Mid Semester Examination :-

Date:- 05.06.2021

1. maximize: $x_1 + x_2$

Subject to: $-x_1 + x_2 \leq 2$

$$x_2 \leq 4 \qquad x_1, x_2 \geq 0.$$

$$x_1 + x_2 \leq 9$$

$$x_1 \leq 6$$

$$x_1 - x_2 \leq 5$$

So, introducing the slack variables.

$$-x_1 + x_2 + S_1 = 2$$

$$x_2 + S_2 = 4$$

$$x_1 + x_2 + S_3 = 9$$

$$x_1 + S_4 = 6$$

$$x_1 - x_2 + S_5 = 5.$$

So, finding the initial basis w.r.t the non basic variable. is a feasible ~~tableau~~. initial tableau.

So,

$$S_1 = 2 + x_1 - x_2$$

$$S_2 = 4 - x_2$$

$$S_3 = 9 - x_1 - x_2$$

$$S_4 = 6 - x_1$$

$$S_5 = 5 - x_1 + x_2$$

$$z = x_1 + x_2$$

BFS = $\langle 0, 0, 2, 4, 9, 6, 5 \rangle$

cost = 0.

$$\begin{aligned} s_1 &= 2 + x_1 - x_2 \\ s_2 &= 4 \quad\;\; - x_2 \\ s_3 &= 9 - x_1 - x_2 \\ s_4 &= 6 - x_1 \\ s_5 &= 5 - x_1 + x_2 \\ \hline z &= x_1 + x_2 \end{aligned}$$

$$\xrightarrow[s_5\downarrow]{x_1\uparrow}$$

$$\begin{aligned} s_1 &= 7 - s_5 \\ s_2 &= 4 \qquad\quad - x_2 \\ s_3 &= 4 + s_5 - 2x_2 \\ s_4 &= 1 + s_5 - x_2 \\ x_1 &= 5 - s_5 + x_2 \\ \hline z &= 5 - s_5 + 2x_2 \end{aligned}$$

$$s_4\downarrow \Big\downarrow x_2\uparrow$$

$$\begin{aligned} s_1 &= 7 - s_5 \\ s_2 &= 3 - s_5 + s_4 \\ s_3 &= 2 - s_5 + 2s_4 \\ x_2 &= 1 + s_5 - s_4 \\ x_1 &= 6 \qquad\quad - s_4 \\ \hline z &= 7 + s_5 - 2s_4. \end{aligned}$$

$$s_3\downarrow \Big\downarrow s_5\uparrow$$

$$\begin{aligned} s_1 &= 5 + s_3 - 2s_4 \\ s_2 &= 1 + s_3 - s_4 \\ s_5 &= 2 - s_3 + 2s_4 \\ x_2 &= 3 - s_3 + s_4 \\ x_1 &= 6 \qquad\quad - s_4. \\ \hline z &= 7 + 2 - s_3 + \cancel{2s_4} - \cancel{2s_4} \\ &= 9 - s_3. \end{aligned}$$

BFS. $\langle 6, 3, 5, 1, 0, 0, 2 \rangle$.

cost $= 9$.

2. Dual of the following LP.

Maximize ; $x_1+x_2+2x_3+15$

Subject to : $2x_1+9x_2+8x_3 \geq 25$

$x_1-6x_2+3x_3=15$

$4x_1+7x_2-20x_3 \geq 4.$

$x_1 \geq 0$

$x_2 \leq 0$

x_3 = unrestricted.

So, $x_3 = x_3^+ - x_3^-$ $x_3^+ \geq 0$

$x_3^- \geq 0.$

$2x_1+9x_2+8x_3 \geq 25 \quad \cdots (1)$

and $x_1-6x_2+3x_3 \geq 15 \quad \cdots (2)$

$x_1-6x_2+3x_3 \leq 15$

or, $-x_1+6x_2-3x_3 \geq -15 \quad \cdots (3)$

$4x_1+7x_2-20x_3 \geq 4 \quad \cdots (4).$

Putting $x_3 = x_3^+ - x_3^-$

constraints are becoming

$y_1 \times \quad 2x_1+9x_2+8x_3^+-8x_3^- \geq 25$ $\quad x_1 \geq 0$

$y_2 \times \quad x_1-6x_2+3x_3^+-3x_3^- \geq 15$ $\quad x_2 \leq 0$

$y_3 \times \quad -x_1+6x_2-3x_3^++3x_3^- \geq -15$ $\quad x_3^+ \geq 0$

$y_4 \times \quad 4x_1+7x_2-20x_3^++20x_3^- \geq 4.$ $\quad x_3^- \geq 0.$

$y_1, y_2, y_3, y_4 \leq 0.$

the objective function becomes.

