CMI - 2020-2021: DS-Analysis

Calculus-1

Limits and continuity

We will begin by reviewing some basic aspects of convergence of a sequence of real numbers. We will denote the set of all real numbers by $\mathbb{R}$.

Let $\{x_n : n = 1, 2, 3, \cdots\}$ be a sequence of real numbers. So $x_n \in \mathbb{R}$ for each $n \geq 1$.

Example 1: Let $x_n = \frac{1}{n}$, $n = 1, 2, 3, \cdots$ It is intuitively clear that this sequence converges to 0 as $n \to \infty$; that is, $|x_n - 0| = \frac{1}{n}$ can be made as small as we want, if we take sufficiently large n. We denote this fact by writing $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 2: In a similar way clearly $\lim_{n\to\infty}(1 - \frac{1}{n^2}) = 1$.

Example 3: Consider the sequence $\{1, 2, 1, 2, 1, 2, \cdots\}$, that is,

$$\begin{aligned} x_n &= 1, \text{ if } n \text{ is odd,} \\ &= 2, \text{ if } n \text{ is even.} \end{aligned}$$

In this case, it can be seen without difficulty that there is no $x \in \mathbb{R}$ such that $|x_n - x|$ can be made as small as we want, however large n might be.

Example 4: Let $x_n = n$, $n \geq 1$. Clearly here also there is no $x \in \mathbb{R}$ such that $|x_n - x|$ can be made as small as we want, however large n might be.

To have a clear idea about convergence we have the following definition.

Definition 1 Let $\{x_n : n = 1, 2, 3, \cdots\}$ be a sequence of real numbers. We say that $\{x_n\}$ is *convergent* if there is $x \in \mathbb{R}$, such that for each $\epsilon > 0$ we can get an integer N so that $|x_n - x| < \epsilon$ for all $n \geq N$. (The choice of N can depend on ϵ.) In such a case we say that $\{x_n\}$ converges to x, denoting it by $x_n \to x$, or $\lim_{n\to\infty} x_n = x$.

If $\{x_n\}$ is not convergent, we say it is *divergent.*

Exercise: Using the above definition, show that the sequences in Examples 1 and 2 are convergent. You will also see that the choice of N depends on ϵ. It is a fairly simple exercise.

Let $\{x_n\}$ be a sequence in $\mathbb{R}$.

(i) We say $\{x_n\}$ is *bounded* if there is a number $M > 0$ such that $|x_n| \leq M$ for all $n \geq 1$. If it is not bounded, it is unbounded.

(ii) We say $\{x_n\}$ is *increasing* if $x_n \leq x_{n+1}$ for all $n \geq 1$.

(iii) We say $\{x_n\}$ is *decreasing* if $x_n \geq x_{n+1}$ for all $n \geq 1$.

(iv) We say $\{x_n\}$ is *monotonic* if it is increasing or if it is decreasing.

Exercise: Identify the increasing, decreasing, bounded and unbounded sequences in Examples 1 - 4.

We now state a basic result on convergence of sequences without proof.

Proposition 2 *(i) Any convergent sequence is bounded.*

(ii) If $\{x_n\}$ is convergent, then the limit is unique; that is, it can not happen that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} x_n = y$, but $x \neq y$.

(iii) Let $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$. Then the following are true: (a) $\lim_{n\to\infty} x_n + y_n = x + y$; (b) $\lim_{n\to\infty} cx_n = cx$ for any constant $c \in \mathbb{R}$; (c) $\lim_{n\to\infty} x_n^2 = x^2$; (d) $\lim_{n\to\infty} x_n y_n = xy$; (e) if $x \neq 0$, then $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$.

Suggestion: Proof of the above is not difficult. You may try it. (Hint: To prove (i): If $x_n \to x$, then with $\epsilon = 1$, conclude that only finitely many x_j's lie outside the interval $(x-1, x+1)$. To prove (ii): Suppose $x \neq y$, then $a := |x-y| > 0$. Take any $\epsilon < \frac{1}{3}a$; use the definition of convergence, to conclude except for finitely many x_j's, all of them lie in $(x-\epsilon, x+\epsilon)$, as well as in $(y-\epsilon, y+\epsilon)$; as these intervals are disjoint, you get a contradiction. So it follows that $x = y$.)

