G.E. (continued) ;

Note Title 17-04-2021

Last time - $m \times m$ matrix $A$

- (Basic, every pivot is nonzero) invertible matrix. G.E. amounts to find lower $\Delta^r$ matrices $L_1, \dots, L_{m-1}$ such that

$$\underbrace{L_{m-1} \cdots L_2 L_1}_{L^{-1}} A = \mathcal{U} \text{ (upper } \Delta^r\text{)}. \quad \therefore A = L\mathcal{U}.$$

- G.E. with pivoting (i) GEPP : amounts to find lower $\Delta^r$ matrices $L_1, \dots, L_{m-1}$ as well as permutation matrices $P_1, \dots, P_{m-1}$ such that -

$$\underbrace{L_{m-1} P_{m-1} \cdots L_2 P_2 L_1 P_1}\, A = \mathcal{U}$$

$$\begin{bmatrix} a_{11} & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \xrightarrow[\text{(replace } a_{11} \text{ by } \max\{|a_{i1}|\})]{P_1 A} \begin{bmatrix} \star & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \xrightarrow{L_1(P_1 A)} \begin{bmatrix} \star & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

$$L^{-1} P A = \mathcal{U}$$
$$PA = L\mathcal{U}$$

(ii) GECP : amounts to -

$$\underbrace{L_{m-1} P_{m-1} \cdots L_2 P_2 L_1 P_1}\, A\, \underbrace{Q_1 Q_2 \cdots Q_{m-1}} = \mathcal{U}$$

$$\underset{A}{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}} \longrightarrow \underset{L_1 P_1 A Q_1}{\begin{bmatrix} \star & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}} \longrightarrow \underset{L_2 P_2 L_1 P_1 A Q_1 Q_2}{\begin{bmatrix} \star & * & * \\ 0 & \star & * \\ 0 & 0 & * \end{bmatrix}}$$

$$L^{-1} P A Q = \mathcal{U}$$
$$PAQ = L\mathcal{U}$$

**Theorem:** Let $A$ be $n\times n$ nonsingular matrix.
$A$ has an $LU$ factorization $\iff$ $\det \Delta_k \neq 0$ for each $1 \leq k \leq n$.

Moreover, this factorization is unique.

$\Delta_k = k\times k$ top-left submatrix of $A$
$= \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$

**Proof:** If $A = LU$ is an $LU$ factorization,
then $\det(\Delta_k^A) = \det(\Delta_k^L)\cdot\det(\Delta_k^U) \neq 0$.

Conversely, we proceed by induction on $k$.

$k=1$: $a_{11} \neq 0$, we can choose $P_1 = I$

(Enough to show that $P_k = I$)

Suppose $P_1, \ldots, P_{k-1} = I$, so that $L_{k-1}L_{k-2}\cdots L_1 A = A_k$

$$\begin{pmatrix} \overset{k\times k}{\begin{matrix} 1 & & 0 \\ & 1 & \\ * & & \ddots \end{matrix}} & 0 \\ * & \ddots \; 1 \end{pmatrix} \begin{pmatrix} \overset{k\times k}{\begin{matrix} a_{11} & \cdots & a_{1k} \\ \vdots & \Delta_k & \\ a_{k1} & \cdots & a_{kk} \end{matrix}} & * \\ * & * \end{pmatrix} = \begin{pmatrix} \begin{matrix} a_{11}^1 & \cdots & a_{1k}^1 \\ & \ddots & \\ 0 & & a_{kk}^k \end{matrix} & * \\ & \begin{matrix} * \\ * \end{matrix} & \end{pmatrix}$$

$$0 \neq \det \Delta_k = \underbrace{a_{11}^1 \cdots}\, a_{kk}^k$$

so $a_{kk}^k \neq 0$, so it can be chosen as pivot.
$\therefore P_k = I$

□