Final Semester Examination :-

1. Let $f(x) = x^2 + ax + b$ $x \in \mathbb{R}$, where a, b are ~~constants~~ constants.

Then $f'(x) = 2x + a$ ~~$= 0$~~ ~~at~~

the line $y = 7x + 3$ is tangent to the graph of f at the point $(3, 24)$

then $f'(x)\Big|_{(3,24)} = 2 \times 3 + a = 6 + a$.

$(y - y_1)\Big|_{(3,24)} = f'(x)\Big|_{(3,24)} \cdot (x - x_1)$ [the general equation of the tangent]

$(y - 24) = (6 + a) \cdot (x - 3)$

or, $y - 24 = 6x - 18 + ax - 3a$

or, $y = (6+a)x + (6 - 3a)$ ... ①

and the tangent is given $y = 7x + 3$ ... ②

comparing ① and ②

$7 = 6 + a$ so, $a = 1$

$3 = 6 - 3a$

and $f(x) = y = x^2 + x + b$ putting the value of a.

and the functional value of $f(x)$ at $x = 3$ is 24.

so, $f(3) = 24 = 3^2 + 3 + b$.

or, $b = 24 - 12 = 12$

Ans:- $a = 1$, $b = 12$.

2. The area under the curve $y = |x^3 - 6x^2 + 8x|$ between $x = 0$ and $x = 4$ ~~or~~ is A.

So, $A = \int_0^4 |x^3 - 6x^2 + 8x| \, dx$

factorising, $x^3 - 6x^2 + 8x$.

$= x(x^2 - 6x + 8)$

$= x.(x^2 - 4x - 2x + 8)$

$= x\,[x(x-4) - 2(x-4)]$.

$= x\,(x-2)(x-4)$

then $A = \int_0^4 |x(x-2)(x-4)| \, dx$

$= \int_0^2 x(x-2)(x-4)dx + \int_2^4 -x(x-2)(x-4)\,dx$

as when $0 < x < 2$ $\quad (x-2) < 0$

and $(x-4) < 0$

so, $x(x-2)(x-4) > 0$

when $2 < x < 4$ $\quad (x-2) > 0$

$(x-4) < 0$

$x(x-2)(x-4) < 0$

So,

$$A = \int_0^2 x(x-2)(x-4)\,dx + \int_2^4 -x(x-2)(x-4)\,dx$$

$$= \int_0^2 (x^3-6x^2+8x)\,dx - \int_2^4 (x^3-6x^2+8x)\,dx$$

$$= \left[\frac{x^4}{4} - 6.\frac{x^3}{3} + 8.\frac{x^2}{2}\right]_0^2 - \left[\frac{x^4}{4} - 6.\frac{x^3}{3} + 8.\frac{x^2}{2}\right]_2^4$$

$$= \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 - \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_2^4$$

$$= \left[\frac{2^4}{4} - 2\times 8 + 4\times 2^2\right] - \left[\frac{4^4}{4} - 2.4^3 + 4.4^2 - \frac{2^4}{4} + 2.2^3 - 4.2^2\right]$$

$$= \left[4 - \cancel{16} + \cancel{16}\right] - \left[\cancel{64} - \cancel{128} + \cancel{64} - 4 + \cancel{16} - \cancel{16}\right]$$

$$= 4 + 4 = 8$$

Ans:- The area under the curve $y = |x^3-6x^2+8x|$ in between $x = 0$ to 4 is 8 sq. unit.

8. $E = \{(x,y,z) \in \mathbb{R}^3 : x+2y+3z = 27\}$

we need to find the point in E which is closest to the point $(1,1,1)$.

let $g(x) = x+2y+3z-27$

and $(x-1)^2+(y-1)^2+(z-1)^2$ is the square of the distance of the point (x,y,z) from $(1,1,1)$

So, the problem ~~boils~~ boils down to minimizing

$f(x,y,z) = (x-1)^2+(y-1)^2+(z-1)^2$, with constraint to $g(x) = x+2y+3z-27 = 0$

now $\nabla f(x,y,z) = (2(x-1), 2(y-1), 2(z-1))$

$\nabla g(x,y,z) = (1,2,3)$

So, $\nabla g(x,y,z) \neq 0 \quad (x,y,z) \in E$

Let $(x_0, y_0, z_0) \in E$ be a local Extremum.

Then implementing Lagrange multiplier method,

$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0, y_0, z_0)$

$2(x_0-1), 2(y_0-1), 2(z_0-1) = \lambda(1,2,3)$

$2z_0 - 2 = 3\lambda$

$2x_0 - 2 = \lambda$

$2y_0 - 2 = 2\lambda$

So, $x_0 = \frac{\lambda+2}{2} = \frac{\lambda}{2}+1 \quad \cdots \text{①}$

$y_0 = \frac{2\lambda+2}{2} = \lambda+1 \quad \cdots \text{②}$

$z_0 = \frac{3\lambda+2}{2} = \frac{3}{2}\lambda+1 \quad \cdots \text{③}$

Since $(x_0, y_0, z_0) \in E$ so,

$x_0 + 2y_0 + 3z_0 - 27 = 0$

So, $\frac{\lambda}{2}+1 + 2\lambda+2 + \frac{9}{2}\lambda + 3 - 27 = 0$

or, $7\lambda - 21 = 0$

or, $\boxed{\lambda = 3.}$

So, putting the value of λ in ①, ② and ③.

$x_0 = 5/2$

$y_0 = 4$

$z_0 = \frac{11}{2}.$

So, the point in the set $E = \{(x,y,z) \in \mathbb{R}^3 : x+2y+3z = 27\}$

is closest to the point $(1,1,1)$ is.

