25/02/2020

MSc. Data Science

Linear Algebra and its applications - Midterm

SOLUTIONS

Time allowed: 3 hours | Total: 100 points

Instructions:

- Show all your work; writing only the answer will not carry any points.
- Any instances of copying will be severely dealt with.
- There is no partial marking for bonus points; you get all or none.

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1. (5 points) Let $A = U\Sigma V^*$ be the SVD of an $m \times n$ matrix A of rank r.
Define $\hat{A} = V\hat{\Sigma}U^*$, where $\hat{\Sigma} = diag(1/\sigma_1, \dots, 1/\sigma_r, 0, \dots, 0)$ ($\hat{A}$ is called the pseudoinverse of A). Prove that if A is invertible, then $\hat{A}$ is the inverse of A.
Answer: *A invertible implies A must be square and A must be full rank. So $m = n = r$. Further,*
$$A^{-1} = (U\Sigma V^*)^{-1} = V^{*-1}\Sigma^{-1}U^{-1} = V\hat{\Sigma}U^* = \hat{A},$$
since U and V are unitary and $\Sigma^{-1} = diag(1/\sigma_1, \dots, 1/\sigma_r)$ is the inverse of $\Sigma = diag(\sigma_1, \dots, \sigma_r)$.

2. (25 points) Consider the matrix $A = \begin{pmatrix} -2 & 11 \\ -10 & 5 \end{pmatrix}$.
   (a) Find an SVD $A = U\Sigma V^*$ of A.
   (b) List the right singular vectors and left singular vectors of A.
   (c) Calculate the pseudoinverse $\hat{A}$.
   (d) What are the 1, 2, ∞ and Frobenius norms of A?
   (e) Draw a labelled picture of the unit ball in $\mathbb{R}^2$ and its image under A. What are the images of the unit vectors $[-1, 0]$ and $[0, -1]$?

   Answer:

   (a)
$$\begin{pmatrix} -2 & 11 \\ -10 & 5 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}$$

   (b) The right singular vectors are the columns of V and left singular vectors of A are the columns of U.

   (c) Since A is invertible, the pseudoinverse of A is same as the inverse of A, which is:
$$\frac{1}{100}\begin{pmatrix} 5 & -11 \\ 10 & -2 \end{pmatrix}$$

   (d)
$$||A||_1 = 16; \quad ||A||_2 = 10\sqrt{2}; \quad ||A||_\infty = 15; \quad ||A||_F = 5\sqrt{10}.$$

(e) *The image of the unit ball in* $\mathbb{R}^2$ *under* A *is an ellipse with the length of the semi-major axis being of* $10\sqrt{2}$ *and that of the semi-minor axis being* $5\sqrt{2}$.

3. (10 points) Consider the following system of equations:

$$\begin{aligned} x + y &= 2 \\ x + 1.0001y &= 2 \end{aligned}$$

   (i) Solve the linear system $Ax = b$ associated to the above system of equations.

   (ii) Solve the linear system $Ax = b'$, where $b' = (2,\ 2.0001)^t$

   (iii) Explain the variation in the solutions obtained in (i) and (ii) using $\kappa(A)$. Use the 1-norm for your calculations.

   Answer:

   (i) $x = 0, y = 2$.

   (ii) $x = y = 1$.

   (iii) Note that $A^{-1} = \frac{1}{10^{-4}} \begin{pmatrix} 1.0001 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 10001 & -10000 \\ -10000 & 10000 \end{pmatrix}$, *so that*

$$\kappa_1(A) = ||A||_1 ||A^{-1}||_1 = (2.0001)(20001) \approx 40002.$$

   The fact that the condition number of A *is so large explains the large variation in the solutions even when there is such little perturbation in the data* b.

4. (25 points) Let $A \in \mathbb{C}^{m \times m}$ (unless otherwise indicated). For each of the following statements, prove that it is true or give an example to show it is false.

   (a) The rank of a square matrix A is the number of non-zero singular values of A.

   Answer: *TRUE.*
   Let $A = U\Sigma V^*$, *then* $AV = U\Sigma$ *i.e.* $Av_i = \sigma_i u_i$. *The right singular vectors* $\{v_i\}$ *form an orthonormal basis of the domain, hence the range space of* A *is spanned by* $\{Av_i\} = \{\sigma_i u_i\}$ *where* $\sigma_i \neq 0$. *Hence rank*$(A) =$ *dimension of the range space of* $A =$ *number of non-zero singular values of* A.

   (b) If all the eigenvalues of A are zero, then $A = 0$.

