LPCO 2019 **Mid-sem Exam (50 marks)**

1. Does Gaussian elimination preserve the column space of a matrix? Justify. **(3 marks)**

2. In the simplex method discussed in class, we made an assumption that the feasible region is bounded. Give a modified simplex algorithm that handles unbounded feasible regions. Explain in a few sentences why your modifications work. **(5 marks)**

3. Write the dual of the following LP: **(4 marks)**

$$\begin{array}{lrcrcrl}
\text{Maximize} & 8x_1 & + & 3x_2 & - & 2x_3 & \\
\text{Subject to} & x_1 & - & 6x_2 & + & x_3 & \geq 2 \\
 & 5x_1 & + & 7x_2 & - & 2x_3 & = -4 \\
 & & & & & x_1 & \leq 0 \\
 & & & & & x_2 & \geq 0
\end{array}$$

4. Exhibit a primal-dual pair such that both are infeasible. **(5 marks)**

5. Consider the following problem. **(8 marks)**

$$\begin{array}{lrcrcrcrcrll}
\text{Maximize} & 10x_1 & + & 24x_2 & + & 20x_3 & + & 20x_4 & + & 25x_5 & & \\
\text{Subject to} & x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & + & 5x_5 & \leq 19 & (C1) \\
 & 2x_1 & + & 4x_2 & + & 3x_3 & + & 2x_4 & + & x_5 & \leq 57 & (C2) \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq 0 &
\end{array}$$

   (a) Write its dual with two variables w_1, w_2 (corresponding to the constraints (C1) and (C2)) and verify that $(w_1, w_2) = (4, 5)$ is a feasible solution.

   (b) Use complementary slackness to show that $(w_1, w_2) = (4, 5)$ gives the optimal solution to the dual.

6. (a) Write the ILP for finding min-cost perfect matching in a bipartite graph with edge weights.

   (b) Show that in the relaxed LP, every non-integral feasible point can be expressed as a convex combination of two distinct feasible points.

   (c) Use (b) to show that every k-regular bipartite graph has a perfect matching[1]. **(10 marks)**

7. Here is the set cover problem:

   Input. A universe D consisting of finite number of elements, and a family $S_1, S_2, \ldots, S_m$ of sets, with each $S_i \subseteq D$. Assume that $\bigcup_{i \in \{1,\ldots,m\}} S_i = D$.

   Goal. Find minimum size subset $W \subseteq \{1, \ldots, m\}$ such that $\bigcup_{i \in W} S_i = D$

   (a) Write an ILP for the set cover problem. **(3 marks)**

   (b) Give the relaxed LP and its dual. **(4 marks)**

   (c) Design a primal-dual algorithm which gives an f-approximation where f is the maximum *frequency* of an element, i.e. $f = \max_{e \in D}\{\text{Number of sets containing element } e\}$. In other words, show that the answer given by the algorithm is at most f times the optimal solution. Describe the algorithm and its analysis. **(8 marks)**

[1] Graph theoretic answers not using (b) will not be accepted

min $19w_1 + 57w_2$

$w_1 + 2w_2 \geq 10$

$w_1 + 4w_2 \geq 24$

$2w_1 + 3w_2 \geq 20$

$3w_1 + 2w_2 \geq 20$

$5w_1 + w_2 \geq 25$

min $\sum s_i$

max $\sum e_i$

$\sum_{e \in S_i} e_i \leq 1 \quad \forall S_i$

max $8x_1 + 3x_2 - 2x_3$

$8x_1 + 3x_2 - 2x_3 \le$

$\max \quad 8x_1 + 3x_2 - 2x_3$

subject to $\quad x_1 - 6x_2 + x_3 \ge 2$

$5x_1 + 7x_2 - 2x_3 = -4$

$x_1 \le 0$

$x_2 \ge 0.$

$y_1 x_1 - 6y_1 x_2 + y_1 x_3 \ge 2y_1$

$5y_2 x_1 + 7y_2 x_2 - 2y_2 x_3 = 4y_2$

$\min \ 2y_1 + 4y_2$

$y_1 + 5y_2 \le 8$

$-6y_1 + 7y_2 \ge 3$

$8x_1 \le 7x_1$

$y_1 - 2y_2 = -2$

$y_1 \ge 0.$

$\max \ x_1 + x_2$

$x_1 \le -1$

$x_2 \le -1$

$x_1 \ge 0$

$x_2 \ge 0$

$\min \sum c_e x_e$

subject to $\sum_{e \text{ incident on } v} x_e = 1 \quad \forall v \in V.$

$x_e \in \{0,1\}$

relaxed $x_e \ge 0.$

$\le y_1 x_1 + y_2 x_2$

$\min \ -y_1 - y_2$

$y_1 \ge 1$

$y_2 \le 1$

$y_1 \ge 0$

$y_2 \ge 0$

$+x_2 \le y_2 x_2 \le -y_2$

$y_2 \le 1$

non-integral

$\sum_{e \text{ incident on } u} x_e = 1.$

$(x_e, \ldots)$

$x_e \ge 0$

$\max \quad x_1 + x_2$

$x_1 \le -1$

$x_2 \ge -1$

$x_1 \ge 0$

$y_2 \ge 0$

$\min \ -y_1 - y_2$

$y_1 \ge 1$

$y_2 \ge 1$

$y_1 \ge 0$

$y_2 \le 0$

$x_2 \le y_2 x_2 \le -y_2$

$y_2 \ge 1$