Gaussian elimination

Note Title 16-04-2021

Let A be a square $m \times m$ matrix.

Idea - to transform A into an upper triangular matrix by introducing zeroes below the diagonal.

Suppose $A = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & \cdots & a_m \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \end{array}$ such that A is invertible & $a_{ii} \neq 0 \ \forall i$.

Step 1 : To annihilate $a_{21}, \dots, a_{m1}$; assume $a_{11} \neq 0$.

- $R_2 - \frac{a_{21}}{a_{11}} R_1 \longleftrightarrow$ multiplying A on the left by

$$L_{21} = \begin{bmatrix} 1 & & & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

- $R_3 - \frac{a_{31}}{a_{11}} R_1 \longleftrightarrow$ left multi. by $\begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ -\frac{a_{31}}{a_{11}} & & \ddots & \\ \vdots & & & 1 \end{bmatrix} = L_{31}$

$\vdots$

- $R_m - \frac{a_{m1}}{a_{11}} R_1 \longleftrightarrow$ L.M. by $\begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & \\ -\frac{a_{m1}}{a_{11}} & & & 1 \end{bmatrix} = L_{m1}$

Let $l_{21} = \frac{a_{21}}{a_{11}}$, $l_{31} = \frac{a_{31}}{a_{11}}$ $\dots$, $l_{m1} = \frac{a_{m1}}{a_{11}}$.

$$L_1 = L_{m1} \cdots L_{31} L_{21} = \begin{bmatrix} 1 & & & 0 \\ -l_{21} & \ddots & & \\ -l_{31} & & \ddots & \\ \vdots & & & 1 \\ -l_{m1} & & & \end{bmatrix}$$

$$L_1 A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & \boxed{a^1_{22}} & \cdots & a^1_{2m} \\ 0 & a^1_{32} & \cdots & a^1_{3m} \\ \vdots & \vdots & & \vdots \\ 0 & & & a^1_{mm} \end{bmatrix}$$

Assuming $a^1_{22} \neq 0$, $L_{32} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -l_{32} & \ddots & \\ & 0 & & \\ & \vdots & & 1 \end{bmatrix}$ & so on.

$l_{32} = \frac{a^1_{32}}{a^1_{22}}$

Finally get L_2.

$$\underline{\text{Step } k}: \quad A_{k-1} = \begin{array}{c c} & \begin{array}{ccccccc} \phantom{a_{11}} & \phantom{a_{12}} & \phantom{a_{13}} & & \phantom{a_{1k-1}} & k & \phantom{a_{1m}} \end{array} \\ k & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k-1} & a_{1k} & \cdots\ a_{1m} \\ 0 & a_{22}^1 & a_{23}^1 & & \cdot & \vdots & \cdots\ a_{2m}^1 \\ 0 & 0 & a_{33}^2 & \ddots & & \vdots & \\ \vdots & \vdots & & & a_{k-1,k-1}^{k-1} & \vdots & \\ & & & & 0 & a_{kk}^{k-1} & \\ & \cdot & & & \vdots & & \\ & & & & 0 & \vdots & \end{bmatrix} \end{array}$$

$$L_k = \begin{array}{c} \begin{array}{ccccc} & & k & & \end{array} \\ \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & 1 & & \\ & & l_{k+1,k} & \ddots & \\ & & \vdots & & \ddots \\ & & l_{m,k} & & 1 \end{bmatrix} \end{array}, \quad \text{where } l_{k+1,k} = \frac{a_{k+1,k}^{k-1}}{a_{kk}^{k-1}}$$

$$l_{jk} = \frac{a_{jk}}{a_{kk}}$$

$$\& \quad A_k = L_k A_{k-1}$$

Continuing in this way, $(L_{m-1} L_{m-2} \cdots L_2 L_1) A = \mathcal{U}$ (say)
(upper Δ^r)

Let $L = L_1^{-1} L_2^{-1} \cdots L_{m-1}^{-1}$, then $A = L\mathcal{U}$.
(the $L\mathcal{U}$-factorization of A).

Note that $L = \begin{bmatrix} 1 & & & 0 \\ l_{21} & 1 & & \\ l_{31} & l_{32} & \ddots & \\ \vdots & \vdots & \ddots & \\ l_{m1} & \cdots & l_{m,m-1} & 1 \end{bmatrix}$; $\det L = 1$.

