CMI - 2020-2021: DS-Analysis

Calculus-6

Riemann integration

Objectives: (i) Given a function f, to find a function F such that $F'(x) = f(x)$. This is the inverse of differentiation, called *integration.*

(ii) Given a function f, with $f(x) \geq 0$, to have a definition of *area under the curve* of $y = f(x)$.

It turns out that these two problems are related; we will consider a method called *Riemann integration.*

Let f be a function defined on an interval I.

Definition 1 An *indefinite integral* of f is a function F such that $F'(x) = f(x)$ for all $x \in I$. This is usally denoted by

$$F = \int f, \quad \text{or} \quad F(x) = \int f(x)dx.$$

Note: If G is another indefinite integral of f, then $G'(x) = f(x)$. Hence it follows that $(F - G)'(x) = 0$ for all $x \in I$. By Theorem 5 of Calculus - 4, there exists a constant C such that $F(x) = G(x) + C,\ x \in I$.

As we know, an advantage of the above definition is that it immediately leads to indefinite integrals of many functions.

Let g be a continuous function on the interval $[a, b]$. Assume that $g(x) \geq 0,\ x \in [a, b]$. For $x \in [a, b]$ define $G(x) =$ area under the graph of $y = g(x)$ between a and x. Of course, we take $G(a) = 0$. Then using geometric ideas, the following result can be proved.

Theorem 2 *Let $g,\ G$ be as above. Then G is differentiable, and its derivative is g, that is, $G'(x) = g(x),\ a < x < b$.*

Let $g,\ G$ be as above. Let H be any indefinite integral of g. By preceding result, we have $H'(x) = G'(x)$ for all $x \in (a, b)$. Also $G(x) = H(x) + C$ for some constant C. As $G(a) = 0$ we get $H(a) = -C$. Hence $G(b) = H(b) + C = H(b) - H(a)$.

Thus the area under the curve between $x = a$ and $x = b$ is $H(b) - H(a)$.

If $\tilde{H}$ is another indefinite integral of g, from the above it follows that $\tilde{H}(b) - \tilde{H}(a) = H(b) - H(a)$.

Summarising, we have

Proposition 3 *Let f be a continuous function on $[a, b]$ such that $f(x) \geq 0,\ x \in [a, b]$. Denote the area under the graph of f between $x = a$ and $x = b$ by $\int_a^b f(x)dx$. Let F be* any *indefinite integral of f. Then*

$$\int_a^b f(x)dx \quad = \quad F(b) - F(a).$$

Note that heuristic/ intuitive idea of area under the graph of a non-negative continuous function f, lead to the notion of $\int_a^b f(x)dx$. Since we can often guess the function F, it is very useful in practice.

We now state a very basic theorem without proof. The proof is lengthy and tedious. It does not require any notion of indefinite integral. It uses the idea of approximating a continuous function by functions that are piecewise constants.

Theorem 4 *Let f be a continuous real valued function on a closed and bounded interval $[a, b]$. Then*

$$\int_a^b f(x)dx$$

is well-defined; that is, it is a unique real number. The unique number $\int_a^b f(x)dx$ is called the definite integral of f between a and b.

If, in addition, $f(x) \geq 0,\ x \in [a, b]$, then $\int_a^b f(x)dx$ is the same as the area under the graph of f between $x = a$ and $x = b$.

Example 1: To find the area under the curve $y = x^2$ between $x = 1$ and $x = 2$.

Note that we need to find $\int_1^2 x^2 dx$. So

$$\int_1^2 x^2 dx = \frac{x^3}{3} \mid_1^2 = \frac{7}{3}.$$

Example 2: To find $\int_0^\pi \sin x dx$.

Note that the indefinite integral of $\sin x$ is $(-\cos x)$. Hence

$$\int_0^\pi \sin x dx = -\cos x \mid_0^\pi = 2.$$

Exercise: Find $\int_1^3 \frac{1}{x} dx$.

We now state two elementary facts, which are similar to those satisfied by area.

