Problems: Limits and Differentiation

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Limit Problems I

Problem 1:

$$\lim_{x \to \infty} \frac{x^2 - 4x + 5}{x^3 - 2}$$

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$$\lim_{x\to\infty} \frac{x^2 - 4x + 5}{x^3 - 2}$$

Direct substitution gives $\frac{0}{0}$ indeterminate form.
Divide numerator and denominator with the highest power of x.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2 - 4x + 5}{x^3 - 2} &= \lim_{x\to\infty} \frac{\frac{x^2-4x+5}{x^3}}{\frac{x^3-2}{x^3}} \\
&= \lim_{x\to\infty} \frac{\frac{x^2}{x^3} - \frac{4x}{x^3} + \frac{5}{x^3}}{\frac{x^3}{x^3} - \frac{2}{x^3}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{4}{x^2} + \frac{5}{x^3}}{1 - \frac{2}{x^3}} \\
&= \frac{0 - 0 + 0}{1 - 0} = 0
\end{aligned}$$

Limit Problems II

Problem 2:

$$\lim_{x \to \infty} \frac{\sin x}{x}$$

[Hint: Try to use Sandwich Theorem]

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[Hint: Try to use Sandwich Theorem]
We know that:

$$\begin{aligned}
-1 &\le \sin x \le 1 \\
\Rightarrow \frac{-1}{x} &\le \frac{\sin x}{x} \le \frac{1}{x} \\
\Rightarrow \lim_{x\to\infty} \frac{-1}{x} &\le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x} \\
\Rightarrow 0 &\le \lim_{x\to\infty} \frac{\sin x}{x} \le 0
\end{aligned}$$

$$\Rightarrow \lim_{x\to\infty} \frac{\sin x}{x} = 0$$

Limit Problems III

Example 2:

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$$

Limit Problems III

Example 2:

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$$

- Direct substitution gives $\frac{0}{0}$.
- Multiply numerator and denominator by conjugate: $\sqrt{x+1} + 1$.

$$\lim_{x \to 0} \frac{(\sqrt{x+1} - 1)(\sqrt{x+1} + 1)}{x(\sqrt{x+1} + 1)} = \lim_{x \to 0} \frac{x}{x(\sqrt{x+1} + 1)}$$

$$= \frac{1}{2}$$

Limit Problems IV

Problem: Find the vertical and horizontal asymptotes of the function

$$f(x) = \frac{2}{x+4}.$$

Definition:

- A line $x = a$ is called a **vertical asymptote** of $f(x)$ if

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty.$$

- A line $y = L$ is called a **horizontal asymptote** of $f(x)$ if

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

Solution:
We have

$$f(x) = \frac{2}{x+4}.$$

(i) Vertical Asymptote: The denominator vanishes at $x = -4$. Check limits:

$$\lim_{x \to -4^+} \frac{2}{x+4} = +\infty, \qquad \lim_{x \to -4^-} \frac{2}{x+4} = -\infty.$$

Hence, the vertical asymptote is

$$x = -4.$$

(ii) Horizontal Asymptote:

$$\lim_{x\to\infty} \frac{2}{x+4} = 0, \qquad \lim_{x\to-\infty} \frac{2}{x+4} = 0.$$

Hence, the horizontal asymptote is

$$y = 0.$$

Final Answer: The vertical asymptote is $x = -4$ and the horizontal asymptote is $y = 0$.

Derivative I

Problem:

Suppose functions f and g satisfy the relation

$$f(g(3x)) = 2x, \quad \text{for all real } x.$$

It is known that

$$g(3) = 2 \quad \text{and} \quad g'(3) = 4.$$

Find the value of $f'(2)$.

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Find the value of $f'(2)$.