Let $\{x_n\}$ be such that $x_n \leq x_{n+1}$, and $x_n \leq b$ for all $n = 1, 2, \cdots$, where b is a constant; that is, $\{x_n\}$ is an increasing sequence which is bounded above. For notational convenience, we put $E = \{x_n : n = 1, 2, \cdots\}$. The number b is an 'upper bound' for the set E. Let λ be the 'least upper bound' for the set E; that is, λ is an upper bound for E, and if c is any upper bound for E then $\lambda \leq c$. The nature/ structure of $\mathbb{R}$ guarantees the existence of a unique least upper bound for any nonempty set which is bounded above. The least upper bound for E is called the *supremum* of E, and is denoted $\sup E$, or $\sup\{z : z \in E\}$. (See Remark 5 below.) With this notation $\lambda = \sup\{x_n :$

$n = 1, 2, 3, \cdots\}$. It can be shown that $\lim_{n\to\infty} x_n = \lambda$. Thus, if $\{x_n\}$ is an increasing sequence which is bounded above, then $\{x_n\}$ is convergent, and

$$\lim_{n\to\infty} x_n = \sup\{x_n : n = 1, 2, \cdots\}.$$

Next, let $\{x_n\}$ be such that $x_n \geq x_{n+1}$, and $x_n \geq \ell$ for all $n = 1, 2, \cdots$, where ℓ is a constant; that is, $\{x_n\}$ is an decreasing sequence which is bounded below. One can now proceed in an analogous manner, noting that ℓ is a 'lower bound' for the set $E = \{x_n : n = 1, 2, \cdots\}$. In this case, the 'greatest lower bound' for the set E would be the limit. The greatest lower bound for E is called the *infimum* of E, and is denoted $\inf E$, or $\inf\{z : z \in E\}$. So, if $\{x_n\}$ is a decreasing sequence which is bounded below, then $\{x_n\}$ is convergent, and

$$\lim_{n\to\infty} x_n = \inf\{x_n : n = 1, 2, \cdots\}.$$

Therefore monotonic sequences are easier to handle. We may state

Proposition 3 *A monotonic sequence is convergent if and only if it is bounded.*

In Example 1, observe that $\lim_{n\to\infty} \frac{1}{n} = \inf\{\frac{1}{n} : n = 1, 2, \cdots\}$, while in Example 2, $\lim_{n\to\infty}(1 - \frac{1}{n^2}) = \sup\{1 - \frac{1}{n^2} : n = 1, 2, \cdots\}$.

Remark 4 A sequence $\{x_n\}$ in $\mathbb{R}$ is called a *Cauchy sequence*, if for every $\epsilon > 0$, there is an integer $N > 0$ such that $|x_n - x_m| < \epsilon$ for all $n, m > N$.

It is not difficult to see that any convergent sequence is a Cauchy sequence.

It is also true that : "Any Cauchy sequence in $\mathbb{R}$ is convergent". The proof of this assertion is beyond the scope of this course. We have mentioned this to indicate that: "To show that a given sequence is convergent, it is enough to prove that the given sequence is a Cauchy sequence".

Remark 5 As indicated earlier, supremum and infimum may make sense for subsets of $\mathbb{R}$ which need not arise from sequences. The structure of $\mathbb{R}$ guarantees the existence of a unique least upper bound (resp. greatest lower bound) for any nonempty set which is bounded above (resp. below).

Let $E_1 = \{x \in \mathbb{R} : x < 10\}$; clearly E_1 is bonded above with 12 as an upper bound. But E_1 is not bounded below. Note that $\sup\{x : x \in E_1\} = 10$. But no $y \in \mathbb{R}$ can be an infimum of E_1. Also $(\sup E_1) \notin E_1$.

Let $E_2 = \{x \in \mathbb{R} : x \leq 10\}$; in this case, clearly $(\sup E_2) \in E_2$.

Let $E_3 = \{x \in \mathbb{R} : x > 3\}$; note that E_3 is bounded below, but not bounded above; clearly $(\inf E_3) = 3 \notin E_3$.

Now $F = E_2 \cap E_3 = \{x \in \mathbb{R} : 3 < x \leq 10\} = (3, 10]$. Clearly the interval $(3, 10]$ is bounded; also $(\sup F) \in F$, but $(\inf F) \notin F$.

Intuitively 'supremum' is very similar to 'maximum'; the term $\sup E$ is used to signify that $\sup E$ need not always be a member of the set E. A similar comment applies to 'infimum' and 'minimum'.

We now review some basic aspects of continuity of functions.

Let $I \subseteq \mathbb{R}$ denote an interval. We will usually take it to be a bounded interval of the form (a, b) or $[a, b]$ with $a < b$, or unbounded interval of the form $[a, \infty)$ or $I = \mathbb{R}$.

