equational form.

$$\begin{array}{llllllll}
\text{Maximize} & 5x_1 & - & x_2 & + & 2x_3 & \\
\text{Subject to} & x_1 & - & 6x_2 & + & x_3 & \leq 2 \\
& 5x_1 & + & 7x_2 & - & 2x_3 & \leq -4 \\
& 8x_1 & - & 10x_2 & + & 19x_3 & \leq 75 \\
& & & 5x_2 & + & 14x_3 & \leq 30
\end{array}$$

   (b) Convert the following equational form LP to general form.

$$\begin{array}{llllllll}
\text{Maximize} & 2x_1 & - & 3x_2 & + & 4x_3 & \\
\text{Subject to} & -x_1 & + & 2x_2 & - & x_3 & = 14 \\
& 5x_1 & - & 6x_2 & + & 12x_3 & = 20 \\
& x_1 & , & x_2 & , & x_3 & \geq 0
\end{array}$$

   (c) Show that if a general form LP is degenerate, then the corresponding equational form LP is also degenerate.

   (d) Show that if an equational form LP is degenerate, then the corresponding general form LP is also degenerate. **(14 marks)**

6. Assume we are given a non-degenerate LP in equational form. Show that the set of basic feasible solutions is exactly the set of extreme points. **(6 marks)**