LOPT End Sem – 2021 24.07.2021

3.

min →

	y_1	y_2	y_3	y_4
x_1	3	-1	2	5
x_2	7	13	-2	12
x_3	-5	2	0	9

↑ Max

Max-min LP.

maximize x_0

Subject to,

$x_0 \leq 3x_1 + 7x_2 - 5.x_3.$

$x_0 \leq -1.x_1 + 13.x_2 + 2x_3.$

$x_0 \leq 2x_1 - 2.x_2 + 0.x_3.$

$x_0 \leq 5x_1 + 12x_2 + 9x_3.$

$x_1 + x_2 + x_3 = 1$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

Min-max LP.

minimize y_0

Subject to

$y_0 \geq 3y_1 - 1y_2 + 2y_3 + 5y_4$

$y_0 \geq 7y_1 + 13y_2 - 2y_3 + 12y_4$

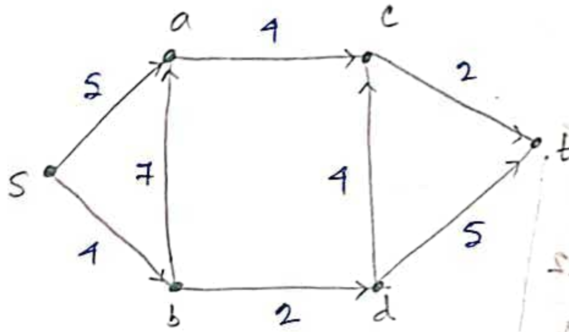
