Mathematical Methods Analysis

Mid Sem.

1. (i) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

We need to check that if the series is convergent or divergent.

Applying ratio test :- If $\sum a_n$ be a series with nonnegative terms,
then if $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ < 1 then converges, > 1 diverges, $= 1$ inconclusive.

here $a_n = \frac{n!}{n^n}$.

So, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}\right]$

$$= \lim_{n\to\infty} \left[\frac{\cancel{(n+1)} \cdot n^n}{(n+1)^n \cdot \cancel{(n+1)}}\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{\left(1+\frac{1}{n}\right)^n}\right]$$

$$= \frac{1}{e} \qquad \left[\text{as } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.\right]$$

$$< 1$$

Hence the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent.

1.(ii) $f(x) = x|x| \qquad x \in (-1, 1)$

$$f(x) = \begin{cases} x^2 & x > 0 \\ -x^2 & x < 0. \end{cases}$$

So, we will check ; if

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x)$$

So, $\lim\limits_{x\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{x\to 0^+} f'(x)$ (Right hand derivative)

or, $\lim\limits_{x\to 0^+} f'(x) = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - x^2}{h}$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} (2x + h)$$

$= 2x = 0.$ [putting $x = 0$.]

$\lim\limits_{x\to 0^-} f'(x) = \lim\limits_{h\to 0} \dfrac{f(x-h) - f(x)}{h}$ (Left hand derivative)

$$= \lim_{h\to 0} \frac{(x-h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} - 2xh + h^2 - \cancel{x^2}}{h}$$

$= \lim\limits_{h\to 0} (-2x + h) = -2x = 0$ (putting $x=0$)

So, $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^-} f'(x)$

Hence the function $f(x)$ is differentiable function on $(-1, 1)$..

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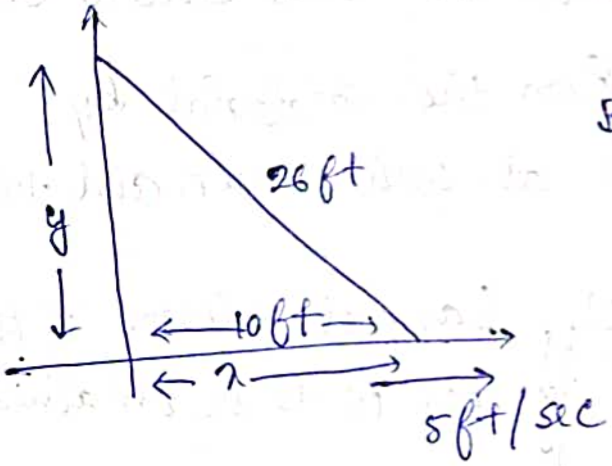
2. A ladder 26 ft. long leans against a vertical wall. The lower end (of the ladder) is being moved away from the wall at a rate of 5 ft./sec.

~~Let~~, $\frac{dx}{dt} = 5$ ft/sec.



By Pithagoras theorem;

$$x^2 + y^2 = 26^2$$

$$y = \sqrt{26^2 - x^2}$$

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$2x \cdot \frac{dx}{dt} + 2 \cdot \sqrt{26 - x^2} \cdot \frac{dy}{dt} = 0$$

or, $$\frac{dy}{dt} = -\frac{x}{\sqrt{26 - x^2}} \cdot \frac{dx}{dt}$$

So, $$\left. \frac{dy}{dt} \right|_{x=10} = -\frac{10}{\sqrt{26^2 - 10^2}} = \frac{-10 \times 5}{24} = -\frac{25}{12} \text{ ft/sec}$$

$$= -2.083 \text{ ft/sec}$$

-ve sign indicates the top of the ladder is coming down the wall and thereby decreasing the y length.

so, the hight of the top end from ground is changing at 2·083 ft/sec when $x = 10$.

