Improper Integral

01 March 2021 20:08

1. $\int_0^\infty \sin(x)\,dx$ diverges.

$$\lim_{n\to\infty}\int_0^n \sin(x)\,dx = \lim_{n\to\infty}\Big[-\cos x\Big]_0^n = \lim_{n\to\infty}(1-\cos n)$$

$$= 1 - \lim_{n\to\infty}\cos n.$$

$\cos n$ is bounded but not monotonic function. So, $\lim_{n\to\infty}\cos(n)$ diverges.

So, $\int_0^\infty \sin x\,dx$ diverges

2. $\int_0^\infty \frac{1}{\sqrt{e^x}}\,dx = \int_0^\infty e^{-x/2}\,dx.$

$$\lim_{n\to\infty}\int_0^n e^{-x/2}\,dx = \lim_{n\to\infty}\Big[e^{-x/2}\times(-2)\Big]_0^n$$

$$= -2\lim_{n\to\infty}\Big[e^{-n/2}-1\Big]$$

$$= \lim_{n\to\infty} 2\Big[1-e^{-n/2}\Big] = 2$$

then the improper integral converges. and the value of the integral is 2

3. $\int_1^4 (x-1)^{-2/3}\,dx$ This function is undefined at $x=1$.

So, the integral will be —

x	4	1
y	$4-n$	$1-n$

$$\lim_{n\to 1}\int_n^4 (x-n)^{-2/3}\,dx = \lim_{n\to 1}\int_{1-n}^{4-n} y^{-2/3}\,dy$$

$$= \lim_{n\to 1} 3\Big[y^{1/3}\Big]_{1-n}^{4-n}.$$

$$= 3\Big[\lim_{n\to 1}(4-n)^{1/3}-(1-n)^{\frac{1}{3}}\Big]$$

$$= 3 \times 3^{\frac{1}{3}} = 3^{\frac{4}{3}}$$

4. Let $B > 2$

$$\lim_{B\to\infty}\int_2^B e^{-2x}\,dx = \lim_{B\to\infty}\frac{1}{2}\int_4^{2B} e^{-y}\,dy$$

$2x = y$

x	2	B
y	4	2B.

$$= \lim_{B\to\infty}\frac{1}{2}\left[\frac{e^{-y}}{-1}\right]_4^{2B.}$$

$$= \lim_{B\to\infty}\frac{1}{2}\left[-e^{-2B}+e^{-4}\right]$$

$$= \frac{e^{-4}}{2}$$

5. Let $\mu \in \mathbb{R}$, $\sigma^2 > 0$. Let g be given by

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < y < \infty,$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\int_{-\infty}^{\infty} x\,g(x)\,dx = \int_{-\infty}^{\infty} x\cdot\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx$$

$$\int_0^\infty x^{a-1}e^{-x}\,dx = \Gamma a \quad [a>0]$$

$$= \int_{-\infty}^{\infty}(\sigma y+\mu)\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}y^2}\,dy$$

$\frac{x-\mu}{\sigma} = y$

$\frac{dx}{\sigma} = dy$

$$= \frac{2\mu}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}y^2}dy + \sigma\int_{-\infty}^{\infty} y\,e^{-\frac{1}{2}y^2}\,dy$$

$\to 0.$

$\frac{1}{2}y^2 = z$

$y\,dy = dz$

$y = \sqrt{2z}$

$$= \frac{\sqrt{2}\,\mu}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{2}}\,z^{-\frac{1}{2}}e^{-z}\,dz$$

$$= \frac{\cancel{\sqrt{2}}\mu}{\sqrt{\pi}}\,\frac{1}{\cancel{\sqrt{2}}}\int_0^\infty z^{-\frac{1}{2}}e^{-z}\,dz$$

$$= \frac{\mu}{\sqrt{\pi}}\int_0^\infty z^{\left(-1+\frac{1}{2}\right)}e^{-z}\,dz$$

$$= \frac{\mu}{\sqrt{\pi}}\cdot\Gamma\tfrac{1}{2} = \frac{\mu}{\sqrt{\pi}}\cdot\sqrt{\pi} = \mu.$$

$$= \frac{\mu}{\sqrt{\pi}} \cdot \Gamma\tfrac{1}{2} = \frac{\mu}{\sqrt{\pi}} \cdot \sqrt{\pi} = \mu.$$

$$\int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} (\sigma y+\mu)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

