CMI - 2020-2021: DS-Analysis

Calculus -10

Maxima and minima of functions of several variables - 2

We will first consider some problems of maximising/ minimising on certain elementary *closed and bounded subsets* of $\mathbb{R}^n$, like a closed rectangle, or a closed ball that we have seen in Calculus - 8.

Consider for example, the closed rectangle C in $\mathbb{R}^2$, that is,

$$[a,\ b] \times [c,\ d] \quad = \quad \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$$

where $a < b,\ c < d$. Clearly the open rectangle $U = (a,\ b) \times (c,\ d)$ is a subset of C. The open set U is referred as the *interior* of C. If $(x,y) \in C$, but $(x,y) \notin U$, then (x,y) is called a *boundary point* of C; (also called a boundary point of U.) In this example, if (x,y) is a boundary point, then one of the following will hold: (i) $x = a,\ c \leq y \leq d$; (ii) $x = b,\ c \leq y \leq d$; (iii) $a \leq x \leq b,\ y = c$; or (iv) $a \leq x \leq b,\ y = d$.

In a similar manner, one can get the boundary points of a closed rectangle or a closed ball in $\mathbb{R}^n$.

The crucial or defining aspect of a boundary point z of a set A : Any open ball around z will intersect A, as well as its complement A^c.

The defining aspect of a closed set: A closed set contains all its boundary points.

A set A is called bounded, if there is a number $k > 0$ such that $||x|| \leq k$ for all $x \in A$.

We now state a basic result without proof; it is the analogue of Theorem 8 of Calculus - 1.

Theorem 1 *Let $S \subset \mathbb{R}^n$ be a closed and bounded set. Let $f : S \to \mathbb{R}$ be a continuous function. Then there exist points $p,\ q \in S$ such that $f(p) = \inf\{f(x) : x \in S\}$, and $f(q) = \sup\{f(x) : x \in S\}$. In other words, a continuous real valued function on a closed and bounded set attains both its maximum and minimum values.*

We will look at a few examples.

Example 1: Let $S = [0,1] \times [0,1]$, the 'unit square' in $\mathbb{R}^2$. Let

$$f(x,y) = x^3 + xy,\ (x,y) \in S.$$

Note that S is a closed and bounded set in $\mathbb{R}^2$. We will try to find the maximum and minimum of f on S.

So the maximum/ minimum may be attained in the interior $U = (0,1) \times (0,1)$, or on the boundary. Clearly the boundary consists of 4 lines:

S_1 is the part of $y-$axis between $(0,0)$ and $(0,1)$;

S_2 is the part of $x-$axis between $(0,0)$ and $(1,0)$;

S_3 is part of the line parallel to the $y-$axis between $(1,0)$ and $(1,1)$;

S_4 is part of the line parallel to the $x-$axis between $(0,1)$ and $(1,1)$.

First we will look at the interior U.

Observe that $\triangledown f(x,y) = (3x^2 + y,\ x),\ (x,y) \in S$. So

$$\triangledown f(x,y) = (0,0) \Leftrightarrow 3x^2 + y = 0,\ x = 0 \Leftrightarrow x = 0,\ y = 0.$$

But the point $(0,0) \notin U$. That is, there is no critical point of f in the interior U of S. Hence by the first-order necessary condition, f does not have any local maximum or local minimum of f in U. Therefore the maximum and minimum of f must occur on the boundary $S_1 \cup S_2 \cup S_3 \cup S_4$ of S.

Clearly $S_1 = \{(0,y) : 0 \leq y \leq 1\}$. So $f(0,y) = 0$ on S_1.

Note that $S_2 = \{(x,0) : 0 \leq x \leq 1\}$. So $f(x,0) = x^3$ on S_2. Hence f has minimum value 0, and maximum value $1 = f(1,0)$ on S_2.

Next, $S_3 = \{(1,y) : 0 \leq y \leq 1\}$. So $f(1,y) = 1 + y$ on S_3. Hence f has minimum value $f(1,0) = 1$, and maximum value $f(1,1) = 2$ on S_3.

