CMI - 2020-2021: DS-Analysis

Calculus-4

Maxima and minima

Differentiation can be used to locate maximum and minimum of a function under suitable conditions. We will assume that functions are defined on intervals.

Definition 1 A function f is said to have *local maximum* (or *relative maximum*) at a point c, if there is an interval (a_1, b_1) such that $a_1 < c < b_1$, and $f(c) \geq f(x)$ for all $x \in [a_1, b_1]$.

Similarly, f is said to have *local minimum* (or *relative minimum*) at a point c, if there is an interval (a_1, b_1) such that $a_1 < c < b_1$, and $f(c) \leq f(x)$ for all $x \in [a_1, b_1]$.

A 'local extreme' means a local maximum or a local minimum.

If f is a continuous function defined on a closed bounded interval $[a, b]$, we know that there is a point in $[a, b]$ where f has a maximum, and that there is a point in $[a, b]$ where f has a minimum. (See Theorem 8 in Calculus - 1.) As any differentiable function is continuous, we can hope to locate such points.

The following result relates local extrema of a function to horizontal tangents of its graph.

Theorem 2 *Let f be a differentiable function on (a, b). Suppose f has a local maximum or a local minimum at $c \in (a, b)$. Then $f'(c) = 0$. (Here note that c is an 'interior' point of (a, b).)*

To get an idea of the proof, put

$$\begin{aligned} g(x) &= \frac{f(x) - f(c)}{x - c}, \ x \neq c, \\ &= f'(c), \ x = c. \end{aligned}$$

As $f'(c)$ exists, note that g is continuous. Since c is a local extremum, it can then be shown that $f'(c)$ has to be 0.
Example 1: Let f be defined on the closed bounded interval $[-\frac{1}{2}, 2]$ by

$$f(x) \quad = \quad x(1-x)^2, \ -\frac{1}{2} \leq x \leq 2.$$

Note that $f'(x) = (1-x)^2 - 2x(1-x) = (1-x)(1-3x)$ for $x \in (-\frac{1}{2}, 2)$. So $f'(x) = 0$ only if $x = 1$ or $x = 1/3$. Also $f(-\frac{1}{2}) = -9/8$, $f(0) = 0$, $f(1/3) = 4/27$, $f(1) = 0$, and $f(2) = 2$. By drawing a graph of f over the interval $[-\frac{1}{2}, 2]$ the following can be seen:

(i) f has a local maximum at $x = 1/3$;

(ii) f has a local minimum at $x = 1$;

(iii) $f(-\frac{1}{2}) = -9/8 = \inf\{f(x) : x \in [-\frac{1}{2}, 2]\}$; so the function f attains its minimum on $[-\frac{1}{2}, 2]$ at the point $x = -\frac{1}{2}$;

(iv) $f(2) = 2 = \sup\{f(x) : x \in [-\frac{1}{2}, 2]\}$; so the function f attains its maximum on $[-\frac{1}{2}, 2]$ at the point $x = 2$.

It is clear that $-\frac{1}{2}$ and 2 are *not* interior points of $[-\frac{1}{2}, 2]$.
Example 2: Let $f(x) = x^2$, $x \in [-1, 1]$. Clearly $f'(x) = 0$ only at $x = 0$. Note that $f(0) = 0$; clearly $x = 0$ is a local minimum of f; moreover, f attains its minimum on $[-1, 1]$ at the interior point $x = 0$. Also $f(-1) = f(1) = 1 = \sup\{f(x) : x \in [-1, 1]\}$. So f attains its maximum on $[-1, 1]$ at two points $x = -1$, and $x = 1$; note that neither of them are interior points of $[-1, 1]$.
Example 3: Take $f(x) = \sin(x)$, $0 \leq x \leq \pi$. Find the local extrema. Find also the points where f attains its maximum/ minimum on $[0.\pi]$. Also find which are interior points and which are not.
Caution! The following aspects should be kept in mind while using Theorem 2. Let f be a continuous function on closed and bounded interval $[a, b]$ such that f is differentiable on (a, b). Note the following:

(i) Let a be an extreme. Then, even if f is defined and differentiable outside $[a, b]$, it need be the case that $f'(a) = 0$. You can note this in the examples above. It is essential that the local extreme $c \in (a, b)$.

(ii) Let $c \in (a, b)$. Suppose $f'(c) = 0$. It need not be the case that c is a local extreme. For example, consider the function $f(x) = x^3$ on the interval $(-1, 1)$. Though $f'(0) = 0$ it is easily seen that $x = 0$ is not a local maximum or a local minimum. So the converse of Theorem 2 is not true.

(iii) Let $f(x) = |x|,\ x \in [-1, 1]$. Clearly f has its minimum at the point $x = 0$. But f is not differentiable at $x = 0$. Also, though the right derivative and the left derivative exist at 0, neither of them is 0. Note that Theorem 2 assumes that f is differentiable at the interior point c, which is a local extreme.

Note: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose f is differentiable on (a, b). Assume also that $f'(y) = 0$ only at finitely many points in (a, b), say, at $y_1, y_2, \cdots, y_k$. Then by comparing the values $f(a),\ f(b),\ f(y_i),\ 1 \leq i \leq k$, one can locate the points where f attains maximum and minimum on the interval $[a, b]$. Thus we get a nice way of narrowing down the search.

