24/04/2018

MSc. Applications of Mathematics

Numerical Linear Algebra - Final Exam

Time allowed: 3 hours | Total: 70 points

Instruction: Show all your work; writing only the answer will not fetch any points.

SOLUTIONS

1. (15 points) Suppose we want to reduce the following matrix to Hessenberg form. Perform the first step of reduction i.e. $Q_1^* A Q_1$. Describe what should be done next to obtain the complete reduction.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 5 & 3 \\ 2 & 3 & 7 & 2 \\ 1 & 0 & 4 & 5 \end{pmatrix}.$$

Solution: $u_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $v_1 = u_1 + ||u_1||e_1 = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$.

$$H_{v_1} = I - \frac{2v_1v_1^*}{v_1^*v_1} = \begin{pmatrix} -0.6666667 & -0.6666667 & -0.3333333 \\ -0.6666667 & 0.7333333 & -0.1333333 \\ -0.3333333 & -0.1333333 & 0.9333333 \end{pmatrix}.$$

$$Q_1^* A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -3 & -3.33333 & -9.333324 & -4.999995 \\ 0 & 0.866667 & 1.266669 & -1.199997 \\ 0 & -1.066665 & 1.133336 & 3.4 \end{pmatrix}.$$

$$Q_1^* A Q_1 = \begin{pmatrix} 1 & -4.6666667 & 0.3333333 & 2.6666667 \\ -3 & 10.111112 & -3.9555554 & -2.3111113 \\ 0 & -1.022222 & 0.5111109 & -1.5777776 \\ 0 & -1.177778 & 1.0888891 & 3.3777776 \end{pmatrix}.$$

To get the complete reduction, calculate the Householder matrix H corresponding to the vector $[-1.022, -1.177]$. Then form the matrix $Q_2^* = \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix}$ and calculate $Q_2^*(Q_1^* A Q_1)Q_2$.

2. (10 points) Show that the Householder reflector $H = 1 - 2ww^T$, with $||w|| = 1$, is symmetric and orthogonal. Find the eigenvalues and eigenvectors of H.

Solution:

Symmetry:

$$H^T = (I - 2ww^T)^T = I - 2(ww^T)^T - I - 2ww^T = H.$$

Orthogonality:

$$H^T H = H^2 = (I - 2ww^t)(I - 2ww^T) = I - 4ww^t + 4ww^T ww^T - = I - 4ww^T + 4wIw^T = I.$$

Eigenvalues: Let us look for vectors v_i and numbers λ_i obeying $Hv_i = \lambda_i v_i$. First, suppose $w \parallel v$:

$$Hv - v - 2ww^T v = v - 2v = -v.$$

Hence the matrix H has one eigenvalue -1 with the corresponding eigenvector being parallel to w.

Now, suppose $w \perp v$:

$$Hv = v - 2ww^T v = v$$

Hence any vector $w \perp v$ is an eigenvector of H, and hence H has an eigenvalue 1 of multiplicity $n-1$ where n is the dimension of H.

3. (10 points) Apply 2 steps of the Rayleigh quotient iteration to locate an approximate eigenvalue/eigenvector pair for the matrix $\begin{pmatrix} 7 & 3 \\ 1 & 0 \end{pmatrix}$. Use initial vector $v^{(0)} = [0, 1]^t$ and the 1-norm for your calculations.

   Solution: Algorithm for Rayleigh quotient iteration:

   Start with $v^{(0)}$ such that $||v|| = 1$.

   $\lambda^{(0)} = v^{(0)^T} A v^{(0)}$.

   for $k = 0$ to convergence,

   Solve $(A - \lambda^{(k-1)} I)w = v^{(k-1)}$ for w using inverse iteration;

   $v^{(k)} = w/||w||$;

   $\lambda^{(k)} = R_A(v^{(k)})$.

   For the given matrix and starting vector $v^{(0)}$, we get,

   1st iteration: $\lambda^{(0)} = 0$, so $A - \lambda^{(0)} I = A$ and solving the system $Aw = v^{(0)}$ for w gives $w = \begin{pmatrix} 1 \\ -7/3 \end{pmatrix}$.

   The 1-norm of w is its maximum column sum which is $10/3$ so that $v^{(1)} = \begin{pmatrix} 3/10 \\ -7/10 \end{pmatrix}$. Then

   $$\lambda^{(1)} = R_A(v^{(1)}) = \frac{v^{(1)t} A v^{(1)}}{v^{(1)t} v^{(1)}} = \frac{-21/10}{29/50} = \frac{-105}{29}.$$

   2nd iteration: $A - \lambda^{(1)} I = \begin{pmatrix} 308/29 & 3 \\ 1 & 105/29 \end{pmatrix}$. Solving $(A - \lambda^{(1)} I)w = v^{(1)}$ for w gives $w = \begin{pmatrix} 0.0333 \\ 0.2181 \end{pmatrix}$.

   The 1-norm of w in this case is $0.0333 + 0.2181 = 0.2514$ so that $v^{(2)} = w/||w|| = \begin{pmatrix} 0.1324 \\ 0.8675 \end{pmatrix}$.