   Answer: *FALSE.*
   Eg. the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *is a nonzero matrix having all eigenvalues zero. Nilpotent matrices have all eigenvalues zero.*

   (c) If A is hermitian and λ is an eigenvalue of A, then $|\lambda|$ is a singular value of A.

   Answer: *TRUE.*
   Let $Av = \lambda v$, *then* $A^*Av = A^2 v = \lambda^2 v$ *implies* λ^2 *is an eigenvalue of* A^*A, *i.e.* $|\lambda|$ *is a singular value of* A.

(d) If A is diagonalizable and all its eigenvalues are equal, then A is diagonal.

Answer: *TRUE.*
Suppose $A = P^{-1}\Lambda P$ *is the eigenvalue decomposition of* A. *If all eigenvalues of* A *are equal (to* λ*, say) then* $\Lambda = \lambda I$ *so that* $A = P^{-1}\Lambda P = P^{-1}\lambda I P = \lambda P^{-1} I P = \lambda I$. *Hence* A *is diagonal.*

(e) Two square matrices A and B are *unitarily equivalent* (i.e. there exists a unitary matrix Q such that $A = QBQ^*$) iff they have the same singular values.

Answer: *FALSE.*
Let $A = U_1\Sigma_1 V_1^*$ *and* $B = U_2\Sigma_2 V_2^*$ *be SVDs.*

$$\begin{aligned} A = QBQ^* &\implies A = Q(U_2\Sigma_2 V_2^*)Q^* \\ &\implies A = U\Sigma_2 V^* \quad \text{where } U = QU_2,\ V = QV_2 \\ &\implies U_1\Sigma_1 V_1^* = U\Sigma_2 V^* \\ &\implies \Sigma_1 = \Sigma_2. \end{aligned}$$

However, the converse is not true. Consider the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *They have the same singular values, namely* $1, 1$ *(check!), but they are not unitarily equivalent. (For if they were unitarily equivalent, the eigenspaces of* A *and* B *corresponding to the same eigenvalue (i.e.* 1*) would have to have the same dimension; but* $\dim\ker(A - 1\cdot I) = 2$*, while* $\dim\ker(B - 1\cdot I) = 1$*.)*

(f) A zero pivot is encountered while applying Gaussian elimination without pivoting to A. So A must be singular.

Answer: *FALSE.*
The matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ *has the first pivot as zero, but is invertible since* $\det A = -1 \neq 0$.

5. (20 points)

(a) Let A be a $m \times n$ matrix and B be a $n \times k$ matrix. Write a pseudocode for the matrix multiplication $C = AB$. What is the operation count for the matrix multiplication using your code?

(b) Write a pseudocode for back substitution, i.e. solving the system of equations $Ux = y$, where U is a $m \times m$ upper triangular matrix. What is the operation count?

(c) Let A be an invertible matrix. In solving the linear system $A^2u = b$, is it more advantageous (in terms of flops) to calculate the matrix A^2 or to solve the two linear systems $Av = b$ and $Au = v$ in turn making use of the same LU factorization?

Answer:

(a) Let $A = (a_{ij}), B = (b_{ij}), C = (c_{ij})$.
Pseudocode:

for *i=1 to m* **do**
  for *j=1 to k* **do**
    $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$
  end
end

Operation count:
Number of flops required to calculate one entry of the product matrix $= 2n-1$ *(*n *multiplications and* $n-1$ *additions).*
Number of entries in the product matrix $= m \times k$.
Therefore, total number of flops required to multiply a $m \times n$ *matrix with a* $n \times k$ *matrix equals*

$$m \times k \times (2n-1).$$

(b) *Suppose*

$$\begin{pmatrix} u_{11} & u_{12} & \dots & u_{1m} \\ 0 & u_{22} & \dots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{mm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

Back Substitution:

$$\begin{aligned} x_m &= \frac{y_m}{u_{mm}} \\ x_{m-1} &= \frac{y_{m-1} - u_{m-1,m}x_m}{u_{m-1,m-1}} \\ &\vdots \\ x_j &= \frac{y_j - \sum_{k=j+1}^{m} u_{jk}x_k}{u_{jj}} \\ &\vdots \\ x_1 &= \frac{y_1 - \sum_{k=2}^{m} u_{1k}x_k}{u_{11}} \end{aligned}$$

