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Linear Optimization Mid Sem

1. $2x_1 + x_2 + 4x_3 = 8$ $\quad x_1, x_2, x_3, x_4 \geq 0.$

$x_2 + x_4 = 4$

So, $\begin{bmatrix} 2 & 1 & 4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

So, $Ax = b.$ $\quad A = \begin{bmatrix} 2 & 1 & 4 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

the independent columns are. $B = \{1, 2\}$

$A_B x_B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ $\quad 2x_1 + x_2 = 8$ $\quad x_1 = 2$

$x_2 = 4$ $\quad x_2 = 4$

$2x_1 = 4.$

So, the BFS = $[2 \; 4 \; 0 \; 0]$

let $B = \{2, 3\}$

$A_B x_B = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ $\quad x_2 + 4x_3 = 8$ $\quad 4x_3 = 4$

$x_2 = 4$ $\quad x_3 = 1$

So, the BFS is $[0 \; 4 \; 1 \; 0]$

let $B = \{3, 4\}$

$A_B x_B = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ $\quad 4x_3 = 8$ $\quad x_3 = 2$

$x_4 = 4$

So, the BFS is $[0 \; 0 \; 2 \; 4]$

Let $B = \{1, 4\}$

$$A_B x_B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \qquad \begin{matrix} 2x_1 = 8 \\ x_4 = 4 \end{matrix} \qquad x_1 = 4$$

the BFS is = $[4\ 0\ 0\ 4]$

let $B = \{2, 4\}$

$$A_B x_B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \qquad \begin{matrix} x_2 = 8 \\ x_2 + x_4 = 4 \end{matrix} \qquad \begin{matrix} x_4 = (4-8) \\ = -4 \end{matrix}$$

So, the solution is $[0\ 8\ 0\ -4]$ $\times$ this is not a Feasible solution as $x_4 < 0$.

So, the Basic feasible solutions are,

$[2\ 4\ 0\ 0]$, $[0\ 4\ 1\ 0]$, $[0\ 0\ 2\ 4]$, $[4\ 0\ 0\ 4]$

2. Let S be the subspace given by the solutions of $x_1 + x_2 + x_3 = 0$.

let $x_3 = t$ then $x_1 + x_2 = -t$ $\qquad s, t \in \mathbb{R}$

let $x_2 = s$ then $x_1 = -s-t$

$$\text{so, } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So, the basis for $S = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ as these 2 vectors are spanning the solution set of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

2.(b) Let T be the affine subspace formed by the solutions $x_1+x_2+x_3=1$. Lets take one solution. $\begin{bmatrix}1\\0\\0\end{bmatrix}$

$\dim(S) = 2$.

adding it to the Previous basis. S. such that we get the basis. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

so, $|\dim(S)|+1 = 3$.

let the 3 points on the plane be, $\begin{array}{ccc} x_1 & x_2 & x_3 \\ \begin{bmatrix}1\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\1\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\1\end{bmatrix} \end{array}$

so, let $\exists$ some $a_1, a_2, a_3 \in \mathbb{R}$ such that,

$$a_1\begin{bmatrix}1\\0\\0\end{bmatrix}+a_2\begin{bmatrix}0\\1\\0\end{bmatrix}+a_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

hence $\underline{a_1+a_2+a_3=1}$ and it is an affine combination.

and it is spanning the solution space of $x_1+x_2+x_3=1$.

hs.

3. minimize:- $x_1+2|x_2|+3|x_3-10|$

subject to: $|x_1|+|x_2+x_3| \le 10$.

let $|x_2| \le t$ | $|x_3-10| = s$.

so, $-t \le x_2 \le t$ | $-s \le (x_3-10) \le s$.

$-t \le x_2 \le t$ | $-s+10 \le x_3 \le s+10$

$-s-t+10 \le x_3+x_2 \le s+t+10$

so, minimize $x_1+2t+3s$.

where $|x_1|+|x_2+x_3| \le 10$ where $-t \le x_2 \le t$

$-s+10 \le x_3 \le s+10$

3. minimize :-

$$x_1 + 2|x_2| + 3|x_3 - 10|$$

Subject to $|x_1| + |x_2 + x_3| \leq 10.$

let $x_1 = y_1 - z_1$ $\quad\quad y_1, z_1, y_2, z_2, y_3, z_3 \geqslant 0.$

$x_2 = y_2 - z_2$

$x_3 - 10 = y_3 - z_3.$

then. minimize, $\boxed{y_1 - z_1 + 2y_2 - 2z_2 + 3y_2 - 3z_3}$.

subject to,

$$|x_1| + |x_2 + x_3| \leq 10.$$

$$|x_1 + x_2 + x_3| \leq |x_1| + |x_2 + x_3| \leq 10.$$

$\rightarrow$ from $|X+Y| \leq |X| + |Y|$

so,

$$-10 \leq x_1 + x_2 + x_3 \leq 10$$

so, $\quad y_1 - z_1 + y_2 - z_2 + y_3 - z_3 + 10 \leq 10.$

$\boxed{y_1 - z_1 + y_2 - z_2 + y_3 - z_3 \leq 0.}$ .... ①

$$x_1 + x_2 + x_3 \geqslant -10$$

or, $y_1 - z_1 + y_2 - z_2 + y_3 - z_3 + 10 \geqslant -10$

or, $\boxed{y_1 - z_1 + y_2 - z_2 + y_3 - z_3 \geqslant -20.}$ ---- ②

4. Let C_{2n} for $n \geq 1$ denote the undirected cycle with $2n$ edges. Vertices are $\{v_1, v_2, .., v_{2n}\}$ and edges are the unorderd pairs $\{(v_1,v_2),(v_2,v_3)\ldots,(v_{2n},v_1)\}$.

