Instructions

- *The question paper has* ***two sides****. There are* ***eight*** *questions.*
- *The duration of the exam is 4 hours. You can refer to the prescribed textbook and lecture notes.*
- *No doubts/clarifications would be entertained during the examination. If needed, make appropriate assumptions and state them in your answers.*

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1. Solve the following LP using the simplex tableau method. **(6 marks)**

$$
\begin{array}{lrcrl}
\text{maximize} & x_1 & + & x_2 & \\[1ex]
\text{subject to} & -x_1 & + & x_2 & \leq 2 \\
& & & x_2 & \leq 4 \\
& x_1 & + & x_2 & \leq 9 \\
& x_1 & & & \leq 6 \\
& x_1 & - & x_2 & \leq 5 \\[1ex]
& x_1 & , & x_2 & \geq 0
\end{array}
$$

2. Write the dual for the following LP: **(4 marks)**

$$
\begin{array}{lrcrcrl}
\text{Maximize} & x_1 & + & x_2 & + & 2x_3 & +\ 15 \\[1ex]
\text{Subject to} & 2x_1 & + & 9x_2 & + & 8x_3 & \geq 25 \\
& x_1 & - & 6x_2 & + & 3x_3 & = 15 \\
& 4x_1 & + & 7x_2 & - & 20x_3 & \geq 4 \\
& & & & & x_1 & \geq 0 \\
& & & & & x_2 & \leq 0 \\
& & & & & x_3 & \text{unrestricted}
\end{array}
$$

3. Answer True or False for each of the following statements. Provide a justification. **(16 marks)**
   (a) If an LP in equational form is feasible, it has a basic feasible solution.
   (b) If an LP with constraints given as $Ax \leq b$ is feasible, it has an extreme point.
   (c) Consider an LP with constraints $Ax = b, x \geq 0$. If there are two basic feasible solutions giving the optimum value, then there are infinitely many feasible solutions giving this optimum value.

4. Can a variable which just left the basis in a simplex tableau reenter in the very next pivot? Explain your answer. **(6 marks)**

5. Consider the linear program: minimize $c^T x$ subject to $Ax \leq b$, $x \geq 0$. Assume c is a non-zero vector. Suppose there is a point x_0 satisfying $Ax_0 < b$ and $x > 0$.

   Show that x_0 cannot be an optimal solution. **(8 marks)**

6. Consider LPs of the form maximize $c^T x$ subject to $Ax = b$, $x \geq 0$, where A is $m \times n$. Is it possible for an optimal solution to have more than m positive variables (that is, more than m variables that are strictly greater than 0).

   If your answer is yes, then you should exhibit a concrete LP that satisfies this condition. If your answer is no, then you should give a proof. **(6 marks)**

7. Let $X = \{x : Ax = b, x \geq 0\}$ where A is $m \times n$ with rank m. Let y be a feasible solution such that $y_1, \ldots, y_q$ are > 0 and $y_{q+1}, \ldots, y_n$ are equal to 0.

   Assume that the columns $A_1, A_2, \ldots, A_q$ corresponding to the positive variables are linearly dependent. Construct feasible points y' and y'' such that y is a convex combination of these points. **(8 marks)**

8. Show by duality that if the problem - minimize $c^T x$ subject to $Ax = b, x \geq 0$ - has a finite optimal solution, then the new problem - minimize $c^T x$ subject to $Ax = b', x \geq 0$ - cannot be unbounded, no matter what value the vector b' might take. **(6 marks)**