$x_1+x_2+2x_3^+-2x_3^+ +15.$

so, $(2y_1+y_2-y_3+4y_4)x_1$
$+(9y_1-6y_2+6y_3+7y_4)x_2$
$+(8y_1+3y_2-3y_3-20y_4)x_3^+$
$+(-8y_1-3y_2+3y_3+20y_4)x_3^-$
$\geqslant 25y_1+15y_2-15y_3+4y_4+15$

and. as $x_2 \le 0$.

then. $2y_1+y_2-y_3+4y_4 \geqslant 1$
$9y_1-6y_2+6y_3+7y_4 \le 1$
$8y_1+3y_2-3y_3-20y_4 \geqslant 2$

$$\begin{cases} -8y_1-3y_2+3y_3+20y_4 \geqslant -2 \\ \text{or, } 8y_1+3y_2-3y_3-20y_4 \le 2. \end{cases}$$

$y_1 \le 0$
$y_2 \le 0$
$y_3 \le 0$
$y_4 \le 0.$

so, $2y_1+y_2-y_3+4y_4 \geqslant 1$
$9y_1-6y_2+6y_3+7y_4 \le 1$
$8y_1+3y_2-3y_3-20y_4 = 2.$

~~$y_1 \geqslant 0$~~ $y_1 \le 0$
~~$y_2 \geqslant 0$~~ $y_2 \le 0$
~~$y_3 \geqslant 0$~~ $y_3 \le 0$
~~$y_4 \geqslant 0.$~~ $y_4 \le 0.$

cost function
~~maximize~~ minimize: $25y_1+15y_2-15y_3+4y_4+15$

if ~~we consider $(y_2-y_3)=y'$ then y' = unconstrained.~~

3. (a) If an LP in equational form is feasible, it has basic feasible solution.

For x to be feasible, we must have $Ax = b$. Which is the equational form of an LP.

The LHS can be written as $Ax = A_B x_B + A_N x_N$.

Where $B \subseteq \{1, 2, .., n\}$, ~~Where $x_j \neq 0 \; \forall j \in B$.~~

$x_j \in x_N$ where $x_j = 0 \; \forall j \notin B$.

A_B is $m \times m$. $x_B : m \times 1$.

the rank of $A = m$.

let, A_B is the non singular matrix consisting of columns $j_1, j_2 .., j_m$ of A which are L.I. and x_N are set to 0 they are called non basic. The BFS is the solution x is the unique vector satisfying $Ax = b$ subject to the condition $x_N = 0$. If x satisfies the sign constraints $x \geq 0$, x is a bfs solution.

Here the equational form is already feasible then all of the solutions will be non negative. Then ~~there~~ it ~~must~~ has a basic feasible solution.

3. (b) If an LP with constraints given as $Ax \le b$ is feasible, it has an extreme point

The statement is false.

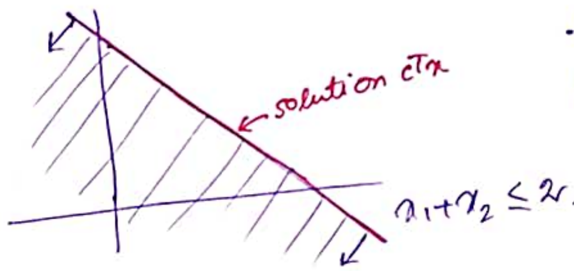
<u>counter example</u> :-

maximize :- $x_1 + x_2$

constraints $x_1 + x_2 \le 2$

x_1, x_2 unbounded.

then it doesn't have any extreme point



the LP is feasible But it has no extreme point.

3. (c) Consider an LP with constraints $Ax = b$, $x \geq 0$. If there are two basic feasible solutions giving the optimum value; then there are infinitely many feasible solutions giving this optimum value.

This statement is true.

Let x_1 and x_2 be BFS satisfying $Ax = b$, $x \geq 0$.

Let x is a convex combination of x_1 and x_2.

then $x = \lambda x_1 + (1-\lambda) x_2$ $\quad \lambda \in [0,1]$.

then $Ax = \lambda A x_1 + (1-\lambda) A x_2 = \lambda b + (1-\lambda) b = b$.

Let, cost = $c = c^T x_1$ and $c^T x_2$

and $c^T x_1 = c^T x_2$

for the 2 BFS.

$$(c^T x) = \lambda\, c^T x_1 + (1-\lambda)\, c^T x_2$$

$$= c.$$

So, x is also a ~~BFS~~ feasible solution which gives optimum value

Since there are infinitely λ, there are infinitely many feasible solutions which gives the optimum value..

4. A variable which just left the basis in a simplex tableau, can't reenter in the very next pivot.

The statement is true.

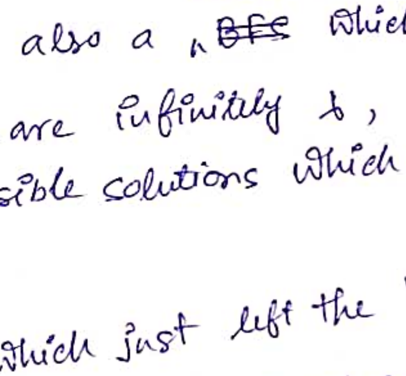
Let x_i is basic in lth row and x_j replaces x_i in the basis.

$\cancel{x_i}$ $\cancel{x_i}$

maximization problem.

Tableau B.

