24/04/2018

MSc. Applications of Mathematics

Numerical Linear Algebra - Final Exam

Time allowed: 3 hours — Total: 70 points

Instruction: Show all your work; writing only the answer will not fetch any points.

1. (10 points) Construct a Givens' rotation that maps $x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ to $x' = \begin{pmatrix} * \\ * \\ * \\ 0 \end{pmatrix}$.

2. (15 points) Suppose we want to reduce the following matrix to Hessenberg form. Perform the first step of reduction i.e. $Q_1^* A Q_1$. Describe what should be done next to obtain the complete reduction.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 5 & 3 \\ 2 & 3 & 7 & 2 \\ 1 & 0 & 4 & 5 \end{pmatrix}.$$

3. (10 points) Show that the Householder reflector $H = 1 - 2ww^T$, with $||w|| = 1$, is symmetric and orthogonal. Find the eigenvalues and eigenvectors of H.

4. (10 points) Apply 2 steps of the Rayleigh quotient iteration to locate an approximate eigenvalue/eigenvector pair for the matrix $\begin{pmatrix} 7 & 3 \\ 1 & 0 \end{pmatrix}$. Use initial vector $v^{(0)} = [0, 1]^t$ and the 1-norm for your calculations.

5. (12 points) Match each of the following situations with the method you will use for them:

   Situations:

   (i) Solve a system of equations iteratively when lesser memory usage is preferred.

   (ii) Solve a system of equations $Ax = b$ when A is symmetric positive-definite.

   (iii) Compute the largest eigenvalue (in absolute value) and corresponding eigenvector of a given matrix.

   (iv) Solve a system of equations iteratively on a parallel system when memory is not an issue.

   (v) Project the data space onto a subspace spanned by the variables that most affect the data.

   (vi) Compute the first few eigenvalues of a large sparse matrix.

   Methods:

   (a) Power iteration.

   (b) Jacobi method.

   (c) Principal component analysis.

   (d) Arnoldi's algorithm.

(e) Gauss-Seidel method.

(f) Cholesky factorization.

6. (10 points) In which of the following situations will finding the SVD be your preferred method of solution? Explain.

   (a) Calculate the rank of a matrix.

   (b) Compute the inverse of a square matrix

   (c) Compute the Jordan canonical form of a square matrix.

   (d) Compute the 2-norm of a matrix.

   (e) Compute the Frobenius norm of a matrix.

7. (3 points) Suppose we have the QR factorization of a $m \times n$ matrix A, given by $A = QR$. How can we use this to obtain the QR factorization of $\tilde{A} = \begin{pmatrix} w^T \\ A \end{pmatrix}$, where $w \in \mathbb{R}^n$?

SOLUTIONS

1. (15 points) Suppose we want to reduce the following matrix to Hessenberg form. Perform the first step of reduction i.e. $Q_1^* A Q_1$. Describe what should be done next to obtain the complete reduction.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 5 & 3 \\ 2 & 3 & 7 & 2 \\ 1 & 0 & 4 & 5 \end{pmatrix}.$$

Solution: $u_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $v_1 = u_1 + ||u_1|| e_1 = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$.

$$H_{v_1} = I - \frac{2v_1 v_1^*}{v_1^* v_1} = \begin{pmatrix} -0.6666667 & -0.6666667 & -0.3333333 \\ -0.6666667 & 0.7333333 & -0.1333333 \\ -0.3333333 & -0.1333333 & 0.9333333 \end{pmatrix}.$$

$$Q_1^* A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -3 & -3.33333 & -9.333324 & -4.999995 \\ 0 & 0.866667 & 1.266669 & -1.199997 \\ 0 & -1.066665 & 1.133336 & 3.4 \end{pmatrix}.$$

$$Q_1^* A Q_1 = \begin{pmatrix} 1 & -4.6666667 & 0.3333333 & 2.6666667 \\ -3 & 10.111112 & -3.9555554 & -2.3111113 \\ 0 & -1.022222 & 0.5111109 & -1.5777776 \\ 0 & -1.177778 & 1.0888891 & 3.3777776 \end{pmatrix}.$$

To get the complete reduction, calculate the Householder matrix H corresponding to the vector $[-1.022, -1.177]$. Then form the matrix $Q_2^* = \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix}$ and calculate $Q_2^*(Q_1^* A Q_1) Q_2$.

2. (10 points) Show that the Householder reflector $H = 1 - 2ww^T$, with $||w|| = 1$, is symmetric and orthogonal. Find the eigenvalues and eigenvectors of H.

Solution:

Symmetry:

$$H^T = (I - 2ww^T)^T = I - 2(ww^T)^T - I - 2ww^T = H.$$

Orthogonality:

$$H^T H = H^2 = (I - 2ww^t)(I - 2ww^T) = I - 4ww^t + 4ww^T ww^T - = I - 4ww^T + 4wIw^T = I.$$

Eigenvalues: Let us look for vectors v_i and numbers λ_i obeying $Hv_i = \lambda_i v_i$. First, suppose $w \parallel v$:

$$Hv - v - 2ww^T v = v - 2v = -v.$$

Hence the matrix H has one eigenvalue -1 with the corresponding eigenvector being parallel to w.

Now, suppose $w \perp v$:

$$Hv = v - 2ww^T v = v$$

Hence any vector $w \perp v$ is an eigenvector of H, and hence H has an eigenvalue 1 of multiplicity $n - 1$ where n is the dimension of H.

3. (10 points) Apply 2 steps of the Rayleigh quotient iteration to locate an approximate eigenvalue/eigenvector pair for the matrix $\begin{pmatrix} 7 & 3 \\ 1 & 0 \end{pmatrix}$. Use initial vector $v^{(0)} = [0, 1]^t$ and the 1-norm for your calculations.

Solution: Algorithm for Rayleigh quotient iteration:

Start with $v^{(0)}$ such that $||v|| = 1$.

$\lambda^{(0)} = v^{(0)^T} A v^{(0)}$.

for $k = 0$ to convergence,

Solve $(A - \lambda^{(k-1)} I)w = v^{(k-1)}$ for w using inverse iteration;

$v^{(k)} = w/||w||$;

$\lambda^{(k)} = R_A(v^{(k)})$.

For the given matrix and starting vector $v^{(0)}$, we get,

1st iteration: $\lambda^{(0)} = 0$, so $A - \lambda^{(0)} I = A$ and solving the system $Aw = v^{(0)}$ for w gives $w = \begin{pmatrix} 1 \\ -7/3 \end{pmatrix}$.

The 1-norm of w is its maximum column sum which is $10/3$ so that $v^{(1)} = \begin{pmatrix} 3/10 \\ -7/10 \end{pmatrix}$. Then

$$\lambda^{(1)} = R_A(v^{(1)}) = \frac{v^{(1)t} A v^{(1)}}{v^{(1)t} v^{(1)}} = \frac{-21/10}{29/50} = \frac{-105}{29}.$$

2nd iteration: $A - \lambda^{(1)} I = \begin{pmatrix} 308/29 & 3 \\ 1 & 105/29 \end{pmatrix}$. Solving $(A - \lambda^{(1)} I)w = v^{(1)}$ for w gives $w = \begin{pmatrix} 0.0333 \\ 0.2181 \end{pmatrix}$.

The 1-norm of w in this case is $0.0333 + 0.2181 = 0.2514$ so that $v^{(2)} = w/||w|| = \begin{pmatrix} 0.1324 \\ 0.8675 \end{pmatrix}$.

Then

$$\lambda^{(2)} = R_A(v^{(2)}) = \frac{v^{(2)t} A v^{(2)}}{v^{(2)t} v^{(2)}} =$$