CMI - 2020-2021: DS-Analysis

Calculus-7

Improper integrals

We begin with an example.

Example 1: Let $\lambda > 0$ be a constant. Define

$$f(x) \quad = \quad e^{-\lambda x}, \ x > 0.$$

For $b > 0$, putting $I(b) = \int_0^b f(x)dx$, note that

$$I(b) \quad = \quad \int_0^b e^{-\lambda x} dx \quad = \quad \frac{e^{-\lambda b} - 1}{-\lambda}.$$

From the above, it is clear that $\lim_{b\to\infty} I(b) = \frac{1}{\lambda}$. In other words

$$\lim_{b\to\infty} \int_0^b e^{-\lambda x} dx = \frac{1}{\lambda}.$$

Let $a \in \mathbb{R}$. Let f be a function on $[a, \infty)$. Suppose $\int_a^b f(x)dx$ exists for all $b \geq a$. Define $I(b) = \int_a^b f(x)dx, \ b \geq a$. Suppose $\lim_{b\to\infty} I(b) = A$, with $A \in \mathbb{R}$. Then we say that the *improper integral* $\int_a^\infty f(x)dx$ converges or exists; in such a case we write

$$\int_a^\infty f(x)dx = A.$$

If the limit does not exist, or if the limit is not finite, then we say that the improper integral diverges.

So in the preceding example we have $\int_0^\infty e^{-\lambda x} dx = 1/\lambda$.

Let $b \in \mathbb{R}$, and f a function on $(-\infty, b]$ such that $J(a) = \int_a^b f(x)dx$ exists for all $a \leq b$. If $\lim_{a\to-\infty} J(a) = B$, with $B \in \mathbb{R}$, we say that the improper integral $\int_{-\infty}^b f(x)dx$ converges/ exists, and write $\int_{-\infty}^b f(x)dx = B$.

Let f be a function on $\mathbb{R}$. If there is $a \in \mathbb{R}$ such that the improper integrals $\int_a^\infty f(x)dx$, and $\int_{-\infty}^a f(x)dx$ both exist, then we say the improper integral

$\int_{-\infty}^{\infty} f(x)dx$ exists; this integral may also be denoted by $\int_{\mathbb{R}} f(x)dx$. Also, in such a case

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx.$$

Improper integrals can arise in a different way as well. Again we look at an example first.

Example 2: Let $b > 0$. Let $g(x) = x^{-s},\ x \in (0, b]$, where s is a fixed real number. For $a \in (0, b]$, note that

$$\begin{aligned} \int_{a}^{b} x^{-s}dx &= \frac{b^{(1-s)} - a^{(1-s)}}{(1-s)},\ s \neq 1, \\ &= \log(b) - \log(a),\ s = 1. \end{aligned}$$

Note that $\lim_{a \to 0+} \int_{a}^{b} g(x)dx$ converges to a finite limit if and only if $s < 1$; in such a case we write $\int_{0}^{b} x^{-s}dx = \frac{1}{(1-s)} b^{(1-s)}$.

Improper integrals are similar to infinite series. For simplicity we consider improper integrals on $[0, \infty)$. Let f be a function on $[0, \infty)$ such that $I(b) = \int_{0}^{b} f(x)dx$ exists for all $b \geq 0$. It can be seen that $I(b), b > a$ are similar to partial sums. (Note that $I(n) = \sum_{k=1}^{n} \int_{k-1}^{k} f(x)dx,\ n = 1, 2, \cdots$) The following useful result, stated without proof, is the analogue of the comparison test for improper integrals.

Theorem 1 *(i) Let $f,\ g$ be functions on $[0, \infty)$ such that $f(x) \geq 0,\ g(x) \geq 0,\ x \geq 0$, and $\int_{0}^{b} f(x)dx,\ \ \int_{0}^{b} g(x)dx$ exist for all $b > 0$. Suppose $f(x) \leq g(x)$ for all $x \geq 0$, and the improper integral $\int_{0}^{\infty} g(x)dx$ exist. Then the improper integral $\int_{0}^{\infty} f(x)dx$ also exists, and*

$$\int_{0}^{\infty} f(x)dx \leq \int_{0}^{\infty} g(x)dx.$$

(ii) If

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = c, \text{ with } \ c \neq 0,$$

then both the integrals $\int_{0}^{\infty} f(x)dx$ and $\int_{0}^{\infty} g(x)dx$ converge or both the integrals diverge.