The next result is called *Rolle's theorem*. It is easy to see it graphically.

Theorem 3 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$; also let f be differentiable on (a, b). If $f(a) = f(b)$, then there is atleast one point $c \in (a, b)$ such that $f'(c) = 0$.*

An idea of proof: Suppose $f'(x) \neq 0$ for every $x \in (a, b)$. By Theorem 8 of Calculus - 1, f attains its maximum and minimum in $[a, b]$. This can not be at any $x \in (a, b)$, because in such a case, $f'(y) = 0$ for some $y \in (a, b)$, by Theorem 2. So it has to be at the end points. But as $f(a) = f(b)$, this would imply f is a constant function. Then $f'(x) = 0,\ x \in (a, b)$, a contradiction.

Next, we consider an important result called the *mean value theorem,* whose proof is an exercise!

Theorem 4 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$; also let f be differentiable on (a, b). Then there is at least one point $c \in (a, b)$ such that*

$$f(b) - f(a) \quad = \quad f'(c)(b - a).$$

Exercise: Take

$$h(x) \quad = \quad f(x)(b - a) - x(f(b) - f(a)),\ x \in [a, b].$$

Note that $h(a) = h(b) = bf(a) - af(b)$. Show that h satisfies all the hypotheses of Rolle's theorem, and derive the mean value theorem.
Note: The mean value theorem makes no assertion about the exact location of the one or more of the 'mean values' c, except that they all must be in (a, b). In many cases it might be quite difficult to find the exact location of the mean values. Its importance is due to the fact, existence of at least one 'mean value' helps in getting important conclusions. The next result is one such.

Theorem 5 *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$*; also let* f *be differentiable on* (a, b)*. Then the following hold:*

(i) If $f'(x) > 0$ *for all* $x \in (a, b)$*, then* f *is* strictly increasing *on* $[a, b]$*; that is,* $y, z \in [a, b],\ y < z$ *imply* $f(y) < f(z)$*.*

(ii) If $f'(x) < 0$ *for all* $x \in (a, b)$*, then* f *is* strictly decreasing *on* $[a, b]$*; that is,* $y, z \in [a, b],\ y < z$ *imply* $f(y) > f(z)$*.*

(iii) If $f'(x) = 0$ *for all* $x \in (a, b)$*, then* f *is a constant on* $[a, b]$*.*

An idea of proof: To prove (i) we need to show that $f(y) < f(z)$ whenever $a \le y < z \le b$. Suppose $a \le y < z \le b$. Apply the mean value theorem to the closed interval $[y, z]$. We get $f(z) - f(y) = f'(c)(z - y)$, for some $y < c < z$. As $f'(c) > 0,\ (z - y) > 0$, the required conclusion follows. Proof of (ii) is similar. For proving (iii) take $y = a$.

The next result is an easy consequence of Theorem 5.

Theorem 6 *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$*; also let* f *be differentiable on* (a, b)*. Let* $c \in (a, b)$*. Then the following hold:*

(i) If $f'(x) > 0$ *for all* $x < c$*, and* $f'(x) < 0$ *for all* $x > c$*, then* f *has a local maximum at* c*.*

(ii) If $f'(x) < 0$ *for all* $x < c$*, and* $f'(x) > 0$ *for all* $x > c$*, then* f *has a local minimum at* c*.*

Thus, a local extreme occurs whenever the derivative changes sign.

An interior point c at which the derivative $f'(c) = 0$ may be called a *critical point* of f.

The next result gives a very useful *second derivative test for local extreme* at a critical point.

Theorem 7 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Assume also that the second derivative f'' exists and is continuous on the interior (a,b). Let $c \in (a,b)$. Then the following hold.*

(i) If $f'(c) = 0$ and $f''(c) > 0$, then c is a local minimum of f.

(ii) If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum of f.

A sketch of proof: (i) Since f'' is continuous, $f''(c) > 0$ implies that $f''(x) > 0$ in an open interval around c; for notational convenience, take that interval as (a,b) itself. Then by Theorem 5, applied to the function f', we get f' is strictly increasing. But as $f'(c) = 0$, this means that f' changes sign from negative to positive at c. Hence by part (ii) of Theorem 6, it follows that f has a local minimum at the point c. Proof of assertion (ii) is entirely analogous.

Definition 8 Let $f : [a,b] \to \mathbb{R}$ be continuous. Suppose for any $x, y \in [a,b]$ and any $0 < \alpha < 1$, we have

$$f(\alpha x + (1-\alpha)y) \quad \leq \quad \alpha f(x) + (1-\alpha)f(y).$$

Then we say that f is a *convex* function on $[a,b]$.

A function f is said to be *concave* if $(-f)$ is convex.

We now state the following useful result without proof; this can be proved using the mean value theorem.

Theorem 9 *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$; also let f be differentiable on (a,b). If f' is increasing on (a,b), then f is convex on $[a,b]$. In particular, f is convex, if f'' exists and is nonnegative on (a,b).*

Exercise: Let $f_1(x) = x^2$; $f_2(x) = e^x$. Show that $f_1,\ f_2$ are convex functions on $\mathbb{R}$.

Let $g(x) = \log x,\ x > 0$. Show that g is a concave function.