$(5/2, 4, 11/2)$ (Ans:-)

4. $f(x,y) = 5x^2 + 5y^2 - xy - 11x + 11y + 11$

$x, y \in \mathbb{R}^2$

$\frac{\partial f}{\partial x} = 10x - y - 11 = 0$

$\frac{\partial f}{\partial y} = 10y - x + 11 = 0$

[at critical points the partial derivatives are 0.

$10x - y = 11 \quad \cdots (1)$

$-x + 10y = -11 \quad \cdots (2)$

or,
$$\begin{aligned} 10\cancel{x} - y &= 11 \\ -10\cancel{x} + 100y &= -110 \\ \hline 99y &= -99 \end{aligned}$$

$\boxed{y = -1}$

so, $10x + 1 = 11$

$10x = 10$

$\boxed{x = 1}$

so, the critical point is $(1,-1)$.

Now, for finding the extremum we need to check the double derivatives and Hessian.

so $\frac{\partial^2 f}{\partial x^2} = 10$ $\qquad \frac{\partial^2 f}{\partial y^2} = 10$

$$\frac{\partial^2 f}{\partial x \partial y} = -1$$

$$\frac{\partial^2 f}{\partial y \partial x} = -1$$

so $H(x,y) = \begin{bmatrix} 10 & -1 \\ -1 & 10 \end{bmatrix}$

$$\det H(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial f}{\partial y \partial x}\right)^2$$

$$= 10 \times 10 - 1$$

$$= 99 > 0$$

so, at $(1,-1)$ extremum exists.

and, as $\frac{\partial^2 f}{\partial x^2} > 0$

then at $(1,-1)$ minimum exists.

the functional value at $(1,-1)$ is $5+5+1-11-11+11$
$= 0.$

$$f(x,y) = 5x^2 + 5y^2 - xy - 11x + 11y + 11$$

$$= \left[\frac{x^2}{2} - 2\,\frac{x}{2}\cdot\frac{y}{2} + \frac{y^2}{2}\right] + \left(\frac{9}{2}x^2 + \frac{9}{2}y^2 - 11x + 11y + 11\right)$$

$$= \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)^2 + \left[\frac{9}{2}x^2 - 11x + \left(\frac{11\sqrt{2}}{6}\right)^2\right]$$

$$+ \left[\frac{9}{2}y^2 + 11y + \left(\frac{11\sqrt{2}}{6}\right)^2\right] - \frac{22}{9}.$$

$$= \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)^2 + \left[\frac{3x}{\sqrt{2}} - \frac{11\sqrt{2}}{6}\right]^2 + \left[\frac{3y}{\sqrt{2}} + \frac{11\sqrt{2}}{6}\right]^2 - \frac{22}{9}$$

when $x \to \infty$ and $x \to -\infty$

$y \to \infty$ $y \to -\infty$

$f(x,y) \to \infty$.

so, so, local minimum is the point where the function takes the minimum value 0.

so, that there exist only one minima so, that will be the global minimum.

So, The global minimum of $f = 0$.

1. $I = \int_{\mathbb{R}} x^2 e^{-b|x|} dx \qquad b > 0$

$f(x)$ is an even function.

$$\lim_{a\to\infty} \int_{-a}^{a} x^2 e^{-b|x|} dx$$

$$= 2 \lim_{a\to\infty} \int_0^a x^2 e^{-b(x)} dx \qquad \left[\begin{array}{l} |x| = x \\ \text{when } x \geq 0 \end{array}\right]$$

$$I' = \lim_{a\to\infty} \int_0^a x^2 e^{-bx} dx$$

let $bx = z$

or, $b\,dx = dz$

or, $dx = \frac{dz}{b}$

$$= \lim_{a\to\infty} \frac{1}{b} \int_0^{ab} \frac{z^2}{b^2} e^{-z} dz$$

$$= \lim_{a'\to\infty} \frac{1}{b^3} \int_0^{a'} z^2 e^{-z} dz \qquad \left[\lim_{a\to\infty} ab = \lim_{a'\to\infty} a' \text{ where } ab = a'\right]$$

$$= \lim_{a'\to\infty} \frac{1}{b^3} \left[\left(-z^2 e^{-z}\right)\Big]_0^{a'} + \int_0^{a'} 2z \cdot e^{-z} dz \right]$$

$$= \lim_{a'\to\infty} \frac{1}{b^3} \left[\left(-a'^2 e^{-a'}\right) + 2\left(-z e^{-z}\right)\Big|_0^{a'} + 2 \int_0^{a'} e^{-z} dz \right]$$

$$= \lim_{a'\to\infty} \frac{1}{b^3} \left[-a'^2 e^{-a'} + 2\left(-a' e^{-a'}\right) - 2 e^{-z}\Big|_0^{a'} \right]$$

$$= \lim_{a'\to\infty} \frac{1}{b^3} \left[\left(-a'^2 e^{-a'}\right) + 2\left(-a' e^{-a'} - 2\left(e^{-a'} - 1\right)\right) \right]$$

Now, $\lim_{a' \to \infty} \left(-a'^2 e^{-a'}\right) = 0$

$\left[\lim_{x \to \infty} \frac{x^p}{e^x} = 0\right]$

and, $\lim_{a' \to \infty} \left(-a' e^{-a'}\right) = 0$

So, $I' = \lim_{a' \to \infty} \frac{1}{b^3} \left[-2\left(e^{-a'} - 1\right)\right]$

$= \frac{2}{b^3}$

So, $I = 2 \cdot \lim_{a \to \infty} \int_0^a x^2 e^{-bx}\, dx$

$= 2I' = \frac{4}{b^3}$ (Ans:-)

So, the limit exists and it is finite.
So, the value of the improper integral exists and the value is $\frac{4}{b^3}$. As the limit is finite, the improper integral converges.