Definition 6 Let $f : I \to \mathbb{R}$ be a function. Let $x \in I$. The function f is said to be *continuous at* x if and only if whenever $\{x_n\}$ is a sequence in I with $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} f(x_n) = f(x)$.

If f is continuous at every $x \in I$, then f is called a *continuous function* on I.

Fact: Composition of continuous functions is also continuous, whenever it is well defined. In other words, if f and g are continuous functions, then the composite function h, given by $h(x) = g(f(x))$, is also continuous whenever h makes sense. (Observe that for $g(f(x))$ to make sense, the value $f(x)$ should be in the domain of definition of g.)
Some examples of continuous functions: (i) Constant function: $f(x) = c$, $x \in \mathbb{R}$, where c is a constant real number. (ii) Identity function: $f(x) = x, x \in \mathbb{R}$. (iii) $f(x) = |x|$, $x \in \mathbb{R}$. (iv) Any polynomial on $\mathbb{R}$. (v) The familiar functions of basic calculus, such as the exponential, trigonometric and logarithmic functions are continuous wherever the definition makes sense.

Continuity of such functions justifies the common practice of evaluation certain limits, like

$$\lim_{y\to 0} e^y = e^0 = 1;$$

the above is in the sense that: If $\{y_n\}$ is any sequence converging to 0, then $\lim_{n\to\infty} e^{y_n} = e^0 = 1$.

We mention below two results on continuous functions without proofs. The first result is called the 'intermediate value theorem'.

Theorem 7 *Let $f : I \to \mathbb{R}$ be a continuous function, where I is an interval. Let $\alpha, \beta \in I$ be such that $\alpha < \beta$ and $f(\alpha) \neq f(\beta)$. If c is a point between $f(\alpha)$ and $f(\beta)$, then there is a point θ such that $\alpha < \theta < \beta$ and $f(\theta) = c$.*

An immediate consequence: If f is a continuous function on I, and $f(\alpha) > 0$, $f(\beta) < 0$, with $\alpha, \beta \in I$, then there is a point θ between α and β such that $f(\theta) = 0$.

Another easy consequence is the following: Let $f(I) = \{y \in \mathbb{R} :$ there is some $x \in I$ with $f(x) = y\}$; that is, $f(I)$ is the *image of* I under the function f. Then the image $f(I)$ of an interval I under a non-constant real valued continuous function f is again an interval. If f is a constant function then $f(I)$ is just a point.

Theorem 8 *Let $[a, b]$ be a closed and bounded interval. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exist points $p, q \in [a, b]$ such that $f(p) = \inf\{f(x) : x \in [a, b]\}$ and $f(q) = \sup\{f(x) : x \in [a, b]\}$. In other words, a continuous real valued function on a closed bounded interval attains both its minimum and maximum values.*

Using both the results above we get the following: If $I = [a, b]$ is a closed bounded interval, and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is the closed bounded interval $[c, d]$ where $c = \inf\{f(x) : x \in [a, b]\}$, and $d = \sup\{f(x) : x \in [a, b]\}$.

Note: In Theorem 8, it is necessary that I is a closed and bounded interval; otherwise, the result need not hold.

(i) Take $I = (0, 1)$, $f(x) = x^2$. Then note that $\inf\{f(x) : x \in (0, 1)\} = 0$, and $\sup\{f(x) : x \in (0, 1)\} = 1$. But at no point $x \in (0, 1)$ we can have $f(x) = 0$, or $f(x) = 1$. So neither minimum nor maximum is attained in I.

If $I = (0, 1]$ with the same f, it is clear that the minimum is not attained. What if $I = [0, 1)$?

(ii) Suppose $I = [0, \infty)$. Clearly I is an unbounded interval. Let $f(x) = e^{-x}$, $x \geq 0$. We know that $\inf\{f(x) : x \in I\} = \inf\{e^{-x} : x \geq 0\} = 0$. But we can not have $f(x) = 0$ at any $x \in I$. Why ?

Next, we look at discontinuities of functions on $\mathbb{R}$. For this, first we consider two related notions.

Let $f : I \to \mathbb{R}$ be a function; assume that $I = (a, b)$, a bounded open interval. Let $x \in I$. Suppose for any decreasing sequence $\{x_n\}$ in I with

$\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} f(x_n) = f(x)$. Then we say that f is *right continuous* at x; this is indicated by writing

$$\lim_{y\to x+} f(y) = f(x),$$

that is, the limit exists and is $f(x)$, whenever $y \to x$ only through values greater than x. This is also denoted by $f(x+) = f(x)$. If f is right continuous at every $x \in I$, we say that f is right continuous on I.

