CMI - 2020-2021: DS-Analysis

Calculus-3

Differentiation

Definition 1 Let $f : (a,b) \to \mathbb{R}$ be a function. The *derivative* of f at $x \in (a,b)$, denoted by $f'(x)$ or $\frac{df(x)}{dx}$, is defined as

$$f'(x) = \frac{df(x)}{dx} := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

provided the limit exists. The derivative $f'(x)$ is also called the *rate of change* of f at x.

If the derivative $f'(x)$ exists at every point $x \in (a,b)$, then we say that f is *differentiable* on (a,b).

It can easily be shown that if f is differentiable on (a,b) then it is continuous on (a,b).

Remark 2 If f is a differentiable function on (a,b), then its derivative f' is also a function on (a,b); so one gets new function from f. This process is called *differentiation;* also f' may be called the *first derivative* of f. If f' is differentiable, the derivative of f', denoted by f" or $\frac{d^2f}{dx^2}$, is called the *second derivative* of f. Similarly, the $n-th$ *derivative* of f, denoted by $f^{(n)}$ or $\frac{d^nf}{dx^n}$, is defined to be the first derivative of $f^{(n-1)}$.

The following basic result must already be familiar.

Proposition 3 *Let f, g be real valued differentiable functions on (a,b). Then $f+g$, cf, $f.g$ are also differentiable functions on (a,b); here $c \in \mathbb{R}$ is a constant, and $f.g$ denotes the function $f.g(x) = f(x)g(x)$. Also $\frac{f}{g}$ is also differentiable at any $x \in (a,b)$ where $g(x) \neq 0$. Moreover we have*

$$\begin{aligned}
(f+g)'(x) &= f'(x)+g'(x),\ x \in (a,b);\\
(cf)'(x) &= cf'(x),\ x \in (a,b);\\
(f.g)'(x) &= f(x)g'(x)+g(x)f'(x),\ x \in (a,b);\\
(\frac{f}{g})'(x) &= \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2},\ x \in (a,b),\ g(x) \neq 0.
\end{aligned}$$

Example 1: Consider the function $(3x^3 - 5x^2 + 2x - 7)/(x^2 - 1)$. Clearly both the numerator and the denominator are polynomials; both are defined on $\mathbb{R}$, and differentiable on $\mathbb{R}$. However, $x^2 - 1 = 0$ when $x = -1$ or $x = 1$. So the above function is differentiable at all $x \in \mathbb{R}$, except at $x = -1, 1$. Find the derivative at points where it is differentiable.

In the definition of derivative, note that h can be positive or negative. However, as in the case of continuity, it is convenient to consider when h varies only over positive or only over negative values.

Definition 4 Let $f : (a, b) \to \mathbb{R}$ and $x \in (a, b)$. The *right derivative* of f at x is defined as

$$\lim_{h \to 0,\ h > 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Similarly, the *left derivative* of f at x is defined as

$$\lim_{h \to 0,\ h < 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Note: Clearly f is differentiable at x, if and only if both the right and the left derivatives of f exist at x, and are equal; in such a case, the common value is $f'(x)$.

Example 2: Let $g(x) = |x|,\ x \in \mathbb{R}$. Clearly g is a continuous function on $\mathbb{R}$. But g is not differentiable at $x = 0$. Why?

Example 3: Let f be given by

$$\begin{aligned} f(x) &= x^2,\ x \geq 0, \\ &= x^3,\ x < 0. \end{aligned}$$

Show that f is a continuous and differentiable function on $\mathbb{R}$. Show that the derivative f' is also continuous on $\mathbb{R}$. Next show that f' is not differentiable at $x = 0$. So f is *not* a twice differentiable function on $\mathbb{R}$. Draw the graphs of f and f'.

Next, we look at *chain rule,* a very useful result.

Let $g,\ f$ be functions such that $\frac{dg(x)}{dx}$ exists for all x where g is defined, and $\frac{df(u)}{du}$ exists for all u where f is defined; moreover suppose that the composite function $f \circ g$ given by

$$f \circ g(x) \quad = \quad f(g(x))$$

makes sense.

Theorem 5 *Let $f,\ g$ be as above. Then the composite function $f \circ g$ is differentiable and*

$$(f \circ g)'(x) \quad = \quad f'(g(x))g'(x).$$

Example 4: Using chain rule, note that

$$\frac{d}{dx}(\sin x)^9 \quad = \quad 9(\sin x)^8 \cos x.$$

Example 5: In a similar fashion show that

$$\frac{d}{dx}e^{\sin 2x} \quad = \quad 2e^{\sin 2x} \cos 2x.$$

Example 6: Let $a > 0$ be a constant. Let $h(x) = a^x,\ x \in \mathbb{R}$. Observe that $a^x = (e^{\log a})^x = e^{(\log a)x}$. So put $g(x) = (\log a)x$, and $f(u) = e^u$. Thus we have $h(x) = f(g(x))$. Therefore by the chain rule we get

$$\begin{aligned} \frac{d}{dx}(a^x) \quad &= \quad f'(g(x))g'(x) \quad = \quad e^{(\log a)x}(\log a) \\ &= \quad a^x(\log a). \end{aligned}$$

Now we look at an example of the technique of *implicit differentiation,* an important application of the chain rule.