(iii) If $c = 0$ in (ii), we can conclude only that convergence of $\int_0^\infty g(x)dx$ implies convergence of $\int_0^\infty f(x)dx$.

Example 3: The improper integral $\int_1^\infty x^{-s}dx$ converges if and only if $s > 1$. If $s > 1$ then

$$\int_1^\infty x^{-s}dx = \frac{1}{s-1}.$$

Proof is left as an exercise.

Example 4: Consider the integral $\int_1^\infty x^s e^{-x}dx$. where s is any fixed real number.

By the preceding example $\int_1^\infty x^{-2}dx = 1$. Recall that

$$\lim_{x\to\infty} \frac{x^t}{e^x} = 0, \text{ for any } t \in \mathbb{R}.$$

Consequently it follows that

$$\lim_{x\to\infty} \frac{x^s e^{-x}}{x^{-2}} = 0.$$

Hence by part (iii) of the preceding theorem the improper integral $\int_1^\infty x^s e^{-x}dx$ exists for any $s \in \mathbb{R}$.

Definition 2 Define the *gamma function* for $\alpha > 0$, by

$$\Gamma(\alpha) \quad = \quad \int_0^\infty e^{-x}x^{\alpha-1}dx.$$

To show that $\Gamma(\alpha)$ is finite for $\alpha > 0$, write

$$\int_0^\infty e^{-x}x^{\alpha-1}dx = \int_0^1 e^{-x}x^{\alpha-1}dx + \int_1^\infty e^{-x}x^{\alpha-1}dx.$$

Note that both the integrals on the right side are improper integrals. By Example 4, it follows that the second integral is finite for any α. To test the convergence of the first integral, consider $\int_a^1 e^{-x}x^{\alpha-1}dx,\ a \in (0,1]$. Take $x = 1/u$. Then $dx = -u^{-2}du$. Consequently we get

$$\int_a^1 e^{-x}x^{\alpha-1}dx \quad = \quad \int_1^{1/a} e^{-(1/u)}u^{-(\alpha+1)}du.$$

Observe that $0 < e^{-(1/u)}u^{-(\alpha+1)} \leq u^{-(\alpha+1)}$ for all $u \in [1, \infty)$. By Example 3, it follows that the improper integral $\int_1^\infty u^{-(\alpha+1)}du$ converges if and only if $\alpha > 0$. So by part (i) of the preceding theorem we get $\int_1^\infty e^{-(1/u)}u^{-(\alpha+1)}du$ converges if $\alpha > 0$. Therefore $\lim_{a\to 0+} \int_a^1 e^{-x}x^{\alpha-1}dx$ is finite if $\alpha > 0$. Thus we obtain that $\Gamma(\alpha)$ is finite for every $\alpha > 0$.
Exercise: (i) Prove that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha),\ \alpha > 0$. (Hint: Use integration by parts.)

(ii) Show that $\Gamma(n + 1) = n!$ for non-negative integers n, using induction.

Remark 3 We will now consider the improper integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx.$$

(There is no expression for the indefinite integral $\int e^{-x^2}dx$ in terms of standard functions: powers of x, trigonometric functions, exponential function, logarithmic function, sums, products or composites of these.)

As the integrand e^{-x^2} is an even function of x, note that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^{\infty} e^{-x^2} dx.$$

So we consider the improper integral on the right side.

Take $x = \sqrt{u}$. Then $dx = \frac{1}{2}u^{-\frac{1}{2}}du$. This gives

$$\begin{aligned} \int_0^{\infty} e^{-x^2} dx &= \int_0^{\infty} \frac{1}{2}e^{-u}u^{-\frac{1}{2}}du \\ &= \frac{1}{2}\int_0^{\infty} e^{-u}u^{\frac{1}{2}-1}du = \frac{1}{2}\Gamma(\frac{1}{2}). \end{aligned}$$

It is known that

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

We will assume this fact.

