Problem Set 5

DS Analysis

February 2021

Problem 1

$$\int_0^{\pi} (cosx + 5x^4)\, dx$$
$$= \int_0^{\pi} cosx\, dx + \int_0^{\pi} 5x^4\, dx$$
$$= [sinx]_0^{\pi} + [x^5]_0^{\pi}$$
$$= \pi^5$$

Problem 2

$$\int_0^{4\pi} |\sin x|\ dx$$
$$= \int_0^{\pi} \sin x\ dx + \int_{\pi}^{2\pi} -\sin x\ dx + \int_{2\pi}^{3\pi} \sin x\ dx + \int_{3\pi}^{4\pi} -\sin x\ dx$$

(N.B. where are $\sin x$ positive and negative.)

Problem 3

$$\int \frac{x}{x+1}\, dx$$
$$= \int (\frac{x+1}{x+1} - \frac{1}{x+1})dx$$
$$= \int 1\, dx - \int \frac{1}{x+1}\, dx$$
$$= x + c - log(x+1) + k$$

Where c and k are constants of integration. Taking $c + k = g$
We have,

$$= x - log(x+1) + g$$

Problem 4

$$\int \frac{e^x}{e^x + 1}\, dx$$

Notice, $\frac{d}{dx}(e^x + 1) = e^x$

Take, $(e^x + 1) = t$. The integral becomes,

$$\int \frac{1}{t}\ dt = log(t)$$

Problem 5

$$\int x^2 e^x\, dx$$

Integrating by parts, we have

$$= x^2 e^x - 2 \int x e^x dx$$

Integrating by parts again the right integral, we have,

$$= x^2 e^x - 2(x e^x - e^x)$$
$$= e^x (x^2 - 2x + 2) + c$$

Where c is the constant of integration.

Problem 6

$$\int x^3 \cos x^2\ dx$$

take, $x^2 = t \Rightarrow 2x\ dx = dt$ The integral becomes

$$\frac{1}{2} \int t \cos t\ dt$$

use,integration by parts.

Problem 7

$$\int_2^3 \frac{x+1}{\sqrt{x^2 + 2x + 3}} dx$$
$$= \int_2^3 \frac{x+1}{\sqrt{(x+1)^2 + 2}} dx$$

Now, taking $(x+1)^2 + 2 = z^2$, $(x+1)dx = zdz$
Thus, using substitution, we get..

$$\int_{\sqrt{11}}^{\sqrt{18}} \frac{z}{z} dz$$
$$= \sqrt{18} - \sqrt{11}$$

Problem 8

$$\begin{aligned} f(x) &= \sin x,\ 0 \geq x \geq \frac{\pi}{2} \\ &= \cos x,\ \frac{\pi}{2} \geq x \geq \pi \end{aligned}$$
$$\int_0^{\pi} f(x)\ dx$$
$$= \int_0^{\frac{\pi}{2}} \sin x\ dx + \int_{\frac{\pi}{2}}^{\pi} \cos x\ dx$$

If $F(x) = \int_0^x f(t)\ dt$
Then,

$$F(x) = \int_0^x \sin t\ dt,\ 0 \geq x \geq \frac{\pi}{2}$$
$$= \int_0^{\frac{\pi}{2}} \sin t\ dt + \int_{\frac{\pi}{2}}^{x} \cos t\ dt,\ \frac{\pi}{2} \geq x \geq \pi$$

Problem 9

$$\int_0^1 (x - x^2)\, dx$$
$$= \int_0^1 x\, dx - \int_0^1 x^2\, dx$$
$$= [\frac{x^2}{2} - \frac{x^3}{3}]_0^1$$
$$= \frac{1}{6}$$

Problem 10

Hint: take $x^2 = t. \Rightarrow d(x^2) = x\ dx$.
So, the problem becomes $y = \int_0^{\pi} sin(t)\ dt$

Problem 11

$$\int \sin^2 x dx$$

We can write it as,

$$\int \sin^2 x dx = \int \sin x \sin x dx$$

Integrating by parts, we have,

$$\sin x \int sinx dx - (\int \cos x (\int sinx dx) dx)$$

$$= -\sin x \cos x + \int \cos x \cos x dx$$

$$= -\sin x \cos x + \int \cos^2 x dx$$

Thus our first part of the problem is proved.
Now using $\cos^2 x = 1 - \sin^2 x$, we get

$$\int \sin^2 x dx = -\sin x \cos x + \int (1 - \sin^2 x) dx$$

Now, we know that $\sin 2x = 2 \sin x \cos x$

$$2 \int \sin^2 x dx = -\frac{2 \sin x \cos x}{2} + x$$

$$2 \int \sin^2 x dx = -\frac{\sin 2x}{2} + x$$

$$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4}$$

Problem 12

Use this formula, $\max\{f, g\} = \frac{1}{2}(f + g) + \frac{1}{2}|f + g|$
then, $\max\{0, f\} = \frac{1}{2}(0 + f) + \frac{1}{2}|0 + f| = \frac{1}{2}(f + |f|)$

f & $|f|$ are both continuous $\Rightarrow \max\{0, f\}$ is also continuous.
Similarly,$\max\{0, -f\} = \frac{1}{2}(0 - f) + \frac{1}{2}|0 - f| = \frac{1}{2}(-f + |f|)$

f & $|f|$ are both continuous $\Rightarrow \max\{0, -f\}$ is also continuous.

Now,

$$f(x) = \frac{1}{2}f(x) + \frac{1}{2}f(x)$$

$$= \frac{1}{2}f(x) + \frac{1}{2}|f(x)| - \frac{1}{2}|f(x)| + \frac{1}{2}f(x)$$

$$= f^+ + f^-$$

$$\int_a^b f(x) = \int_a^b f^+(x) + \int_a^b f^-(x)$$