Matrix norms.

Note Title — 10-04-2021

- Vector norm : a function $\|\cdot\| : V \to \mathbb{R}$ satisfying
  - $\|x\| \geq 0$ & $\|x\| = 0 \Leftrightarrow x = 0$
  - $\|x+y\| \leq \|x\| + \|y\|$
  - $\|\alpha x\| = |\alpha| \cdot \|x\|$.

- An important class of vector norms is the class of p-norms defined for $p \geq 1$:

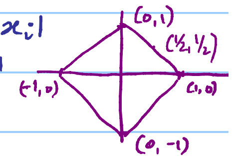
$(x = (x_1, \dots, x_n))$

$$\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}.$$

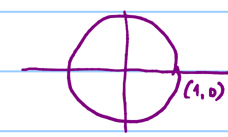
Closed unit discs in some p-norms:

$n = 2$, $x = (x_1, x_2)$

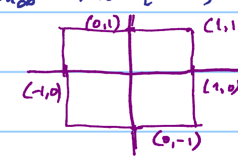
$p = 1$: $\|x\|_1 = \sum |x_i| = |x_1| + |x_2|$



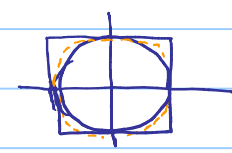
$p = 2$: $\|x\|_2 = \left(\sum |x_i|^2\right)^{1/2} = \left(x_1^2 + x_2^2\right)^{1/2}$



$p = \infty$: $\|x\|_\infty = \max\{|x_1|, |x_2|\}$.



$2 \leq p < \infty$



Defn: $\|x\|_\infty = \max_i \{|x_i|\}$

Another class: weighted p-norms
$\|x\|_W := \|Wx\|_p$ for any norm $\|\cdot\|$ & any non-singular matrix W.

Matrix norms -

Defn: A matrix norm is a function $\|\cdot\| : \mathbb{C}^{m \times n} \to \mathbb{R}$ satisfying -

- $\|A\| \geq 0 \;\; \forall A$ & $\|A\| = 0 \Leftrightarrow A = 0$
- $\|A + B\| \leq \|A\| + \|B\|$
- $\|\alpha A\| = |\alpha| \cdot \|A\|$
- $\|AB\| \leq \|A\| \cdot \|B\|$ (desirable ppty, whenever multiplication exists i.e. $m = n$)

Important examples -

① Induced matrix norms : Suppose $A \in \mathbb{C}^{m \times n}$, Consider $A : \underset{\|\cdot\|_n}{\mathbb{C}^n} \to \underset{\|\cdot\|_m}{\mathbb{C}^m}$

The induced matrix norm $\|A\|_{(m,n)}$ is the smallest scalar C such that -

$$\|Ax\|_m \leq C \|x\|_n$$

i.e. $\dfrac{\|Ax\|_m}{\|x\|_n} \leq C.$ $\left\{ \begin{array}{l} C \text{ is the maximum} \\ \text{factor by which } A \text{ can} \\ \text{'stretch' } x. \end{array} \right.$

$\|\alpha x\| = |\alpha| \cdot \|x\|$, so $\dfrac{\|A\alpha x\|}{\|\alpha x\|} = \dfrac{|\alpha| \, \|Ax\|}{|\alpha| \, \|x\|} = \dfrac{\|Ax\|}{\|x\|}$

So it is sufficient to consider vectors x of norm 1.

$$\|A\|_{(m,n)} = \sup_{\substack{x \neq 0 \\ x \in \mathbb{C}^n}} \frac{\|Ax\|_m}{\|x\|_n}$$

$$= \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_n = 1}} \|Ax\|_m$$