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Date:- 09.07.21

Linear Algebra and its Application

Roll- MDS202035.

6. Considering an algorithm for QR factorization of a given Matrix A, let $\tilde{Q}$ and $\tilde{R}$ denote the computed matrices.

For Householder's transformation we know that the algorithm is backward stable. so,

For showing <u>backward Stability</u>:- If A is slightly ~~pert~~ perturbed by δA. then also we can find its accurate QR factorization.

$$\tilde{Q}\tilde{R} = A + \delta A \quad , \text{ then } \frac{\|\delta A\|}{\|A\|} = O(\epsilon_{machine})$$

$\boxed{\text{Backward Stability} \Rightarrow \text{Forward Stability}}$

So, no need to check it separately.

<u>For Accuracy</u>:- In Householder's orthogonalization, $\tilde{Q}$ is almost orthogonal in the sense that $\|I - \tilde{Q}^T\tilde{Q}\| \leq c\epsilon$.

ϵ = machine precision.

c depends only on the dimension of A.

So, for Accuracy, due to rounding off errors, we have

$A = \tilde{Q}\tilde{R} + E$, where $\tilde{Q}$ is no longer orthogonal but

Somewhat orthogonal. We can show that $\|E\| \leq c\epsilon\|A\|$. Which we have written earlier that $\|I - \tilde{Q}^T\tilde{Q}\| \leq c\epsilon$.

By the way Householder compute an accurate R-factor $\tilde{R}$ in the sense of backward error and there exist an orthogonal matrix $\hat{Q}$ and residual matrix $\hat{E}$, such that $\|\hat{E}\| \leq c\epsilon\|A\|$. and $A = \hat{Q}\tilde{R} + \hat{E}$.

2. (a) In solving a system of equations $Ax = b$, of course $LUx = b$ is more advantageous than computing $x = A^{-1}b$.

For $LUx = b$.

we can perform 2-step solution procedure.

1. Solve the lower triangular system $Ly = b$. for y by forward substitution.

2. Then solve the upper triangular system $Ux = y$ for x by backward substitution.

We won't solve it using inverse method because ~~A^{-1} is~~ finding A^{-1} is more computationally expensive.

$$\cancel{A^{-1} \neq (LU)^{-1} \neq U^{-1}L^{-1}}$$

$$\cancel{x = A^{-1}b = U^{-1}L^{-1}b.}$$

$$\cancel{\text{so, } y_1 = (L^{-1}b)}$$

$$\cancel{\text{and. } x = U^{-1}y_1}$$

~~in 2 steps we need to compute.~~

and finding inverse is difficult, so, we use LU.

inverse map is only well defined iff $Ax = b$ has unique solution. So, if A is singular A^{-1} doesn't exist but we can find LU.

more precisely,

1. Computing the inverse takes a lot of time.
2. In cases when the given matrix actually ~~have~~ has many 0 entries, computing the inverse is more diffficult.
3. Sometimes the inverse is so inaccurate that it is not worth the trouble to multiply by the inverse to get the solution.

But LU factorization is unstable. So, we can use complete pivoting for making it backward stable.
LU can be computed for singular matrices too.
So, Implementing LU is more ~~effi~~ efficient and accurate numerically.

(b). Flop count: $\boxed{x = A^{-1} b}$
Constructing A^{-1} takes $O(n^3)$ for a $n \times n$ matrix.

Flop count:- $LUx = b$.
LU factorization takes. $O(\frac{2}{3}n^3)$.
and for back substitution it takes $O(n^2)$.
So, the overall flop count is $O(\frac{2}{3}n^3)$.

proof of the flop count of LU factorization is in the next page.

k	i	add-sub flops.	mult-div flops.
1	$2:n = (n-1)$ rows.	$(n-1)\times n$	$(n+1)\times(n-1)$
2	$3:n = (n-2)$ rows.	$(n-2)(n-1)$	$n\times(n-2)$
⋮	⋮	⋮	⋮
$n-1$	$n:n$, 1 rows.	1×2	1×3.

Now, add the add sub flops,

$$\sum_{k=1}^{n-1}(n-k)(n+1-k) = \sum_{k=1}^{n-1}\left(n^2+n-2nk-k+k^2\right) = \frac{1}{3}n^3-\frac{1}{3}n.$$

and mult div flops.

$$\sum_{k=1}^{n-1}(n-k)(n+2-k) = \sum_{k=1}^{n-1}\left(n^2+2n-2nk+2k+k^2\right) = \frac{1}{3}n^3+\frac{5}{2}n^2-\frac{17}{6}.$$

So, the flop count for LU factorization is $O(\frac{2}{3}n^3)$.

