CMI - 2020-2021: DS-Analysis

Calculus-5

Maxima and minima: contd.

Some problems on maxima and minima will be considered.

Example 1: Among all rectangles of given perimeter, the square has the largest area.

Let x, y denote lengths of adjacent sides of a rectangle; assume $x \leq y$. Let $2P$ be the given perimeter, where $P > 0$ is a constant. Clearly $2(x+y) = 2P$. So $y = P - x$ where $0 \leq x \leq P$. Area of the rectangle is $xy = x(P-x)$, which has to be maximised. Hence take $f(x) = x(P-x) = Px - x^2,\ 0 \leq x \leq P$. Verify that $f'(x) = P - 2x$, and $f''(x) = -2$ for all $0 < x < P$. By Theorem 7 in Calculus - 4, it now follows that f has a local maximum at any critical point, that is, where $f'(x) = 0$. Note that $f = 0$ at the end points. Clearly $f'(x) = 0$ if and only if $x = \frac{1}{2}P$, in which case $y = x = \frac{1}{2}P$. So the function f attains its maximum only at $y = x = \frac{1}{2}P$. The required result now follows.

Example 2: To find the shortest distance from a given point $(0, b)$ on the $y-$axis to the parabola $x^2 = 4y$. (Here b may any given real number.)

In other words, given the point $(0, b)$ on the $y-$axis, we need the point on the parabola $x^2 = 4y$ which is closest to the given point. Note that the distance of a point (x, y) on the parabola from $(0, b)$ is

$$d = \sqrt{x^2 + (y-b)^2},$$

that is to be minimised subject to the constraint $x^2 = 4y$. Observe that in the above we take the non-negative square root on the right side.

From a rough diagram, if $b < 0$, it is clear that the minimum distance is $|b|$, that is, the distance between the origin $(0, 0)$, which is on the parabola, to the point $(0, b)$ on the $y-$axis.

So let $y \geq 0$. As $x^2 = 4y$, note that d can be expressed as a function of y alone. Also, as distance non-negative, clearly d is minimised if only if d^2 is minimised. Hence it is convenient to take

$$f(y) = d^2 = 4y + (y-b)^2,\ y \geq 0.$$

So, $f'(y) = 4 + 2(y - b)$. Then verify that $f'(y) = 0 \Leftrightarrow y = (b - 2)$.

Now we need to consider 2 possibilities.

Suppose $0 \leq b < 2$. Then verify that $f'(y) > 0$ when $y \geq 0$. So by Theorem 5 in Calculus - 4, the function f is strictly increasing on $y \geq 0$. Hence f attains minimum at the end point $y = 0$. Consequently the minimum distance in this case is $d = \sqrt{b^2} = |b|$.

Next, let $b \geq 2$. In this case, $f'(y) = 0$ if and only if $y = (b - 2)$. Observe that $f''(y) = 2$ for all $y \geq 0$. So by Theorem 5 in Calculus - 4, the function f' is strictly increasing on $y \geq 0$. So it follows that $f' < 0$ on $(0, b - 2)$, and $f' > 0$ on $y > (b - 2)$. Hence by Theorems 5 and 7 of Calculus - 4, note that f attains its minimum only at the critical point $y = (b - 2)$. Consequently the minimum distance in this case is $d = 2\sqrt{b - 1}$.

Hence the minimum distance is $|b|$ if $b < 2$, and it is $2\sqrt{b - 1}$ if $b \geq 2$.

Example 3: The cost of producing and marketing x units of an item is given by

$$f(x) \quad = \quad 0.02x^2 + 160x + 4,00,000, \;\; x \geq 0.$$

(It is assumed that x can take any value in $[0, \infty)$.) One unit of the item is sold for Rs. 400. How many items should be sold for maximising profit?

The profit arising from selling x units of the item is clearly

$$\begin{aligned} P(x) \;&=\; 400x - f(x) \;=\; 400x - (0.02x^2 + 160x + 4,00,000) \\ &=\; -002x^2 + 240x - 4,00,000, \;\; x \geq 0. \end{aligned}$$

Of course, $P(x) < 0$ would indicate loss. Need to maximise $P(x)$ over $[0, \infty)$.

Verify that $P'(x) = -0.04x + 240$, and $P''(x) = -0.04$. So by Theorem 5 of Calculus - 4, the function P' is strictly decreasing on $x \geq 0$.

It is easy to see that $P'(x) = 0$ if and only if $x = \frac{240}{0.04} = 6,000$. So $x = 6,000$ is the only critical point in $(0, \infty)$.

Note that $P(0) < 0$. Note also that $P'(x) < 0, \;\; x > 6,000$; hence by Theorem 5 of Calculus - 4, the function P is strictly decreasing on $x > 6,000$.

Thus it follows that P has a maximum at the only critical point $x = 6,000$. Hence $6,000$ units should be sold to maximise profit.

Remark: The following facts may also be used to simplify arguments when the functions involved are polynomials on infinite intervals.

