Review of Functions and Their Graphical Visualization

Atanu Saha

FLAME University, Pune
School of Computing and Data Sciences

August 22, 2025

Outline

1. Introduction to Calculus
2. Functions
3. Composition and Inverse Functions

Introduction

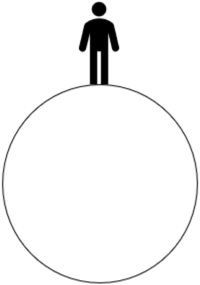
Calculus, one of the most important branches of mathematics, was independently developed in the late 17^{th} century by **Sir Isaac Newton** and **Gottfried Wilhelm Leibniz**.

Newton humbly acknowledged the influence of earlier scholars, famously stating:

"If I have seen further, it is by standing on the shoulders of giants."

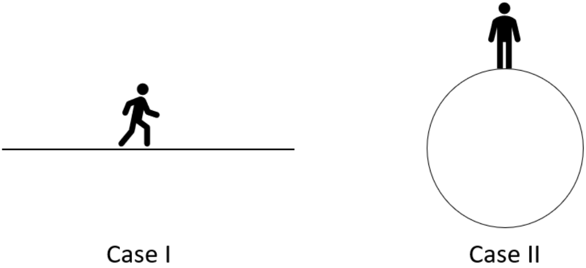
This sentiment reflects the cumulative nature of mathematical progress, where each generation builds upon the achievements of those before.

Case I



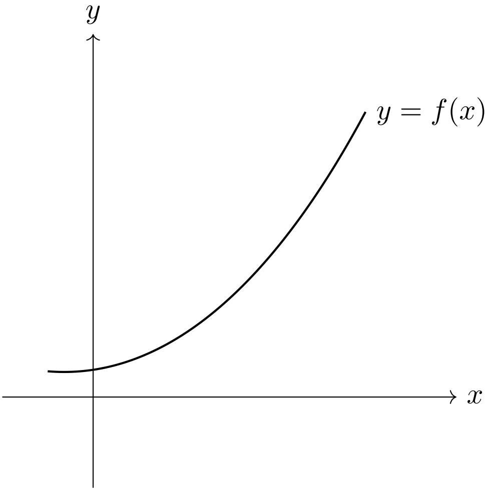
Case II

Flat Here, Round There

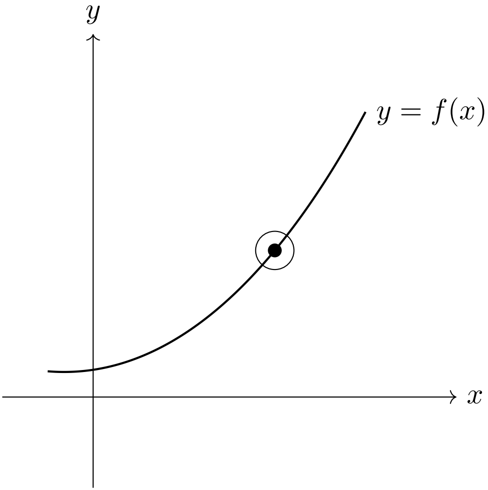


When a function is observed locally, over a very small interval, it often resembles a simpler function (typically a linear one). Differential calculus is concerned with rigorously identifying this local behavior of the function.

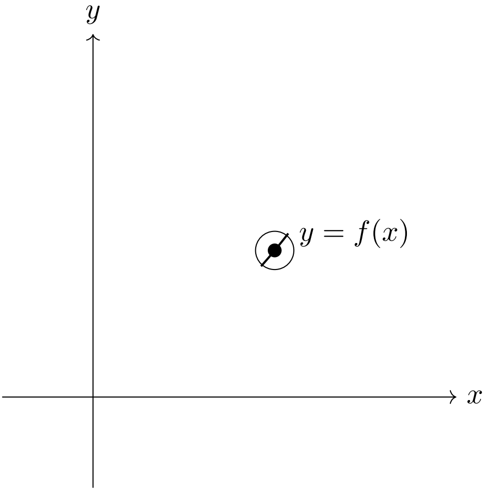
Graph I : The curve of a function $y = f(x)$