Pseudocode:

for *i=0 to m-1* **do**

  j=m-i

  $x_j = \left(y_j - \sum_{k=j+1}^{m} u_{jk}x_k \right) / u_{jj}$

end

Operation count: The i-th step in the loop requires $(1 + 2(m-i))$ *floating point operations. So total number of flops required equals-*

$$\sum_{i=1}^{m}(1+2(m-i)) = m + 2\sum_{i=1}^{m}(m-i) = m + 2\sum_{i=0}^{m-1} i = m + \frac{2(m-1)(m)}{2} = m^2.$$

(c) *Solving* $A^2u = b$ *would involve:*

i. *calculating* A^2: $n^2(2n-1) = 2n^3 - n^2$ *flops;*

ii. *LU factorization for* A^2: $\approx \frac{2n^3}{3}$ *flops;*

iii. *solving* $(LU)u = b$: *involves solving* $Lx = b$ *for* x *and then* $Uu = x$ *for* u*, so a total of* $2n^2$ *flops.*

Thus the operation count for this method would be $\approx (2n^3 - n^2 + \frac{2n^3}{3} + 2n^2) = \frac{8n^3}{3} + n^2$ *flops.*

On the other hand, the second method would involve:

i. *LU factorization of A:* $\approx \frac{2n^3}{3}$ *flops;*

ii. *2 forward and 2 back substitutions:* $\approx 4n^2$ *flops.*

So the operation count would be $\approx \frac{2n^3}{3} + 4n^2$ *flops, which is definitely cheaper than the first method.*

6. (5 points) Consider an algorithm for computing the full SVD $U\Sigma V^*$ of a given matrix A. Let $\tilde{U}$, $\tilde{\Sigma}$ and $\tilde{V}$ denote the computed matrices. Explain what it would mean for this algorithm to be:

   (a) accurate;

   (b) forward stable;

   (c) backward stable.

   Answer: *The algorithm* $\tilde{f}$ *takes data* A *and produces 3 matrices* $\tilde{U}, \tilde{V}$ *and* $\tilde{\Sigma}$.
   Let $\tilde{A}$ *denote the product* $\tilde{U}\tilde{\Sigma}\tilde{V}^*$.

   (a) The algorithm would be said to be **accurate** *if*

   $$\frac{||\tilde{A} - A||}{||A||} = O(\epsilon_{mach}).$$

   (b) The algorithm would be said to be **(forward) stable** *if*

   $$\frac{||\tilde{A} - A'||}{||A'||} = O(\epsilon_{mach}),$$

   for some $A' = A + \delta A$ *such that* $\frac{||\delta A||}{||A||} = O(\epsilon_{mach})$.

   (c) The algorithm would be said to be **backward stable** *if*

   $$\tilde{A} = A',$$

   for some $A' = A + \delta A$ *such that* $\frac{||\delta A||}{||A||} = O(\epsilon_{mach})$.

7. (5 points) Let $A = \begin{pmatrix} 7 & 6 \\ 9 & 8 \end{pmatrix}$.

   (a) Calculate the LU factorization of A in exact arithmetic.

   (b) Calculate the LU factorization of A in base 10, precision 2, floating point arithmetic. Let these computed matrices be denoted by $\tilde{L}$, $\tilde{U}$.

   (c) (5 points bonus) Write $\tilde{L}\tilde{U}$ as $A + \delta A$. What can you say about $\frac{||\delta A||}{||L||\cdot||U||}$? Can you draw conclusions about the stability of LU factorization in this case?

   Answer:

   (a) In exact arithmetic:

   $$\begin{pmatrix} 7 & 6 \\ 9 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 9/7 & 1 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 0 & 2/7 \end{pmatrix}.$$

(b) In base 10*, precision* 2 *floating point arithmetic:*

$$\begin{pmatrix} 7 & 6 \\ 9 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1.2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 0 & 2.8 \times 10^{-1} \end{pmatrix}.$$

(c) $\tilde{L}\tilde{U} = \begin{pmatrix} 7 & 6 \\ 8.4 & 7.4 \end{pmatrix} = A + \delta A$*, where* $\delta A = \begin{pmatrix} 0 & 0 \\ -0.6 & -0.6 \end{pmatrix}$*. So* $||\delta A||_1 = 0.6$*.*
Now $||L||_1 = 16/7, ||U||_1 = 7$*, so that* $||L||_1||U||_1 = 16$*, so*

$$\frac{||\delta A||}{||L|| \cdot ||U||} = \frac{0.6}{16} = 0.0375 < \frac{1}{20} = \epsilon_{mach},$$

which is as expected. The growth factor $\rho = \frac{max|u_{ij}|}{max|a_{ij}|} = \frac{7}{9} < 1$*, so stability is expected.*