SVD - Singular value decomposition.

Note Title 09-04-2021

Defn: Given $A \in \mathbb{C}^{m\times n}$, a singular value decomposition (SVD) of A is a factorization $A = U\Sigma V^*$ where U & V are unitary & Σ is diagonal.

The diagonal entries of Σ are called the singular values of A & are usually arranged in decreasing order.

Dimension considerations: let $m \geq n$ (WLOG); for simplicity, assumed rank $= n$.

- Reduced SVD -

$$\underset{m\times n}{A} = \underset{m\times n}{\tilde{U}}\ \underset{n\times n}{\Sigma}\ \underset{n\times n}{V^*}$$

- Full SVD -

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\begin{bmatrix}\Sigma \\ 0 \\ \vdots \\ 0\end{bmatrix}}\ \underset{n\times n}{V^*}$$

(extra $m-n$ columns of U)

$$\Sigma = \begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & & 0 \\ & & \vdots\ 0\end{bmatrix}_{m\times m}$$

Columns of U are called "left singular vectors" of A
V - - "right singular vectors" of A.

- If $A = U\Sigma V^*$, then $AV = U\Sigma$ $\quad \left(\Sigma = \begin{bmatrix}\sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots\ \sigma_n\end{bmatrix}\right)$

$$A\left[v_1 | \cdots | v_n\right] = \left[u_1 | \cdots | u_n\right]\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_n\end{bmatrix}$$

$$\left[Av_1 | Av_2 | \cdots | Av_n\right] = \left[\sigma_1 u_1 | \cdots | \sigma_n u_n\right]$$

i.e. $\boxed{Av_i = \sigma_i u_i}$ $\quad \forall\ 1 \leq i \leq n.$

Consequences -

(1) The e-values of A^*A are σ_i^2. The right singular vectors v_i are the corr. orthonormal e-vectors i.e.

$$\overbrace{\underset{n\times m}{A^*}\ \underset{m\times n}{A}}^{n\times n}$$

$$A^*A\, v_j = \sigma_j^2 v_j \qquad 1 \leq j \leq n.$$

Proof: $A^*A = (U\Sigma V^*)^*(U\Sigma V^*)$

$$= V\Sigma^* \underbrace{U^* U}\Sigma V^*$$

$A^*A = \underline{V}\,\Sigma^2\,\underline{V}^*$ $\rightarrow$ this is the e-value decomposition of A^*A. $\square$

② The e-values of AA^* are σ_i^2 $(1 \le i \le n)$ & $(m-n)$ zeroes.

$\overbrace{\underset{m\times n}{A}\ \underset{n\times m}{A^*}}^{m\times m}$

The left singular vectors u_i are the corresponding orthonormal e-vectors i.e. $AA^* u_j = \sigma_j^2 u_j$ $(1 \le j \le m)$

Proof: If U is $m\times n$, choose $(m-n)$ orthogonal vectors $\{\tilde{u}_1, \dots, \tilde{u}_{m-n}\}$ & let $\tilde{U} = [U \mid \tilde{u}_1 \mid \cdots \mid \tilde{u}_{m-n}]$.

$$AA^* = (U\Sigma \underbrace{V^*)(V}\Sigma^* U^*)$$

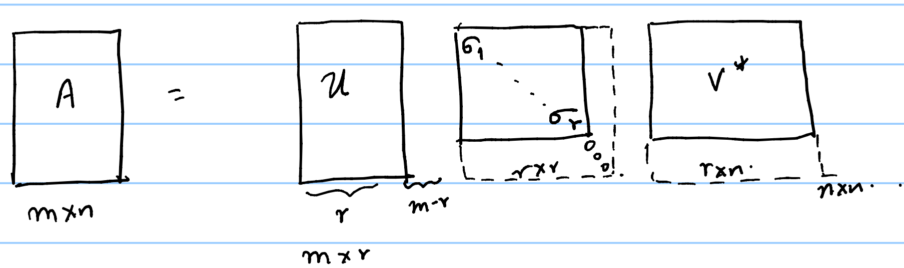
$$= U\Sigma^2 U^*$$

$$= \tilde{U}\left[\begin{array}{c|c} \Sigma^2 & 0 \\ \hline 0 & 0 \end{array}\right]\tilde{U}^*$$

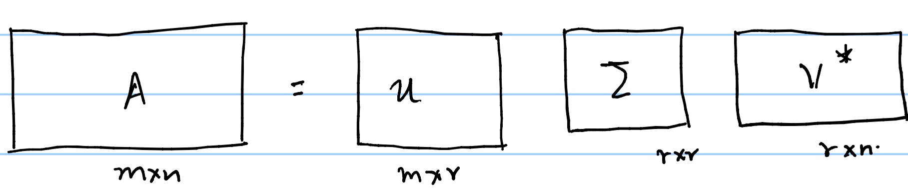
$\rightarrow$ this is the e-value decomp. of AA^*. $\square$

