9 May 2021

M.Sc. Data Science
LAA - Homework 2

1. Prove that $||A||_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$, where σ_i are the singular values of A.

   According to the definition of Frobenius Norm,

$$||A||_F^2 = \text{tr } (A^*A) = \sum_{i=1}^{m}(A^*A)_{ii} = \sum_{i=1}^{m}(\sum_{j=1}^{n} a_{ij}^* a_{ij}) = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2$$

   Frobenius Norm is not altered by a pre or post Orthogonal Transformation.

   Hence,

$$||A||_F^2 = \text{tr}(A^*A) = \text{tr } (U\Sigma V^*)^*(U\Sigma V^*) = \text{tr } (V\Sigma^* U^* U\Sigma V^*) = \text{tr } (V\Sigma^*\Sigma V^*) \text{ as, } U^*U = I$$

   Since, tr $(AB) =$ tr (BA),

$$\text{tr } (U\Sigma V^*)^*(U\Sigma V^*) = \text{tr } (V\Sigma^*\Sigma V^*) = \text{tr } (V^*V\Sigma^*\Sigma)$$

   As, V is Unitary, $V^*V = VV^* = I$.

$$\text{tr } (U\Sigma V^*)^*(U\Sigma V^*) = \text{tr } (V^*V\Sigma^*\Sigma) = \text{tr } (I\Sigma^*\Sigma) = \text{tr } (\Sigma^*\Sigma) = ||\Sigma||_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2} \text{ (Proved.)}$$

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2. Calculate the full and reduced singular value decompositions for the matrix

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

   full singular value decomposition

   Singular value decomposition of a matrix A is a factorization $A = U\Sigma V^*$, where U, V are unitary and Σ is diagonal. The diagonal entries of Σ are called singular values of A and usually arranged in decreasing order.

   Let us calculate AA^T.

$$AA^T = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} \times \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}$$

   Finding the eigen values and eigen vectors of AA^T,

$$\det(AA^T - \lambda I) = 0$$

$$\det \begin{pmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{pmatrix} = 0$$

$$(17-\lambda)^2 - 64 = 0$$

$$\lambda = 25, 9$$

   Now, finding the eigen vectors with respect to the eigen values.

When, $\lambda = 25$

$$\begin{pmatrix} 17-25 & 8 \\ 8 & 17-25 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving, this we get,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

When, $\lambda = 9$

$$\begin{pmatrix} 17-9 & 8 \\ 8 & 17-9 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving, this we get,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence, dividing with the norm of the vectors, $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Now, let us calculate the eigen values and eigen vectors for $A^T A$.

$$A^T A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix} \times \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{pmatrix}$$

Finding the eigen values and eigen vectors of $A^T A$,

$$\det(A^T A - \lambda I) = 0$$

$$\det \begin{pmatrix} 13-\lambda & 12 & 2 \\ 12 & 13-\lambda & -2 \\ 2 & -2 & 8-\lambda \end{pmatrix} = 0$$

$$-\lambda^3 + 34\lambda^2 - 225\lambda = 0$$

$$\lambda = 25, 9, 0$$

When, $\lambda = 25$

$$\begin{pmatrix} 13-25 & 12 & 2 \\ 12 & 13-25 & -2 \\ 2 & -2 & 8-25 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving, this we get,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

When, $\lambda = 9$

$$\begin{pmatrix} 13-9 & 12 & 2 \\ 12 & 13-9 & -2 \\ 2 & -2 & 8-9 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving, this we get,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.25 \\ -0.25 \\ 1 \end{pmatrix}$$

When, $\lambda = 0$

$$\begin{pmatrix} 13-0 & 12 & 2 \\ 12 & 13-0 & -2 \\ 2 & -2 & 8-0 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving, this we get,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

Hence, dividing with the norm of the vectors, $V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \end{pmatrix}$

So, the final decomposition is, $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}$

Reduced singular value decomposition

The eigen values are eigen vectors are already calculated. Here the given matrix is 2×3. and we found the rank of A is 2. Hence the dimension of the diagonal matrix Σ will ne 2×2. And the diagonal elements are thr square root of the common non zero eigen values of AA^T and A^TA. So, the reduced SVD is,

$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \end{pmatrix}$

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3. Prove that for a $n \times n$ matrix A,

$$|\det A| = \prod_{i=1}^{n} \sigma_i$$

where σ_i are the singular values of A.

