Successive differentiation I

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Example 1: Successive Differentiation

Problem

Find the first, second, and third derivatives of

$$y = x^3 - 3x^2 + 2x - 1.$$

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$$y = x^3 - 3x^2 + 2x - 1.$$

Solution

$$\begin{aligned} y' &= 3x^2 - 6x + 2, \\ y'' &= 6x - 6, \\ y''' &= 6. \end{aligned}$$

Example 2: Trigonometric Function

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Observation: After every 4 differentiations, the function repeats.

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Example 3: Exponential Function

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Find the n^{th} derivative of

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Solution

$$\begin{aligned} y' &= ae^{ax}, \\ y'' &= a^2e^{ax}, \\ y''' &= a^3e^{ax}, \\ &\vdots \\ y^{(n)} &= a^ne^{ax}. \end{aligned}$$

Example 4: Product of Functions

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Find the second derivative of

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Problem

Find the second derivative of

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Solution

$$y' = 2x \sin x + x^2 \cos x,$$

$$y'' = 2 \sin x + 2x \cos x + 2x \cos x - x^2 \sin x,$$

$$y'' = 2 \sin x + 4x \cos x - x^2 \sin x.$$

n^{th} Derivative: Overview

Concept

The n^{th} derivative, denoted by $f^{(n)}(x)$, represents the derivative of order n. Different types of functions follow distinct patterns for their n^{th} derivatives.

We will find formulas for:

1. Polynomial functions
2. Exponential functions
3. Trigonometric functions
4. Logarithmic functions
5. Product of exponential and trigonometric

1. Polynomial Function

Function

$$y = x^m$$

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Function

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Successive Differentiation

$$\begin{aligned} y' &= mx^{m-1} \\ y'' &= m(m-1)x^{m-2} \\ &\vdots \\ y^{(n)} &= m(m-1)(m-2)\dots(m-n+1)x^{m-n} \end{aligned}$$

$$\boxed{y^{(n)} = \frac{m!}{(m-n)!}x^{m-n}}, \quad \text{for } n \leq m$$

If $n > m$, then $y^{(n)} = 0$.

2. Exponential Function

Function

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Function

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Derivatives

$$y' = ae^{ax},$$
$$y'' = a^2e^{ax},$$
$$\vdots$$
$$y^{(n)} = a^n e^{ax}.$$

$$\boxed{y^{(n)} = a^n e^{ax}}$$

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3. Trigonometric Functions

Function

For $y = \sin(ax)$ and $y = \cos(ax)$

$$y^{(n)}(\sin ax) = a^n \sin\left(ax + n\frac{\pi}{2}\right),$$

$$y^{(n)}(\cos ax) = a^n \cos\left(ax + n\frac{\pi}{2}\right).$$

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Examples

$$\frac{d^3}{dx^3}(\sin 2x) = 2^3 \sin\left(2x + \tfrac{3\pi}{2}\right) = -8\cos 2x,$$

$$\frac{d^4}{dx^4}(\cos x) = 1^4 \cos(x + 2\pi) = \cos x.$$

I. $\mathbf{n^{th}}$ Derivative of $y = \sin(ax + b)$

The Process

Let $y = \sin(ax + b)$

- $y_1 = a\cos(ax + b) = a\sin\left(ax + b + \frac{\pi}{2}\right)$
- $y_2 = a^2\cos\left(ax + b + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + \frac{2\pi}{2}\right)$
- $\vdots$

The Result

The n^{th} derivative is:

$$y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$

Note: Similarly, for $y = \cos(ax + b)$, the result is:

$$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$

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II. $\mathbf{n^{th}}$ Derivative of $y = e^{ax}\sin(bx+c)$

The Process (Using $a = r\cos\alpha$ and $b = r\sin\alpha$)

Let $y = e^{ax}\sin(bx+c)$

- $y_1 = e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$
- $y_1 = e^{ax}[r\cos\alpha\sin(bx+c) + r\sin\alpha\cos(bx+c)]$
- $y_1 = e^{ax}r\sin(bx+c+\alpha)$
- $\vdots$

The Result

The n^{th} derivative is:

$$y_n = e^{ax}r^n\sin(bx+c+n\alpha)$$

Where $r^2 = a^2 + b^2$ and $\tan\alpha = \frac{b}{a}$.