ILP for the Maximum independent set.

<u>Maximize</u> $\sum_{v \in \{v_1,v_2,..,v_{2n}\}} x_v$.

Choosing variables we want the subset of vertices. so we choose x_v. for each $v \in \{v_1, v_2, \ldots, v_{2n}\}$. if we choose x_v it will be 1 and if we don't takit it will be 0.

<u>Constraints</u>:-

For every edge at ~~least~~ most one vertices is chosen

so, $x_u + x_v \leq 1$ for every ~~$u, v \in \{v_1, v_2, \ldots, v_{2n}\}$~~ ~~and~~ $u,v \in \{(v_1,v_2),(v_2,v_3)\cdots,(v_{2n},v_1)\}$

$0 \leq x_v \leq 1$ for every $v \in \{v_1, v_2, .., v_{2n}\}$.

$x_v \in \mathbb{Z}$.

the LP relaxation of the problem is.

Maximize. $\sum_{v \in \{v_1,v_2,..,v_{2n}\}} x_v$.

$2n$ no of edges.

Constraints $x_u + x_v \leq 1$ $\forall\ uv \in \{(v_1,v_2)\ldots,(v_{2n},v_1)\}$.

$0 \leq x_v \leq 1$ $\forall\ v \in \{v_1, v_2, .., v_{2n}\}$.

so, summing all the constraints we get $2.\sum_{v=1}^{2n} x_v = 2n$

so, $\sum_{v=1}^{2n} x_v = n$

For any maximization problem, we know that

ILP optimum $\leq$ LP optimum.

Here in this problem of C_{2n} LP optimum is n. Which is an integer value. Because no. of vertices cannot be fractional. as the total number of vertices are $2n$.

so, LP optimum occurs at an integral point n.

so, $\Rightarrow$ ILP optimum $\geq$ $c(n)$ [cost at n]

so, ILP optimum = LP optimum.

5. A universe D consisting of finite number of elements and a family $S_1, S_2, \ldots, S_m$ of sets with $S_i \subseteq D$ and $|S_i| = 3$ for every $i \in \{1, \ldots, m\}$.

Goal:- Find a minimum size subset $W \subseteq D$ of the universe that intersects with each S_i; that is. $W \cap S_i$ is non empty for every $i \in \{1, \ldots, m\}$.

Let D has n elements. $D = \{d_1, d_2, \ldots, d_n\}$

Let x_i is the indication variable if d_j ~~belon~~ $\in W$

~~otherwise d_j~~ = then $x_i = 1$

otherwise $x_i = 0$. if $d_i \notin W$.

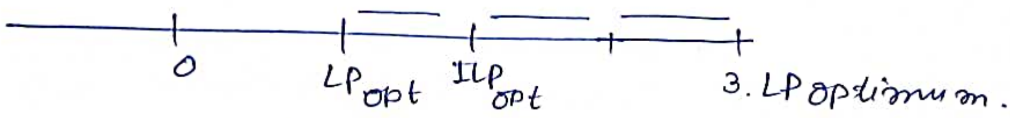
so, minimize :- $\sum_{j=1}^{m} x_i$.

Subject to :- for each $i \in \{1, 2, \ldots, m\}$

$$\sum_{d_i \in S_i} x_i \geq 1 \qquad x_i \in \{0, 1\}$$

$$i \in \{1, 2, 3\}$$

To show $\boxed{\text{ILP optimum} \leq 3 \text{ LP optimum}}$



Suppose x^* is the LP optimum.

We construct y^* as follows.

$$y^*_v = \begin{cases} 1 & \text{if } x^*_v \geq \frac{1}{3} \\ 0 & \text{otherwise.} \end{cases}$$

$\boxed{\text{claim 1:}}$ y^* is feasible solution to both LP and ILP.

So, $x_{v_1} + x_{v_2} + x_{v_3} \geq 1$.

any feasible solution should have at least one of either x_{v_1}, x_{v_2} or, x_{v_3}.

So, either of $x_{v_1} \geq \frac{1}{3}$ or, $x_{v_2} \geq \frac{1}{3}$ or, $x_{v_3} \geq \frac{1}{3}$.

$\Rightarrow x^*_{v_1} \geq \frac{1}{3}$ or, $x^*_{v_2} \geq \frac{1}{3}$ or, $x^*_{v_3} \geq \frac{1}{3}$.

$\Rightarrow y^*_{v_1} = 1$ or, $y^*_{v_2} = 1$ or, $y^*_{v_3} = 1$

therefore for every x_v we have $y^*_{v_1} + y^*_{v_2} + y^*_{v_3} \geq 1$.

$\boxed{\text{claim-2}}$ $\boxed{\sum_v x^*_v \leq \sum_v y^*_v \leq 3 \sum_v x^*_v}$

as. x^* is the LP optimum and no solution can be better than this for a minimization problem.

and we know that $y^*_v \le 3 x^*_v$.

then $\sum_v y^*_v \le 3 \sum_v x^*_v$. for every v.

Last claim :- $\boxed{\text{ILP optimum} \le \sum_v y^{*}_v}$

y^* is a feasible solution of ILP.

so, ~~we can get atmost~~ ILP optimum will always be less than any ILP solution. [as it is minimization problem]

Hence ILP optimum $\le \sum_v y^*_v$.

so, from all the previous problems.

ILP optimum $\le \sum_v y^*_v \le 3 \sum_v x^*_v \le 3$. LP. optimum.