(Ans)

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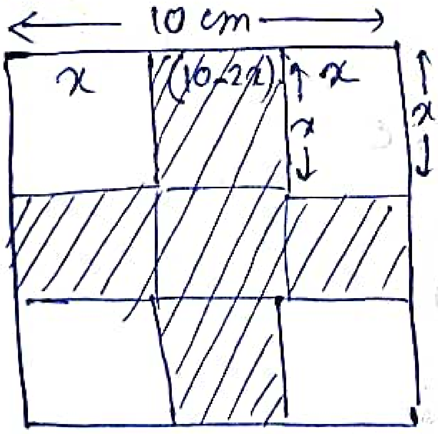
3. A square piece of a material has side 10 cm. An open box is made from the material by removing equal squares at each corner. and turning up the sides.



largest volume of the box is to be obtained.

Let the squares of x side is removed from the piece of metal.
then the hight of the box is x and the length and bredth is $(10-2x)$ each.

Hence the volume is $V = (10-2x)^2 \cdot x$.

So, we need maximum volume.

By differentiating V w.r.t x,

$$\frac{dV}{dx} = \frac{d}{dx}(10-2x)^2 \cdot x.$$

$$= 2(10-2x)\cdot(-2)\cdot x + (10-2x)^2$$

$$= -4(10x-2x^2) + (10-2x)^2$$

$$= -40x + 8x^2 + 100 - 40x + 4x^2$$

$$= 12x^2 - 80x + 100.$$

So, $\frac{dV}{dx} = 0$.

or, $(10-2x)(10-2x-4x) = 0$.

or, $(10-2x)(10-\cancel{8}6x) = 0$

So, either, $2x = 10$
$x = 5$

or, $x = \frac{10}{6} = \frac{5}{3}$.

lets check the double derivative at the points $x = \frac{5}{3}, 5$.

$$\frac{d^2V}{dx^2} = 24x - 80.$$

$$\left.\frac{d^2v}{dx^2}\right|_{x=5} = (24\times5-80) = 40 > 0.$$

then at $x=5$ it attains minima

$$\left.\frac{d^2v}{dx^2}\right|_{x=5/3} = \left(24\times\frac{5}{3}-80\right) = -40 < 0$$

then at $x=5/3$ it attains maxima

So, The dimension of box is, hight $= 5/3$ cm

length $= \frac{20}{3}$ cm

bredth $= \frac{20}{3}$ cm.

(Answer)-

4. $\int e^{-4x} \cos(2x)\, dx = I$

or, $\cos(2x) \cdot \int e^{-4x}\, dx - \int\left[-\sin(2x) \cdot 2 \cdot \int e^{-4x}\, dx\right] dx = I$
(using integration by parts)

or, $\cos 2x \cdot \left(\frac{e^{-4x}}{-4}\right) + \int\left[2 \cdot \sin 2x \cdot \frac{e^{-4x}}{-4}\right] dx = I$

or, $-\frac{1}{4}\cos 2x \cdot e^{-4x} - \frac{1}{2}\int e^{-4x} \cdot \sin 2x\, dx = I$

or, $-\frac{1}{4}\cos 2x \cdot e^{-4x} - \frac{1}{2}\left[\sin 2x \cdot \frac{e^{-4x}}{-4} - \int \cos(2x) \cdot 2 \cdot \frac{e^{-4x}}{-4}\, dx\right] = I$
(integration by parts)

or, $-\frac{1}{4}\cos(2x) \cdot e^{-4x} + \frac{1}{8}\sin 2x \cdot e^{-4x} + \frac{1}{2}\int\left(-\frac{1}{2}\right) \cdot \cos 2x \cdot e^{-4x}\, dx = I$

or, $-\frac{1}{4}\cos(2x)\, e^{-4x} + \frac{1}{8}\sin 2x \cdot e^{-4x} - \frac{1}{4}\int \underbrace{\cos 2x \cdot e^{-4x}\, dx}_{= I} = I$

or, $I \cdot \left(1 + \frac{1}{4}\right) = \frac{1}{8}\sin 2x\, e^{-4x} - \frac{1}{4}\cos 2x \cdot e^{-4x} + c$

or, $I = \frac{4}{5}\left(\frac{1}{8}\sin 2x\, e^{-4x} - \frac{1}{4}\cos 2x \cdot e^{-4x}\right) + c$

$= \frac{1}{10}\sin 2x\, e^{-4x} - \frac{1}{5}\cos 2x\, e^{-4x} + c$

$= \frac{1}{10} e^{-4x}\left(\sin 2x - 2\cos 2x\right) + c$ (Answer).

where c is integration constant.