# Review of Functions

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**Problem1:** Find the domain and range of $f(x) = \sqrt{x - 1}$.

# Problems: Domain and Range

**Problem1:** Find the domain and range of $f(x) = \sqrt{x-1}$.
**Solution:** For the square root to be defined,

$$x - 1 \geq 0 \quad \Rightarrow \quad x \geq 1$$

So,

$$\text{Domain} = [1, \infty)$$

Since $\sqrt{x-1} \geq 0$ for all $x \geq 1$,

$$\text{Range} = [0, \infty)$$

# Problems: Domain and Range

**Problem2:** Find the domain of $y = \sqrt{1 - x^2}$.

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**Solution:** Inside the square root must be non-negative:

$$1 - x^2 \geq 0$$

Factorize:

$$(1 - x)(1 + x) \geq 0 \quad \Leftrightarrow \quad (x - 1)(x + 1) \leq 0$$

This inequality holds when $x \in [-1, 1]$.

$$\therefore \text{ Domain} = [-1, 1]$$

**Problem3:** Find the domain of

$$f(x) = \frac{x^2}{(x-1)x}$$

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$$f(x) = \frac{x^2}{(x-1)x}$$

**Solution:** The denominator must be non-zero:

$$x(x-1) \neq 0$$

So,

$$x \neq 0 \quad \text{and} \quad x \neq 1$$

Thus,

$$\text{Domain} = \mathbb{R} \setminus \{0, 1\}$$

# Problems: Domain and Range

**Problem4:** Find the domain of

$$f(x) = \sqrt{\frac{x-2}{x-3}}$$

# Problems: Domain and Range

**Problem4:** Find the domain of

$$f(x) = \sqrt{\frac{x-2}{x-3}}$$

**Solution:** 1. Denominator cannot be zero:

$$x \neq 3$$

2. Expression inside the root must be non-negative:

$$\frac{x-2}{x-3} \geq 0$$

Critical points: $x = 2$, $x = 3$. By sign analysis on intervals $(-\infty, 2]$, $(2, 3)$, and $(3, \infty)$:

$$\frac{x-2}{x-3} \geq 0 \quad \Rightarrow \quad (-\infty, 2] \cup (3, \infty)$$

Thus,

$$\text{Domain} = (-\infty, 2] \cup (3, \infty)$$

# Procedure for Finding the Range of a Function

For a function $y = f(x)$, the steps are:

1. Express $x$ explicitly in terms of $y$.
2. Find the possible values for $y$ (similar to finding domain for $x$).
3. Eliminate any values of $y$ that are not valid with respect to $x$, if applicable.

**Problem1:**Find the domain and range of

$$f(x) = \frac{1}{x+3}$$

# Problems: Domain and Range

**Problem1:**Find the domain and range of

$$f(x) = \frac{1}{x+3}$$

**Domain:** Denominator $\neq 0 \quad \Rightarrow x \neq -3.$

$$\text{Domain} = \mathbb{R} \setminus \{-3\}$$

**Range:** Let $y = \frac{1}{x+3} \ \Rightarrow \ x = \frac{1-3y}{y}, \ y \neq 0.$

$$\text{Range} = \mathbb{R} \setminus \{0\}$$

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**Problem2:**Find the domain and range of

$$f(x) = \frac{2x+3}{x}$$

# Problems: Domain and Range

**Problem2:** Find the domain and range of

$$f(x) = \frac{2x+3}{x}$$

**Domain:** $x \neq 0 \;\; \Rightarrow \;\; \mathbb{R} \setminus \{0\}$

**Range:** Let $y = \dfrac{2x+3}{x} \;\; \Rightarrow \;\; x = \dfrac{3}{y-2}$, so $y \neq 2$.

$$\text{Range} = \mathbb{R} \setminus \{2\}$$

**Problem3:**Find the domain and range of

$$f(x) = \sqrt{x-2}$$

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$$f(x) = \sqrt{x-2}$$

**Domain:** $x - 2 \geq 0 \;\; \Rightarrow \;\; x \geq 2.$

$$\text{Domain} = [2, \infty)$$

**Range:** Let $y = f(x)$.

$$y = \sqrt{x-2}$$

Since the square root symbol denotes the principal (non-negative) square root, $y \geq 0$. Square both sides to solve for $x$:

$$y^2 = x - 2$$

$$x = y^2 + 2$$

Since $y^2 + 2$ is defined for all real $y$, and we already established $y \geq 0$, the range is all non-negative real numbers.

$$Range = [0, \infty)$$

**Problem 4:** Find the domain and range of

$$f(x) = \sqrt{4 - x^2}$$

# Problems: Domain and Range

**Problem 4:** Find the domain and range of

$$f(x) = \sqrt{4 - x^2}$$

**Domain:** $4 - x^2 \geq 0 \;\Rightarrow\; (2 - x)(2 + x) \geq 0 \;\Rightarrow\; -2 \leq x \leq 2.$

$$\text{Domain} = [-2, 2]$$

**Range:** Let $y = f(x)$.

$$y = \sqrt{4 - x^2}$$

Again, since it's a square root, $y \geq 0$. Square both sides:

$$y^2 = 4 - x^2$$

Rearrange to solve for $x^2$:

$$x^2 = 4 - y^2$$

For $x^2$ to be a real number, it must be non-negative:

$$x^2 \geq 0 \Rightarrow 4 - y^2 \geq 0$$

$$4 \geq y^2 \Rightarrow -2 \leq y \leq 2$$

Combining with $y \geq 0$, the range is $[0, 2]$.