4. We want to reduce the following matrix to Hessenberg form.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 2 \\ 3 & 4 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \qquad u_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$\therefore\ v_1 = u_1 + \|u_1\| e_1$

$$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \sqrt{1^2+3^2+0^2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + 3.1623\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 4.1623 \\ 3 \\ 0 \end{bmatrix}$$

$$H_{v_1} = I - \frac{2 v_1 v_1^*}{v_1^* v_1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{26.3246} \begin{bmatrix} 4.1623 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 4.1623 & 3 & 0 \end{bmatrix}$$

$$H_{v_1} = \begin{bmatrix} -0.3162 & -0.9487 & 0 \\ -0.9487 & 0.3162 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q_1^* A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.316 & -0.948 & 0 \\ 0 & -0.948 & 0.3162 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 2 \\ 3 & 4 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ -3.162 & -4.447 & -1.5811 & -2.5298 \\ 0 & -0.6326 & -1.5812 & -1.265 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$Q_1^* A Q_1 = \begin{bmatrix} 1 & -0.948 & 0.316 & 0 \\ -3.1623 & 2.9 & 3.7 & -2.5298 \\ 0 & 1.697 & 0.0916 & -1.265 \\ 0 & -1.264 & -0.632 & 2 \end{bmatrix}$$

for complete reduction, we need to calculate the householders' matrix H corresponding to the vector $\begin{bmatrix} 1.697 & -1.264 \end{bmatrix}$

— We then form the matrix $Q_2^* = \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix}$ and

calculate $Q_2^*(Q_1^* A Q_1) Q_2$

3. (a) let $(A - \lambda I)$ is also a tridigonal and all its off diagonal entries are nonzero. We can write $(A - \lambda I)$ as a block matrix $\begin{bmatrix} V^T & 0 \\ B & u \end{bmatrix}$ where B is an $(m-1)\times(m-1)$ matrix as. A we have taken as $m \times m$ matrix. u, v are vectors of length $m-1$.

Now, we can see that B is upper triangular and with nonzero diagonal entries. Thus B is non-degenerate. If $|A - \lambda I| = 0$, then rank $(A - \lambda I) = m-1$. This implies $\dim(\text{null}(A - \lambda I)) = 1$ i.e the geometric multiplicity of A is 1. As A is Hermitian, the GM coinsides with AM. Thus we can show that the eigen values of A are distinct.

(b) $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

we know that no. of eigen values of A in $[1, 2]$

= No. of eigen values in $(-\infty, 2)$ − No. of eigen values in $(-\infty, 1)$

= No. of neg eigen values of $(A - 2I)$ − No of negative eigen values of $(A - I)$.

Now, let

$$B = A-2I = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

taking the principal submatrices.

$$|B^{(1)}| = -1$$

$$|B^{(2)}| = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0$$

$$|B^{(3)}| = \begin{vmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 1$$

$$|B^{(4)}| = \begin{vmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= -1\begin{vmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1$$

the changes in the determinant is —

$1, |B^{(1)}|, |B^{(2)}|, |B^{(3)}|, |B^{(4)}|$ is $1, -1, 0, 1, 1$ respectively.

so, there are 2 sign changes. So, No. of negative eigen values of $B = 2$.

Again $C = A - I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$$|c^{(1)}| = 0$$

$$|c^{(2)}| = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$|c^{(3)}| = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1(1 \times 1) = -1$$

$$|c^{(4)}| = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -1\left[1(2-1) - 1 \times 0\right]$$
$$= -1$$

Now, we see the changes of sign in the determinant.

1	1
$\lvert c^{(1)} \rvert$	0
$\lvert c^{(2)} \rvert$	-1
$\lvert c^{(3)} \rvert$	-1
$\lvert c^{(4)} \rvert$	-1

There are only 1 sign change.

$\therefore$ No. of negative eigen values of $c = 1$.

$\therefore$ No. of eigen values of A in $[1,2]$ = No. of negative eigen values of B − No. of negative eigen value of $c = 2 - 1 = 1$

5. Jacobi's method for finding eigenvalues of the matrix.

$$A = \begin{bmatrix} -3 & -5 \\ -5 & 7 \end{bmatrix}$$

A is a real symmetric matrix. The off diagonal element is -5 which is highest by magnitude.