Solution:
Differentiate both sides of

$$f(g(3x)) = 2x$$

with respect to x:

$$f'(g(3x)) \cdot g'(3x) \cdot 3 = 2.$$

$$f'(g(3x)) \cdot g'(3x) = \frac{2}{3}.$$

Derivative I

Now we need $f'(2)$.
Since $g(3) = 2$, set $3x = 3 \implies x = 1$.
At $x = 1$,

$$g(3x) = g(3) = 2.$$

So plugging $x = 1$ into the derivative relation:

$$f'(2) \cdot g'(3) = \frac{2}{3}.$$

Given $g'(3) = 4$:

$$f'(2) \cdot 4 = \frac{2}{3}.$$

$$f'(2) = \frac{2}{12} = \frac{1}{6}.$$

$$\boxed{\frac{1}{6}}$$

Limit Problem V

Problem : Find

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Step 1: Take natural logarithm

Let

$$L = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$$

Take natural log:

$$\Rightarrow \ln L = \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right)$$

Step 2: Rewrite in standard form

Write as:

$$\ln L = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}$$

Now it is in the form

$$\frac{\ln(1+y)}{y} \quad \text{with } y = \frac{2}{x}.$$

Step 3: Apply the standard limit

As $x \to \infty$, $y = \frac{2}{x} \to 0$ and

$$\lim_{y \to 0} \frac{\ln(1+y)}{y} = 1$$

So,

$$\ln L = \lim_{x \to \infty} 2 \cdot \frac{\ln(1 + 2/x)}{2/x} = 2 \cdot 1 = 2$$

$$L = e^{\ln L} = e^2$$

$$\boxed{e^2}$$

Derivative II

Problem: Using the definition of derivative, find $f'(x)$ if

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Solution:
The definition of derivative is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Substitute $f(x) = x^3 + 5x$:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 + 5(x+h) - (x^3 + 5x)}{h}.$$

Expand:

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3,$$

$$5(x+h) = 5x + 5h.$$

Derivative II

So the numerator becomes:

$$x^3 + 3x^2h + 3xh^2 + h^3 + 5x + 5h - (x^3 + 5x).$$

Simplify:

$$= 3x^2h + 3xh^2 + h^3 + 5h.$$

Thus,

$$f'(x) = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 + 5h}{h}.$$

Factor h:

$$f'(x) = \lim_{h \to 0} \left(3x^2 + 3xh + h^2 + 5\right).$$

Now take the limit as $h \to 0$:

$$f'(x) = 3x^2 + 5.$$

Final Answer:

$$f'(x) = 3x^2 + 5$$

Problem: Differentiability of a Piecewise Function

Check if the function

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$$

is differentiable at $x = 1$.

Step 1: Check Continuity at $x = 1$

For differentiability, the function must be continuous at $x = 1$.

$$f(1^-) = \lim_{x \to 1^-} f(x) = 1^2 = 1$$

$$f(1^+) = \lim_{x \to 1^+} f(x) = 2(1) - 1 = 1$$

$$f(1) = 1$$

Since $f(1^-) = f(1^+) = f(1)$, the function is continuous at $x = 1$.

Step 2: Left-hand Derivative at $x = 1$

Left-hand derivative:

$$f'_{-}(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)^2 - 1}{h}$$

Expand numerator:

$$(1+h)^2 - 1 = 1 + 2h + h^2 - 1 = 2h + h^2$$

$$f'_{-}(1) = \lim_{h \to 0^-} \frac{2h + h^2}{h} = \lim_{h \to 0^-} (2 + h) = 2$$

Step 3: Right-hand Derivative at $x = 1$

Right-hand derivative:

$$
\begin{aligned}
f'_+(1) &= \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h \to 0^+} \frac{2(1+h) - 1 - 1}{h} \\
&= \lim_{h \to 0^+} \frac{2 + 2h - 2}{h} \\
&= \lim_{h \to 0^+} \frac{2h}{h} = 2
\end{aligned}
$$

Step 4: Conclusion

Since

$$f'_-(1) = f'_+(1) = 2$$

the function is differentiable at $x = 1$.

$$\boxed{f \text{ is differentiable at } x = 1 \text{ with } f'(1) = 2}$$