③ If rank $A = r < \min\{m,n\}$, the reduced SVD of A takes the form:

(Reduced)



④ If $m \le n$ to begin with, then SVD of A is of the form- (rank $A = r$)



(5) SVD of A^* : If $A = \mathcal{U} \Sigma V^*$, $A^* = \underline{V \Sigma^* \mathcal{U}^*}$, $\Sigma = \Sigma^*$

[are $\overset{\text{non-zero}}{\text{singular}}$ values of A^* = $\overset{\text{non-zero.}}{\text{singular}}$ values of A? why?]

$$(A^*)^* A^* = AA^*$$

(non-zero eigenvalues of A^*A =non-zero e-values of AA^*)

$$A^*(A^*)^* = A^*A$$

Columns of V are left singular vectors of A^*

Columns of $\mathcal{U}$ are right singular vectors of A^*.

$$A^* = V \Sigma \mathcal{U}^*$$

$$A^* \mathcal{U} = V \Sigma$$

$$\boxed{A^* u_j = \sigma_j v_j}$$

Is $\Sigma = \Sigma^*$? $\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$, $\Sigma^* = \begin{bmatrix} \bar{\sigma}_1 & & & & \\ & \ddots & & & \\ & & \bar{\sigma}_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$

(1) the matrices A^*A & AA^* are symmetric positive semidef.
(real e-value) (non-neg e-values).

Consider $\langle A^*Av, v\rangle$
$= \langle Av, Av\rangle = \|Av\| \geq 0$ $\forall v \in V$.

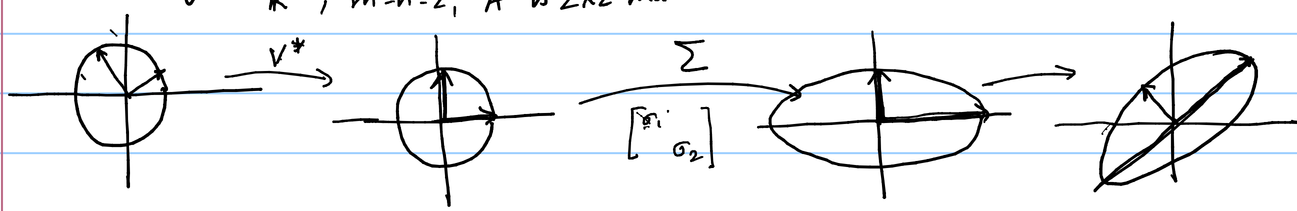
[T is pos. semidef if $\langle Tv, v\rangle \geq 0$ $\forall v \neq 0$.

Similarly, $\langle AA^*v, v\rangle = \langle A^*v, A^*v\rangle = \|A^*v\| \geq 0$ $\forall v \in V$.

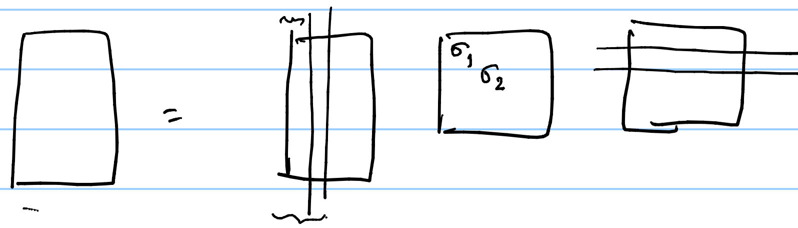
$\therefore$ The e-values of A^*A & AA^* are real & non-neg.
Their +ve sq. roots are the singular values of A (& A^*).

- Existence of SVD — after matrix norms is done.

- Geometry of the SVD : $A \in \mathbb{C}^{m\times n}$, $A = \mathcal{U} \Sigma V^*$
R^2, $m=n=2$, A is 2×2 matrix.



$$Ax = (U\Sigma V^*)x = U\Sigma(V^*x)$$
$$= U(\Sigma(V^*x))$$



$$A \approx \underbrace{\underbrace{\underbrace{u_1\sigma_1 v_1^*}_{\text{rank}1} + \underbrace{u_2\sigma_2 v_2^*}_{\text{rank}1}}_{\text{rank } 2} + \cdots + u_k\sigma_k v_k^*}_{\text{rank } k} \qquad A_k.$$

$\|A - A_k\| \le \|A - B\|$ for rank k matrix B