Fact 1: Let f be continuous function on $[a, b]$. Suppose that there exist two numbers $m,\ M$ such that $m \leq f(x) \leq M,\ x \in [a, b]$. Then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a).$$

Fact 2: Let $a < b < c$. Let f be a continuous function on $[a, c]$. Then

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx.$$

Remark 5 Let f be a function on $[a, b]$ such that the following hold. There exist points $\{a_i : 1 \leq i \leq k\}$ and functions $\{f_i : 1 \leq i \leq k\}$ such that

$$a = a_0 < a_1 < a_2 < \cdots < a_{k-1} < a_k = b,$$

f_j is a bounded continuous function on $(a_{j-1}, a_j),\ 1 \leq j \leq k$, and $f(x) = f_j(x),\ x \in (a_{j-1}, a_j),\ 1 \leq j \leq k$. (The values of f at the finite number of points $a_i,\ i = 1, 2, \cdots k$ may not matter.) Then one can define $\int_a^b f(x) dx$ by putting

$$\int_a^b f(x) dx = \sum_{j=1}^{k} \int_{a_{j-1}}^{a_j} f_j(x) dx.$$

That is, definite integrals can be defined even for bounded piecewise continuous functions on bounded intervals. (Such situations may arise in probability theory.)

Example 3: Let f be a function on $[-5,5]$ given by

$$\begin{aligned} f(s) &= 0, \ -5 \le s \le 0, \\ &= 1, \ 0 < s < 1, \\ &= 0, \ 1 \le s \le 5. \end{aligned}$$

That is f is *piecewise continuous.* Put $F(x) = \int_{-5}^{x} f(s)ds, \ x \in [-5,5]$. By the above remark, note that $F(x)$ is well defined. It can be checked that

$$\begin{aligned} F(x) &= 0, \ -5 \le x \le 0, \\ &= x, \ 0 \le x \le 1, \\ &= 1, \ 1 \le x \le 5. \end{aligned}$$

These arise in connection with the *uniform distribution* on $(0,1)$ in probability theory. (In fact, $f(s) = 1, \ s \in (0,1)$ and $f(s) = 0, \ s \notin (0,1)$. So $F(x) = 0, \ x \le 0$, and $F(x) = 1, \ x \ge 1$. Note that f and F are defined for all $x \in \mathbb{R}$.)

Fact 3: Another elementary fact: Suppose $\int_a^b f(y)dy$, and $\int_a^b g(y)dy$ are well defined. Let $\alpha, \ \beta \in \mathbb{R}$ be constants. Then

$$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx;$$

that is, integration is a linear operation.

We now state a theorem without proof; this result, called the *fundamental theorem of calculus,* indicates the connection between integration and differentiation.

Theorem 6 *Let f be a continuous function on $[a,b]$. Let*

$$F(x) = \int_a^x f(s)ds, \ x \in [a,b].$$

Then F is differentiable on (a,b), and $F'(x) = f(x), \ x \in (a,b)$.

Suppose f is piecewise continuous as in Remark 5. Then F is a continuous function on $[a,b]$. Also F is differentiable at any point $x \in (a,b)$ where f is continuous, and $F'(x) = f(x)$ at points where f is continuous.

In both cases

$$\int_a^b f(s)ds = F(x) \mid_a^b = F(b) - F(a).$$

Exercise: Verify the second part of the above theorem in Example 3.

Now we shall briefly recall technique of integration using substitution. This procedure can be stated as follows:

Fact: Let g be a differentiable function on $[a, b]$; also let g' be continuous. Let f be a continuous function on an interval containing the values of g. Then

$$\int_a^b f(g(x))\frac{dg(x)}{dx}dx = \int_{g(a)}^{g(b)} f(u)du.$$

Example 4: To find

$$\int_0^3 (x^2+1)^3(2x)dx.$$

Take $f(u) = u^3$ and $g(x) = (x^2 + 1)$. Clearly $g'(x) = 2x$; hence the integrand is of the form $f(g(x))g'(x)$. First we shall find the indefinite integral. Now

$$\begin{aligned} \int (x^2+1)^3(2x)dx &= \int u^3 du = \frac{u^4}{4} \\ &= \frac{(x^2+1)^4}{4}; \end{aligned}$$

in the last step, we have expressed all the terms in terms of x. At this stage, it is a good practice to differentiate the indefinite integral to check if it agrees with the integrand. Also $g(0) = 1,\ g(3) = 10$. Therefore we have

$$\begin{aligned} \int_0^3 (x^2+1)^3(2x)dx &= \int_{g(0)}^{g(3)} f(u)du \\ &= \int_1^{10} u^3 du = \frac{u^4}{4} \mid_1^{10} \\ &= \frac{9999}{4}. \end{aligned}$$

Example 5: To find the indefinite integral $\int xe^{x^2}dx$, and $\int_0^2 xe^{x^2}dx$.

