2020

MATHEMATICS — HONOURS

Sixth Paper

(Module - XI)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meanings.

Group - A

[Vector Calculus - II]

(Marks : 10)

1. Answer ***any one*** question :

(a) Verify Stoke's theorem for a vector field defined by $\vec{F} = \left(x^2 - y^2\right)\hat{i} + 2xy\,\hat{j}$ in the rectangular region in xy-plane bounded by the straight lines $x = 0,\ x = 5,\ y = 0,\ y = 8$; 10

(b) Prove that for any scalar function φ (x, y, z),

$$\iiint \vec{\nabla}\varphi\, dv = \iint \varphi\,\hat{n}\, dS$$

where $\hat{n}$ is the outward drawn unit normal vector to the surface S. 10

(c) If V is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$, then show that

(i) $\displaystyle\iiint_V \vec{\nabla}\times\vec{F}\, dV = -\frac{8}{3}\hat{k}$

(ii) $\displaystyle\iiint_V \vec{\nabla}\cdot\vec{F}\, dV = \frac{16}{3}$ where $\vec{F} = \left(3x^2 - 8z\right)\hat{i} - 2xy\,\hat{j} - 8x\,\hat{k}$ 6+4

(d) Verify Green's theorem for the line integral $\displaystyle\oint_C \left(x^2 + xy\right)dx + xdy$, where C is the bounding curve of the region traced by $y = x^2$ & $y = x$. 10

Please Turn Over

Group - B
[Analytical Statics - II]
(Marks : 20)

Answer ***question no.*** **2** and ***any one*** question from the rest.

2. (a) Find the centre of gravity of the arc of the parabola $y^2 = 16x$ included between the lines $x = 0$ and $x = 4$. 6

Or,

(b) Find the condition of stability of equilibrium of a mechanical system having one degree of freedom. 6

3. A solid frustum of a paraboloid of revolution of height h unit and latus rectum 8 unit rests with its vertex on the vertex of a paraboloid of revolution whose latus rectum is 4 unit. Show that the equilibrium is stable if $h < 2$. 14

4. Forces $\vec{X}, 2\vec{X}, 3\vec{X}$ act along the vectors $\hat{i}+\hat{j}-\hat{k}$, $\hat{i}-\hat{j}+\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively. Find the resultant wrench, pitch and intensity. 14

5. A force $\vec{P}$ acts along the axis of x and another force $n\vec{P}$, where n is a positive integer, acts along a generator of the cylinder $x^2 + y^2 = a^2$. Show that the central axis lies on the cylinder

$$n^2(nx - z)^2 + \left(1 + n^2\right)^2 y^2 = n^4 a^2. \qquad 14$$

6. State and establish the principle of virtual work for a system of co-planar forces acting on a rigid body. 14

Group - C
[Analytical Dynamics of a Particle - II]
(Marks : 20)

Answer ***question no.*** **7** and ***any one*** question from the rest.

7. (a) A particle moves with central acceleration μ/r^3. Where r is the distance of particle from centre of force. If it be projected from an apse at a distance 'a' from the centre of force with a velocity equal to $\sqrt{2}$ times that in a circle, find the path. 4

Or,

(b) Classify the equilibrium point for the linear system $AX = \dot{X}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\dot{X} = \dfrac{dX}{dt}$, for different values of the scalars a, b, c and d. 4

8. A particle of mass M is at rest and begins to move under the action of a constant force $\vec{F}$ in a fixed direction. It encounters the resistance of a stream of fine dust moving in the opposite direction with velocity $\vec{u}$, which deposits matter on it at a constant rate σ. Show that its mass will be m, when it has travelled a distance $\frac{k}{\sigma^2}\left[m - M\left\{1+\log\left(\frac{m}{M}\right)\right\}\right]$ where $k = F - \sigma u$.

[F and u are the magnitudes of the force $\vec{F}$ and the velocity $\vec{u}$ respectively]. 16

9. A small bead starts sliding down a semicircular wire of radius *'a'* with coefficient of friction μ. If it starts with a velocity *'u'* from one extreme point in the upper end, find the time taken to slide down to the lowest point (Assume that the wire is fixed in a horizontal base with its centre upwards and the diameter of the free ends is horizontal). Also find the increased velocity at that point. 10+6

10. A particle describes an ellipse under inverse square law about a focus. If it is projected with a velocity of magnitude V from a point at a distance l from the centre of force, find the periodic time. 16

11. Determine the eigenvalues and corresponding eigenvectors of the following linear dynamical system :

$$\frac{dx}{dt} = 2x + y$$

$$\frac{dy}{dt} = x + 2y$$

Classify its equilibrium points. 6+6+4

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Group - A

[Hydrostatics]

(Marks : 25)

Answer ***any one*** question.