$\frac{x-\mu}{\sigma} = y$

$x = \sigma y + \mu$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 y^2 e^{-\frac{1}{2}y^2} dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu^2 e^{-\frac{1}{2}y^2} dy$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^{\infty} \sigma^2 y^2 e^{-\frac{1}{2}y^2} dy + \frac{\mu^2}{\sqrt{\cancel{2\pi}}} \times \frac{\cancel{2}}{\sqrt{\cancel{2}}} \cancel{\sqrt{\pi}}$$

$\int_{-\infty} e^{-y} dy$

$$= \sqrt{\frac{2}{\pi}} \sigma^2 \int_0^{\infty} y^2 e^{-\frac{1}{2}y^2} dy + \mu^2$$

$\frac{1}{2}y^2 = z$

$y\,dy = dz$

$dy = \frac{1}{y} dz$

$= \frac{1}{\sqrt{2z}} dz$

$$= \sqrt{\frac{2}{\pi}} \sigma^2 \int_0^{\infty} 2z\, e^{-z} \frac{dz}{\sqrt{2z}} + \mu^2$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} \sqrt{z}\, e^{-z} dz + \mu^2$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \int_0^{\infty} z^{\left(\frac{3}{2}-1\right)} e^{-z} dz + \mu^2$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\tfrac{3}{2} = \boxed{\sigma^2 + \mu^2}$$

6-8

$$f(x) = P_n(x) + R_{n+1}(x)$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}a}{k!}(x-a)^k + R_{n+1}(x)$$

$$= P_n(x) + R_{n+1}(x)$$

$$R_{n+1}(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$$

$$R_{\cdot\cdot}(x) = \int^{x} (x-t)^n f^{n+1}(t)\, dt$$

$$R_{n+1}(x) = \int_a^x \frac{(x-t)^n}{n!} f^{n+1}(t)\, dt$$

$$\left| f^{(n+1)}(x) \right| \le M_{n+1} \qquad \forall x \in I$$

$$\left| R_{n+1}(x) \right| \le \frac{M_{n+1} |x-a|^{n+1}}{(n+1)!}, \; x \in I$$

$f(x) = \cos x.$ $\qquad f(x) = \sum_{n=0}^{n} \frac{f^n(a)}{n!} (x-a)^n$ around a.

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

$$f(x) = \cos 0 + \frac{-\sin 0}{1!} x + \frac{-\cos 0}{2!} x^2 + \frac{\sin 0}{3!} x^3 + \frac{\cos 0}{4!} x^4 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{f^{n+1}(c) \cdot x^{n+1}}{(n+1)!}$$

$$\frac{f^{(2n+1)}(c) \cdot x^{(2n+1)}}{(2n+1)!} = \left| \frac{-\sin(c) \quad x^{2n+1}}{(2n+1)!} \right| \le + \frac{x^{2n+1}}{(2n+1)!}$$

$f(x) = \frac{1}{1+x}$ $\qquad |x| < 1 \qquad -1 < x < 1$

$= (1+x)^{-1}$ $\qquad$ on $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} \cdot x^k$

$$= 1 + \frac{\frac{d}{dx} \frac{1}{1+x} \Big|_{x=0}}{1!} \cdot x + \frac{\frac{d^2}{dx^2} \frac{1}{1+x} \Big|_{x=0}}{2!} x^2 + \cdots$$

$$= 1 + \frac{(-1) \cdot (1+0)^{-2}}{1!} x + \frac{+2 \cdot (1+0)^{-3}}{2!} x^2 + \frac{3 \times 2}{3!} x^3$$

$$\frac{\quad}{1!} \qquad \frac{\quad}{2!}\,x + \frac{\quad}{3!}$$

$$+ \cdots + \frac{f^{n+1}(x)}{(n+1)!}\,x^{n+1}$$

$$= 1 + x + x^2 + x^3 + \cdots + \frac{(n+1)!}{(n+1)!}\,(1+x)^{-(n+2)}\Big|_{x=0}\,x^{n+1}$$

$$\left|\frac{f^{(n+1)}(x)}{(n+1)!}\,x^{n+1}\right| < \left|\frac{x^{n+1}}{1-c}\right| \qquad |x| < 1$$

$$\boxed{0 < c < x}$$