25 September 2020 Time allowed: 4 hours

M.Sc. Data Science

Linear algebra and its applications - Final Exam

Note:

- *Answers must be clear and complete to receive grades. Write proper intermediate steps in order to avoid losing marks.*
- *Use the ∞-norm in all your calculations, unless otherwise specified.*

1. (10 points) Let A be a Hermitian, tri-diagonal matrix

$$A = \begin{pmatrix} a_1 & b_1 & 0 & \dots & \dots & 0 \\ b_1 & a_2 & b_2 & 0 & \dots & 0 \\ 0 & b_2 & a_3 & b_3 & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & b_{m-2} & a_{m-1} & b_{m-1} \\ 0 & \dots & \dots & 0 & b_{m-1} & a_m \end{pmatrix}$$

with all off-diagonal entries b_i being non-zero. Show that the eigenvalues of A are distinct.

2. (10 points) You are given a matrix

$$A = \begin{pmatrix} -4 & 1 & 1 \\ 0 & 3 & 1 \\ -2 & 0 & 15 \end{pmatrix}$$

and the information that a desired eigenvalue of A is close to 3 (for example, we can know this by looking at the Gershgorin discs). Using the initial vector $v^{(0)} = [1, 1, 1]^t$, apply 2 iterations of the method you would use to find the desired eigenvalue. You may leave the eigenvalue estimates displayed in the correct form without simplifying to the final answer.

3. (10 points) Suppose we want to reduce the following matrix to Hessenberg form. Perform the first step of reduction i.e. $Q_1^* A Q_1$. Describe what should be done next to obtain the complete reduction.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 0 & 1 \end{pmatrix}.$$

4. (15 points) Consider the Jacobi's method for finding eigenvalues applied to the symmetric 2×2 matrix:

$$A = \begin{pmatrix} a & d \\ d & b \end{pmatrix}.$$

   (a) Prove that the diagonalisation can be obtained if the angle of rotation θ satisfies

$$\tan 2\theta = \frac{2d}{b-a}.$$

   (b) Suppose now that $A \in \mathbb{R}^{m \times m}$, A is symmetric. How many flops are required for executing the computation $J^T A J$ once? How many flops for zeroing all the off-diagonal entries (i.e. one sweep)?

(c) How does the above operation count compare with the total operation count for tridiagonalizing a real symmetric matrix and finding its eigenvalues using the QR iteration?

5. (10 points)

   (a) Find the projection matrix P that projects any vector in $\mathbb{R}^3$ onto the column space of $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   (b) Find the projection of vector $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ on the column space of A.

6. (10 points)

   (a) Describe an algorithm (or write a pseudocode) for computing the inverse of an invertible $n \times n$ matrix A using Gaussian elimination. What is the operation count?

   (b) In solving a system of equations $Ax = b$, which of the following is more advantageous: using the LU factorization of A and solving $LUx = b$, or computing A^{-1} and then $x = A^{-1}b$?

7. (15 points) Approximate the solution of the following linear system upto one iteration using the

   (a) Jacobi method,

   (b) Gauss-Seidel method

   (c) SOR method with $\omega = 2$.

$$\begin{aligned} 2x_1 + 7x_2 + x_3 &= 19 \\ 4x_1 + x_2 - x_3 &= 3 \\ x_1 - 3x_2 + 12x_3 &= 31 \end{aligned}$$

Let the initial approximation be $(x_0, y_0, z_0) = (0, 0, 0)$. In each case state the matrix of the method. (Hint: First change the order of equations to obtain a strictly diagonally dominant coefficient matrix.)

8. (10 points) Let A be a $m \times n$ matrix with SVD $A = U\Sigma V^T$. Compute the SVDs of the following matrices in terms of U, Σ and V:

   (a) $(A^T A)^{-1}$

   (b) $(A^T A)^{-1} A^T$

   (c) $A(A^T A)^{-1}$

   (d) $A(A^T A)^{-1} A^T$