Finally, $S_4 = \{(x,1) : 0 \leq x \leq 1\}$. So $f(x,1) = x^3 + x$ on S_4. Now, using the procedure for finding maximum/ minimum in the one-dimensional case, verify that on S_4 the function f has the minimum value $f(0,1) = 0$, and the maximum value $f(1,1) = 2$.

Thus, in S, the maximum value 2 is taken only at $(1,1)$. But the minimum value 0 is taken at every point in S_1.

(Note: As $0 \leq x,\ y \leq 1$, it is easy to observe that $0 \leq f(x,y) \leq 2$ for all $(x,y) \in S$, and also that $f(1,1) = 2,\ f(0,0) = 0$.

However, our discussion above indicates the procedure to be followed in general.)

Example 2: Let $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, the 'closed unit ball' in $\mathbb{R}^2$. Let

$$f(x,y) = xy - \sqrt{(1 - x^2 - y^2)},\ (x,y) \in S.$$

We will find the maximum/ minimum of f on the closed and bounded set S.

First, we will look at the interior $U = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, of S. Verify that

$$\begin{aligned} D_1 f(x,y) &= y + \frac{x}{(1 - x^2 - y^2)^{1/2}}, \\ D_2 f(x,y) &= x + \frac{y}{(1 - x^2 - y^2)^{1/2}}. \end{aligned}$$

For notational simplicity, write $r^2 = x^2 + y^2$. Observe that $\bigtriangledown f(x,y) = (0,0)$ if and only if

$$-y = \frac{x}{(1 - r^2)^{1/2}}, \quad \text{and} \quad -x = \frac{y}{(1 - r^2)^{1/2}}.$$

If $y \neq 0$, this is equivalent to

$$\frac{-x}{y} = (1 - r^2)^{1/2}, \quad \text{and} \quad \frac{-x}{y} = \frac{1}{(1 - r^2)^{1/2}}.$$

But in the interior U, note that $0 \leq r < 1$. So $(1 - r^2) < 1$, and $\frac{1}{1-r^2} > 1$, and hence the above requirements are impossible. This implies that $y = 0$, and hence $x = 0$. Thus in the interior U, the origin $(0,0)$ is the only critical point. Clearly $f(0,0) = -1$.

Verify that $D_{11} f(0,0) = 1 = D_{22} f(0,0)$, and $D_{12} f(0,0) = 1 = D_{21} f(0,0)$. So $\det Hf(0,0) = 0$. Hence, Theorem - 7 of Calculus - 9 (that is, second order sufficient condition) is not helpful in this case to determine if f has a local minimum or a local maximum at the critical point $(0,0)$.

Next we consider the values of f on the boundary of S. Note that the boundary of S is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, that is, the 'unit circle' in $\mathbb{R}^2$. Here $r = 1$.

Using polar coordinates, any (x, y) in the boundary of S can be written as $x = \cos(\theta)$, $y = \sin(\theta)$, with $0 \leq \theta \leq 2\pi$. Then $f(x, y) = \sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$.

So the maximum of f on the boundary is when $\sin(2\theta) = 1$. This happens only at two points: P_1 when $\theta = \pi/4$, and P_2 when $\theta = 5\pi/4$. At these points $f(P_1) = f(P_2) = 1/2$. It can be seen that P_1 corresponds to $(x, y) = (\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}})$; and P_2 corresponds to $(x, y) = (-\frac{1}{\sqrt{2}},\ \ -\frac{1}{\sqrt{2}})$. At these points $f(x, y) = \frac{1}{2}$. Hence f attains its maximum value on S as follows:

$$f(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}) \ = \ f(-\frac{1}{\sqrt{2}},\ \ -\frac{1}{\sqrt{2}}) \ = \ 1/2.$$