Let

$$p(x) \quad = \quad a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \ x \in \mathbb{R},$$

be a polynomial of degree n; the coefficient of x^n, that is, a_n is called the leading coefficient.

Let $n = 2k, \ k \geq 1$ be even. If the leading coefficient $a_n = a_{2k} > 0$, then

$$\begin{aligned} \lim_{x \to +\infty} p(x) &= +\infty, \\ \lim_{x \to -\infty} p(x) &= +\infty. \end{aligned}$$

If the leading coefficient $a_{2k} < 0$, then

$$\begin{aligned} \lim_{x \to +\infty} p(x) &= -\infty, \\ \lim_{x \to -\infty} p(x) &= -\infty. \end{aligned}$$

Write

$$p(x) \quad = \quad [x^n] \ \ [a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}], \ x \in \mathbb{R}.$$

Separately taking limits in the two terms on the right side, the above assertions can be proved.

Let $n = 2k+1, \ k \geq 0$ be odd. If the leading coefficient $a_n = a_{2k+1} > 0$, then

$$\begin{aligned} \lim_{x \to +\infty} p(x) &= +\infty, \\ \lim_{x \to -\infty} p(x) &= -\infty. \end{aligned}$$

If the leading coefficient $a_{2k+1} < 0$, then

$$\begin{aligned} \lim_{x \to +\infty} p(x) &= -\infty, \\ \lim_{x \to -\infty} p(x) &= +\infty. \end{aligned}$$

The above are not difficult to prove. You may try with $n = 1, 2, 3, 4$. You may also get an idea by drawing graphs of polynomials like $p(x) = x,\ (-x),\ x^2,\ (-x^2),\ x^3,\ (-x^3),\ x^4,\ (-x^4)$, etc.

In Example 2, note that f is a polynomial of degree 2, with leading coefficient > 0. In Example 3, note that P is a polynomial of degree 2, with leading coefficient < 0. You may draw graphs of f and P choosing appropriate scales; and observe the behaviour as x or y go becomes larger in positive or negative direction. Using these, make the arguments shorter in both the examples.

Remark: Suppose a function of the form $q(x) = \frac{c}{x^k}$, where $k \geq 1$ is an integer, $c \in \mathbb{R}$ is a constant, occurs in a maximization/ minimization problem over an interval $(0, b]$. If $c > 0$, then

$$\lim_{x \to 0+} q(x) = +\infty;$$

if $c < 0$, then

$$\lim_{x \to 0+} q(x) = -\infty.$$

The above facts can also be used to advantage.

Several such useful facts may appear as one proceeds.

Example 4: Find the point on the graph of the equation $y^2 = 4x$ which is nearest to the point $(2, 1)$.

The distance d between $(2, 1)$ and a point (x, y) on the graph of $y^2 = 4x$, is $d = \sqrt{(x-2)^2 + (y-1)^2}$. Since minimising d is the same as minimising d^2, and using $x = \frac{y^2}{4}$, we may take the function to be minimised as

$$\begin{aligned} f(y) &= (\frac{y^2}{4} - 2)^2 + (y-1)^2 \\ &= \frac{y^4}{16} - 2y + 5,\ y \in \mathbb{R}. \end{aligned}$$

As f is a polynomial of degree 4, with leading coefficient $\frac{1}{16} > 0$, we know that

$$\lim_{y \to +\infty} f(y) = \lim_{y \to -\infty} f(y) = +\infty.$$

Hence f will have the minimum value only at a critical point.

Easily seen that

$$f'(y) = \frac{y^3}{4} - 2, \qquad f''(y) = \frac{3}{4}y^2, \quad y \in \mathbb{R}.$$

It is clear that $f'(y) = 0$ if and only if $y = 2$; also $f''(2) = 3 > 0$. So $y = 2$ is the only critical point, and f attains the minimum value when $y = 2$. Clearly $x = 1$ when $y = 2$.

Thus $(1, 2)$ is the point on the graph of $y^2 = 4x$ nearest to $(2, 1)$; also the minimum distance is $\sqrt{2}$.

Example 5: Consider a success-failure experiment with probability of success p in any individual trial, where $0 \leq p \leq 1$. Suppose that in n trials s successes have been observed. The likelihood function L is given by

$$L(p) \quad = \quad p^s(1-p)^{n-s}.$$

Find the value of p which maximises the likelihood function. (Here $n,\ s$ are taken to be integer constants, with $n \geq 2,\ 0 < s < n$; in this context, it is realistic to assume that n is not a small integer.)

It is clear that $L(0) = L(1) = 0$, and $L(p) > 0,\ 0 < p < 1$. So the maximum can be reached only at a critical point. Verify that

$$L'(p) \quad = \quad p^{s-1}(1-p)^{n-s-1}(-np+s),\ 0 < p < 1.$$

Note that the first two terms on the right side can not be 0; so $L'(p) = 0$ if and only if $p = \frac{s}{n}$. As this is the only critical point, and as $L(s/n) > L(0),\ L(s/n) > L(1)$, it follows that L attains maximum value at s/n.

One may also find that $L''(\frac{s}{n}) < 0$ at the critical point.