Note that
$$\int xe^{x^2}dx = \frac{1}{2}\int 2xe^{x^2}dx.$$
So take $u = x^2$, and hence $du = 2xdx$. This gives
$$\begin{aligned}\int xe^{x^2}dx &= \frac{1}{2}\int 2xe^{x^2}dx \\ &= \frac{1}{2}\int e^u du = \frac{1}{2}e^u = \frac{1}{2}e^{x^2},\end{aligned}$$
giving the indefinite integral. Easy to verify that $\frac{d}{dx}(\frac{1}{2}e^{x^2}) = xe^{x^2}$.

Therefore
$$\int_0^2 xe^{x^2}dx = \frac{1}{2}e^{x^2}\mid_0^2 = \frac{e^4-1}{2}.$$
Exercise: Show that $\int \cos(3x)dx = \frac{1}{3}\sin(3x)$.

Now we consider briefly the technique of integration by parts.

Let f, g be two differentiable functions. We know that $(fg)'(x) = f(x)g'(x) + g(x)f'(x)$; so $f(x)g'(x) = (fg)'(x) - g(x)f'(x)$. Note that the indefinite integral of $(fg)'$ is fg; that is, $\int(fg)'(x)dx = f(x)g(x)$. Hence we obtain
$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx,$$
which is called *integrating by parts.* To find the integral on the left side, this method is very useful if finding the integral on the right side is simpler.

Example 6: To find $\int \log(x)dx$.

Take $f(x) = \log(x),\ g(x) = x$. As $g'(x) = 1$, note that the integrand is in the form $f(x)g'(x)$. So integrating by parts, we get
$$\int \log(x)dx = x\log(x) - \int x\frac{1}{x}dx = x\log(x) - x.$$
Example 7: To find $\int xe^{-x}dx$.

Note that the choice of f, g should make it easier/ simpler to solve the problem. So take $f(x) = x,\ g(x) = -e^{-x}$; then $f'(x) = 1,\ g'(x) = e^{-x}$. Clearly the integrand is in the form $f(x)g'(x)$. Integrating by parts,
$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -(x+1)e^{-x}.$$

In the above example, if we had chosen f, g such that $g'(x) = x$, then it would have complicated the problem rather than simplifying it.

We will look at an example where we first make a substitution before integrating by parts.

Example 8: To find $\int e^{2x}\sin(3x)dx$.

Put $I = \int e^{2x}\sin(3x)dx$.

Using substitution show that $\int \sin(3x)dx = -\frac{1}{3}\cos(3x)$, similar to the exercise above. Take $f(x) = e^{2x},\ g(x) = -\frac{1}{3}\cos(3x)$. Integrating by parts, as $f'(x) = 2e^{2x}$, we get

$$I = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\int e^{2x}\cos(3x)dx.$$

Now apply the same procedure to the integral $J = \int e^{2x}\cos(3x)dx$. By the exercise above $\int \cos(3x)dx = \frac{1}{3}\sin(3x)$. In this case take $f(x) = e^{2x},\ g(x) = \frac{1}{3}\sin(3x)$. So again, integrating by parts, we get

$$J = \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\int e^{2x}\sin(3x)dx.$$

Thus we have

$$\begin{aligned} I &= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}J \\ &= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}[\frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}I] \end{aligned}$$

That is,

$$\frac{13}{9}I = \frac{1}{9}e^{2x}[2\sin(3x) - 3\cos(3x)].$$

Therefore

$$\int e^{2x}\sin(3x)dx = \frac{1}{13}e^{2x}[2\sin(3x) - 3\cos(3x)].$$

Note: In the 3 examples above, verify if derivatives of the indefinite integrals agree with the respective integrands.