   Then

   $$\lambda^{(2)} = R_A(v^{(2)}) = \frac{v^{(2)t} A v^{(2)}}{v^{(2)t} v^{(2)}} =$$

4. (10 points) Match each of the following situations with the method you will use for them:

   Situations:

(i) Solve a system of equations iteratively when lesser memory usage is preferred.

(ii) Compute a particular eigenvalue and the corresponding eigenvector of a given matrix.

(iii) Compute a set of largest eigenvalues (in absolute value) and corresponding eigenvectors of a given matrix.

(iv) Solve a system of equations iteratively on a parallel system when memory is not an issue.

(v) Compute the largest eigenvalue (in absolute value) and the corresponding eigenvector of a given matrix.

Methods:

(a) Power iteration.

(b) Jacobi method.

(c) Inverse power iteration.

(d) Gauss-Seidel method.

(e) Simultaneous iteration.

Solution: (i)-(d), (ii)-(c), (iii)-(e), (iv)-(b), (v)-(a).

5. (10 points) Suppose the $n \times n$ real matrix A admits a Cholesky factorization $A = BB^T$. Show that:

$$\kappa(B) = \kappa(A)^{\frac{1}{2}}.$$

Solution 1:

- $$\begin{aligned} ||A||^2 &= \lambda_{max}^{AA^T} \qquad (by\ definition) \\ &= \lambda_{max}^{A^2} \qquad (since\ A\ is\ symmetric) \\ &= (\lambda_{max}^{A})^2 \qquad (since\ A\ is\ positive\ definite). \\ \text{Thus } ||A|| &= \lambda_{max}^{A}. \end{aligned}$$

- $||B||^2 = \lambda_{max}^{BB^T} = \lambda_{max}^{A} = ||A||$.
- Similarly, $||B^{-1}|| = ||A^{-1}||$.
- $\kappa(A) = ||A|| \cdot ||A^{-1}|| = ||B||^2 \cdot ||B^{-1}||^2 = \kappa(B)^2$

Solution 2 : Let the SVD of B be $B = U\Sigma V^*$. Since condition number is preserved under unitary operations,

$$\kappa(B) = \kappa(\Sigma).$$

Further $A = BB^T$ implies

$$\kappa(A) = \kappa(U\Sigma V^*(V^*)^T\Sigma^T U^T) = \kappa(\Sigma^2).$$

Suppose $\Sigma = $diag$(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, then $\Sigma^2 = $diag$(\sigma_1^2, \ldots, \sigma_n^2)$. So

$$\kappa(B) = \kappa(\Sigma) = \frac{\sigma_1}{\sigma_n}, \quad \kappa(A) = \kappa(\Sigma^2) = \frac{\sigma_1^2}{\sigma_n^2}.$$

Thus $\kappa(B) = \kappa(A)^{\frac{1}{2}}$.

6. (10 points) Approximate the solution of the following linear system upto two iterations using Gauss-Seidel method with initial approximation $(x_0, y_0, z_0) = (0, 0, 0)$. Also state the Gauss-Seidel matrix.

$$\begin{pmatrix} 5 & 2 & -1 \\ 3 & 7 & 3 \\ 1 & -4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Solution: A splitting of $A = \begin{pmatrix} 5 & 2 & -1 \\ 3 & 7 & 3 \\ 1 & -4 & 6 \end{pmatrix}$ is given by $A = D - E - F$ where

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 6 \end{pmatrix}, E = \begin{pmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ -1 & 4 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & -2 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$(D - E)^{-1} = \begin{pmatrix} 1/5 & 0 & 0 \\ -3/35 & 1/7 & 0 \\ -19/210 & 4/42 & 1/6 \end{pmatrix}.$$

The Gauss-Seidel matrix $L_1 = (D-E)^{-1}F = \begin{pmatrix} 0 & -2/5 & 1/5 \\ 0 & 6/35 & -18/35 \\ 0 & 38/210 & -79/210 \end{pmatrix} = \begin{pmatrix} 0 & -0.4 & 0.2 \\ 0 & 0.1714286 & -0.5142857 \\ 0 & 0.1809524 & -0.3761905 \end{pmatrix}$.

The Gauss-Seidel iteration is given by

$$X^{(k+1)} = L_1 X^{(k)} + (D - E)^{-1} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

- First iteration: $X^{(0)} = [0, 0, 0]^T$.

$$X^{(1)} = L_1 X^{(0)} + (D - E)^{-1} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/5 \\ -11/35 \\ -23/210 \end{pmatrix} = \begin{pmatrix} 0.4 \\ -0.3142857 \\ -0.1095238 \end{pmatrix}.$$

- Second iteration:

$$X^{(2)} = L_1 X^{(1)} + (D - E)^{-1} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 529/1050 \\ 2292/7350 \\ 5521/44100 \end{pmatrix} = \begin{pmatrix} 0.5038095 \\ -0.3118367 \\ -0.1251927 \end{pmatrix}.$$