23 April 2021

MSc. Data Science
LAA - Homework 1

1. Let A be an $m \times n$ matrix. Define

$$||A||_F = (\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}}.$$

(i) Prove that this is indeed a matrix norm.

(ii) Evaluate $||A||_1$, $||A||_2$, $||A||_\infty$ and $||A||_F$ for

$$A = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}.$$

(a) A be an $m \times n$ matrix and $||A||_F = (\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}}$.. To prove that this is indeed a matrix norm we need to verify the properties of the Matrix Norm.

- First, $||A||_F$ is positive definite as we are squaring all the entries and then taking square root of it. Hence, it is following the positive definite property.
- Second, We need to prove the triangle inequality. Let us take 2 norms, $||A||_F$ and $||B||_F$. Need to show, $||A+B||_F \leq ||A||_F + ||B||_F$.

$$||A+B||_F^2 = ||A||_F^2 + ||B||_F^2 + 2||AB||_F$$

Now,

$$||AB||_F = (\sum_{i=1}^{m}\sum_{j=1}^{n}(|a_{ij}||b_{ij}|)^2)^{\frac{1}{2}}$$

using Cauchy-Schwartz Inequality,

$$(\sum_{i=1}^{m}\sum_{j=1}^{n}(|a_{ij}||b_{ij}|)^2)^{\frac{1}{2}} \leq (\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}}(\sum_{i=1}^{m}\sum_{j=1}^{n}|b_{ij}|^2)^{\frac{1}{2}} = ||A||_F||B||_F$$

Hence,

$$2||AB||_F \leq 2||A||_F||B||_F$$

So,

$$||A+B||_F^2 \leq ||A||_F^2 + ||B||_F^2 + 2||A||_F||B||_F = (||A||_F + ||B||_F)^2$$

By applying Square Root on both side,

$$||A+B||_F \leq ||A||_F + ||B||_F \text{ (Proved)}$$

- We have already shown by CS inequality that, $||AB||_F \leq ||A||_F||B||_F$

As, it is following all the properties of a Matrix Norm, Frobenius norm is indeed a matrix norm.

(b)

$$A = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}.$$

- Calculating 1-Norm,

$$||A||_1 = \max_{1\leq j\leq 3}\left[\sum_{i=1}^{3}|a_{ij}|\right] = \max{[10,5,10]} = \mathbf{10}$$

- Calculating Infinity Norm,

$$||A||_\infty = \max_{1\le i\le 3}\left[\sum_{j=1}^{3}|a_{ij}|\right] = \max\,[10,5,10] = \mathbf{10}$$

- Calculating Frobenius Norm,

$$||A||_F = (\sum_{i=1}^{3}\sum_{j=1}^{3}|a_{ij}|^2)^{\frac{1}{2}} = [\ |4|^2+|-2|^2+|4|^2+|-2|^2+|1|^2+|-2|^2+|4|^2+|-2|^2+|4|^2\ ]^{\frac{1}{2}} = \mathbf{9}$$

- Now for calculating 2-norm, we need to evaluate the matrix A^*A.

$$A^*A = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}^* \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} = \begin{pmatrix} 36 & -18 & 36 \\ -18 & 9 & -18 \\ 36 & -18 & 36 \end{pmatrix}$$

  Now, calculating the eigen value of the A^*A,

$$\begin{vmatrix} 36-\lambda & -18 & 36 \\ -18 & 9-\lambda & -18 \\ 36 & -18 & 36-\lambda \end{vmatrix} = 0$$

  Solving this equation we get,

$$4\lambda^2 \times (81-\lambda) = 0$$

  Hence, the eigen values are, 0,0,81. As by the definition the 2-norm is the square root of of the maximum eigen value of the matrix A^*A,

$$||A||_2 = \sqrt{81} = \mathbf{9}$$

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2. Evaluate $||I_{n\times n}||_F$.

   According to the definition of Frobenius Norm,

$$||A||_F = (\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}}$$

   As $I_{n\times n}$ is Identity Matrix, then the diagonal entries are 1 and all the off diagonal entries are 0. Mathematically, $I_{ii} = 1$ and $I_{ij} = 0$, where $i \neq j$.

   Hence,

$$||I_{n\times n}||_F = [1^2 + 1^2 + ...\text{ n times}]^{\frac{1}{2}} = \sqrt{n}$$

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3. Prove that for the induced 2-norm and the Frobenius norm on matrices are invariant under multiplication by unitary matrices.

   A matrix norm is unitary invariant if, $||UAV|| = ||A||\ \forall\ U^*U = I, V^*V = I$.

- **Induced 2-Norm**

  So,

  $$(UAV)^*(UAV) = V^*A^*U^*UAV = V^*A^*AV \text{ as, } U^*U = I$$

  We know that for an Unitary Matrix,$V^*V = I = VV^*$. Hence, $V^* = V^{-1}$

  $$(UAV)^*(UAV) = V^{-1}A^*AV$$

  From the above equation, it is clear that, $(UAV)^*(UAV)$ and A^*A are similar. Hence both of them will have same eigen values.

