Design and Analysis of Algorithms 2020
Problem Set 2

Chennai Mathematical Institute

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> *An algorithm must be seen to be believed.*
>
> Donald Knuth, *Computer Scientist and Mathematician*

Problem 1. A binary tree is *full* if all of its vertices have either zero or two children. Let B_n denote the number of full binary trees with n vertices.

a. For general n, derive a recurrence relation for B_n.

b. Show by induction that B_n is $2^{\Omega(n)}$.

Problem 2. Given a sorted array A of n distinct integers, give a divide-and-conquer algorithm that runs in time $O(\log n)$, and finds out whether there is an index i for which $A[i] = i$.

Problem 3. We find the *greatest common divisor (gcd)* of two positive integers, using divide-and-conquer.

a. Show that the following rule is true

$$\gcd(a, b) = \begin{cases} 2\gcd(a/2, b/2) & \text{if } a, b \text{ are even} \\ \gcd(a, b/2) & \text{if } a \text{ is odd, and } b \text{ is even} \\ \gcd((a-b)/2, b) & \text{if } a, b \text{ are odd} \end{cases}$$

b. Using part a, or otherwise, give an efficient divide-and-conquer algorithm for greatest common divisor.

c. Compare the efficiency of your algorithm to Euclid's algorithm if a and b are n-bit integers.

Problem 4. Given an array A of length n, where each element is at most k positions away from its position in the sorted list, give an algorithm that sorts the array in time $\mathcal{O}(n \log k)$.

Problem 5. Given an array A of length n, give an algorithm to find the largest k elements, that runs in time $\mathcal{O}(nk)$, using heaps.

***Problem 6.** A *positive sequence* is a finite sequence of positive integers. *Sum of a sequence* is the sum of all the elements in the sequence. We say that a sequence A *can be embedded into another sequence* B, if there exists a strictly increasing function $\phi : \{1, 2, \ldots, |A|\} \to \{1, 2, \ldots, |B|\}$, such that $\forall i \in \{1, 2, \ldots, |A|\}, A[i] \leqslant B[\phi(i)]$, where $|S|$ denotes the length of the sequence S.

Given a positive integer n, construct a positive sequence U with sum $\mathcal{O}(n \log n)$, such that all the positive sequences with sum n, can be embedded into U.

> *Fancy algorithms are slow when* N *is small, and* N *is usually small.*
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> Robert "Rob" Pike, *Programmer and author*