Problem set 4

DS ANALYSIS

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Problem 1

$$\begin{aligned} f(x) &= x^2 - 2x + 1 \\ &= (x-1)^2 \end{aligned}$$

Differentiate the function w.r.t. x to find the slope at each point. Positive slope implies function is increasing, negative slope implies function is decreasing. 0 slope implies critical point.
The function is increasing in $x \in [1, \infty)$
The function is decreasing in $x \in [-\infty, 1]$
The minimum is at $x = 1$

Problem 2

$$f(x) = -x^3 + 3x - 5$$

Differentiating w.r.t x we get:

$$f'(x) = -3x^2 + 3$$

$$\begin{aligned} f'(x) &> 0 \ \forall x \in (-1, 1) \\ &< 0 \ \forall x \in \mathbb{R} \backslash (-1, 1) \end{aligned}$$

$\Rightarrow f$ is increasing in (-1,1) and decreasing everywhere else.

$$f'(x) = 0 \Rightarrow x = \pm 1$$

So, -1 and 1 are the critical pts of f

$$f''(x) = -6x \ \ < 0 \ for \ x = 1$$

$$> 0 \ for \ x = -1$$

$\Rightarrow f(x)$ has a local minima at $x = -1$ and local maxima at $x = 1$

Problem 3

$$f(x) = (x-4)^5$$

Following the same method as question 1, the slope is increasing for all $x \in \mathbf{R}$
Thus, minimum at $-\infty$
Maximum at $+\infty$
Critical point at x = 4

Problem 4

$$f(x) = 2\sin x + 3\cos x,\ x \in [-2\pi, 2\pi]$$
$$f'(x) = 2\cos x - 3\sin x = 0$$
$$\Rightarrow \tan x = \frac{2}{3}$$
$$\Rightarrow x = n\pi + \tan^{-1}\frac{2}{3}$$

n can take values -2,-1,0,1 as $0 < \tan^{-1}\frac{2}{3} < \frac{\pi}{2} \Rightarrow$ The critical pts of $f(x)$ are $-2\pi + \tan^{-1}\frac{2}{3}$,$-\pi + \tan^{-1}\frac{2}{3}$,$\tan^{-1}\frac{2}{3}$,$\pi + \tan^{-1}\frac{2}{3}$

$$0 < \tan^{-1}\frac{2}{3} < \frac{\pi}{2}$$
$$\Rightarrow -2\pi < -2\pi + \tan^{-1}\frac{2}{3} < -\frac{3\pi}{2}$$
$$\Rightarrow -\pi < -\pi + \tan^{-1}\frac{2}{3} < -\frac{\pi}{2}$$
$$\Rightarrow \pi < \pi + \tan^{-1}\frac{2}{3} < \frac{3\pi}{2}$$

Notice,in the interval,$(-2\pi, -\frac{3\pi}{2})$ $\sin x > 0, \cos x > 0$.
in the interval,$(-\pi, -\frac{\pi}{2})$ $\sin x < 0, \cos x < 0$
in the interval,$(0, \frac{\pi}{2})$ $\sin x > 0, \cos x > 0$
in the interval,$(\pi, \frac{3\pi}{2})$ $\sin x < 0, \cos x < 0$

$$f''(x) = -2\sin x - 3\cos x < 0 \ \textit{at}\ x = -2\pi + \tan^{-1}\frac{2}{3}\ \textit{so local maxima.}$$

similarly , calculate at all 4 pts.

Problem 5

$$f(x) = x + sin(x)$$

Differentiating w.r.t. x,

$$f'(x) = 1 + cos(x)$$

Now, for all $x \in \mathbf{R}$
$1 + cos(x) \geq 0$ since $-1 \leq cos(x) \leq 1$
Thus, the function is increasing for all $x \in \mathbf{R}$
Thus, minimum at $-\infty$
Maximum at $+\infty$
Critical points at $x = \pi, 3\pi, ...$

Problem 7

$$f(x) = \frac{x^2}{\sqrt{x+1}}$$

Increasing from $x \in (0, \infty)$
Decreasing from $x \in (-1, 0)$
Minimum at 0

Problem 8

$$f(x) = \frac{1}{(x-1)(x-3)} \quad 0 \geq x \geq 10$$

Notice,

$$\lim_{x \to 1^-} f(x) = \infty$$

$$\lim_{x \to 1^+} f(x) = -\infty$$

$$\lim_{x \to 3^-} f(x) = -\infty$$

$$\lim_{x \to 3^+} f(x) = \infty$$

$f(x)$ has discontinuity at $x = 1, 3$ Check for critical pts at 3 separate intervals,i.e [0,1),(1,3),(3,10].

Problem 9

Given $S > 0$.
$x + y = S$, to find minimum $x^2 + y^2$
Let $f(x, y) = x^2 + y^2$. Take $y = S - x$.
To find minimum of the function $f(x) = x^2 + (S - x)^2$
Differentiating the function w.r.t. x, we get,

$$f'(x) = 4x - 2S$$

Equating to 0, we get $x = \frac{S}{2}$
(Note that, if we differentiate $f'(x)$ w.r.t. x, we get $f''(x) > 0$. Thus the function has a minimum and no maximum).
Taking $x = \frac{S}{2}$, we have $y = \frac{S}{2}$. Thus, $f(x, y)$ is smallest when $x = y$.

Problem 10

Notice, It is mentioned that the rectangle is of given area.Let that area be A. By definition A is constant.

Let, the smaller side be x.$\Rightarrow$the longer side is $\frac{A}{x}$.

Perimeter(S)=$2x + 2\frac{A}{x}$.

Have to minimize $f(x) = 2x + 2\frac{A}{x}$

$$f'(x) = 2 - 2\frac{A}{x^2} = 0$$

$$\Rightarrow x = \pm\sqrt{A}$$

$$f''(x) = +4\frac{A}{x^3} > 0 \ for x = \sqrt{A}$$

$f(x)$ is minimum at $x = \sqrt{A}$.$\Rightarrow$All the sides are of length $\sqrt{A}$

The perimeter is minimum for square.

Problem 11

Cost of producing x units =

$$f(x) = 10x^2 + 200x + 6000$$

Price per unit for x units sold =

$$p(x) = 1000 - 10x$$

To maximize the profit, we need to maximize the function

$$g(x) = xp(x) - f(x)$$

$$= -20x^2 + 800x - 6000$$

We find the critical point at x = 20 and the maximum profit to be 2000.