$y_0 \geq -5y_1 + 2y_0 + 0.y_3 + 9y_4.$

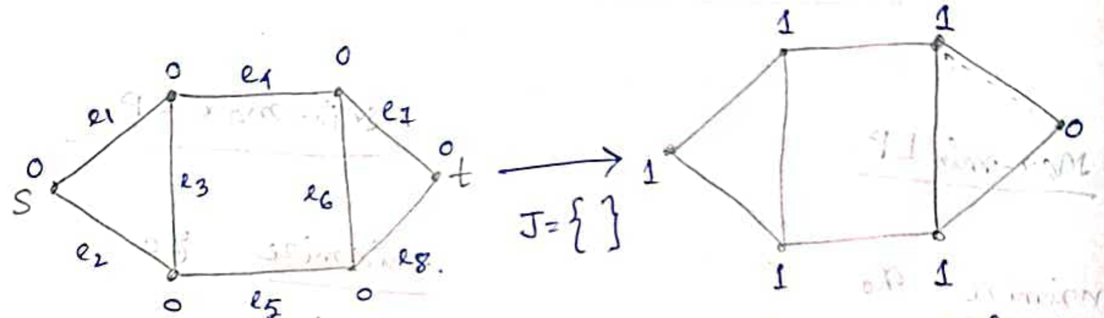
$y_1 + y_2 + y_3 + y_4 = 1$

$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0.$

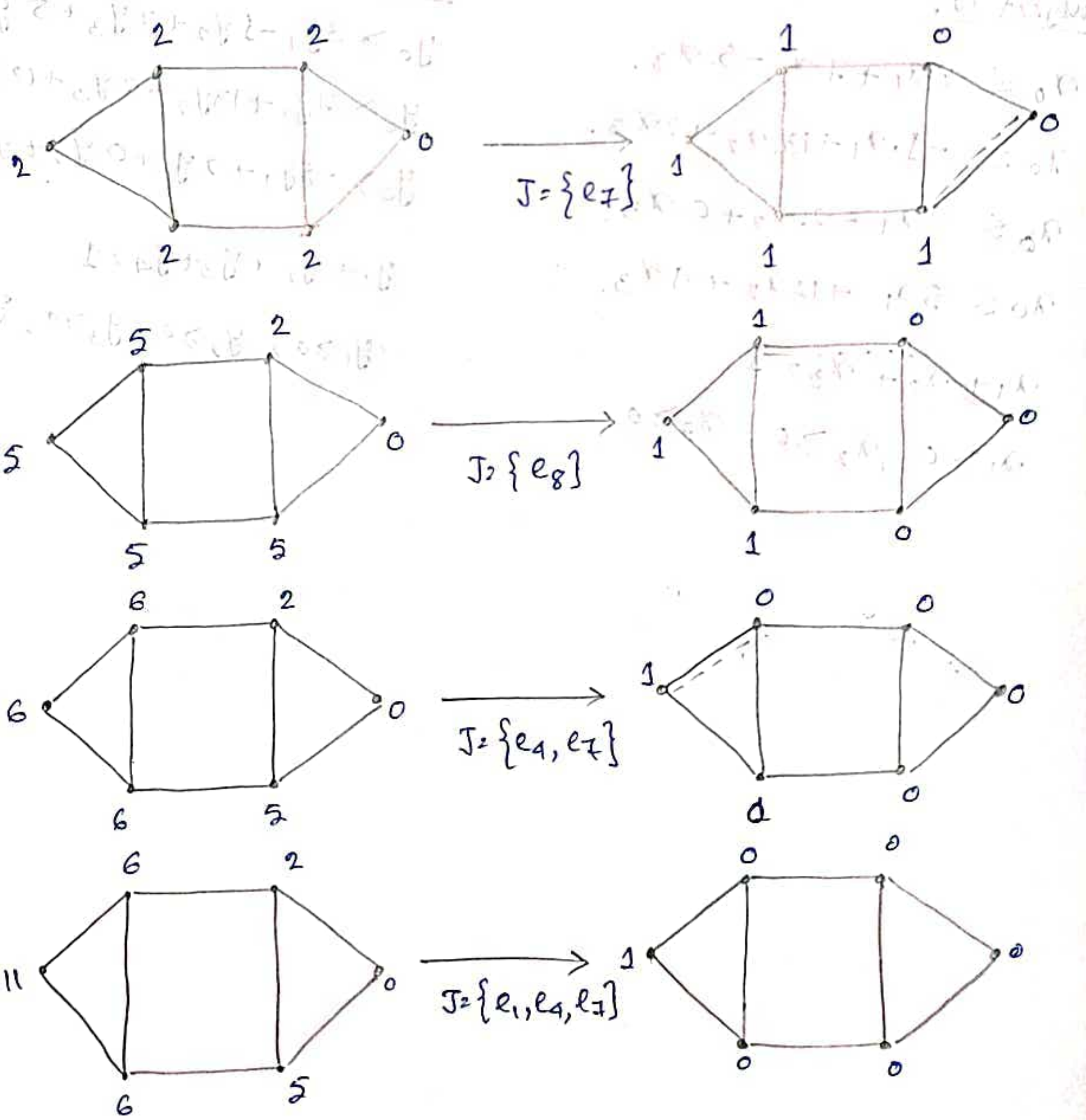
H.

4.

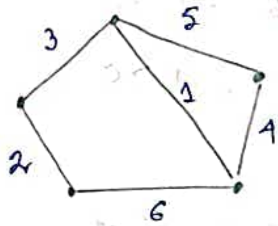




t is reachable from t.

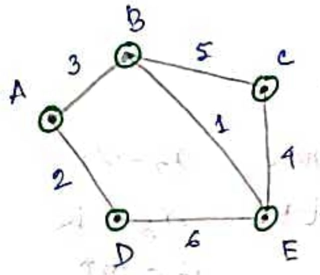


5.



$E_0 = \Phi$ $\quad G_0 = (V, E_0)$

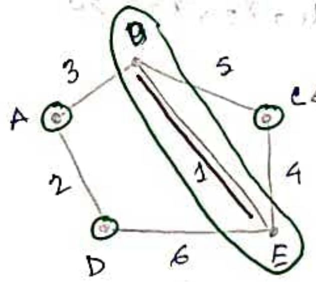
π_0



y

$E_1 = \{BE\}$

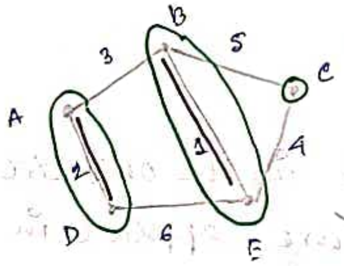
π_1



y^2

$E_2 = \{AD, BE\}$.

