LPCO Problem Set 1

January 17, 2021

Problem 1. You have seen in class that an LP might be *unbounded*, i.e., there will be feasible points with arbitrarily large values for the objective function. It might also happen that an LP is *infeasible*, i.e., there are no feasible points at all for the LP. Give an example each of an unbounded LP and an infeasible LP.

Problem 2. You saw in class an LP for the maximum matching problem. Try to construct LPs for the following two problems:

Definition 1 (Max-flow)**.** Suppose you are given a graph $G = (V, E)$ with two special vertices, a source vertex s and a sink vertex t. Assume that each edge (i, j) has an upper bound c_{ij} called it's capacity. Find the maximum flow possible from s to t.

Think of each edge as a pipe having maximum capacity c_{ij} that can pass through it. We send oil into the system at the vertex s, and oil exits the system at vertex t. Oil cannot enter or exit the system through any of the other nodes. We need to find the maximum possible oil that can be transported from s to t through this graph. Hint: Every vertex other than s and t should have amount of incoming oil = amount of outgoing oil.

Definition 2 (Min-cut)**.** Suppose you are given a graph $G = (V, E)$ with two special vertices, a source vertex s and a sink vertex t. Assume that each edge (i, j) has a cost c_{ij}. Find the minimum cut of s and t.

Consider the slightly different problem we have here. We still have a source vertex s and a sink vertex t, but now we for each pipe (i, j), we have a cost c_{ij} associated with destroying the pipe. We need to find the minimum total cost of pipes we need to destroy to stop any oil flowing from s to t.

This problem has an interesting history dealing with a secret report for the US Air Force, the Cold War and the Soviet railway network: Link to Schrijver's paper. Schrijver incidentally also has a 2000 page book on Combinatorial Optimization that might help you out with the course :)

Problem 3. While you will learn how to solve general LPs next in the course, you can actually solve an LP with two variables by hand. Just take the variables to be the x and y axis and plot, intersect, and shade on a graph to find the solution! Try solving the following LP using the graphical method:

$$\begin{aligned}
\text{maximize: } & x_1 + x_2 \\
\text{subject to: } & x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_2 - x_1 \leq 1 \\
& x_1 + 6x_2 \leq 15 \\
& 4x_1 - x_2 \leq 10.
\end{aligned}$$

Problem 4. While LPs have variables take any real number, a restricted version called Integer Linear Programming (ILP) allows the variables to only be integers. ILPs are very useful in modelling combinatorial problems, in fact, an even stricter restriction called 0/1 Integer Linear Program can be used to model a huge class of problems. A 0/1 ILP is one in which each variable can only take a value in $\{0, 1\}$.

Definition 3 (3-SAT)**.** A 3-SAT formula is a boolean formula which is an AND of clauses, where each clause is an OR of 3 literals. For example, $(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_4 \vee \overline{x_5})$ is a 3-SAT formula.

Show that 0/1-ILP is NP-hard. Do this by exhibiting, for each instance of 3-SAT, an instance of 0/1-ILP such that the solution to the ILP helps you produce a solution to the 3-SAT problem.

Because $\{0, 1\} \subset \mathbb{Z}$, if 0/1-ILP is NP-hard, then ILP is NP-hard as well right? *Riiiight?*

Problem 5. If the feasible set is unbounded, is the solution necessarily unbounded? If the feasible set is bounded, is the solution bounded?