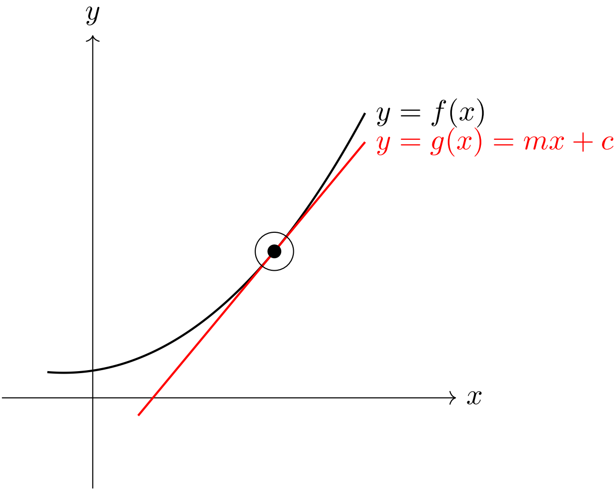
Graph II : Focus on a tiny piece of the curve.



Graph III : Up close, the curve looks nearly straight.



Graph IV : That "straightness" is captured by a tangent line.



Why Calculus?

- Describes **change** and **motion**.
- Used in Physics, Engineering, Economics, Biology.
- Two pillars:
  1. **Differential Calculus** – rates of change, slopes of curves.
  2. **Integral Calculus** – accumulation, area under curves.
- Bridges the gap between algebra and real-world applications.

Definition of a Function

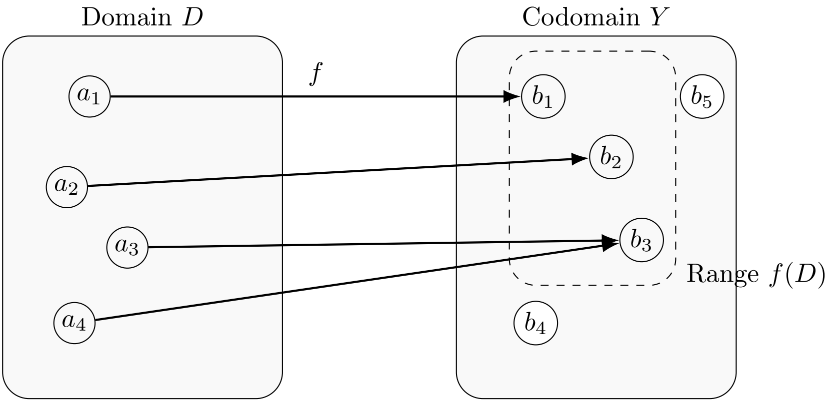
Function

A *function* f from a set D to a set Y is a rule that assigns a **unique** element $f(x) \in Y$ to each element $x \in D$.

$$y = f(x)$$

- x: **independent variable** (input)
- y: **dependent variable** (output)
- D: **domain** (all possible input values)
- R: **range** (all possible output values)

Function



Function $f : D \to Y$

$b_4, b_5 \in Y \setminus f(D)$

Examples of Functions

Domains and Ranges

Function	Domain	Range
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = \frac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Increasing and Decreasing Functions

Definition

Let f be a function on an interval I and let x_1 and x_2 be any two points in I.

- f is **increasing** on I if

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

- f is **decreasing** on I if

$$x_1 < x_2 \implies f(x_1) > f(x_2)$$

Intuition:

- Increasing $\rightarrow$ graph rises left to right.
- Decreasing $\rightarrow$ graph falls left to right.