In exactly similar manner, *left continuity* at $x \in I$ can be defined, which is indicated by writing

$$\lim_{y\to x-} f(y) = f(x),$$

that is, the limit exists and is $f(x)$, whenever $y \to x$ only through values less than x. This is also denoted by $f(x-) = f(x)$.
Note: Suppose $I = [a, b]$ and $f : I \to \mathbb{R}$ a function. Without any additional information, we know only that f is defined on $[a, b]$. Hence it makes sense to talk only about right continuity at $x = a$. Also it makes sense to consider only left continuity at $x = b$. (What if I is $[a, b),\ (a, b],\ [a, \infty)$, or $(-\infty, b]$?)
Exercise: A function $f : I \to \mathbb{R}$ is continuous at $x \in I$, if and only if f is right continuous as well as left continuous at x, that is, if and only if $f(x+) = f(x-) = f(x)$.

We say $x \in I$ is a *discontinuity of* f, or a point of discontinuity of f, if x is not continuous at x.

Suppose $x \in I$ is a discontinuity of f. From the above it is clear that one of the following must be true:

a) Either $f(x+)$ or $f(x-)$ does not exist.

b) Both $f(x+)$ and $f(x-)$ exist but $f(x+) \neq f(x-)$.

c) Both $f(x+)$ and $f(x-)$ exist and $f(x+) = f(x-)$, but $f(x+) = f(x-) \neq f(x)$.

Some examples are given below.

Example 5: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x) &= \frac{x}{|x|},\ x \neq 0, \\ &= A,\ x = 0, \end{aligned}$$

where A is a fixed real number. Convince yourself that f is continuous at any $x \neq 0$, whereas 0 is a discontinuity of f, whatever be the value of A. Find $f(0+),\ f(0-)$. Draw a graph of f.

Example 6: Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$\begin{aligned} f(x) &= \frac{1}{x^2},\ x \neq 0, \\ &= 0,\ x = 0. \end{aligned}$$

What can you say about $f(0+),\ f(0-)$? What are the continuity points of f? Draw a graph of f.

Example 7: Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$\begin{aligned} g(x) &= \sin(\frac{1}{x}),\ x \neq 0, \\ &= A,\ x = 0, \end{aligned}$$

where A is a fixed real number. For $x = 1/(n\pi)$, where n in non-zero integer, we have $g(x) = \sin(n\pi) = 0$. Between two such points the function values rise to $+1$ and drop back to 0, or else drop to -1 and rise back to 0. Hence between any $x = 1/(n\pi)$ and $x = 0$, the function has an infinite number of oscillations. Is it possible to choose A so that g is a continuous function? Draw a graph of g.

Analogous to monotonic sequences, we have monotonic functions.

A function $f : I \to \mathbb{R}$ is said to be *increasing* on I if $x, y \in I$ with $x \leq y$, then $f(x) \leq f(y)$; it is *strictly increasing* on I if $x < y$ implies $f(x) < f(y)$. *Decreasing, strictly decreasing* functions are similarly defined. A function is called *monotonic* on I if it is increasing on I or decreasing on I.

We now state a fairly obvious result.

Theorem 9 *Let* $f : [a, b] \to \mathbb{R}$ *be an increasing function on* $[a, b]$*. For any* $c \in (a, b)$ *both* $f(c+)$ *and* $f(c-)$ *exist, and*

$$f(c-) \;\; \leq \;\; f(c) \;\; \leq \;\; f(c+).$$

Moreover, at the end points, $f(a) \leq f(a+)$*, and* $f(b-) \leq f(b)$*. In addition, if* f *is continuous on* $[a, b]$*, then* f *is a continuous function from* $[a, b]$ *onto* $[f(a), f(b)]$*; that is, the Range* (f) *is* $[f(a), f(b)]$*. If* f *is a strictly increasing and continuous function on* $[a, b]$*, then it is a one-to-one, onto and continuous function from* $[a, b]$ *onto* $[f(a), f(b)]$*. Consequently, its inverse* $f^{-1} : [f(a), f(b)] \to [a, b]$ *can be defined; also* f^{-1} *is a continuous and strictly increasing function on* $[f(a), f(b)]$*.*

Exercise: Formulate the corresponding version for decreasing functions.