LPCO Problem Set 2

January 27, 2021

Problem 1. Recall Max-flow and Min-cut from the previous problem set. It has been reproduced below. Show that max-flow is dual to min-cut, i.e., the LP of one is dual to the other.

Definition 1 (Max-flow)**.** Suppose you are given a graph $G = (V, E)$ with two special vertices, a source vertex s and a sink vertex t. Assume that each edge (i, j) has an upper bound c_{ij} called it's capacity. Find the maximum flow possible from s to t.

Think of each edge as a pipe having maximum capacity c_{ij} that can pass through it. We send oil into the system at the vertex s, and oil exits the system at vertex t. Oil cannot enter or exit the system through any of the other nodes. We need to find the maximum possible oil that can be transported from s to t through this graph. Hint: Every vertex other than s and t should have amount of incoming oil = amount of outgoing oil.

Definition 2 (Min-cut)**.** Suppose you are given a graph $G = (V, E)$ with two special vertices, a source vertex s and a sink vertex t. Assume that each edge (i, j) has a cost c_{ij}. Find the minimum cut of s and t.

Consider the slightly different problem we have here. We still have a source vertex s and a sink vertex t, but now we for each pipe (i, j), we have a cost c_{ij} associated with destroying the pipe. We need to find the minimum total cost of pipes we need to destroy to stop any oil flowing from s to t.

Problem 2. Try your hand at working out the simplex method on the following problems

$$\begin{aligned}
\text{minimize: } & -x_1 - x_2 \\
\text{subject to: } & x_1 - x_2 \geq -1 \\
& -x_1 \geq -3 \\
& -x_2 \geq -2 \\
& x_1, x_2 \geq 0.
\end{aligned}$$

$$\begin{aligned}
\text{minimize: } & -x_1 \\
\text{subject to: } & -x_1 + x_2 \geq -1 \\
& x_1 - x_2 \geq -2 \\
& x_1, x_2 \geq 0.
\end{aligned}$$

$$\begin{aligned}
\text{minimize: } & -x_2 \\
\text{subject to: } & x_1 - x_2 \geq 0 \\
& -x_1 \geq -2 \\
& x_1, x_2 \geq 0.
\end{aligned}$$

While you are at it, also try finding the dual of the following LP:

$$\begin{aligned}
\text{minimize: } & -2x_1 - 3x_2 \\
\text{subject to: } & -4x_1 - 8x_2 \geq -12 \\
& -2x_1 - x_2 \geq -3 \\
& -3x_1 - 2x_2 \geq -4 \\
& x_1, x_2 \geq 0.
\end{aligned}$$

Problem 3. Show that if x and y are feasible solutions of the primal and dual and $c^T x = b^T y$, then x and y must be optimal solutions to the primal and dual.

Problem 4. Suppose you have an LP formulation with n unconstrained variables. Show that you can reformulate the LP with $n+1$ non-negative variables.

Problem 5. Show that finding a single feasible solution is as hard as finding an optimal feasible solution. That is, suppose we have an oracle that returns an arbitrary feasible solution (if one exists, returns None otherwise) given an LP.

$$\begin{aligned}
\text{maximize: } & c^t x \\
\text{subject to: } & Ax \leq b \\
& x \geq 0.
\end{aligned}$$

How do we solve the above LP if we are only allowed queries to the oracle?

Problem 6. Give an example of an LP where both primal and dual are infeasible.