$A_{12} = -5 \qquad A_{21} = -5$

So $A_{12} = A_{21}$

$p = 1 \qquad q = 2 \qquad A_{pp} = -3, \quad A_{qq} = 7$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2A_{pq}}{A_{pp}-A_{qq}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2(-5)}{-3-7}\right) = \frac{1}{2}\tan^{-1}(1) = \frac{\pi}{8}$$

$$J_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{8} & -\sin\frac{\pi}{8} \\ \sin\frac{\pi}{8} & \cos\frac{\pi}{8} \end{bmatrix} = \begin{bmatrix} 0.9238 & -0.30268 \\ 0.38268 & 0.92388 \end{bmatrix}$$

$$J_1 = J_1^T A J_1 = \begin{bmatrix} -5.071068 & 0 \\ 0 & 9.07168 \end{bmatrix}$$

The eigen value of A are ~~o~~ (-5.71068) and (9.071).

7. (i) Solve a system of equations iteratively when lesser memory usage is preferred

(d) Gauss Seidel method

~~Th~~ It requires the least allocation among all the methods we have studied in the class. So, it is preferable.

(ii) compute a particular eigenvalue and the corresponding eigen vector of a given matrix.

(c) Inverse power iteration.

This method converges to a particular eigenvalue As here in the question we are required to find a particular eigen value and its corresponding eigen vector, it is preferrable to use inverse iteration.

(iii) compute a set of largest eigenvalues (in absolute value) and corresponding eigenvectors of a given matrix

(e) Simultaneous iteration.

This simultaneous iteration converges to a set of eigenvectors. here we need to find the number of ~~largest~~ eigenvalues from these converged vectors which are largest.

(iv) Solve a system of equations iteratively on a parallel system when memory is not an issue.

(b) Jacobi method.

Jacobi method ~~are~~ has some parts that are independent to eachother, thus it is suitable for solving a system of equations iteratively on a parallel system when memory is not an issue.

(v) compute the largest eigen value (in absolute value) and the corresponding eigen vector of a given ~~g~~ matrix.

(a) power iteration.

In power iteration method convergence is ~~guaranteed~~ inevitable to the maximum eigen value. So, for computing the largest absolute eigen value and the corresponding eigen vector of a given matrix

P.T.O.

8. $A = \begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix}$ $\quad v^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Here we'll use power iteration method as it finds the dominant eigenvector/eigen value of A.

$$P_1 = A v^{(0)} = \begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 8 \end{bmatrix}$$

$$v^{(1)} = \frac{P_1}{\| P_1 \|_2} = \frac{1}{\sqrt{49+64}} \begin{bmatrix} -7 \\ 8 \end{bmatrix} = \begin{bmatrix} -0.659 \\ 0.753 \end{bmatrix}$$

$$\lambda^{(1)} = v^{(1)T} A v^{(1)}$$

$$= \begin{bmatrix} -0.659 & 0.753 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} -0.659 \\ 0.753 \end{bmatrix}$$

$$= \begin{bmatrix} 10.637 & -2.071 \end{bmatrix} \begin{bmatrix} -0.659 \\ 0.753 \end{bmatrix} = -8.569.$$

After the first iteration $\lambda^{(1)} = -8.569$.

$$P_2 = A v^{(1)} = \begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} -0.659 \\ 0.753 \end{bmatrix} = \begin{bmatrix} 6.119 \\ -6.025 \end{bmatrix}$$

$$v^{(2)} = \frac{P_2}{\| P_2 \|_2} = \frac{1}{8.587} \begin{bmatrix} 6.119 \\ -6.025 \end{bmatrix} = \begin{bmatrix} 0.713 \\ -0.702 \end{bmatrix}$$

$$\lambda^{(2)} = v^{(2)T} A v^{(2)} = \begin{bmatrix} 0.713 & -0.702 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 0.713 \\ -0.702 \end{bmatrix}$$

$$= -9.057$$

After 2nd iteration $\lambda^{(2)} = -9.057$

$$P_3 = A v^{(2)} = \begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 0.713 \\ -0.702 \end{bmatrix} = \begin{bmatrix} -6.395 \\ 6.406 \end{bmatrix}$$

$$v^{(3)} = \frac{P_3}{\|P_3\|_2} = \frac{1}{9.052} \begin{bmatrix} -6.395 \\ 6.406 \end{bmatrix} = \begin{bmatrix} -0.706 \\ 0.708 \end{bmatrix}$$

$$\lambda^{(3)} = v^{(3)T} A v^{(3)} = \begin{bmatrix} -0.706 & 0.708 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} -0.706 \\ 0.708 \end{bmatrix}$$

$$= \begin{bmatrix} -0.706 & 0.708 \end{bmatrix} \begin{bmatrix} 6.358 \\ -6.356 \end{bmatrix} = -8.989.$$

