Limit

When studying a function $y=f(x)$, we find ourselves interested in the function's behaviour near a perticular point c, but not at c.

For example:–

i) If c is an irrational number. [appoximated by close rational number]

ii) Zero division [Undefined]

Question

How does the function $f(x) = \dfrac{x^2-4}{x-2}$ behave near $x=2$?

→ If we take $x=2$

Then $f(2) = \dfrac{2^2-4}{2-2} = \dfrac{0}{0}$ [Undefined]

Now Let's take some values of x, which are close to 2.

$$f(2.5) = \frac{(2.5)^2-4}{2.5-2} = 4.5$$

In this way we can take some more values.

x	$f(x)$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001
2.00001	4.00001

x	$f(x)$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

From this we can observe that,

As $x \to 2$ $f(x) \to 4$

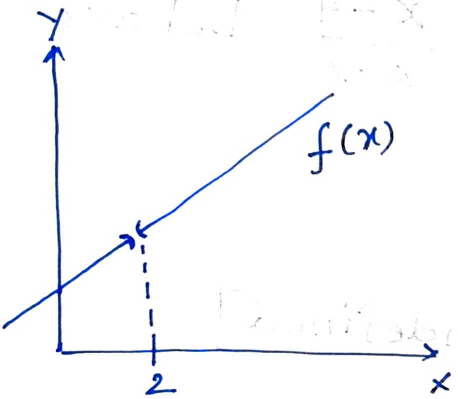
Let's draw the graph of $f(x)$

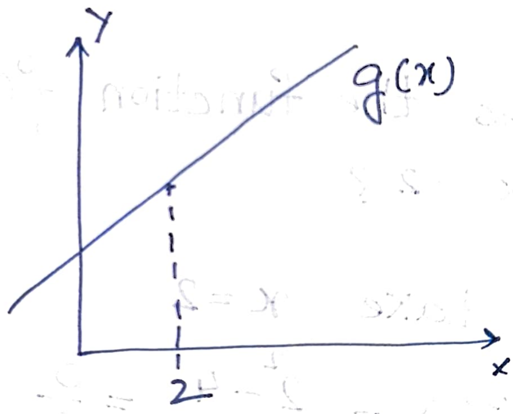
$$f(x) = \frac{x^2 - 4}{x - 2} \quad x \neq 2$$

$$= \frac{(x^2+2)(x-2)}{(x-2)} \quad x \neq 2$$

$$= x + 2 \quad , \quad x \neq 2$$

$g(x) = x + 2$





The heights of these two endpoints are called limits.

Left hand Limit :-

$$\lim_{x \to c^-} f(x) = \lim_{h \to 0} f(c-h)$$

Right hand Limit :-

$$\lim_{x \to c^+} f(x) = \lim_{h \to 0} f(c+h)$$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x \to 2} x + \lim_{x \to 2} 2$$

$$= 2 + 2 = 4$$

Example:

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}$$

$$= \lim_{x \to 2} \frac{x^2 - 3x - 2x + 6}{x - 2}$$

$$= \lim_{x \to 2} \frac{x(x-3) - 2(x-3)}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x-3)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x \to 2} x - \lim_{x \to 2} -3$$

$$= 2 - 3 = -1$$

Try :-

① $\lim_{x \to 2} (x^2 + 4x - 3)$

② $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$

③ $\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$

The Limit Laws

To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use several fundamental rules.

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

In words, the Sum Rule says that the limit of a sum is the sum of the limits. Similarly, the

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(\sqrt{x^2+100}-10\right)\left(\sqrt{x^2+100}+10\right)}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \lim_{x\to 0} \frac{\left(\sqrt{x^2+100}\right)^2-10^2}{x^2 \times \left(\sqrt{x^2+100}+10\right)}$$

$$= \lim_{x\to 0} \frac{x^2+\cancel{100}-\cancel{100}}{x^2 \times \left(\sqrt{x^2+100}+10\right)}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}\left(\sqrt{x^2+100}+10\right)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{10+10} = \frac{1}{20}$$