1. Given a graph $G = (V, E)$, the goal is to find a largest possible subset S of vertices of G such that there is no edge between any pair of vertices in S.
   (a) (3 points) Give a simple ILP for this problem with non-negative variables x_v for each vertex, and constraints for each edge $(u, v) \in E$, and write its LP relaxation.
   (b) (5 points) Show that the integrality gap for this formulation is large by giving a class of graphs for which $\frac{Opt(LP)}{Opt(ILP)} \geq \frac{n}{2}$.
   (c) (7 points) To fix this, give an LP with an added constraint for each odd cycle in the graph, bounding the sum of x_i inside each odd cycle. You do not need to show any approximation guarantees here.
   (d) (8 points) Now there are exponentially many constraints in your relaxed LP. Show that the LP can be solved in time polynomial in $|V|, |E|$ using ellipsoid algorithm.

2. Given: an $m \times n$ matrix A such that

$$A_{ij} \in [0,1] \text{ for each } i, j, 1 \leq i \leq m, 1 \leq j \leq n \text{ and} \tag{1}$$

$$\sum_i A_{ij} = \sum_j A_{ij} = 1 \tag{2}$$

   (a) (4 points) Is it necessary that $m = n$?
   (b) (5 points) Show that each matrix A described above can be written as a convex combination of matrices $B_1, \ldots, B_\ell$ for some ℓ, where each $(B_k)_{ij} \in \{0, 1\}$, and (2) above holds for each B_k, $1 \leq k \leq \ell$.
   (c) (8 points) Describe an algorithm to find such a convex combination.