Substituting r and α: $y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

4. Logarithmic Function

Function

$$y = \ln x$$

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Function

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Successive Derivatives

$$\begin{aligned}
y' &= \frac{1}{x}, \\
y'' &= -\frac{1}{x^2}, \\
y^{(3)} &= 2!\frac{1}{x^3}, \\
&\vdots \\
y^{(n)} &= (-1)^{n-1}(n-1)!\,x^{-n}.
\end{aligned}$$

$$\boxed{y^{(n)} = (-1)^{n-1}(n-1)!\,x^{-n}}$$

Summary: n^{th} Derivatives of Common Functions

Function	$n^{\textbf{th}}$ **Derivative Formula**
x^m	$\dfrac{m!}{(m-n)!}x^{m-n}$
e^{ax}	$a^n e^{ax}$
$\sin(ax)$	$a^n \sin(ax + n\pi/2)$
$\cos(ax)$	$a^n \cos(ax + n\pi/2)$
$\ln x$	$(-1)^{n-1}(n-1)!\, x^{-n}$
$e^{ax}\sin(bx)$	$e^{ax} r^n \sin(bx + n\theta)$
$e^{ax}\cos(bx)$	$e^{ax} r^n \cos(bx + n\theta)$

$$r = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Summary

Function (y)	$\mathbf{n^{th}}$ Derivative ($\mathbf{y_n}$)	Conditions
e^{ax+b}	$a^n e^{ax+b}$	
$(ax+b)^m$	$\frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$	$m > n$
$(ax+b)^m$	0	$m < n$
$(ax+b)^m$	$m!a^n$	$m = n$
$\ln(ax+b)$	$(-1)^{n-1}\frac{(n-1)!a^n}{(ax+b)^n}$	
$\sin(ax+b)$	$a^n \sin(ax+b+\frac{n\pi}{2})$	
$\cos(ax+b)$	$a^n \cos(ax+b+\frac{n\pi}{2})$	
$e^{ax}\sin(bx+c)$	$e^{ax}(a^2+b^2)^{\frac{n}{2}}\sin(bx+c+n\tan^{-1}\frac{b}{a})$	
$e^{ax}\cos(bx+c)$	$e^{ax}(a^2+b^2)^{\frac{n}{2}}\cos(bx+c+n\tan^{-1}\frac{b}{a})$	

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Problem: Find y_n for $y = \frac{1}{1-5x+6x^2}$.

Solution: Partial Fractions

1. Resolve into partial fractions:

$$y = \frac{1}{(1-3x)(1-2x)} = \frac{3}{1-3x} - \frac{2}{1-2x}$$

2. Apply the formula for y_n of $\frac{1}{ax+b}$ (where a is -3 and -2 respectively):

$$y_n = 3\left[\frac{(-1)^n n!(-3)^n}{(1-3x)^{n+1}}\right] - 2\left[\frac{(-1)^n n!(-2)^n}{(1-2x)^{n+1}}\right]$$

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Problem 1: $\mathbf{n^{th}}$ derivative of $\frac{1}{1-5x+6x^2}$

Final Result

The simplified n^{th} derivative is:

$$\Rightarrow y_n = (-1)^{n+1} n! \left[\left(\frac{3}{1-3x} \right)^{n+1} - \left(\frac{2}{1-2x} \right)^{n+1} \right]$$

Problem 2: $\mathbf{n^{th}}$ derivative of $\sin \mathbf{6x} \cos \mathbf{4x}$

Problem: Find y_n for $y = \sin 6x \cos 4x$.

Solution: Product-to-Sum Identity

1. Use the identity $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$ to linearize the expression:

$$y = \frac{1}{2}(\sin 10x + \sin 2x)$$

2. Apply the formula for y_n of $\sin(ax+b)$, which is $a^n \sin(ax + b + \frac{n\pi}{2})$.

Problem 2: $\mathbf{n^{th}}$ derivative of $\sin \mathbf{6x} \cos \mathbf{4x}$

Final Result

The n^{th} derivative is:

$$y_n = \frac{1}{2}\left[10^n \sin\left(10x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right)\right]$$

(Note: The function for the second term in the source's result is cos, though the derivation suggests sin)

Problem 3: Second-Order Differential Equation Proof

Problem: If $y = \log(x + \sqrt{x^2+1})$, show that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$.

Solution Steps

1. **First Derivative (y_1):** Differentiate y:

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$

2. **Rearrange:** Clear the fraction:

$$(\sqrt{1+x^2})\frac{dy}{dx} = 1$$

3. **Second Derivative (y_2):** Differentiate again w.r.t x:

$$(\sqrt{1+x^2})\frac{d^2y}{dx^2} + \left(\frac{x}{\sqrt{1+x^2}}\right)\frac{dy}{dx} = 0$$

Problem 3: Second-Order Differential Equation Proof

Final Proof

Simplify: Multiply the entire equation by $\sqrt{1+x^2}$ to clear the denominator.

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$