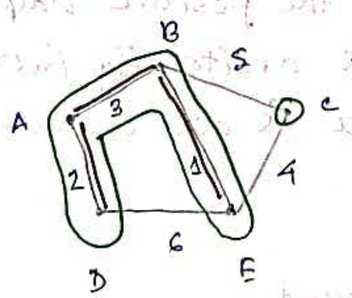
π_2



y^3

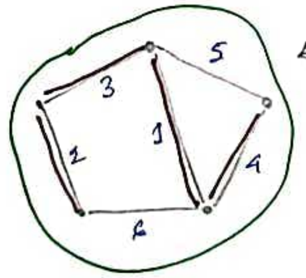
$E_3 = \{AB, AD, BE\}$.

π_3



$E_4 = \{AB, AD, BE, EC\}$.

π_4



$E_4 = \{AB, AD, BE, EC\}$

Minimum Spanning Tree

the minimum cost is 10.

2. • Maximize $c^T x$ subject to $Ax > b$, $x \geq 0$.

• Maximize $-c^T x$ subject to $Ax = b$, $x \geq 0$.

we have to give an example of A, c, b such that the above 2 LPs are unbounded.

So,

maximize $x_1 - x_2$ subject to $x_3 = k$, $k \in \mathbb{R}^+$, $x_1, x_2, x_3 \geq 0$.

maximize $x_2 - x_1$ subject to $x_3 = k$, $k \in \mathbb{R}^+$, $x_1, x_2, x_3 \geq 0$.

So, here $c^T = [1 \ -1 \ 0]$.

$-c^T = [-1 \ 1 \ 0]$.

and $A = [0 \ 0 \ 1]$

$b = [k]$. $k \in \mathbb{R}^+$.

So, here the only difference is in the objective function. But the constraints are applied in the 3rd dimension where 1st and 2nd dimensions are free to take any values in the positive half of the region. So, the cost is arbitrarily increasing and cost is unbounded. $-c^T x$ is also unbounded for the same reason.

Hence,

The above 2 LP's are unbounded.

1. primal

minimize: $c^T x$

subject to: $Ax = b$

$x \geq 0.$

Dual

maximize $b^T y$

subject to $A^T y \leq c$

y unrestricted.

x^* is the optimum of the primal and y^* is the optimum of the dual.

so, $Ax = b.$

$Ax^* = b.$

so, $A_1 x^* = b_1$

so, this will be feasible for any feasible solution x.

and $b^T y = b_1 y_1 + b_2 y_2 + \dots + b_m y_m.$

$= \cancel{b}.(A_1 x) y_1 + (A_2 x) y_2 + \dots + (A_m x) y_m.$

$= (A_1 y_1) x + (A_2 y_2) x + \dots + (A_m y_m) x.$

as y^* is optimum $b^T y^* \geq b^T y.$ for all feasible solution y.

so, $b^T y^* = (y_1^* A_1) x + (y_2^* A_2) x + \dots + (y_m^* A_m) x.$

$b^T y^* = \sum_{k=1}^{m} (y_k^* A_k) x.$

x^* is optimum for the primal. so, $c^T x^* \leq c^T x.$ x is any feasible sol.

So, $c^T x - y_k^* A_k x \geq c^T x^* - b^T y^*$

again. ~~min~~ $c^T x - y_k^* A_k x \geq c^T x_0^* - y_k^* A_k x^*$

which is also. $\geq c^T x^* - b^T y^*.$

so, $(c^T x - y_k^* A_k x^*)$ will also have a feasible optimum. such that solution x^* which will be

$A_i x = b_i, \quad i = 1, \dots, m \quad i \neq k$

$x \geq 0.$

6. Input :- A universe D consisting of finite number of elements ; and a family $S_1, S_2, \ldots, S_m$ of sets with $S_i \subseteq D$ and $|S_i| \leq 3$ for every $i \in \{1, \ldots, m\}$

O/P :- Find $W \subseteq D$ such that $W \cap S_i \neq \phi$ for every $i \in \{1, \ldots, m\}$

Consider a variable x_v for each element $v \in D$.

Since $W \cap S_i \neq \phi$, at least one element from S_i has to be selected.

