

$$\begin{array}{r} 24 \\ 14 \\ \hline 96 \\ 24 \\ \hline 336 \\ 29 \\ \hline 361 \end{array}$$



$$y_2, y_2 \geq 42$$

$$x_1 - y_2 \leq 4, x_1 - y_2 x_2 \leq -4, y_2$$

if feasible path

$$-y_2 \leq -4, y_2$$

$$x_2 \geq y_2 x_2, y_1(x_1) + y_2(x_2) \leq -4, y_1 + y_2$$

$$1 \leq y_2$$

$$x_1 \leq 4, y_1$$

1. Does Gaussian elimination preserve the column space of a matrix? Justify. (3 marks)

2. In the simplex method discussed in class, we made an assumption that the feasible region is bounded. Give a modified simplex algorithm that handles unbounded feasible regions. Explain in a few sentences why your modifications work. (5 marks)

3. Write the dual of the following LP: (4 marks)

$$\text{Maximize } 8x_1 + 3x_2 - 2x_3$$

$$\text{Subject to } \begin{array}{l} x_1 - 6x_2 + x_3 \geq 2 \\ 5x_1 + 7x_2 - 2x_3 = -4 \\ x_1 \leq 0 \\ x_2 \geq 0 \end{array}$$

$$x_1 + 2x_2 + 3x_3 \leq 10$$

$$2x_1 + 3x_2 + 2x_3 \leq 40$$

4. Exhibit a primal-dual pair such that both are infeasible. (5 marks)

5. Consider the following problem. (8 marks)

$$\min 19w_1 + 57w_2$$

$$\text{Maximize } 10x_1 + 24x_2 + 20x_3 + 20x_4 + 25x_5$$

$$w_1 + 2w_2 \geq 10$$

Subject to

$$x_1 + x_2 + 2x_3 + 3x_4 + 5x_5 \leq 19$$

$$(C1)$$

$$w_1 + 4w_2 \geq 24$$

$w_2 \times$

$$2x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 \leq 57$$

$$(C2)$$

$$2w_1 + 3w_2 \geq 20$$

(a) Write its dual with two variables w_1, w_2 (corresponding to the constraints (C1) and (C2)) and verify that $(w_1, w_2) = (4, 5)$ is a feasible solution.

$$3w_1 + 2w_2 \geq 20$$

(b) Use complementary slackness to show that $(w_1, w_2) = (4, 5)$ gives the optimal solution to the dual.

$$5w_1 + w_2 \geq 25$$

(c) Use (b) to show that every k -regular bipartite graph has a perfect matching¹. (10 marks)

7. Here is the set cover problem:

Input. A universe D consisting of finite number of elements, and a family S_1, S_2, \dots, S_m of sets, with each $S_i \subseteq D$. Assume that $\bigcup_{i \in \{1, \dots, m\}} S_i = D$.

Goal. Find minimum size subset $W \subseteq \{1, \dots, m\}$ such that $\bigcup_{i \in W} S_i = D$

(a) Write an ILP for the set cover problem.

(b) Give the relaxed LP and its dual.

(c) Design a primal-dual algorithm which gives an f -approximation where f is the maximum frequency of an element, i.e. $f = \max_{e \in D} \{\text{Number of sets containing element } e\}$. In other words, show that the answer given by the algorithm is at most f times the optimal solution.

Describe the algorithm and its analysis.

$$\min \sum S_i$$

$$(8 \text{ marks})$$

$$\sum_{e \in U} x_e - \sum_{e \in S_i} S_i \geq 1 \quad \forall e$$

$$\max \sum e_i$$

$$\sum e_i \leq 1 \quad \forall S_i$$

$$8x_1 + 3x_2 - 2x_3$$

¹Graph theoretic answers not using (b) will not be accepted

$$\max x_1$$

$$\min -y_1$$

$$\max x_1 + x_2$$

$$\min y_1$$

$$x_1 - 6x_2 + x_3 \geq 2$$

$$5x_1 + 7x_2 - 2x_3 = -4$$

$$x_1 \leq 0$$

$$x_1 \leq -1$$

$$y_1 \geq 1$$

$$y_1 \geq 0$$

$$y \geq 0$$

$$x_1 \geq 0$$

$$x_1 \geq 0$$

$$y_1 \geq 0$$

$$y_1 \geq 0$$

$$y \geq 0$$

$$x_1 \geq 0$$

$$x_1 \geq 0$$

$$y_1 \geq 0$$

$$y_1 \geq 0$$

$$y \geq 0$$

$$x_1 \geq 0$$

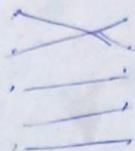
$$\max 8x_1 + 3x_2 - 2x_3$$

$$\text{subject to } x_1 - 6x_2 + x_3 \geq 2$$

$$5x_1 + 7x_2 - 2x_3 = -4$$

$$x_1 \leq 0$$

$$x_2 \geq 0$$



$$\min \sum c_e x_e$$

$$\text{subject to } \sum_{e \text{ incident on } v} x_e = 1 \quad \forall v \in V.$$

$$x_e \in \{0, 1\}$$

$$\text{relaxed } x_e \geq 0$$

non-integer

$$(x_e, \dots) \quad \sum_{e \text{ incident on } v} x_e = 1$$

$$\max x_1 + x_2$$

$$x_1 \leq -1$$

$$x_2 \geq -1$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_2 \leq y_2 x_1 \leq -y_2$$

$$y_2 \geq 1$$

$$8y_1 + 3y_2 - 2y_3 \leq 0$$

$$y_1 x_1 - 6y_1 x_2 + y_1 x_3 \geq 2y_1$$

$$5y_1 x_1 + 7y_2 x_2 - 2y_3 x_3 = 4y_2$$

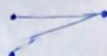
$$\min 2y_1 + 4y_2$$

$$y_1 + 5y_2 \leq 8$$

$$-6y_1 + 7y_2 \geq 3$$

$$y_1 - 2y_2 = -2$$

$$8x_1 \leq 7x_1$$



$$y_1 \geq 0$$

$$\max x_1 + x_2$$

$$x_1 \leq -1$$

$$x_2 \leq -1$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$\leq y_1 x_1 + y_2 x_2$$

$$\min -y_1 - y_2$$

$$y_1 \geq 1$$

$$y_2 \leq 1$$

$$y_1 \geq 0$$

$$y_2 \geq 0$$

$$\min -y_1 - y_2$$

$$y_1 \geq 1$$

$$y_2 \geq 1$$

$$y_1 \geq 0$$

$$y_2 \leq 0$$