LPCO Problem Set 4

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Problem 1. An $m \times n$ matrix A is said to be doubly stochastic if all entries are non-negative and if each row sum and column sum is 1. Show that for a doubly stochastic matrix, $m = n$. Show that every doubly stochastic matrix can be written as a convex combination of permutation matrices. (Hint: Look at the bipartite matching polytope)

Problem 2. Recall that for any graph G, the matching polytope is given by:

$$\sum_{e \ v} x_e \leq 1, \ \forall v \in V$$
$$\sum_{e=(u,v),u,v \in U} x_e \leq \lfloor \tfrac{1}{2}|U| \rfloor, \ \forall \, U \subset V, \ |U| \text{ odd}$$
$$x_e \geq 0, \ \forall e \in E$$

Write the dual LP for finding a max weight matching using the variables y_v for $v \in V$ and z_U for $U \subset V$ with $|U|$ odd. Show that if the weight function is integral, then we can take y and z to be integral as well[1]. Further, we can also satisfy that the set $\{U \mid z_U > 0\}$ is laminar[2].

Problem 3. We try to find optimal strategies in a simple two-player card game. Suppose the game is played over m rounds, and each round has an associated payoff $w_i \geq 0$. Assume wlog that $w_1 \geq w_2 \geq \ldots \geq w_m$. Player 1 has a set $F = \{f_1, f_2, \ldots, f_m\}$ of cards that she has drawn, and player 2 has a set $G = \{g_1, g_2, \ldots, g_m\}$ of cards that he has drawn. There is a total order on all the cards $(F \cup G)$ that says which card wins against which card. A pure strategy for player 1 is a function $p : [m] \to F$ that says which card she plays in which round. Similarly, a pure strategy for player 2 is a function $q : [m] \to G$ which says that he plays his card $q(i)$ in round i. Given a pair of pure strategies (p, q) of both players, the payoff is given as follows:

In each round i, player 1 plays her card $p(i)$ and player 2 plays his card $q(i)$. If $p(i) > q(i)$, then player 1 receives payoff w_i from this round. Else if $q(i) > p(i)$, then player 2 receives payoff w_i. The total payoff for each player is the sum of payoffs over all the rounds. Suppose player 1 knows the strategy q of player 2 beforehand. Give an algorithm for player 1 to find her optimal strategy to obtain the highest payoff possible.

Instead of only pure strategies, each player can also have a probability (p_{ix}) of playing card x in round i. Note that the matrix $P = (p_{ix})$ is doubly stochastic since each card can only be used in one round, and only one card can be used in each round. If both players are allowed to use impure/mixed strategies, then give an LP that returns the optimal payoff for player 1. If the LP has exponentially many constraints, then also give a polynomial time separation oracle for it.

Problem 4. Obtain a 2-approximation algorithm for the minimum multicut problem on trees. Let $G = (V, E)$ be a graph with non-negative capacities $c_e \geq 0$ on each edge. Let $\{(s_1, t_1), \ldots, (s_k t_k)\}$ be a specified set of pairs of vertices where $s_i \neq t_i$ for all $1 \leq i \leq k$. A *multicut* is a set of edges whose removal separates each pair (s_i, t_i). The minimum multicut problem asks for a minimum cost multicut, and we ask for a 2-approximation to the minimum multicut problem when the graph is a tree.

[1] This shows that the matching polytope is totally dual integral.

[2] A collection of sets is said to be laminar if for every pair of sets U, W in the collection, either $U \cap W = \emptyset$ or one of them is contained in the other.

Problem 5. Give an f-approximation for the set cover problem. Given a ground set of elements $E = \{e_1, e_2, \dots, e_n\}$, some subsets of those elements $S_1, S_2, \dots, S_m \subseteq E$, and a non-negative weight w_j for each subset S_j, the set cover problem asks for a minimum-weight collection of subsets that covers the entirety of E: Find an $I \subseteq \{1, 2, \dots, m\}$ with minimum $\sum_{j \in I} w_j$ satisfying $\cup_{j \in I} S_j = E$. We define f as the maximum number of subsets that a single element e belongs to. Concretely, $f = \max_i |\{j : e_i \in S_j\}|$.