Problem Set 1

Ds Analysis

January 2021

Problem 1

$$a_n = \frac{n^2}{n+1} - \frac{n^2+3}{n}$$

$$= \frac{n^2-1}{n+1} - \frac{n^2}{n} - \frac{3}{n} + \frac{1}{n+1}$$

$$= n - 1 - n - \frac{3}{n} + \frac{1}{n}$$

$$= -1 - \frac{3}{n} - \frac{1}{n+1}$$

Since $\frac{3}{n}$ and $\frac{1}{n+1}$ converges to 0 as n $\to \infty$.
The sequence converges to -1

Problem 2

Given the seq.

$$a_n = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{10}$$

Hint: consider the the following 2 sub seq. a_{2n} and a_{2n+1}. check their convergence.

Problem 3

$$a_n = sin(\frac{n\pi}{2})$$

Here we have,

$$\begin{aligned} a_n &= 0 \quad \text{for n = 2,4,6...} \\ &= 1 \quad \text{for n = 1,5,9,...} \\ &= -1 \quad \text{for n = 3,7,11,..} \end{aligned} \tag{1}$$

As is evident, there does not exist any x for which $|x_n - x|$ can be made as small as possible however large n might be.

Problem 4

$$a_n = 2^{\frac{1}{n}}$$

Hint: The seq. is monotonically decreasing and bounded.
Start with:

$$2^{n+1} > 2^n \ \forall \ n \geq \ 0$$

Problem 5

$$a_n = \frac{n\sin(n!)}{n^2+5}$$

Now, we know that,

$$-1 \leq \sin(x) \leq 1 \quad \text{for any x}$$

Thus,

$$-1 \leq \sin(n!) \leq 1$$

Multiplying each side by n, we have

$$-n \leq n\sin(n!) \leq n$$

Dividing each side by n^2+5, we have

$$\frac{-n}{n^2+5} \leq \frac{n\sin(n!)}{n^2+5} \leq \frac{n}{n^2+5}$$

Now, as n $\rightarrow \infty$, $\frac{-n}{n^2+5} \rightarrow 0$ and $\frac{n}{n^2+5} \rightarrow 0$, by Sandwich Theorem, we have:

$$\frac{n\sin(n!)}{n^2+5} \rightarrow 0$$

Thus the sequence a_n converges to 0.

Problem 6

Given,

$$a_n = \frac{3^n + (-2)^n}{3^{n+1} + (-2)^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(3^{n+1} + (-2)^{n+1})^2}{(3^{n+2} + (-2)^{n+2})(3^n + (-2)^n)}$$

$$= \frac{3^{2n+2} + (-2)^{2n+2} - 12(-6)^n}{3^{2n+2} + (-2)^{2n+2} + 13(-6)^n} < 1$$

$\Rightarrow$ The seq is monotonically decreasing and $a_n > 0$.
$\Rightarrow$ The seq is convergent. Now,

$$\lim_{n\to\infty} \frac{3^n + (-2)^n}{3^{n+1} + (-2)^{n+1}}$$

$$= \frac{1 + (\frac{-2}{3})^n}{3(1 + (\frac{-2}{3})^{n+1})} = \frac{1}{3}$$

Problem 7

$$
\begin{aligned}
a_n &= \sqrt{n+7} - \sqrt{n+4} \\
&= \frac{(\sqrt{n+7} - \sqrt{n+4})(\sqrt{n+7} + \sqrt{n+4})}{\sqrt{n+7} + \sqrt{n+4}} \\
&= \frac{3}{\sqrt{n+7} + \sqrt{n+4}}
\end{aligned}
$$

Thus as $n \to \infty$ we have $a_n \to 0$

Problem 8

The Ratio Test for Sequences: If a_n is a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ and if $L < 1$, then a_n converges and $\lim_{n\to\infty} a_n = 0$.

Hint : $\lim_{n\to\infty} \frac{(n+1)a^{n+1}}{na^n} = a < 1$

alternately, Given $a_n = n\ a^n$, $0 < a < 1$, let, $b = \frac{1}{a}.b > 1$ exponential function dominates linear function.