for a $n \times n$ matrix A, we can find it's SVD as, $A = U\Sigma V^*$.

$$|\det A| = |\det(U\Sigma V^*)| = |\det(U)\det(\Sigma)\det(V^*)| = |\det(U)||\det(\Sigma)||\det(V^*)|$$

As, U, V are unitary matrices, $|\det(U)| = |\det\Sigma| = 1$. Hence,

$$|\det A| = |\det(\Sigma)|$$

. We all know that, the singular values of a matrices are real and positive and the matrix Σ is a square diagonal matrix for a $n \times n$ matrix. Hence, $|\det(\Sigma)| = \prod_{i=1}^{n} \sigma_i$. Hence,

$$|\det A| = \prod_{i=1}^{n} \sigma_i \text{ (Proved.)}$$

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4. Write down the differences between the eigenvalue decomposition and the singular value decomposition of a matrix.

If A is any $m \times n$ matrix, then Eigen value decomposition has some problems,

(a) The eigen vectors corresponding to the eigen values are not always orthogonal.

(b) There are not always enough eigen vectors.

(c) And, eigen value calculation requires the matrix to be a square one. The eigen values doesn't exist for any rectangular matrix.

Where, singular value decomposition solve these problems. It can decompose matrices of any order.The singular values are always real and positive. But in case of eigen values, it might be imaginary or negative in some cases. Like consider the rotation matrix. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
the eigen values are $i, -i$. but the singular values are 1,1.

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5. If $x \in \mathbb{C}^m$ and A is a $m \times n$ matrix, prove the following inequalities:

(a) $||x||_\infty \leq ||x||_2$.
(b) $||x||_2 \leq \sqrt{m}||x||_\infty$.
(c) $||A||_\infty \leq \sqrt{n}||A||_2$.
(d) $||A||_2 \leq \sqrt{m}||A||_\infty$.

(a) $||x||_\infty \leq ||x||_2$.

$$||x||_2 = (\sum_{i=1}^{m} |x_i^2|)^{\frac{1}{2}} \tag{1}$$

$$||x||_\infty = \max_{1 \leq i \leq m} |x_i| \tag{2}$$

$$\tag{3}$$

So, as the maximum element is also present in the 2-norm so, definitely, 2-norm will be greater than inf-norm.

$$||x||_\infty \leq ||x||_2 \qquad \text{(Proved.)} \tag{4}$$

(b) $||x||_2 \leq \sqrt{m}||x||_\infty$.

$$||x||_2 = (\sum_{i=1}^{m} |x_i^2|)^{\frac{1}{2}} \tag{5}$$

$$||x||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_m^2} \tag{6}$$

Now, replacing the values with the maximum value say x_m.

$$||x||_2 \leq \sqrt{x_m^2 + x_m^2 + \cdots + x_m^2} \tag{7}$$

$$||x||_2 \leq \sqrt{m \times x_m^2} \tag{8}$$

$$||x||_2 \leq \sqrt{m}\sqrt{x_m^2} \tag{9}$$

$$||x||_2 \leq \sqrt{m}x_m \tag{10}$$

$$||x||_2 \leq \sqrt{m}||x||_\infty \tag{11}$$

Hence, $||x||_2 \leq \sqrt{m}||x||_\infty$ (Proved.)

(c) $||A||_\infty \leq \sqrt{n}||A||_2$.
According to the definition of p-norm,

$$||A||_p = \sup_{x \neq 0} \frac{||Ax||_p}{||x||_p} \tag{12}$$

We already know that, $\forall x \in \mathbb{C}^m$, $||x||_\infty \leq ||x||_2 \leq \sqrt{m}||x||_\infty$

$$||A||_\infty = \sup_{x\neq 0} \frac{||Ax||_\infty}{||x||_\infty} \tag{13}$$
$$\leq \sup_{x\neq 0} \frac{||Ax||_2}{||x||_\infty} \tag{14}$$

As, A in $m \times n$, x has to be $n \times 1$ and Ax has to be $m \times 1$. So, Using, $||x||_2 \leq \sqrt{n}||x||_\infty$

$$||A||_\infty \leq \sup_{x\neq 0} \frac{||Ax||_2}{\frac{1}{\sqrt{n}}||x||_2} \tag{15}$$
$$= \sqrt{n}||A||_2 \tag{16}$$

Hence, $||A||_\infty \leq \sqrt{n}||A||_2$. (Proved.)

