CMI - 2020-2021: DS-Analysis

Calculus-2

Infinite series

We look at infinite series.

Definition 1 Let $a_n \in \mathbb{R},\ n \geq 1$. Write

$$s_n \ = \ \sum_{j=1}^{n} a_i,\ n = 1, 2, 3, \cdots$$

Note that $\{s_n : n = 1, 2, \cdots\}$ is the sequence of partial sums. If $\{s_n\}$ converges, and the limit is s, we say that the *series* $\sum_{n=1}^{\infty} a_n$ *converges,* and the limit s is called *sum* of the series. We will denote this by $\sum_{n=1}^{\infty} a_n = s$.

If $\{s_n\}$ does not converge, we say that the series $\sum_{n=1}^{\infty} a_n$ *diverges.*

Note: The following are easily proved.

(i) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(ii) If $\sum_{n=1}^{\infty} a_n = s$, and $\sum_{n=1}^{\infty} b_n = t$, then $\sum_{n=1}^{\infty} a_n + b_n = s + t$; also $\sum_{n=1}^{\infty} ca_n = cs$, for any constant $c \in \mathbb{R}$.

Some examples:

1) Let $|r| < 1$. Then the geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{(1-r)}$.

If $|r| \geq 1$, then the series does not converge, because $\{r_n\}$ does not converge to 0.

2) The converse of (i) above is not true. Take $a_n = 1/n,\ n \geq 1$. Then $a_n \to 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

3) $\sum_{n=1}^{\infty} \frac{1}{n^k} < \infty$, for $k = 2, 3, \cdots$; that is, $\sum_{n=1}^{\infty} \frac{1}{n^k}$ is convergent for $k = 2, 3, \cdots$.

4) Let $x \in \mathbb{R}$ be fixed. Then $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$. This is an example of a power series expansion.

Some general results on infinite series are discussed below.

Consider a series with non-negative terms, that is, $a_n \geq 0$ for all n. Then clearly the sequence $\{s_n\}$ of partial sums is an increasing sequence of non-negative numbers. If $\{s_n\}$ is bounded above, we know that $\lim_{n\to\infty} s_n =$

$\sup\{s_n : n = 1, 2, \cdots\}$. If $\{s_n\}$ is not bounded above, then $s_n \to +\infty$, that is, $\{s_n\}$ doesn't converge. Thus we have proved the following result.

Theorem 2 *Let $a_n \geq 0,\ n = 1, 2, 3, \cdots$; and $s_n = a_1 + a_2 + \cdots + a_n,\ n \geq 1$, be the sequence of partial sums. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence $\{s_n\}$ is bounded above. If $\{s_n\}$ is bounded above, then*

$$\sum_{n=1}^{\infty} a_n = \sup\{s_n : n = 1, 2, 3, \cdots\}.$$

Definition 3 We say that the series $\sum_{n=1}^{\infty} a_n$ *converges absolutely* if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

We give below two results without proofs.

Theorem 4 *If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges; that is, absolute convergence implies convergence.*

While the result above may not be surprising, the next one, due to Leibniz, may be somewhat surprising.

Theorem 5 *Let $\sum_{n=1}^{\infty} a_n$ be a series such that*

(a) $\lim_{n\to\infty} a_n = 0$;

(b) $|a_{n+1}| \leq |a_n|,\ n \geq 1$;

(c) the terms a_n are alternately positive and negative.

Then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

To get an idea of the proof, without loss of generality, let $a_1 > 0$. Put $s_n = \sum_{k=1}^{n} a_k,\ n \geq 1$. Then note that

$$s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_7 \leq s_5 \leq s_3 \leq s_1.$$

Therefore $\{s_{2k+1}\}$ is a decreasing sequence, while $\{s_{2k}\}$ is an increasing sequence. Both the sequences are in the bounded interval $[s_2, s_1]$. Hence both the sequences converge. Since $a_n \to 0$, it follows that

$$\lim_{k\to\infty} s_{2k+1} = \lim_{k\to\infty} s_{2k} = \sum_{n=1}^{\infty} a_n.$$

Example 1: The series $\sum_{n=1}^{\infty}(-1)^{(n+1)}/n$ is convergent by the above theorem. But it is not absolutely convergent. (Note: It turns out that $\sum_{n=1}^{\infty}(-1)^{(n+1)}/n = \log 2$.)