Similarly, the minimum of f on the boundary is when $\sin(2\theta) = -1$. Again this happens at two points: P_3 when $\theta = 3\pi/4$. and P_4 when $\theta = 7\pi/4$. At these points $f(P_3) = f(P_4) = -1/2$. Note that P_3 corresponds to $(x, y) = (-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}})$; and P_4 corresponds to $(x, y) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. As $-1 < -1/2$, we see that f attains its minimum value on S at the origin $(0, 0)$, and $f(0, 0) = -1$.
Example 3: Let S be as in Example 2. Let

$$f(x, y) = xy + \sqrt{(1 - x^2 - y^2)},\ (x, y) \in S.$$

Proceeding as in Example 2, show that the origin $(0, 0)$ is the only critical point of f in the interior U. Clearly $f(0, 0) = 1$.

Verify that $D_{11}f(0, 0) = -1 = D_{22}f(0, 0)$, and $D_{12}f(0, 0) = 1 = D_{21}f(0, 0)$. So $\det Hf(0, 0) = 0$. In this case also, the second order sufficient condition is not helpful to find if f has a local maximum or a local minimum at the critical point $(0, 0)$.

On the boundary, as in Example 2, $f(x, y) = \frac{1}{2}\sin(2\theta)$. So the maximum value of $\frac{1}{2}$ on the boundary is attained $P_1,\ P_2$; and the minimum value of $-\frac{1}{2}$ at $P_3,\ P_4$. Hence on S, the function f attains maximum value of 1 at $(0, 0)$; and the minimum value of $-\frac{1}{2}$ at P_3 and P_4. (Here $P_i,\ i = 1, 2, 3, 4$ are as in Example 2.) Verify the details.

We will now consider the *method of Lagrange multipliers.*

This method provides a necessary condition for a maximisation/ minimisation problem with constraints, that is, with side conditions.

Let $n,\ m$ be positive numbers with $m < n$. Let $U \subseteq \mathbb{R}^n$ be an open set; let $f : U \to \mathbb{R}$ be a function. For $i = 1, 2, \cdots, m$ let $g_i : U \to \mathbb{R}$ be a function. Suppose we consider the problem:

Maximise $f(x), x \in U$,

subject to $g_i(x) = 0,\ i = 1, 2, \cdots, m$.

Let $E = \{x \in U : g_i(x) = 0,\ 1 \le i \le m\}$.

A point $x \in E$ is called a 'local maximum' for the above problem if there is $r > 0$ such that $f(x) \ge f(y)$ for all $y \in E \cap \{y \in U : ||y - x|| < r\}$.

A 'local minimum' for a minimisation problem with constraints can be similarly defined. A 'local extremum' shall denote a local maximum/ local minimum.

The justification for the method rests on the following result, which we state without proof.

Theorem 2 *Let the notations be as above. Assume the following:*

(a) The functions f and $g_i,\ i = 1, 2, \cdots, m$ have continuous first order partial derivatives.

(b) Let $x_0 \in E$ be a local extremum to the above maximisation/ minimisation problem.

(c) The m vectors $\bigtriangledown g_i(x_0),\ i = 1, 2, \cdots, m$ are linearly independent vectors.

Then there exist m real numbers $\lambda_1, \lambda_2, \cdots, \lambda_m$ such that the following n equations are satisfied:

$$D_k f(x_0) - \big(\sum_{j=1}^{m} \lambda_j D_k g_j(x_0)\big) \;=\; 0,\ k = 1, 2, \cdots, n.$$

The n equations above is equivalent to the vector equation:

$$\bigtriangledown f(x_0) \;=\; \lambda_1 \bigtriangledown g_1(x_0) + \cdots + \lambda_m \bigtriangledown g_m(x_0).$$

The numbers $\lambda_1, \cdots, \lambda_m$ which are introduced to solve such problem are called *Lagrange multipliers.*

Note: When there is only one constraint, say, $g(x) = 0$, then assumption (c) in the above theorem is just $\bigtriangledown g(x_0) \neq 0$, that is, $D_k g(x_0) \neq 0$ for at least one k.