(d) $||A||_2 \leq \sqrt{m}||A||_\infty$.
According to the definition of p-norm,

$$||A||_p = \sup_{x\neq 0} \frac{||Ax||_p}{||x||_p} \tag{17}$$

We already know that, $\forall x \in \mathbb{C}^m$, $||x||_\infty \leq ||x||_2 \leq \sqrt{m}||x||_\infty$

$$||A||_2 = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2} \tag{18}$$
$$\leq \sup_{x\neq 0} \frac{\sqrt{m}||Ax||_\infty}{||x||_2} \tag{19}$$

As, A in $m \times n$, x has to be $n \times 1$ and Ax has to be $m \times 1$. So, Using, $||x||_\infty \leq ||x||_2$

$$||A||_2 \leq \sup_{x\neq 0} \frac{\sqrt{m}||Ax||_\infty}{||x||_\infty} \tag{20}$$
$$= \sqrt{m}||A||_\infty \tag{21}$$

Hence, $||A||_2 \leq \sqrt{m}||A||_\infty$. (Proved.)

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6. Given $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, show that A^*A is nonsingular if and only if A has full rank.

Let us assume that A^*A is singular and A has a full rank. Then for $x \neq 0$, there must be some vector x, for which, $A^*Ax = 0$.

$$\implies \quad A^*Ax = 0 \tag{22}$$
$$\implies \quad x^*A^*Ax = 0 \tag{23}$$
$$\implies (Ax)^*Ax = 0 \tag{24}$$
$$\implies \quad ||Ax||^2 = 0 \tag{25}$$
$$\implies \quad Ax = 0 \tag{26}$$

This implies, $x \in$ Nullity(A). Hence, contradiction that A has a full rank So, A^*A is nonsingular if A has a full rank.

Conversely, Let, A doesn't have full rank but A^*A is nonsingular. Then there must be some $x \neq 0$, for which, $Ax = 0$.So, $A * Ax = 0$ where x is a nonzero vector. Which is a contradiction because, for a non singular matrix the nullity is the zero vector. So, A^*A should be singular.

Hence, A^*A is nonsingular if and only if A has full rank. (Proved.)

Note: *All norms on $\mathbb{C}^n$ are equivalent, i.e. if $||\cdot||_\alpha$ and $||\cdot||_\beta$ are norms on $\mathbb{R}^n$, then there exist positive constants c_1 and c_2 such that*

$$c_1||x||_\alpha \leq ||x||_\beta \leq c_2||x||_\alpha.$$

For example, if $x \in \mathbb{R}^n$, then

$$||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2,$$

$$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty,$$

$$||x||_\infty \leq ||x||_1 \leq n||x||_\infty.$$

In example 5 above, you are proving this inequality for the case $\alpha = 2$ and $\beta = \infty$. Because it holds for vector norms, a similar inequality also holds for matrix norms, you are proving some of these in example 5 too. More inequalities are true:

$$\frac{1}{\sqrt{m}}||A||_1 \leq ||A||_2 \leq \sqrt{n}||A||_1,$$

$$||A||_2 \leq ||A||_F \leq \sqrt{\min\{m,n\}}||A||_2.$$

Going further, for condition numbers defined using matrix norms, the following holds true: any two matrix condition numbers $\kappa_\alpha(\cdot)$ and $\kappa_\beta(\cdot)$ are equivalent i.e.

$$c_1\kappa_\alpha(A) \leq \kappa_\beta(A) \leq c_2\kappa_\alpha(A).$$

For example on $\mathbb{R}^{n\times n}$ we have

$$\frac{1}{n}\kappa_2(A) \leq \kappa_1(A) \leq n\kappa_2(A),$$

$$\frac{1}{n}\kappa_\infty(A) \leq \kappa_2(A) \leq n\kappa_\infty(A),$$

$$\frac{1}{n^2}\kappa_1(A) \leq \kappa_\infty(A) \leq n^2\kappa_1(A).$$

What do all these inequalities mean for us in practice? They mean the following: if you are working with induced matrix norms and matrix condition numbers, and you prove an upper bound or a lower bound w.r.t. one norm, then a similar bound will hold w.r.t. the other norms also, differing by a factor c_1 or c_2, as the case may be.

So you will see in proofs and examples later, that we'll sometimes work with the 1-norm and sometimes with the ∞-norm, whichever is easy for us. However, the 2-norm and the Frobenius norm are preferred in many cases and they have the nicest (most convenient) expression in terms of singular values.