Problem Set 2

Ds Analysis

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Question 1

$$a_n = b_n + b_{n+1}$$

$$a_1 + a_2 + a_3 + ...a_n = b_1 - b_2 + b_2 - b_3 + b_3 - b_4... - b_n$$

$$\sum_{n=1}^{\infty} a_n = b_1 - b_\infty$$

$$\sum_{n=1}^{\infty} a_n = b_1 - \lim_{n\to\infty} b_n$$

Let the sequence b_n be divergent. Let b_n = n. Then $\lim_{n\to\infty} b_n = \infty$. Thus $\sum_{n=1}^{\infty} a_n = b_1 - \infty$. Hence $\sum_{n=1}^{\infty} a_n$ is divergent.

Conversely, let $\sum_{n=1}^{\infty} a_n$ be divergent. Then $b_1 - \sum_{n=1}^{\infty} a_n$ is divergent. Thus b_n is divergent.

Part 1

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1})$$

We take $b_n = \frac{1}{n}$. Then by the previous series convergence, $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges to $\frac{1}{1} - 0$ Thus the series converges to 1.

Part 2

$$\sum_{n=1}^{\infty} \log \frac{n}{n+1} = \sum_{n=1}^{\infty} (\log n - \log n + 1)$$

We take $b_n = \log n$. By the previous argument for convergence, since b_n does not converge, thus the series $\sum_{n=1}^{\infty} \log \frac{n}{n+1}$ does not converge.

Question 2

Notice $0 < a < 10 \Rightarrow \frac{a}{10} < 1 \Rightarrow$ The given series $\sum_{n=1}^{\infty} \frac{a^n}{10^n}$ is a geometric series . A simple ratio test would show

$$r = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$$

$$= \lim_{n\to\infty} \frac{a^{n+1}}{10^{n+1}} \times \frac{10^n}{a^n}$$

$$= \lim_{n\to\infty} \frac{a}{10}$$

$$= \frac{a}{10} < 1$$

Hence the series converges.

Question 3

$$\sum_{n=1}^{\infty} \frac{n!}{(n+2)!} = \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

Comparison test using $b_n = \frac{1}{n^2}$
Since, for all n, $0 < \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, thus by comparison test, $\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$ converges.

Question 4

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+5)}}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \frac{5}{\sqrt{n(n+5)}}$$

$$= \frac{1}{5} \sum_{n=1}^{\infty} \frac{n+5-n}{\sqrt{n(n+5)}}$$

$$= \frac{1}{5} (\sum_{n=1}^{\infty} \sqrt{n} - \sqrt{n+5})$$

The seq $\sqrt{n}$ does not converge.So, The series does not converge.(From $if\ and\ only\ if$ statement given in your problem set 2.)
Alternately,use p-test and comparison test(page: 4) together.
(Hint:

$$n(n+5) < (n+5)^2$$

$$\Rightarrow \frac{1}{n+5} < \frac{1}{\sqrt{n(n+5)}}$$

$$\Rightarrow \sum \frac{1}{n+5} < \sum \frac{1}{\sqrt{n(n+5)}}$$

Question 5

$$a_n = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

Using the ratio test here,

$$r = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \times \frac{2^n}{n^2}$$

$$= \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{2}$$

$$= \frac{1}{2}$$

Since $r < 1$, by the ratio test, we have that the series a_n to be convergent

Question 6

$$\sum_{n=1}^{\infty} \frac{n\cos^2 \frac{n\pi}{3}}{2^n}$$

Use, ratio test. Definition of ratio test:

> **Corollary 7.5.3** (Ratio test)**.** *Let* $\sum_{n=m}^{\infty} a_n$ *be a series of* non-zero *numbers. (The non-zero hypothesis is required so that the ratios* $|a_{n+1}|/|a_n|$ *appearing below are well-defined.)*
>
> - *If* $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$*, then the series* $\sum_{n=m}^{\infty} a_n$ *is absolutely convergent (hence conditionally convergent).*
> - *If* $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$*, then the series* $\sum_{n=m}^{\infty} a_n$ *is not conditionally convergent (and thus cannot be absolutely convergent).*
> - *In the remaining cases, we cannot assert any conclusion.*

Can you use comparison test? (Hint : $2^{\frac{n}{2}} > n$ for $n > 4$)

Corollary 7.3.2 (Comparison test). *Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two formal series of real numbers, and suppose that $|a_n| \leq b_n$ for all $n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, and in fact*

$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Question 7

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

We use the integral test here,

$$\int_1^{\infty} n^2 e^{-n^3}\, dn$$

$$= \frac{1}{3e}$$

Since the integral converges, so does the series.

For more details on integral test may visit This website.

Question 8

Given series :

$$\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$$

Use root test.Root test:

Theorem 7.5.1 (Root test). *Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$.*

(*a*) *If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent (and hence conditionally convergent).*

(*b*) *If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is not conditionally convergent (and hence cannot be absolutely convergent either).*

(*c*) *If $\alpha = 1$, we cannot assert any conclusion.*

Here

$$a_n = (\frac{n}{n+1})^{n^2}$$

$$\Rightarrow \lim sup_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}|(\frac{n}{n+1})^{n^2}|^{1/n} = \frac{1}{e} < 1$$

Question 9

Use ratio test.

$$a_n = \sum_{n=1}^{\infty} \frac{n^n}{n!3^n}$$

$$\lim sup_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}$$

$$= \lim_{n\to\infty}\frac{1}{3}(\frac{n+1}{n})^n = \frac{e}{3} < 1$$