We will look at a few examples.

Example 4: Find the maximum of $f(x, y) = x+y$, on the circle with radius 1. In other words, find the maximum of $f(x, y) = x+y$ subject to the constraint $x^2 + y^2 = 1$.

Take $g(x, y) = x^2 + y^2 - 1$. So $E = \{(x, y) : g(x, y) = 0\}$. Note that

$$\bigtriangledown f(x, y) = (1, 1), \quad \text{and} \quad \bigtriangledown g(x, y) = (2x, 2y).$$

Clearly $\bigtriangledown g(x, y) \neq 0,\ (x, y) \in E$.

Let $(x_0, y_0) \in E$ be a point such that

$$\bigtriangledown f(x_0, y_0) = \lambda \bigtriangledown g(x_0, y_0).$$

In other words,

$$1 = 2x_0\lambda, \quad 1 = 2y_0\lambda.$$

So $x_0 \neq 0,\ y_0 \neq 0$. Hence $\lambda = 1/(2x_0) = 1/(2y_0)$, which in turn implies $x_0 = y_0$. Also (x_0, y_0) must satisfy $g(x_0, y_0) = 0$. Consequently we have 2 possibilities:

$$x_0 = \pm\frac{1}{\sqrt{2}}, \quad y_0 = \pm\frac{1}{\sqrt{2}}.$$

Observe that

$$\begin{aligned} f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) &= \frac{2}{\sqrt{2}}, \\ f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) &= -\frac{2}{\sqrt{2}}. \end{aligned}$$

Consequently, $(1/\sqrt{2}, 1/\sqrt{2})$ is the maximum for f, with $f(1/\sqrt{2}, 1/\sqrt{2}) = 2/\sqrt{2} > 0$.

Example 5: Find the point on the surface $\{(x,y,z) \in \mathbb{R}^3 : x^2+2y^2-z^2=1\}$ which is closest to the origin.

Note that $x^2+y^2+z^2$ is the square of the distance of the point (x,y,z) from the origin. So the problem is equivalent to solving:

Minimise $f(x,y,z) = x^2+y^2+z^2,\ (x,y,z) \in \mathbb{R}^3$,

subject to $g(x,y,z) = x^2+2y^2-z^2-1 = 0$.

So $E = \{(x,y,z) : g(x,y,z) = 0\}$, that is, the given surface. Note that

$$\bigtriangledown f(x,y,z) = (2x, 2y, 2z), \quad \text{and} \quad \bigtriangledown g(x,y,z) = (2x, 4y, -2z).$$

Clearly $\bigtriangledown g(x,y,z) \neq 0,\ (x,y,z) \in E$.

Let $(x_0, y_0, z_0) \in E$ be a local extremum. Then by the preceding theorem, we have $\bigtriangledown f(x_0,y_0,z_0) = \lambda \bigtriangledown g(x_0,y_0,z_0)$. In other words,

$$2x_0 = 2\lambda x_0, \quad 2y_0 = 4\lambda y_0, \quad 2z_0 = -2\lambda z_0.$$

If $z_0 \neq 0$, by the third equation $\lambda = -1$. Then by the first two equations $x_0 = y_0 = 0$. As the constraint must be satisfied, this would imply $z^2 = -1$, which is impossible. Hence $z_0 = 0$ must hold for any solution.

If $x_0 \neq 0$, then the first equation implies $\lambda = 1$. The second and third equation now imply $y_0 = z_0 = 0$. Then the constraint would imply $x_0 = \pm 1$. In such a case we get two solutions: $(1,0,0)$ and $(-1,0,0)$.

Similarly, if $y_0 \neq 0$, we get two solutions: $(0, \sqrt{\frac{1}{2}}, 0)$ and $(0, -\sqrt{\frac{1}{2}}, 0)$.

Thus we have four local extrema for the function f subject to the constraint g.

By a direct computation, minimum value of $1/2$ is attained at the two points: $(0, \sqrt{\frac{1}{2}}, 0)$ and $(0, -\sqrt{\frac{1}{2}}, 0)$.