Even and Odd Functions: Symmetry

Definitions

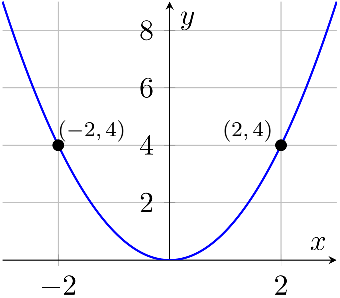
A function $y = f(x)$ is

$$\text{even if } f(-x) = f(x), \quad \text{odd if } f(-x) = -f(x),$$

for every x in the function's domain.

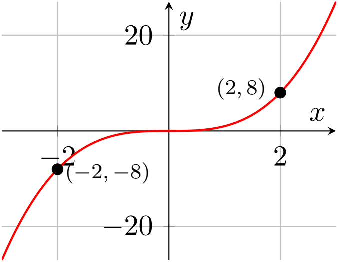
Even and Odd Functions: Symmetry Illustrated

Even Function: $f(x) = x^2$



Here $f(-2) = 4$ and $f(2) = 4$.
$\Rightarrow f(-x) = f(x)$

Odd Function: $f(x) = x^3$



Here $f(-2) = -8$ and $f(2) = 8$.
$\Rightarrow f(-x) = -f(x)$

Composition of Functions

- If $f : A \to B$ and $g : B \to C$, then $(g \circ f)(x) = g(f(x))$.
- Example:

$$\begin{aligned} f(x) &= 2x + 3 \\ g(x) &= x^2 \\ (g \circ f)(x) &= g(f(x)) = (2x+3)^2 \end{aligned}$$

One-to-One (Injective) Functions

Definition

A function $f : D \to R$ is called **one-to-one (injective)** if

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

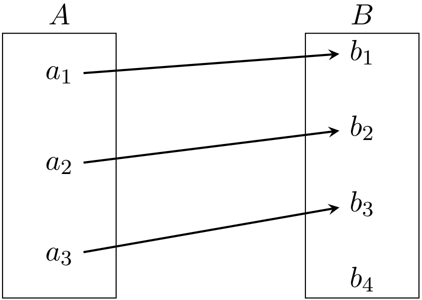
That is, distinct inputs give distinct outputs.

Horizontal Line Test

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

One-One (Injective) Function

$f : A \to B$ is a one-one (injective) mapping



Each element of A maps to a unique element of B. No two inputs share the same output.

Onto (Surjective) Functions

Definition

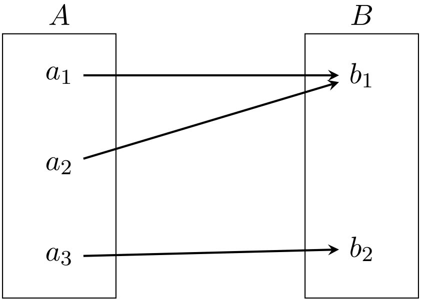
A function $f : D \to R$ is called **onto (surjective)** if

$$\forall y \in R,\ \exists x \in D \text{ such that } f(x) = y.$$

That is, every element of the codomain is an output of the function.

- Surjective $\to$ Range = Codomain.
- Not surjective $\to$ Some $y \in R$ has no preimage.

$f : A \to B$ is an onto (surjective) mapping



Each element of B has at least one pre-image in A. The function "covers" the entire codomain B.

Inverse Functions

Definition

Suppose f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(b) = a \quad \text{if } f(a) = b.$$

- The domain of f^{-1} is R.
- The range of f^{-1} is D.
- Graphically: Reflection of f across the line $y = x$.

Inverse Functions Example

- f^{-1} reverses the effect of f.
- If $f(x) = y$, then $f^{-1}(y) = x$.
- Example:

$$f(x) = 3x + 2$$
$$f^{-1}(y) = \frac{y-2}{3}$$

- Graphically: reflection across the line $y = x$.

Programs and Courses

- **Applied Mathematics**: `gesstalt.github.io/teaching/flameam.html`
- **Computer Science**: `gesstalt.github.io/teaching/flamecs.html`
- **Computer Science and Design**: `gesstalt.github.io/teaching/flamecd.html`
- **Data Science and Economics**: `gesstalt.github.io/teaching/flameds.html`