$$Range = [0, 2]$$

**Problem 5:**Find the domain and range of

$$f(x) = \sqrt{x^2 - 4}$$

# Problems: Domain and Range

**Problem 5:** Find the domain and range of

$$f(x) = \sqrt{x^2 - 4}$$

**Domain:** $x^2 - 4 \geq 0 \;\Rightarrow\; (x-2)(x+2) \geq 0 \;\Rightarrow\; x \leq -2 \;\text{ or }\; x \geq 2.$

$$\text{Domain} = (-\infty, -2] \cup [2, \infty)$$

**Range:** Let $y = f(x)$.

$$y = \sqrt{x^2 - 4}$$

Since it's a square root, $y \geq 0$. Square both sides:

$$y^2 = x^2 - 4$$

Rearrange to solve for $x^2$:

$$x^2 = y^2 + 4$$

For $x^2$ to be defined and real, $x^2 \geq 0$, which is always true for $y^2 + 4$ since $y^2 \geq 0$. Combining with $y \geq 0$, the range is all non-negative real numbers.

$$Range = [0, \infty)$$

# Problems: Domain and Range

**Problem 6:** Find the domain and range of

$$f(x) = \frac{1}{\sqrt{4 - x^2}}$$

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$$f(x) = \frac{1}{\sqrt{4 - x^2}}$$

**Domain:** Inside root
$4 - x^2 > 0 \;\; \Rightarrow \;\; (2 - x)(2 + x) > 0 \;\; \Rightarrow \;\; -2 < x < 2.$

$$\text{Domain} = (-2, 2)$$

**Range:** Let $y = f(x)$.

$$y = \frac{1}{\sqrt{4 - x^2}}$$

Since the denominator is a square root, $\sqrt{4 - x^2} > 0$, which implies $y > 0$. Square both sides:

$$y^2 = \frac{1}{4 - x^2}$$

# Problems: Domain and Range

Rearrange to solve for $4 - x^2$:

$$4 - x^2 = \frac{1}{y^2}$$

$$x^2 = 4 - \frac{1}{y^2}$$

For $x^2$ to be real and non-negative:

$$4 - \frac{1}{y^2} \geq 0$$

$$4 \geq \frac{1}{y^2}$$

Since $y^2 > 0$:

$$4y^2 \geq 1$$

$$y^2 \geq \frac{1}{4}$$

Taking the square root of both sides:

$$|y| \geq \frac{1}{2}$$

So, $y \leq -\frac{1}{2}$ or $y \geq \frac{1}{2}$. Combining this with our earlier condition $y > 0$, the range is $[\frac{1}{2}, \infty)$.

$$Range = [\frac{1}{2}, \infty)$$

# Example: One-One Function

**Problem:** Show that $f(x) = 2x + 3$ is a one-one function.

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**Problem:** Show that $f(x) = 2x + 3$ is a one-one function.
**Solution:** Let $f(x_1) = f(x_2)$. Then,

$$2x_1 + 3 = 2x_2 + 3$$

$$\Rightarrow 2x_1 = 2x_2 \quad \Rightarrow \quad x_1 = x_2$$

Hence, $f(x)$ is **one-one**.

# Bijective Function Problem

**Problem:**Show that the function $f : A \to B$ defined by $f(x) = \frac{x-2}{x-3}$, where $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$, is bijective and to find its inverse, $f^{-1}(x)$.

**Problem:**Show that the function $f : A \to B$ defined by $f(x) = \frac{x-2}{x-3}$, where $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$, is bijective and to find its inverse, $f^{-1}(x)$.

**Proof of Bijectivity**

To prove that a function is bijective, we must show that it is both one-one (injective) and onto (surjective).

# Bijective Function Problem

**One-One (Injective)** A function is one-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
Let's assume $f(x_1) = f(x_2)$.

$$\frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

Cross-multiplying gives:

$$(x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$$

Expanding both sides:

$$x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$$

Subtracting $x_1x_2 + 6$ from both sides:

$$-3x_1 - 2x_2 = -3x_2 - 2x_1$$

# Bijective Function Problem

Rearranging the terms:

$$-3x_1 + 2x_1 = -3x_2 + 2x_2$$

$$-x_1 = -x_2$$

$$x_1 = x_2$$

Since $f(x_1) = f(x_2)$ implies $x_1 = x_2$, the function is one-one.
**Onto (Surjective)** A function is onto if for every $y$ in the codomain $B$, there exists an $x$ in the domain $A$ such that $f(x) = y$. To prove this, we express $x$ in terms of $y$.
Let $y = f(x)$.

$$y = \frac{x-2}{x-3}$$

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# Bijective Function Problem

Multiply both sides by $(x-3)$:

$$y(x-3) = x-2$$

$$xy - 3y = x - 2$$

Group the terms with $x$ on one side:

$$xy - x = 3y - 2$$

$$x(y-1) = 3y - 2$$

Solve for $x$:

$$x = \frac{3y-2}{y-1}$$

$$\forall y \in B, \exists x \in A \text{ such that } f(x) = y.$$

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# Bijective Function Problem

For every $y \in \mathbb{R} - \{1\}$, the expression for $x$ is defined and $x \in \mathbb{R} - \{3\}$. This confirms that the function is onto.
Since the function is both one-one and onto, it is bijective.
**Finding the Inverse Function** $f^{-1}(x)$ The inverse function is found by taking the expression for $x$ in terms of $y$ and swapping the variables. From the onto proof, we have:

$$x = \frac{3y - 2}{y - 1}$$

Replacing $x$ with $f^{-1}(x)$ and $y$ with $x$:

$$f^{-1}(x) = \frac{3x - 2}{x - 1}$$