Example 6: This is an elementary economics/ business application. Suppose a company wants to spend Rs. 90 lakhs to purchase x type-I machines and y type-II machines. Suppose each type-I machine costs Rs. 3 lakhs, and each type-II machine Rs. 5 lakhs. To maximise utility of the purchase, the company wants to maximise xy. How should $x,\ y$ be chosen?

It is easily seen that the problem is:

Maximise $f(x,y) = xy,\ x > 0,\ y > 0,$
subject to $g(x,y) = 3x + 5y - 90 = 0$.
So $E = \{(x,y) : x > 0, y > 0,\ g(x,y) = 0\}$. Clearly

$$\bigtriangledown f(x,y) = (y,x), \qquad \bigtriangledown g(x,y) = (3,5).$$

So $\bigtriangledown g(x,y) \neq 0,\ (x,y) \in E$. If $(x_0,y_0) \in E$ is a local maximum, then by the preceding theorem, we have $\bigtriangledown f(x_0,y_0) = \lambda \bigtriangledown g(x_0,y_0)$. Hence

$$y_0 = 3\lambda, \qquad x_0 = 5\lambda.$$

Substituting these in the constraint, we see that $3(5\lambda) + 5(3\lambda) = 90$, giving $\lambda = 3$. Hence $(15,9)$ is the only local maximum for this problem with constraint.

Thus the company must buy 15 type-I machines and 9 type-II machines to maximise the utility of the purchase.
Note: Due to the constraint, it can be seen that $0 < x < 30,\ 0 < y < 18$, in the problem above. So $0 < xy < (30)(18)$. Hence it follows that $\lim_{x\to 0} f(x,y) = 0$, $\lim_{y\to 0} f(x,y) = 0$. Consequently, as $(15,9)$ is the only local extremum, it follows that f attains the maximum value at $(x_0,y_0) = (15,9)$ for the above problem with constraint.
Example 7: Find the maximum and the minimum of the function $f(x,y) = x + y^2,\ (x,y) \in \mathbb{R}^2$, subject to the constraint $2x^2 + y^2 = 1$.

Take $g(x,y) = 2x^2 + y^2 - 1$. Here $E = \{(x,y) : g(x,y) = 0\}$. Note that

$$\bigtriangledown f(x,y) = (1,2y), \qquad \bigtriangledown g(x,y) = (4x,2y).$$

As $(0,0) \notin E$, note that $\bigtriangledown g(x,y) \neq 0,\ (x,y) \in E$.

Let $(x_0,y_0) \in E$ be a local extremum for the maximisation/ minimisation problem with the constraint. Then by the preceding theorem $\bigtriangledown f(x_0,y_0) = \lambda \bigtriangledown g(x_0,y_0)$. So we get

$$1 = \lambda 4x_0, \quad 2y_0 = \lambda 2y_0.$$

Case (i): Let $y_0 = 0$. Then by the constraint, $x_0^2 = 1/2$, and hence $x_0 = 1/\sqrt{2}$, or $x_0 = -1/\sqrt{2}$. Clearly $f(1/\sqrt{2}, 0) = 1/\sqrt{2}$, and $f(-1/\sqrt{2}, 0) = -1/\sqrt{2}$.

Case (ii): Let $y_0 \neq 0$. As $2y_0 = \lambda 2y_0$, we have $\lambda = 1$. So $x_0 = 1/4$. Then the constraint implies $y_0^2 = 7/8$, implying $y_0 = \pm\sqrt{7/8}$. Note that $f(1/4, \pm\sqrt{7/8}) = 9/8$.

Comparing the values of f at the four local extrema, we get: The maximum value of $(9/8)$ is attained at the 2 points, $(1/4, \sqrt{7/8})$ and $(1/4, -\sqrt{7/8})$, while the minimum value of $(-1/\sqrt{2})$ is attained only at the point $(-1/\sqrt{2}, 0)$.