LPCO 2019 **End-sem Exam (50 marks)**

1. Solve the following LP using the simplex tableau method. **(5 marks)**

$$
\begin{array}{llrcrl}
\text{maximize} & 5x_1 & + & 4x_2 & & \\
\text{subject to} & x_1 & + & 2x_2 & \leq 6 \\
 & -2x_1 & + & x_2 & \leq 4 \\
 & 5x_1 & + & 3x_2 & \leq 15 \\
 & x_1 & , & x_2 & \geq 0
\end{array}
$$

2. Prove or disprove the following statement: if the primal has a unique optimum, then the dual has a unique optimum (an LP is said to have a unique optimum if there is exactly one feasible solution giving the optimum value). **(6 marks)**

3. Suppose x^* is the unique optimum of an LP. Show that the second best extreme point must be adjacent to x^*. **(6 marks)**

4. Show that if the LP maximize $c^T x$ subject to $Ax = b$ (where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$) is unbounded, there is a rational vector $\alpha \in \mathbb{R}^n$ such that (a) $c^T\alpha > 0$ and (b) for every feasible point x and $k > 0$, $x + k\alpha$ is feasible. **(6 marks)**

5. We are given a complete undirected graph with positive costs on edges. For every pair of vertices (i, j) we are given a positive integer t_{ij} which indicates the minimum number of edge-disjoint paths required between i and j.

   The problem is to select a minimum cost subset of edges that satisfies the connectivity criterion.

   (a) Write an LP relaxation for this problem. **(4 marks)**

   (b) Give a polynomial-time separation oracle. **(5 marks)**

6. Consider the Steiner forest problem: given a complete undirected graph with non-negative edge weights and disjoint subsets $S_1, \ldots, S_k$ of its vertices, find a minimum cost subset of edges so that vertices in each S_i are in a connected component induced by the picked edges.

   This problem is NP-hard.

   (a) Write an ILP for this problem. **(4 marks)**

   (b) Give the relaxed LP and its dual. **(3 marks)**

   (c) Describe a primal-dual algorithm which gives a 2-approximation. Provide the analysis of your method. **(6 marks)**

7. Where does the ellipsoid method (discussed in class) fail in the case of ILPs? **(5 marks)**

$\frac{7}{5}x_2 = 3 + \frac{1}{5}x_5 - x_3$

$\boxed{x_2 = \frac{15}{7} + \frac{1}{7}x_5 - \frac{5}{7}x_3}$

$x_1 = 3 - \frac{1}{5}x_5 - \frac{3}{5}x_2$

max $c^T x$

$Ax = b$

min $b^T y$

$y^T A = c$

$10 - \frac{2}{5}x_5 - \frac{11}{5}\cdot\frac{15}{7} - \frac{11}{5}\cdot\frac{1}{7}x_5$

infeasible

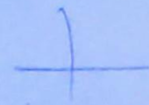
$15 - x_5 + \frac{15}{7} + \frac{1}{7}x_5 - \frac{5}{7}x_3$

min $\sum_{\text{edges}} x_e$

$\forall (i,j),\ \forall S \ni i \in S, j \notin S,\quad \sum_{\substack{e=(u,v) \\ u \in S, v \notin S}} x_e \geq t_{ij}$

$x_e \geq 0$

$(x_e)\quad \exists (i,j)\quad \exists S \text{ s.t. } \sum_{\substack{e(u,v) \\ u \in S, v \notin S}} x_e < t_{ij}$



max $c^T x$

unique optimum

$\Rightarrow c^T \ldots A^{(i)}$

$\neq$

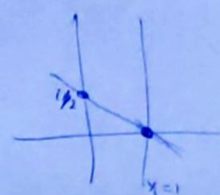
min $\sum c_e x_e$

$\sum_{u \& v} x_e \geq 1 \quad \forall S$ separating $u \& v$ $\quad \forall u,v \in S_i \quad \forall S_i$

$x_e \geq 0$

$x_1 + 2x_2 \leq 1$

max $(0\ 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

min $[1\ 1]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$(y_1, y_2)\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$y_1 + y_2 \geq 0$

$0 + 2y_2 \geq 1$

$y_2 \geq 1/2$

$y_1 + y_2 \geq 0$

$y_2 \geq 1/2 \quad \min y_1 + y_2$

$y_1, y_2 \geq 0$

min $y_1 + y_2$

$(0, 1/2)$