Problem 9

$$
\begin{aligned}
&\lim_{x\to1} \frac{2x^2 - 3x + 1}{x-1} \\
&= \lim_{x\to1} \frac{2x^2 - 2x - x + 1}{x-1} \\
&= \lim_{x\to1} \frac{2x(x-1) - (x-1)}{x-1} \\
&= \lim_{x\to1} \frac{(2x-1)(x-1)}{x-1} \\
&= \lim_{x\to1} (2x-1) \\
&= 1
\end{aligned}
$$

Problem 10

Hint : : $x^2 + 2ax + x^2 \ = \ (x+a)^2$

Note : $x \to a$ but one can assume $x = a$ only when the denominator term does not converg to zero as $x \to a$

Problem 11

$$\lim_{x\to 0} \frac{\sin(x)}{x}$$

Using L'Hospital's rule, since our function is in $\frac{0}{0}$ form, we differentiate numerator and denominator w.r.t. x. Thus,

$$= \lim_{x\to 0} \frac{\cos(x)}{1}$$

$$= 1$$

For a non-L'Hospital rule approach, check out : $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

Problem 12

Hint :

$$\frac{d}{dx}(\sin 5x) \ = 5\cos x$$

$$1 - \cos x \ = \ 2\sin\frac{x^2}{2}$$

Problem 13

$$\lim_{x\to 0} x\sin(\frac{1}{x})$$

Now, similar to problem 5, we have,

$$-1 \leq \sin(x) \leq 1 \quad \text{for any x}$$

Thus,

$$-1 \leq \sin(\frac{1}{x}) \leq 1$$

Multiplying each side by x, we have

$$-x \leq x\sin(\frac{1}{x}) \leq x$$

Now, we have

$$\lim_{x\to 0} x = 0 \quad \text{and} \quad \lim_{-x\to 0} x = 0$$

By Sandwich Theorem,

$$\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$$

Problem 14

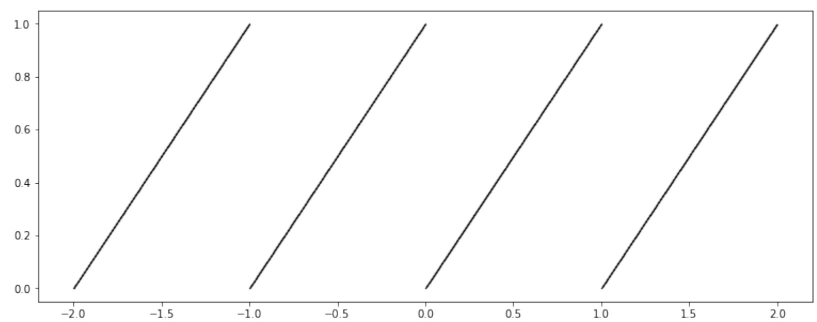
Solving the problem is relatively easy.So instead we want to talk about plotting graphs with discontinuity,as some of you raised the issue in last tutorial class: The function $func14$ is equivalent to $f(x) = x - [x], x \in \mathbb{R}$

```
def func14(x):
    '''takes input an numpy array
        and returns an array of the values
        x - [x]'''
    y = x - np.floor(x)
    return y
```

```
x=np.arange(-2,2,0.001)
y=func14(x)
```

```
plt.figure(figsize=(13,5))
plt.plot(x,y,".k",markersize=0.5)
```

The code for the function x-[x]



The graph of the function x-[x]

Problem 15

Plotting a graph, to find discontinuities, we need to find breaks in the function

