9 May 2021

M.Sc. Data Science
LAA - Homework 2

1. Prove that $||A||_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$, where σ_i are the singular values of A.
2. Calculate the full and reduced singular value decompositions for the matrix
$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$
3. Prove that for a $n \times n$ matrix A,
$$|\det A| = \prod_{i=1}^{n} \sigma_i$$
where σ_i are the singular values of A.
4. Write down the differences between the eigenvalue decomposition and the singular value decomposition of a matrix.
5. If $x \in \mathbb{C}^m$ and A is a $m \times n$ matrix, prove the following inequalities:
   (a) $||x||_\infty \leq ||x||_2$.
   (b) $||x||_2 \leq \sqrt{m}||x||_\infty$.
   (c) $||A||_\infty \leq \sqrt{n}||A||_2$.
   (d) $||A||_2 \leq \sqrt{m}||A||_\infty$.
6. Given $A \in \mathbb{C}^{m\times n}$ with $m \geq n$, show that A^*A is nonsingular if and only if A has full rank.

Note: *All norms on $\mathbb{C}^n$ are equivalent, i.e. if $||\cdot||_\alpha$ and $||\cdot||_\beta$ are norms on $\mathbb{R}^n$, then there exist positive constants c_1 and c_2 such that*
$$c_1||x||_\alpha \leq ||x||_\beta \leq c_2||x||_\alpha.$$
For example, if $x \in \mathbb{R}^n$, then
$$||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2,$$
$$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty,$$
$$||x||_\infty \leq ||x||_1 \leq n||x||_\infty.$$

In example 5 above, you are proving this inequality for the case $\alpha = 2$ and $\beta = \infty$. Because it holds for vector norms, a similar inequality also holds for matrix norms, you are proving some of these in example 5 too. More inequalities are true:
$$\frac{1}{\sqrt{m}}||A||_1 \leq ||A||_2 \leq \sqrt{n}||A||_1,$$
$$||A||_2 \leq ||A||_F \leq \sqrt{\min\{m,n\}}||A||_2.$$

Going further, for condition numbers defined using matrix norms, the following holds true: any two matrix condition numbers $\kappa_\alpha(\cdot)$ and $\kappa_\beta(\cdot)$ are equivalent i.e.
$$c_1\kappa_\alpha(A) \leq \kappa_\beta(A) \leq c_2\kappa_\alpha(A).$$

For example on $\mathbb{R}^{n\times n}$ we have

$$\frac{1}{n}\kappa_2(A) \le \kappa_1(A) \le n\kappa_2(A),$$
$$\frac{1}{n}\kappa_\infty(A) \le \kappa_2(A) \le n\kappa_\infty(A),$$
$$\frac{1}{n^2}\kappa_1(A) \le \kappa_\infty(A) \le n^2\kappa_1(A).$$

What do all these inequalities mean for us in practice? They mean the following: if you are working with induced matrix norms and matrix condition numbers, and you prove an upper bound or a lower bound w.r.t. one norm, then a similar bound will hold w.r.t. the other norms also, differing by a factor c_1 or c_2, as the case may be.

So you will see in proofs and examples later, that we'll sometimes work with the 1-norm and sometimes with the ∞-norm, whichever is easy for us. However, the 2-norm and the Frobenius norm are preferred in many cases and they have the nicest (most convenient) expression in terms of singular values.