Design and Analysis of Algorithms

Duration: 2.5 Hours
Maximum Marks: 40

1. Solve the following recurrence using a recursion tree:

$$T(n) \;=\; T(3n/5) + T(n/3) + n$$

(5 marks)

2. You have large set of identical rods which have $N-1$ notches marked on them. The lengths of the N segments formed by the notches as we move from one end to the other are $x_1, x_2, \ldots x_N$. You are only allowed to break such a rod only once. Further the break has to be at one of the notches, leaving you with a piece of length $x_1+x_2+\ldots+x_i$ and another with length $x_{i+1}+\ldots+x_N$ (if we choose to break at the ith notch). You are given a set of desired lengths $y_1 \leq y_2 \leq \ldots \leq y_m$ in ascending order. We would like an efficient algorithm that determines if it is possible to make pieces with these lengths. How would you solve this problem if N is much much larger than m? What if N and m are roughly the same (i.e. same upto a constant factor)? *(7 marks)*

3. Given a list A of N integers, an equal subsequence is a set of positions which have the same value i.e. $i_1 \leq i_2 \leq \ldots \leq i_k$ such that $A[i_1] = A[i_2] = \ldots = A[i_k]$. Find the length of the longest equal subsequence in A. (Hint: This is almost trivial).

   In addition to A you are also given a number K and we say that an equal sequence $i_1, \ldots i_k$ is K-close if $i_{j+1} - i_j \leq K$. That is, the adjacent positions in the sequence are not farther than K apart. Describe an algorithm to compute the longest K-close equal sequence. What is its complexity? *(6 marks)*

4. You are given list A of N integers, $A[1], \ldots, A[N]$. You are also given a number K, $K \leq N$. Your aim is to output the value $B[i]$, $1 \leq i \leq N-K+1$, where $B[i]$ is the minimum value among $A[i], A[i+1], \ldots, A[i+K-1]$. Describe an algorithm which achieves this in time $\mathcal{O}(Nlog(K))$. [Hint: Minimum of $a_i, a_{i+1} \ldots a_k, a_{k+1} \ldots a_j$ is the same as computing the minimum of $a_i, a_{i+1}, \ldots a_k$ and the minimum of $a_{k+1}, \ldots, a_j$ and then taking the minimum of these two values] *(8 marks)*

5. Let $G = (V, E)$ be any connected weighted graph. Let $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ be two minimum cost spanning trees of G. Let $w_1, w_2, \ldots, w_k$ and $u_1, u_2, \ldots, u_k$ be the weights of the edges of E_1 and E_2 respectively, written in ascending order. Can these sequences be different? If your answer is yes then give an example of such a graph and two minimum cost spanning trees, and if your answer is no then provide a proof. *(8 marks)*

6. You are given a list of values $0 < a_1 < a_2 < \ldots < a_N$. You are allowed to increase or decrease the value of a_i by any amount from $0, \ldots K$. Once this is done, let the resulting sequence be $0 \leq b_1 \leq b_2 \ldots \leq b_N$. You are also given a value M. Your aim is to do this so that

$$Min_{1 \leq i < N}(b_{i+1} - b_i) \geq M$$

Describe an algorithm which checks if this possible and if so outputs a set of values of b_i's that achieves this. *(8 marks)*