Algorithms: EndSem Exam

December 1, 2017

Answer all questions (Maximum marks 100).

1. Let G be a k-regular bipartite graph. Prove that the edge set of G can be partitioned into k perfect matchings. *(15 marks)*

2. Let $G = (V, E)$ be an undirected graph with each edge having unit capacity in either direction. Let $S \subseteq V$ be a set of vertices. The cut $[S, \bar{S}]$ consits of edges with exactly one end point in S. The capacity of a cut is the number of edges contained in it. A global min-cut is a cut $[S, \bar{S}]$ of minimum capacity.

   Prove that using an implementation of the Ford-Fulkerson algorithm that takes time $f(n, m)$ (where, $n = |V|, m = |E|$) it is possible to find the capacity of the global min-cut in time $O(nf(n, m))$. *(15 marks)*

3. Let $G = (V, E, c)$ be a directed capacitated graph with $c : E \to \mathbb{N}$ being the capacity function. Let $s, t \in V$ be distinct vertices. Devise a polynomial time algorithm to determine which edges of the graph do not support non-zero flow in any maximum flow. You may assume that you have access to a library function that computes maximum s, t-flow in a given capacitated network. *(15 marks)*

4. Here are some problems:

   - **3SAT** Given a collection of clauses each containing exactly 3 literals (where each literal is one of n variables or its negation). Does there exist a truth assignment to the variables such that each clause includes a true literal.
   - **Dominating Set**: Given an undirected graph G and a positive integer K does there exist a set $S \subseteq V(G)$ of vertices of size $|S| \leq K$ such that every vertex $v \in V(G)$ is either in S or has a neighbour in S.
   - **Hitting Set** Given a collection C of subsets of S and a positive integer K does there exist a subset $T \subseteq S$ such that every element of C has a nonempty intersection with T. $|T| \leq K$
   - **Set Splitting** Let C be a collection of subsets of a finite set S. Is there a partition of S into two subsets S_1, S_2 such that no ~~subset~~ element of C is contained enirely within S_1 or within S_2.

3SAT $\leq$ Set Splitting

Cover $C_1, C_2, C_3 \ldots$ using $s_1, s_2, s_3 \ldots$

→ $S \subseteq$ Vertices
→ $|S| \leq K$
cover edges

INDUCT

K = int integer

S = vertices

$T = \{v_1, v_2, \dots, v_k\}$

VC $\leq$ HS

- **Vertex Cover**: Given an undirected graph G and a positive integer K does there exist a set $S \subseteq V(G)$ of vertices of size $|S| \leq K$ such that every edge in $E(G)$ has an endpoint in S.

VC $\leq_p$ DS.

(a) Use NP-completeness of **Vertex Cover** to show that **Hitting Set** is NP−complete. *(5 marks)*

(b) Use NP-completeness of **Vertex Cover** to show that **Dominating Set** is NP-complete. *(15 marks)*

(c) Use NP-completeness of **3SAT** to show that **Splitting Set** is NP-complete. *(15 marks)*

5. Let M be a nondeterministic Turing machine (with binary tape and input alphabets) that is restricted to use space $S(n) = O(n^c)$ (for some constant c) on its work tape. Let $L = L(M)$ be the language accepted by the NTM and let $\bar{L}$ be its complement (i.e. $\{0,1\}^* \setminus L$). Prove [1] that there is a deterministic Turing machine running in space $O(n^d)$ for some constant d that accepts $\bar{L}$. How is d related to c? *(15 marks)*

6. Let G be an undirected graph and let T be a DFS-tree of G. Let S be the non-leaf vertices of T. Show that:

(a) G has a matching of size at least $\lceil |S|/2 \rceil$. *(10 marks)*

(b) S forms a vertex cover of the graph of size at most twice the optimal vertex cover. *(5 marks)*

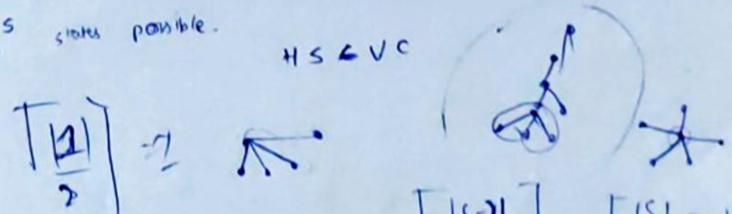
$S(n) = O(n^c)$ = $S(n)$ = S.

stack alphabet = Γ, $|\Gamma| = n$, states = Q, $|Q| = q$.

head can be at any S.

$q S n^S$ states possible.

HS $\leq$ VC



$\left\lceil \frac{|S-2|}{2} \right\rceil = \left\lceil \frac{|S|}{2} - 1 \right\rceil = \left\lceil \frac{|S|}{2} \right\rceil$

[1] You must prove the result from scratch and not assume any results proved in Algorithms class. Anything taught in TOC ought to be ok.