  So, by definition of 2-Norm,

  $$||UAV||_2 = \sqrt{\lambda_{\max}((UAV)^*(UAV))} = \sqrt{\lambda_{\max}(A^*A)} = ||A||_2$$

  Where, $\lambda_{\max}$ is the maximum eigen value.

  Hence, Induced 2-Norm of a Matrix is Unitary Invariant.(Proved)

- **Frobenius Norm**

  According to the definition of Frobenius Norm,

  $$||A||_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2 = \sum_{i=1}^{m}(\sum_{j=1}^{n}a_{ij}^*a_{ij}) = \sum_{i=1}^{m}(A^*A)_{ii} = \text{tr } (A^*A)$$

  Again, to show, the Frobenius Norm to be Unitary Invariant,

  we need to show, $||UAV||_F = ||A||_F \; \forall \; U^*U = I, V^*V = I$.

  Hence,

  $$\text{tr } (UAV)^*(UAV) = \text{tr } (V^*A^*U^*UAV) = \text{tr } (V^*A^*AV) \text{ as, } U^*U = I$$

  Since, $\text{tr } (AB) = \text{tr } (BA)$,

  $$\text{tr } (UAV)^*(UAV) = \text{tr } (V^*A^*AV) = \text{tr } (VV^*A^*A)$$

  As, V is Unitary, $V^*V = VV^* = I$.

  $$\text{tr } (UAV)^*(UAV) = \text{tr } (VV^*A^*A) = \text{tr } (IA^*A) = \text{tr } (A^*A) = ||A||_F$$

  Hence, Frobenius Norm of a Matrix is Unitary Invariant. (Proved)

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4. Study Section 1.3 of Strang's book 'Linear Algebra and Learning from Data' (you can find it here). Then solve problems 1, 2, 4 and 6 from problem set 1.3 (page 20).

   (a) **The Null Space of AB contains the Null Space of B.**

   Let a vector v is in the Null Space of B. Then, by the definition of Null Space, $Bv = 0$. So, multiplying A on the both side,

   $$A(Bv) = 0$$

   $$(AB)v = 0$$

   Hence, v is also in the Null Space of AB. Hence Proved.

(b) **Find a square matrix with rank(A^2) < rank(A). Confirm that the rank(A^TA) = rank(A).**

The nilpotent square matrix of degree 2 will give $\text{rank}(A^2) < \text{rank}(A)$. The example of such matrix is,

$$A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

where, $a \in \mathbb{R}, a \neq 0$.
Hence, $A^2 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
The $\text{rank}(A) = 1$, but the $\text{rank}(A^2) = 0$. Hence, $\text{rank}(A^2) < \text{rank}(A)$.
Now, $A^TA = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \times \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix}$. The $\text{rank}(A^TA) = 1$
Hence, $\text{rank}(A^TA) = \text{rank}(A)$ (Confirmed).

(c) **If row space of A = Column Space of A, and also $N(A) = N(A^T)$, is A symmetric?**

by the definition of null space $N(A)$ = Contains all solutions x to $Ax = 0$.
and the definition of Left null space $N(A^T)$ = Contains all solutions of y to $A^Ty = 0$.

As, $\text{rank}(A) = \text{rank}(A^T)$, by rank-nullity theorem, A is a square matrix.

By, the definition of column space $A = C(A)$ = Contains all combinations of the columns of A.
and the definition of row space $A = C(A^T)$ = Contains all combinations of the columns of A^T.

It is given that, row space of A = Column Space of A, then from the above definitions, $C(A) = C(A^T)$.
So, $\dim C(A) = \dim C(A^T) = \text{rank}(A)$.

Now, here two cases may arise, the rank of A is equal to the dimension of the matrix, or the rank of A is less than equals to the dimension of the matrix.
Hence, If A is of full rank. Then, A is invertible and the null space and the left null space will have the dimension 0.In this case A need not to be symmetric.
Example - $\begin{pmatrix} 1 & 4 \\ 3 & 8 \end{pmatrix}$. This matrix is not symmetric but the row space of A = Column Space of A, and also $N(A) = N(A^T)$.
Now, If, A is not of full rank, then it should be symmetric.
Example - $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

So, A **doesn't necessarily be a symmetric matrix.**

(d) **Show that A^TA has the same null space as A.**

First, let's assume x is a vector in the null space of A. So, $Ax = 0$.
Multiplying A^T on both side,

$$A^T(Ax) = 0 = (A^TA)x$$

Hence, null space of A^TA contains the null space of A.
Now, Let, y is a vector in the null space of A^TA. So, $A^TAy = 0$.
Multiplying y^T on the both side,

$$y^TA^TAy = 0$$
$$\implies (y^TA^T)Ay = 0$$

$$\implies (Ay)^T Ay = 0$$
$$\implies ||Ay||^2 = 0$$
$$\implies Ay = 0$$

hence, null space of A contains the null space of $A^T A$.
Finally we can say that,$A^T A$ has the same null space as A (proved).