We now discuss some useful tests for convergence of series. Comparison test given below is the most commonly used.

Theorem 6 (Comparison test) *(i) Let $a_n \geq 0,\ b_n \geq 0$ for $n = 1, 2, 3, \cdots$ Suppose that there exist constant $C > 0$, such that $a_n \leq C b_n$ for all $n \geq 1$. Then convergence of $\sum_{n=1}^{\infty} b_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n$, and*

$$\sum_{n=1}^{\infty} a_n \ \leq \ C \sum_{n=1}^{\infty} b_n.$$

(ii) Assume that $a_n > 0$, $b_n > 0$ for all n. Suppose that

$$\lim_{n \to \infty} \frac{a_n}{b_n} \ = \ \beta \ \neq \ 0.$$

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

(iii) If $\beta = 0$ in (ii) above, we can conclude only that convergence of $\sum_{n=1}^{\infty} b_n$ implies convergence of $\sum_{n=1}^{\infty} a_n$.

Assertion (i) of the above theorem follows from Theorem 2; assertions (ii) and (iii) can be derived using (i).

Exercise: Give examples to illustrate assertions (i)-(iii) of Theorem 6.

Theorem 7 *(*Ratio test*) Let $a_n > 0,\ n = 1, 2, 3, \cdots$*

(i) Suppose there exist $\beta < 1,\ N \geq 1$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \ \leq \ \beta, \text{ for all } n \geq N.$$

Then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Suppose there exist $\beta > 1,\ N \geq 1$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \ \geq \ \beta, \text{ for all } n \geq N.$$

Then $\sum_{n=1}^{\infty} a_n$ diverges.

Assertion (i) can be derived using comparison test without much difficulty by showing $|a_{N+k}| \leq a_N \beta^k,\ k \geq 1$. Under the hypothesis of assertion (ii), it easily follows that a_n can not converge to 0, and hence the series diverges.
Exercise: Suppose $\lim_{n\to\infty} |(a_{n+1}/a_n)| = \gamma$. If $\gamma < 1$, show that $\sum_{n=1}^{\infty} a_n$ converges. If $\gamma > 1$ show that the series diverges. (In some books this exercise may be given as the ratio test.)
Example 2: Consider $\sum_{n=1}^{\infty} n/(2^n)$. Clearly $\lim_{n\to\infty} |(a_{n+1}/a_n)| = \frac{1}{2}$. So, taking $\beta = 3/4$ in ratio test, it follows that this series converges.
Example 3: Consider $\sum_{n=1}^{\infty} n!a^n$, where $0 < a < 1$. Clearly this series diverges.

Theorem 8 *(*Root test*) Let $a_n > 0,\ n = 1, 2, 3, \cdots$ Suppose*

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \beta$$

(i) If $\beta < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
(ii) If $\beta > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

For proving assertion (i), let $x \in (\beta, 1)$. By definition of β, clearly $|a_n| < x^n$. Use comparison test to get the desired result.

To prove assertion (ii), as $\beta > 1$, note that $a_n > 1$ for infinitely many n. So a_n can not converge to 0, implying the divergence of the series.
Example 4: Using root test show that $\sum_{n=2}^{\infty} (1/\log n)^n$ converges.
Example 5: Consider $\sum_{n=1}^{\infty} (n^n)/(2^n)$. In this case $(a_n)^{\frac{1}{n}} = n/2 \to +\infty$. Hence the series diverges.