Each S_i can contain at most 3 elements.

$S_i = \{v_i^1\}$ or, $\{v_i^1, v_i^2\}$ or, $\{v_i^1, v_i^2, v_i^3\}$

ILP of the problem :-

$$\text{minimize} \sum_{v \in D} x_v$$

Subject to $x_{v_i^1} + x_{v_i^2} + x_{v_i^3} \geq 1$ for each S_i $i \in \{1, \ldots, m\}$

$x_v \in \{0, 1\}$ for every $v \in D$.

Relaxing the ILP to LP :-

$$\text{minimize} \sum_{v \in D} x_v$$

subject to $x_{v_i^1} + x_{v_i^2} + x_{v_i^3} \geq 1$ for each S_i $i \in \{1, \ldots, m\}$

$0 \leq x_v \leq 1$ ~~$x_v \in \{0,1\}$~~ for every $v \in D$.

Writing the dual of the LP :-

$$\text{maximize} \sum_{\substack{\cancel{v \in D} \\ i \text{ s.t } v \in S_i}} y_i$$

Subject to $\forall i \{1, 2, \ldots, m\}$.

$$\sum_{\substack{i \text{ s.t} \\ v \in S_i}} y_i \leq 1$$

$y_i \geq 0$.

Here in the primal we will remove $x_v \le 1$. constraint for the primal dual algorithm. It will be still a relaxed LP and it will contain all the feasible solutions of the ILP.

Primal Dual Algorithm :-

Primal

minimize $\sum_{v \in D} x_v$.

Subj to :- $x_{v_i^1} + x_{v_i^2} + x_{v_i^3} \geq 1 \quad \forall i \in \{1, \ldots, m\}$.

$x_v \geq 0 \quad \forall v \in D$.

Dual

maximize $\sum_{\substack{i \text{ s.t} \\ v \in S_i}} y_i$

Subj to :- $\sum_{\substack{i \text{ s.t} \\ v \in S_i \\ W \cap S_i \neq \phi}} y_i \leq 1 \quad \forall i \in \{1, \ldots, m\}$.

$y_i \geq 0$.

1. Initialise all y_i to 0.

2. Iterative step:- We have a feasible solution y. Let $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ be the sets which are tight for y. ~~So~~ As, we get a ~~D ⊆~~ $W \subseteq D$ such that ~~D can~~ $S_i \cap W \neq \phi \quad \forall i \in \{1, \ldots, k\}$.

- Now we will pick a vertex $v \notin$ ~~$S_{i_1} \cup S_{i_2} \cup \ldots \cup S_{i_k}$~~ $\{S_i \cap W\} \; \forall i \in \{1, \ldots, k\}$ and increase y_v until some dual constraint becomes tight i.e $y_i = 1$.
  We will include that vertex in W. in the next step.

3. Termination :- When all the S_i's are exhausted. like we have found atleast one element from the intersection of S_i and W for all $i \in \{1, \ldots, m\}$.

The Algorithm terminates with $S_{i_1}, S_{i_2}, \dots, S_{i_k}$ being tight for dual.

This gives a primal solution $x_v = 1$ ~~if i∈~~ if $v \in W$.
if $W \cap S_i \neq \phi$.

$= 0$ otherwise.
~~if $v \in W$~~.

cost of primal :-

$$\sum_{v \in D} x_v = \sum_{v \in D} \sum_{\substack{i \text{ s.t } \\ v \in S_i \\ W \cap S_i \neq \phi}} y_i$$

~~Suppose~~ The cost of dual $= \sum_{\substack{i \text{ s.t } \\ v \in S_i \\ W \cap S_i \neq \phi}} y_i$

the intersection of W and S_i is at most 3.

so, $$\sum_{v \in D} \sum_{\substack{i \text{ s.t } \\ v \in S_i \\ W \cap S_i \neq \phi}} y_i \leq 3 \sum_{\substack{i \text{ s.t } \\ v \in S_i \\ W \cap S_i \neq \phi}} y_i$$

so, cost of primal $\leq 3 \times$ (cost of dual)

this shows that this primal dual algorithm gives an 3 approximation.