Consider the equation $x^2 + y^2 = r^2$, where $r > 0$ is a constant. Here x is viewed as the independent variable, and y as a dependent variable; that is, $y = y(x)$ is a function of x. We assume that $y'(x)$ exists. So differentiating both sides of the equation $x^2 + y^2 = r^2$ with respect to x, we get

$$2x + 2y(x)y'(x) = 0.$$

Solve the above equation for $y'(x)$ to get

$$\frac{d}{dx}y(x) = y'(x) = \frac{-x}{y(x)}, \text{ if } y(x) \neq 0.$$

The above relation is also written as $y' = \frac{-x}{y}$, if $y \neq 0$.
Example 7: Suppose $3x^3y - y^4 + 5x^2 = 3$. Assuming that y is a function of x, find $y'(x)$.

We have already mentioned that the derivative is also called the rate of change. Some examples are discussed in that connection.
Example 8: A square is expanding in such a way that its side is changing at 2 cm/ sec. Find the rate of change of its area when the side is 6 cm long.

If the side is x cm, the area is $A = A(x) = x^2$ sq.cm. Here x, and hence A are functions of time. Clearly $A(x(t)) = (x(t))^2$. The rate of change of area is $\frac{d(A(x(t)))}{dt}$. So by chain rule

$$\frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt} = 2x\frac{dx}{dt}.$$

As the side increases by 2 cm /sec we have $\frac{dx}{dt} = 2$. Hence when $x = 6$ cm we get $\frac{dA}{dt} \mid_{x=6} \; = \; 24$ sq. cm. /sec.
Example 9: A cylinder is being compressed from the side and stretched, so that the radius of the base is decreasing at a rate of 2 cm/ sec, and the height is increasing at a rate of 5 cm/ sec. Find the rate at which the volume of the cylinder is changing when the radius is 6 cm and the height is 8 cm.

We know that the volume of the cylinder is $V = \pi r^2 h$, where r is the radius of the base, and h is the height. Given that $\frac{dr}{dt} = -2, \; \frac{dh}{dt} = 5$; the negative sign indicates that the radius is decreasing. Therefore we get

$$\frac{dV}{dt} = \pi\big[r^2\frac{dh}{dt} + h2r\frac{dr}{dt}\big] = \pi[5r^2 - 4hr].$$

Hence when $r = 6, \; h = 8$ we get $\frac{dV}{dt} \mid_{r=6,h=8} \; = \; -12\pi$ cm^3 /sec. The negative sign indicates that the volume of the cylinder is decreasing when $r = 6, \; h = 8$.

Example 10: A balloon is going up, starting at a point P. An observer standing 200 ft away looks at the balloon, and the angle θ which the balloon makes increases at the rate of (1/20) radian/ sec. Find the rate at which the distance of the balloon from the ground is increasing when $\theta = (\pi/4)$.

Let y denote the distance from the balloon to the ground. Observe that $\tan(\theta) = y/200$. Hence $y = 200\tan(\theta)$. (It might be helpful to draw a rough diagram.) We need to find $\frac{dy}{dt}$, when $\theta = \pi/4$. Note that

$$\begin{aligned} \frac{dy}{dt} &= 200\frac{d\tan(\theta)}{dt} \\ &= 200(1+\tan^2(\theta))\frac{d\theta}{dt} \end{aligned}$$

As we are given $\frac{d\theta}{dt} = (1/20)$, and $\tan(\pi/4) = 1$, we get

$$\frac{dy}{dt}\,|_{\theta=(\pi/4)} = 20 \text{ ft/sec.}$$

We look at another interpretation of the derivative.

Let f be a differentiable function on the interval (a, b). Consider the graph $y = f(x)$ of the function. It is a curve in the plane. Let $x_0 \in (a, b)$, and let $y_0 = f(x_0)$; denote by $P = (x_0, y_0) = (x_0, f(x_0))$. So P is point on the graph $y = f(x)$. Then the derivative $f'(x_0)$ is the *slope* of the curve $y = f(x)$ at the point P. That is, the straight line with slope $f'(x_0)$ passing thru' the point P is the tangent line to the curve at P.

Example 11: Find the slope of the graph of the function $f(x) = 2x^2$ at the point whose x coordinate is 3. Also find the equation of the tangent line at that point.

Note that $f'(x) = 4x$. So $f'(3) = 12$. Clearly $f(3) = 18$. Hence it follows that the slope of the graph of f at the point $(3, 18)$ is 12. And the equation of the tangent line passing thru' $(3, 18)$ is

$$y - 18 = 12(x - 3).$$

Example 12: Let $f(x) = 2/(x + 1)$. Find the slope of the graph of f at the point whose x coordinate is 2, and also the equation of the tangent line at that point.

Note that $f'(x) = -2/(x+1)^2$. Clearly $f(2) = 2/3$, and $f'(2) = -2/9$. So the equation of the tangent line passing thru' $(2,\ 2/3)$ is

$$y = \frac{-2}{9}x + \frac{10}{9},$$

that is, $2x + 9y = 10$.
Example 13: Let

$$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 1,\ x \in \mathbb{R}.$$

Find the points on the graph of f at which the tangent line is horizontal.

Note that a line is horizontal only when the slope of the line is 0. So we need to find the points x at which the derivative $f'(x) = 0$.

Observe that $f'(x) = x^2 - 4x + 3 = (x-1)(x-3)$. So $f'(x) = 0$ if and only if $x = 1,\ x = 3$. Note that $f(1) = 7/3$, and $f(3) = 1$. Therefore the points on the graph of f at which the tangent line is horizontal are $(1, \frac{7}{3})$, and $(3, 1)$.
Example 14: Let $f(x) = x + \sin x,\ x \in \mathbb{R}$. Find the points on the graph of f at which the tangent line is horizontal.