Therefore the improper integral on $(-\infty, \infty)$ exists and is given by

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

Using the above, it is clear that

$$\int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2}\int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{2\pi}.$$

Consequently, writing $\exp(s) = e^s$, we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx = 1.$$

Let f be a function on $\mathbb{R}$ such that $f(x) \geq 0$ for all $x \in \mathbb{R}$, and $\int_{-\infty}^{\infty} f(x)dx = 1$. Then we say that f is a *probability density function* on $\mathbb{R}$.

We can now introduce three important classes of probability density functions on $\mathbb{R}$.

Exponential densities: The exponential density with parameter $\lambda > 0$ is given by

$$\begin{aligned} f(x) &= \lambda e^{-\lambda x},\ x > 0, \\ &= 0,\ x \leq 0. \end{aligned}$$

By Example 1, it is clear that f is a probability density function.

Gamma densities: Let $\alpha,\ \lambda > 0$ be fixed. The gamma density with parameters $\alpha,\ \lambda$ is given by

$$\begin{aligned} g(x) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x},\ x > 0, \\ &= 0,\ x \leq 0. \end{aligned}$$

From our discussion on gamma functions it is clear that

$$\int_0^{\infty} x^{\alpha-1}e^{-\lambda x}dx = \frac{\Gamma(\alpha)}{\lambda^\alpha},$$

and hence we have $\int_{\mathbb{R}} g(x)dx = 1$. So g is a probability density function.

Note that the exponential density with parameter λ is the same as the gamma density with parameters $1,\ \lambda$.

Normal densities: The *standard normal density* is given by

$$\varphi(x) \;=\; \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \;\; -\infty < x < \infty.$$

By Remark 3 it is clear that φ is a probability density function.

Let $\mu \in \mathbb{R}$, $\sigma^2 > 0$. The normal density with parameters μ, σ^2 is given by

$$g(y) \;=\; \frac{1}{\sigma\sqrt{2\pi}} \exp\big(-\frac{(y-\mu)^2}{2\sigma^2}\big), \;\; -\infty < y < \infty,$$

where σ is the positive square root of σ^2. Observe that

$$g(y) = \frac{1}{\sigma}\varphi\big(\frac{y-\mu}{\sigma}\big), \; y \in \mathbb{R};$$

hence, taking $x = (y-\mu)/\sigma$, and using Remark 3, it can be shown that $\int_{\mathbb{R}} g(y)dy = 1$. So g is a probability density function.

It is clear that the standard normal density is a normal density with parameters $\mu = 0$, $\sigma^2 = 1$.

In addition to their importance in probability theory and statistics, the above classes of probability densities are useful for modelling situations in physical sciences, engineering, biology and economics.

We will look at one short application of integration in connection with probability and statistics.

Let f be a probability density function on $\mathbb{R}$. We say that the mean (or expectation) corresponding to f exists if $\int_{-\infty}^{\infty} |x| f(x)dx < \infty$; in such case the *mean* or *expectation* m is defined by

$$m = \int_{-\infty}^{\infty} x f(x) dx.$$

This is the analogue of the average value.

Example 5: Let $a, b \in \mathbb{R}$ such that $a < b$. Let f be given by

$$\begin{aligned} f(x) &= \frac{1}{b-a}, \; x \in (a,b), \\ &= 0, \; x \notin (a,b). \end{aligned}$$

