Note Title — Special linear systems. — 07-05-2021

2 considerations :

- is pivoting necessary? • pivoting increases cost of computation (moving data around)
- is pivoting safe? - pivoting destroys exploitable structure.

We will consider 4 kinds of exploitable structures -

- diagonally dominant
- symmetric
- positive def.
- bandedness.

① Diagonal dominance -

consider $A = LU$ without pivoting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d - \left(\frac{c}{a}\right) b \end{pmatrix}$$

If a & d "dominate" b & c in magnitude then the elements of L & U will be bounded. This will make LU factorization stable.

This motivates -

Defn: ① Row diagonally dominant - for $A \in \mathbb{R}^{n \times n}$,

if $|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$ for every $i = 1, \dots, n$ holds.

② Column diagonally dominant -

if $|a_{jj}| \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|$ for $j = 1, \dots, n$, holds.

Theorem: If A is non-singular & CDD, then it has a LU factorization & the entries of $L = (l_{ij})$ satisfy $|l_{ij}| \le 1$.

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② Symmetry & LDL^T factorization.

$n = 2$:
$$\begin{pmatrix} a & c \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & c \\ 0 & d - \left(\frac{c}{a}\right)c \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} a & 0 \\ 0 & d - \left(\frac{c}{a}\right)c \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & c/a \\ 0 & 1 \end{pmatrix}}_{L^T}$$

Theorem: (G-L Section 4.1) If $A \in \mathbb{R}^{n \times n}$ is symmetric & the principal submatrix $A(1:k, 1:k)$ is nonsingular for $k = 1, \dots, n$, then $\exists$ a lower Δ^r matrix L & $D = \text{diag}(d_1, \dots, d_n)$ such that $A = LDL^T$; the factorization is unique.

Proof: A has a LU factorization $A = LU$.

Then $\underbrace{L^{-1} A L^{T^{-1}}}_{\text{symmetric}} = \underbrace{U L^{T^{-1}}}_{\text{upper } \Delta^r}$.

$\therefore$ the above matrix must be diagonal, say D.

$\therefore L^{-1} A L^{T^{-1}} = D$ i.e. $A = LDL^T$.

Next time: • Pos. def. — unsymmetric / symmetric — Cholesky factorization

• Pos. semidef. , • Banded systems.