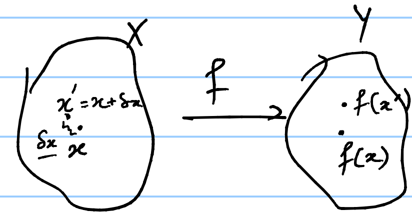
Conditioning of a problem.

Note Title 17-04-2021

Let X = normed v.s. of data
Y = normed v.s. of solutions
$f : X \to Y$



Defn: Absolute condition number - Let δx denote a small perturbation in x & $\delta f = f(x+\delta x) - f(x)$.

The abs. condition number $\hat{\kappa}(x)$ of f at x is defined as -

$$\hat{\kappa}(x) = \lim_{\delta \to 0} \sup_{\|\delta x\| \leq \delta} \frac{\|\delta f\|}{\|\delta x\|} \begin{array}{l} \leftarrow \text{abs. change in solution} \\ \leftarrow \text{abs. change in data} \end{array}$$

If we restrict to infinitesimal δx then we may write -

$$\hat{\kappa}(x) = \sup_{\delta x} \frac{\|\delta f\|}{\|\delta x\|}$$

Defn (Relative condition number) $\kappa(x)$ of a function f at x is defined as -

$$\kappa(x) = \sup_{\delta x} \frac{\|\delta f\| / \|f\|}{\|\delta x\| / \|x\|} \begin{array}{l} \leftarrow \text{rel. change in } f \\ \leftarrow \text{rel. change in } x. \end{array}$$

- If f is differentiable, with its Jacobian being $J(x)$,

then $\delta f \approx J(x) \cdot \delta x$, $J(x) = \lim_{\|\delta x\| \to 0} \frac{\delta f}{\delta x}$

so $\hat{\kappa}(x) = \|J(x)\|$

& $\kappa(x) = \dfrac{\|J(x)\|}{\|f(x)\| / \|x\|}$.

Examples: (1) f is the problem of computing $\sqrt{x}$ for $x>0$.

$\delta x \begin{cases} 2 \mapsto \sqrt{2} \\ 2.005 \mapsto \end{cases}$ $\quad f: x \mapsto \sqrt{x}$, $\quad J(x) = 1/2\sqrt{x}$

$$K(x) = \frac{\|J(x)\|}{\|f(x)\|/\|x\|} = \frac{\|1/2\sqrt{x}\|}{\|\sqrt{x}\|/\|x\|} = \frac{1}{2}.$$

This is a well-conditioned problem.

(2) f is the problem of computing the roots of a monic quadratic polynomial i.e. x^2+bx+c.

$$f: (b,c) \longmapsto \sqrt{b^2-4c}$$

$$\text{Jacobian of } f = J = \begin{bmatrix} \frac{2b}{2\sqrt{b^2-4c}} & \frac{-4}{2\sqrt{b^2-4c}} \end{bmatrix}$$

If $f(b,c)$ has repeated roots, i.e. $b^2-4c=0$, then $\|J\| = \infty$, so $K(x) = \infty$, in which case f is ill-conditioned

eg: lets consider $x^2-2x+1 = (x-1)^2$

Take a small perturbation in coefficients:

$$x^2-2x+0.9999$$
$$= (x-0.99)(x-1.01)$$

<u>Condition of matrix-vector multiplication</u>

2 problems: (1) $x \longmapsto Ax \; (=b)$ $\qquad Ax = b.$

(2) $b \longmapsto A^{-1}b \; (=x)$ $\qquad A^{-1}b = x.$

For the 1st problem, $K(x) = \sup_{\delta x} \left(\frac{\|A(x+\delta x) - Ax\|}{\|Ax\|} \Big/ \frac{\|\delta x\|}{\|x\|} \right)$

$$\sup \frac{\|Ax\|}{\|x\|} \qquad = \sup_{\delta x} \left(\frac{\|A\,\delta x\|}{\|\delta x\|} \Big/ \frac{\|Ax\|}{\|x\|} \right)$$

$$\star \quad K(x) = \|A\| \cdot \frac{\|Ax\|}{\|x\|}$$ (exact formula for K w.r.t. perturbations in x)

$$K(x) \leq \|A\| \cdot \|A^{-1}\| .$$

$$\therefore K(x) \leq K(A).$$

The term $\|A\| \cdot \|A^{-1}\|$ is denoted by $K(A)$ & is called the condition number of A.

For the 2nd problem, x is replaced by b, A is replaced by A^{-1},

So $K(b) = \|A^{-1}\| \cdot \frac{\|A^{-1}b\|}{\|b\|}$ w.r.t. perturbations in b.

In particular,
$$\left(K(b) \leq K(A) \right).$$