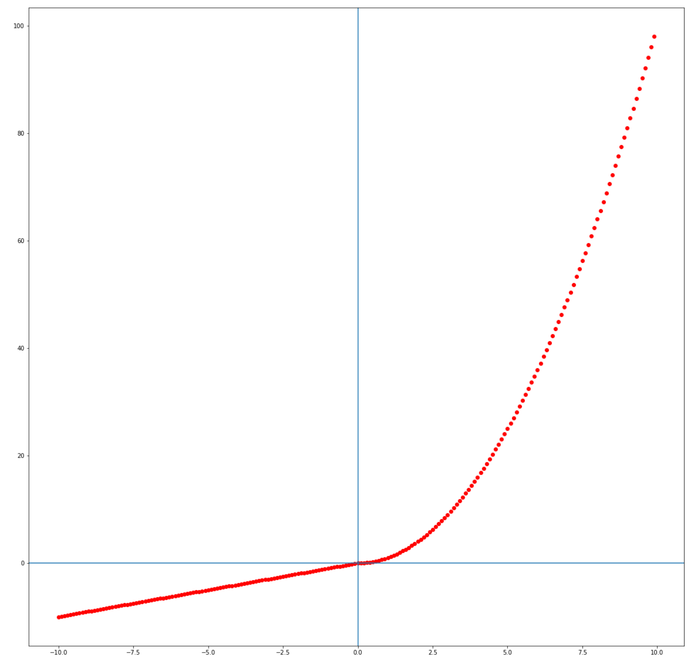
From the graph it's clearly evident that the function is not discontinuous. However, the only point of discontinuity can be at 0. So, let's just double check. Checking the left hand limit, we have,

$$\lim_{x \to 0^-} x = 0$$

Checking the right hand limit, we have,

$$\lim_{x \to 0^+} x^2 = 0$$

Since left hand limit = right hand limit, the function is continuous at 0.



Plotting the graph of the function

Problem 16

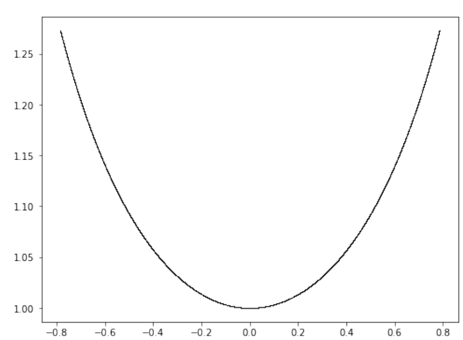


Fig. 1: Plotting the graph of the function

A rough idea of how to plot $\frac{\tan(x)}{x}$ by hand :

$$f(x) = \begin{cases} \dfrac{\tan x}{x} & \text{for } x \in \left[-\pi/4, \dfrac{\pi}{4}\right] \quad x \neq 0 \\ 1 & x = 0 \end{cases}$$

(1) First check for discontinuity

the only pt to check here is $x = 0$

$$\lim_{x \to 0^+} \frac{\tan x}{x} = \lim_{x \to 0^+} \frac{x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots}{x}$$

$$= \lim_{x \to 0^+} 1 + \frac{2x^2}{3!} + \frac{16x^4}{5!} + \cdots$$

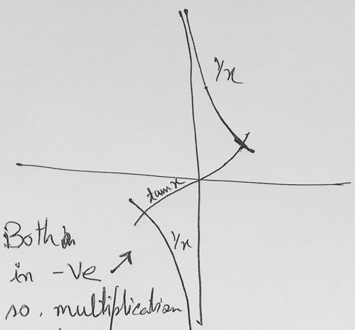
$$= 1$$

similarly for $x \to 0^-$

$\Rightarrow f(x)$ is continious at $x = 0$

$\Rightarrow f(x)$ " " for $x \in [-\pi/4, \pi/4]$

Now, plot $\tan x$ & $1/x$ in $\left[-\pi/4, \pi/4\right]$



Both in -ve ↗
so, multiplication
would be in
positive region

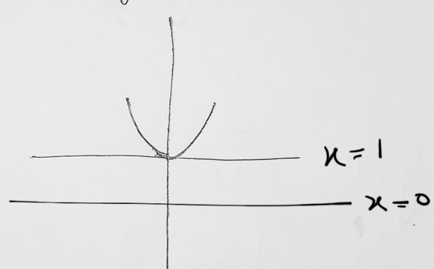
As seen previously
there's no discontinuity
in $\frac{\tan x}{x}$ $\forall x \in \left[-\pi/4, \pi/4\right]$

& since, $\frac{\tan x}{x} = 1 + \frac{2x^2}{3!} + \cdots$

~~⇒ The graph would~~

⇒ $\tan x$ would dominate
x in $\left[-\pi/4, \pi/4\right]$

So, the graph would be



~~if f(x~~

Similarly, check for yourself the graph of

$$f(x) = \frac{\tan x}{|x|} \quad \forall x \in \left[-\pi/4, \pi/4\right], x \neq 0$$

$$= 1 \qquad x = 0$$