$\lambda^{(3)} = -8.989.$

Now, $P_4 = A v^{(3)} = \begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} -0.706 \\ 0.708 \end{bmatrix} = \begin{bmatrix} 6.358 \\ -6.356 \end{bmatrix}$

$$v^{(4)} = \frac{P_4}{\|P_4\|_2} = \frac{1}{8.99} \begin{bmatrix} 6.358 \\ -6.356 \end{bmatrix} = \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}$$

$$\lambda^{(4)} = v^{(4)T} A v^{(4)} = \begin{bmatrix} 0.707 & -0.707 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}$$

$$= \begin{bmatrix} 0.707 & -0.707 \end{bmatrix} \begin{bmatrix} -6.363 \\ 6.363 \end{bmatrix} = -8.997$$

after 4th iteration $\lambda^{(4)} = -8.997.$

$$P_5 = A v^{(4)} = \begin{bmatrix} -7 & 2 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix} = \begin{bmatrix} -6.363 \\ 6.363 \end{bmatrix}$$

$$v^{(5)} = \frac{P_5}{\|P_5\|_2} = \frac{1}{8.999} \begin{bmatrix} -6.363 \\ 6.363 \end{bmatrix} = \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix}.$$

$$\lambda^{(5)} = v^{(5)T} A\, v^{(5)} = \begin{bmatrix} -0.707 & 0.707 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix}$$

$$= -8.997$$

So, after 5th iteration $\lambda^{(5)} = -8.997$

So, $\lambda^{(5)} = \lambda^{(4)}$

and the dominant eigen value $= \lambda = -8.997$.

if we have calculated without floating point errors.

then $\lambda \approx 9$.

So, eigen vector is . $\begin{bmatrix} 6.363 \\ -6.363 \end{bmatrix}$ or, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

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1. Consider the system of equations $\begin{bmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$

$x = (x_1\, x_2)$ that minimizes $\|Ax - b\|_2$.

$$A = \begin{bmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$$

finding the QR factor of A.

$$a_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} -6 \\ -8 \\ 1 \end{bmatrix}$$

So, $q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$

$\hat{q}_2 = a_2 - \langle a_2, q_1 \rangle q_1 = \begin{bmatrix} -6 \\ -8 \\ 1 \end{bmatrix} - \frac{10}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$

$\hat{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = q_2 = \frac{\hat{q}_2}{\|\hat{q}_2\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 3/5 & 0 \\ \frac{4}{5} & 0 \\ 0 & 1 \end{bmatrix}$ $\quad R = \begin{bmatrix} \|a_1\| & q_1^T a_2 \\ 0 & \|\hat{q}_2\| \end{bmatrix} = \begin{bmatrix} 5 & -10 \\ 0 & 1 \end{bmatrix}$

$A = QR = \begin{bmatrix} \frac{3}{5} & 0 \\ \frac{4}{5} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -10 \\ 0 & 1 \end{bmatrix}$

$QRx = b.$

as Q is orthogonal

or, $Rx = Q^T b.$ $\quad Q^{-1} = Q^T$

or, $\begin{bmatrix} 5 & -10 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$

So, solving $\quad 5x_1 - 10x_2 = -\frac{3}{5} + \frac{28}{5} = \frac{25}{5} = 5$

$x_2 = 2.$

So. $5x_1 - 20 = 5$ $\quad$ So, $x = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$x_1 = \frac{25}{5} = 5$

So, $Ax = b$ when $x = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

hence $\|Ax - b\|_2$ is minimized at $x = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

(b) Setting up the normal equation for the given problem.

$$A^T A x = A^T b.$$

So, $\begin{bmatrix} 3 & 4 & 0 \\ -6 & -8 & 1 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 0 \\ -6 & -8 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 25 & -50 \\ -50 & 101 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 25 \\ -48 \end{bmatrix}$

So $\boxed{\begin{aligned} 25x_1 - 50x_2 &= 25 \\ -50x_1 + 101x_2 &= -48 \end{aligned}}$

(c) It is not recommended to use normal equations in computing the solution for the least square problem.
So, If $A^T A$ is not invertible than the normal equation has some issues.

$$\det(A^T A) = \det \begin{vmatrix} 1 + 10^{-20} & 1 \\ 1 & 1 \end{vmatrix} = 1 + 10^{-20} - 1 = 10^{-20} \approx 0.$$

So, finding least square solution will be difficult.