B



here the coefficient of x_j is positive so, it leaves the basis. and we find the most strict constraint on x_j such that $\frac{-p_i}{a_{ij}}$ is minimum for all the rows and $a_{ij} < 0$.

hence after this step, the coefficient of x_j in the cost will be strictly negative. In the next iteration of simplex, only variable with a positive coefficient can enter the basis.

Hence, the variable that just left cannot re-enter the basis in the very next pivot.

5. Consider the LP : minimize $c^T x$ subject to $Ax \leq b$ $x \geq 0$.

Assume c is a non-zero vector.

Suppose ~~that~~ there is a point x_0 satisfying $Ax_0 < b$ and $x > 0$

x_0 is an interior point. So, we can write x_0 interms of ~~two~~ the convex combination of two points. x, y in the polyhedron such that,

$$x_0 = \lambda x + (1-\lambda) y \qquad 0 \leq \lambda \leq 1$$

$$\text{and} \quad c^T x < c^T y \qquad \text{(assume)}$$

We assume that x_0 is the optimal solution.

So, we should get $c^T x_0 < c^T x < c^T y$.

$$\text{now,} \quad c^T x_0 = \lambda c^T x + (1-\lambda)(c^T y) > \lambda c^T x + (1-\lambda) c^T x$$
$$= c^T x$$

So, $c^T x_0 > c^T x$.

Which is a contradiction to our assumption.

So, x_0 cannot be an optimal solution.

6. ~~minimize~~ $c^T x$: maximize

Subject to $Ax \leq b$ $\quad x \geq 0$

$A = m \times n$.

Is it possible to for an optimal solution to have more than m positive variables?

The statement is true.

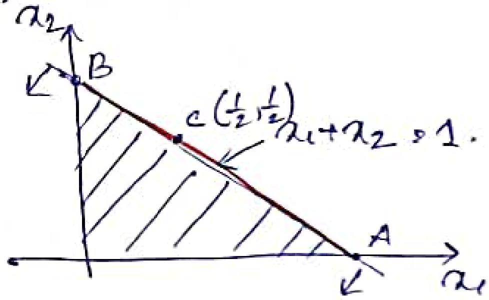
It is possible for an optimal solution to have more than m positive variables.

Consider the case where we have alternative optimal solution. Let $x_1 + x_2$ maximize.

Subject to $x_1 + x_2 \leq 1$.

$x_1 \geq 0$

$x_2 \geq 0$.



So, the maximum value of the LP is 1.

and it is possible for all the points on the line $x_1 + x_2 = 1$

So, here more than ~~2 variables are strictly~~ 2 positive variables. $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}$

So, for any optimal solution it is possible to have more than m positive variables. But this is not true for BFS.

8. If the problem- minimizes $c^T x$ subject to
$Ax = b$ $x \geq 0$ has a finite optimal solution, then the new problem - minimize $c^T x$ subject to $Ax = b'$, $x \geq 0$ cannot be ~~bo~~ unbounded. no matter what value the b' might take.

By duality theorem,
Since the primal LP has an optimal solution, both the primal and dual LP are bounded and has an optimal solution.

Since $A^T y \leq c$ feasible region is bounded, we can change the cost function of the dual without affecting the feasible region.

Dual

maximize: $r^T y$
subject to: $A^T y \leq c$

y unrestricted

Primal

minimize: $c^T x$
subj. to : $Ax = b'$
$x \geq 0$.

put $r = b'$.
Since the dual is both ~~un~~ bounded and feasible, by duality theorem primal is also bounded and feasible and attains its minimum. So, the new problem - min $c^T x$ subject to $Ax = b'$, $x \geq 0$ cannot be unbounded no matter what value the b' might take.

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7\. Let $X = \{x : Ax = b, x \geq 0\}$ where A is $m \times n$ with rank m.

Let y be a feasible solution such that

$y_1, \dots, y_r$ are > 0

$y_{r+1}, \dots, y_n = 0$.

columns of $A_1\ A_2 \dots A_r$ are LD.

So, $A_{11}x_1 + A_{12}x_2 \dots + A_{1n}x_n = b_1$

$A_{21}x_1 + A_{22}x_2 \dots + A_{2n}x_n = b_2$

$\vdots$

$A_{m1}x_1 + A_{m2}x_2 \dots + A_{mn}x_n = b_m$.

~~Let The columns~~

So, we can assume that $m \geq r$.

then, $\cancel{A' x_m = b_m}$.

$A' y_m = b_m$. ← It has unique solution.

There will be a non trivial solution to $A'y = 0$.

So, ~~we have A~~ let the solution be y_0.

then $A' y_0 = 0$.

$A'(y_m + \epsilon y_0) = A' y_m + \epsilon A' y_0$

$= A' y_m$

$= b_m$.

Adding any multiple of y_0 to y_m still satisfy A' constraint.

Now consider.

$$A'(y + \epsilon y_0) = b_m.$$

$$A'(y - \epsilon y_0) = b_m.$$

so, $y = \frac{1}{2}(y + \epsilon y_0) + \frac{1}{2}(y - \epsilon y_0)$

$$= \frac{1}{2} y'' + \frac{1}{2} y'$$

hence we could write y as a convex combination of 2 feasible points.