If A is any square matrix with a_{ii} not necessarily $\neq 0$, then we have to introduce "pivoting".
This is done as follows -

Step 1: check whether $a_{11} \neq 0$. If yes, find L_1 & compute $L_1 A$.

$j\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_{j1} \end{bmatrix}$

Else, suppose a_{j1} is the first nonzero element in the first column.

Then exchange rows j & 1.

i.e. left multiply by $E_{1j} = \begin{array}{c c} & \begin{array}{ccccc} 1 & & & j & \end{array} \\ \begin{array}{c} 1 \\ \\ j \\ \end{array} & \begin{bmatrix} 0 & \cdots & \cdots & 1 & \\ \vdots & \ddots & & \vdots & \\ 1 & 0 & \cdots & 0 & \\ & & & & \ddots \ 1 \end{bmatrix} \end{array}$

So $\underset{P_1 =}{E_{1j}} A$ has a non-zero pivot a_{j1} at the (1, 1) spot.

- Proceed as usual, form the matrix L_1 & $A_1 = L_1 (P_1 A)$

$$A_1 = \left[\begin{array}{c|ccc} \alpha_{11} & \cdots & & \alpha_{1m} \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right]$$

- Check if the next pivot is nonzero.

Step k: $\quad P_k = \begin{cases} I & \text{if } a_{kk}^{k-1} \text{ is the pivot.} \\ E_{kj} & \text{if pivot is } a_{jk}^{k-1}. \end{cases}$

$$(\det P_k = \pm 1).$$

$$A_{k+1} = \left(L_k P_k \cdots \left(L_3 P_3 \left(L_2 P_2 \left(\underbrace{L_1 P_1 A}_{A_1} \right)\right)\right)\right)$$

Continuing:

$$\underbrace{L_{m-1} P_{m-1} L_{m-2} P_{m-2} \cdots L_1 P_1}_{L^{-1}} A = U.$$

(check L^{-1} is actually lower Δ^r)

- Note that $\det U = \pm \det A$ depending on # of permutation matrices reqd.
- At each step $\det(A_k) = \pm \det A \neq 0$, atleast one of the elements $a_{jk}^{k-1} \neq 0$, so a pivot can be chosen.
- $\det A = \pm$ (product of pivots).
- U can be found irrespective of whether A is invertible or not.

If A is not invertible, $\exists A_k$ such that $a_{kj}^{k-1} = 0 \quad \forall k \le j \le n$

Then let $A_k = A_{k+1}$ & set $P_k = L_k = I$.

Algorithms

I] G.E. without pivoting: $\mathcal{U} = A$, $L = I$.

(column) for $k = 1$ to $m-1$

(row) for $j = k+1$ to m

$l_{jk} = u_{jk} / u_{kk}$

working on j^{th} row $\rightarrow$ $u_{j,k:m} = u_{j,k:m} - l_{jk} \cdot u_{k,k:m}$

Operation count: each addition, subtraction, multi., division & sq. root counts as 1 flop.

$\approx \frac{2}{3} m^3$.

II) G.E. with partial pivoting (GEPP).

If at any step k, the pivot element is zero, then choose the largest among subdiagonal elements of the k^{th} column & use it as pivot (i.e. do corr. row exchanges)

Algorithm: $\mathcal{U} = A$, $L = I$.

for $k = 1$ to $m-1$

if $u_{ii} = 0$ { • select $i \geq k$ to maximize $|u_{ik}|$

• interchange rows u_i & u_k

(for $j = 1$ to $k-1$

$u_{kj} \leftrightarrow u_{ij}$)

for $j = k+1$ to m

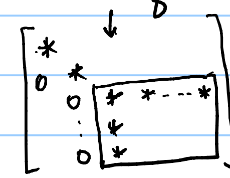
$l_{jk} = u_{jk} / u_{kk}$

$u_{j,k:m} = u_{j,k:m} - l_{jk} u_{k,k:m}$.

Operation count is same as before, but time reqd. is longer.

III) G.E. with complete pivoting (GECP).

At the k^{th} step, pivot is chosen from maximum of $(m-k) \times (m-k)$ elements.



This is rarely done in practice because selection of pivots takes a long time & improvement in stability is not considerable.