1. (a) A hollow sphere of radius *'a'*, half filled with liquid, is made to rotate with angular velocity ω about its vertical diameter. If the lower point of the sphere is just exposed, show that

$$2g = a\omega^2\left(2-\sqrt[3]{4}\right)$$

(b) Prove that the pressure at a point in a perfect fluid in motion is same in every direction. 25

2. (a) A tube in the form of a parabola held with its vertex downwards and axis vertical, is filled with different liquids of densities δ and δ'. If the distance of the free surface of the liquids from the focus of parabola be r and r' respectively, show that the distance of their common surface from the focus is $\dfrac{r\delta - r'\delta'}{\delta - \delta'}$.

(b) One end of a horizontal pipe of circular section is closed by a vertical door hinged to the pipe at the top. Show that the moment about the hinge of the liquid pressure is (i) $\dfrac{5}{4}g\rho\pi a^4$ when full and (ii) $g\rho a^4\left(\dfrac{2}{3}+\dfrac{\pi}{8}\right)$ when half-full of liquid, where 'a' is the radius of the section. 25

3. (a) If the floating solid be a cylinder, with its axis vertical, the ratio of whose specific gravity to that of the fluid is σ, prove that the equilibrium will be stable if the ratio of the radius of the base to the height be greater than $\left[2\sigma(1-\sigma)\right]^{1/2}$.

(b) A given volume V of liquid is acted upon by forces $X = \frac{-\mu x}{a^2}, Y = \frac{-\mu y}{b^2}, Z = \frac{-\mu z}{c^2}$. Find the equation of the free surface. (a, b, c are constants & x, y, z have their usual meanings) 25

4. (a) Write down the general equation of equilibrium for a mass of homogeneous liquid contained in a vessel, revolving about a vertical axis. Find the equation of the surface of equi-pressure when the gravity is the only external force.

(b) If near the earth's surface gravity be assumed to be constant and the absolute temperature in the atmosphere be given by $T = T_0\left(1 - \frac{Z}{nH}\right)$, [where H is the height of the homogeneous atmosphere], show that the pressure in the atmosphere will be given by $P = P_0\left(1 - \frac{Z}{nH}\right)^n$.

(n is a constant integer) 25

Group - B

[Rigid Dynamics]

(Marks : 25)

Answer ***any two*** questions, taking ***one*** from each ***Section***.

Section - I

5. (a) Explain motion of a rigid body in two dimensions, defining properly the variables used, which determine the motion. Find the moment of momentum of a rigid body moving in two dimensions about the centre of inertia.

(b) ABC is a triangular area where AD is perpendicular to BC, AE is a median and O is the middle point of DE. Show that BC is a principal axis of the triangle at O. 20

6. (a) Prove that the time of a small oscillation of a compound pendulum is minimum when the axis of suspension is parallel to the maximum radius vector of the momental ellipsoid at the centre of inertia and the point of suspension is taken such that the centre of inertia bisects the join of the point of suspension and the centre of oscillation.

(b) A solid circular disc of radius 'a' is rolling up a rough plane (along the line of greatest slope) inclined at an angle 'α' to the horizontal. If V be the magnitude of velocity of the centre of the disc at an instant, show that the disc ascends a further distance $\frac{3V^2}{4g\sin\alpha}$ along the plane, before coming to rest. 20

Section - II

7. An elliptic lamina can rotate about a horizontal axis passing through a focus and perpendicular to its plane. If the eccentricity of the ellipse be $\sqrt{\frac{2}{5}}$, prove that the centre of oscillation will be at the other focus. 5

8. A uniform rod of length $2a$ is placed with one end in contact with a small horizontal table and is then allowed to fall; if α be its initial inclination to the vertical, show that its angular velocity when it is inclined at an angle θ, is $\left\{\frac{6g}{a}\left(\frac{\cos\alpha-\cos\theta}{1+3\sin^2\theta}\right)\right\}^{1/2}$. 5

9. A uniform solid cylinder is placed with its axis horizontal on a plane, whose inclination to the horizon is α. Show that the least coefficient of friction between it and the plane so that it may roll without sliding, is $\frac{1}{3}\tan\alpha$. 5

10. AB, BC are two equal similar rods freely hinged at B and lie in a straight line on a smooth table. The end A is struck by a blow perpendicular to AB. Show that the resulting velocity of A is $\frac{7}{2}$ times the velocity that of B. 5