It is easily seen that f is a probability density function; this is the *uniform* density on (a, b). Verify that $\int_{\mathbb{R}} |x| f(x) dx \leq (|a| + |b|)(b - a)$; so the expectation exists. Verify that $m = (a + b)/2$ in this case.
Example 6: Let φ denote the standard normal density. Show that $\int_{\mathbb{R}} |x| \varphi(x) dx = \sqrt{(2/\pi)}$. So the mean exists. Note that $x \mapsto x\varphi(x)$ is an odd function on $\mathbb{R}$. Hence it follows that the mean $m = 0$. Show that $\int_{\mathbb{R}} x^2 \varphi(x) dx = 1$.
Example 7: Consider the exponential density with parameter $\lambda > 0$. Prove that the mean exists, and that $m = 1/\lambda$.
Example 8: Consider the gamma density g with parameters $\alpha > 0,\ \lambda > 0$. Show that the mean exists. To find the mean

$$\begin{aligned} m &= \int_{\mathbb{R}} x g(x) dx \\ &= \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\lambda x} dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{((\alpha+1)-1)} e^{-\lambda x} dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha + 1)}{\lambda^{\alpha+1}} \\ &= \frac{\alpha}{\lambda}, \end{aligned}$$

where we used some of the facts about gamma functions discussed earlier.

Taylor's expansion

As polynomials are nice functions to work with, it may be very useful to have polynomial approximations to other classes of functions that occur in various contexts. Taylor's expansion is one such versatile approximation scheme.

Let f be a function which has derivatives up to order n on an interval I. Let $a \in I$. Denote $f^{(0)}(a) = f(a)$, $f^{(k)}(a) = \frac{d^k f}{dx^k}(a)$ the $k-$th derivative of f at $x = a$, for $k = 1, 2, \cdots, n$. Let

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Clearly P_n is a polynomial of degree $\leq n$. P_n is called the *Taylor polynomial of degree n generated by f at the point $x = a$*. If $a = 0$ then P_n is called the Taylor polynomial of degree n for f.

It can be shown that P_n is the only polynomial of degree $\leq n$ such that $P_n^{(k)}(a) = f^{(k)}(a),\ k = 0, 1, 2, \cdots, n$.

Example 1: Let $f(x) = e^x,\ x \in \mathbb{R},\ a = 0$. Then $f^{(k)}(x) = e^x, x \in \mathbb{R}$, and hence $f^{(k)}(0) = 1$ for all $k = 0, 1, 2, \cdots$ Therefore for $n \geq 1$

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!},\ x \in \mathbb{R}.$$

Example 2: Let $f(x) = \sin(x),\ x \in \mathbb{R},\ a = 0$. Then $f'(x) = \cos(x),\ f''(x) = -\sin(x),\ f^{(3)}(x) = -\cos(x),\ f^{(4)}(x) = \sin(x)$, etc. So $f^{(2k+1)}(0) = (-1)^k$, $f^{(2k)}(0) = 0$. Thus only odd powers of x appear in Taylor polynomials generated by the sine function at 0. Hence for $n = 2m+1,\ m = 0, 1, 2, \cdots$ we have

$$P_{2m+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!},\ x \in \mathbb{R}.$$

Also note that $P_{2m+2} = P_{2m+1},\ m = 0, 1, 2, \cdots$.

Next we want to see how good an approximation P_n gives to f. In that direction we state the following result without proof. The result is called the *Taylor's formula with remainder.*

Theorem 4 *Let f be a function having derivatives up to order $(n+1)$ on an interval I; assume also that the $(n+1)-th$ derivative $f^{(n+1)}$ is also continuous on I. Let $a \in I$. Then for every $x \in I$, we have*

$$\begin{aligned} f(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_{n+1}(x) \\ &= P_n(x) + R_{n+1}(x), \end{aligned}$$

where

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

with c being a point between a and x. The remainder can also be written

$$R_{n+1}(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt.$$

The proof of the above theorem is an application of integration by parts.
Note: Observe that the unknown point c can depend on x, in addition to a and f.
Exercise: Note that $R_{n+1}(x) = f(x) - P_n(x),\ x \in I$. So R_{n+1} is the error in approximating f by P_n. The following error estimate is easy to see. Assume all the hypotheses of the above theorem. Suppose there is a constant M_{n+1} such that $|f^{(n+1)}(x)| \leq M_{n+1}$ for all $x \in I$. Then

$$|R_{n+1}(x)| \leq \frac{M_{n+1}|x-a|^{n+1}}{(n+1)!},\ x \in I.$$