2020

MATHEMATICS — HONOURS

Fifth Paper

(Module - IX)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

($\mathbb{Q}$, $\mathbb{R}$, $\mathbb{N}$ denote the sets of rational numbers, real numbers and natural numbers respectively)

Answer ***question no.*** **1** and ***any two*** questions from the rest.

1. (a) Answer ***any one*** of the following : 6

(i) Prove or disprove total variation of $\sin x+\cos x$ on $\left[0, \frac{\pi}{4}\right]$ is $\sqrt{2}$.

(ii) Correct or justify : A Riemann-integrable function f on $[a, b]$ may be neither continuous nor monotone on $[a, b]$.

(iii) Find the limit function of the sequence $\{f_n\}$ given by $f_n(x)=\frac{[nx]}{n}, 0 \leq x \leq 1;\ n \in \mathbb{N}\cdot$

([y] denote the largest integer less than or equal to y).

(iv) Prove or disprove : The power series $\sum\limits_{n=0}^{\infty} a_n x^n$ and $\sum\limits_{n=1}^{\infty} n.a_n x^{n-1}$ have same radius of convergence.

(b) Prove or disprove ***any one*** of the following : 4

(i) The set $A=\left\{5+x\sqrt{2} : x \in \mathbb{Q}\right\}$ is of measure zero.

(ii) Every bounded enumerable set is compact.

(iii) The function $f(x)=\sum\limits_{n=1}^{\infty} \frac{\sin\left(n^2 x\right)}{n^2}$ is continuous on $\mathbb{R}$.

(iv) Radius of convergence of $1+\frac{x}{2}+\left(\frac{x}{4}\right)^2+\left(\frac{x}{2}\right)^3+\left(\frac{x}{4}\right)^4+\ldots\ldots$ is 2.

2. (a) If S is a bounded, closed subset of $\mathbb{R}$, prove that every infinite open cover of S has a finite subcover.

(b) Choosing a suitable open cover, prove that $A = (0, 1) \cup \{5, 6\}$ is not compact.

(c) If a function f is Riemann-integrable on $[a, b]$, prove that the set

$S = \left\{ x \in [a,b] \middle/ \int_x^b f(t)\,dt \text{ is continuous} \right\}$ is compact. 10+6+4

3. (a) Construct a real valued function on a compact interval which is uniformly continuous but not of bounded variation on that interval.

(b) If $f : [a,b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, prove that its variation function is monotonically increasing on $[a, b]$.

(c) Let $f, g : [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 \cos\frac{\pi}{x^2} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$

and $g(x) = e^{x^2+1}$, $x \in [0,1]$.

Examine whether $\gamma = (f, g)$ is rectifiable. 6+6+8

4. (a) Prove that a bounded function $f : [a,b] \to \mathbb{R}$ is Riemann Integrable on $[a, b]$ if and only if for every $\varepsilon\ (> 0)$ there is a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.

(b) If f, g are Riemann integrable on $[a, b]$ and $|g(x)| > 1$ for all $x \in [a, b]$, use Lebesgue's criterion to show that $\frac{f}{g}$ is Riemann integrable on $[a, b]$.

(c) If $f : [a,b] \to \mathbb{R}$ is a continuous function such that $\int_a^b f^2(x)dx = 0$ then prove that the set $\{x \in [a,b] / f(x) = 0\}$ is uncountable. 8+6+6

5. (a) If $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, prove that $\int_a^b f(x)\,dx = \mu(b-a)$,

where $\inf\limits_{x\in[a,b]} f(x) \le \mu \le \sup\limits_{x\in[a,b]} f(x)$.

(b) Correct or justify : If a real valued function f has a primitive on $[a, b]$, then f is Riemann integrable on $[a, b]$.

(c) Prove that $\frac{\pi}{6} > \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2020}}} > \frac{1}{2}$. 6+6+8

6. (a) Let $\{f_n\}$ be a sequence of functions defined on $[a, b]$ such that $\lim\limits_{n\to\infty} f_n(x) = f(x), \forall x \in [a, b]$ and

$M_n = \sup\limits_{x\in[a, b]} |f_n(x) - f(x)|$.

Show that $\{f_n\}$ converges uniformly to f on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$.

(b) Let $f_n(x) = \begin{cases} \frac{x}{n^2} & \text{if } n \text{ is even} \\ \frac{1}{n^2} & \text{if } n \text{ is odd} \end{cases}$ where $x \in \mathbb{R}$.

Find the limit function of $\{f_n\}$ with proper justification. Is the convergence uniform? Justify.

(c) Let f be a real valued uniformly continuous function on $\mathbb{R}$. If $f_n(x) = f\left(x + \frac{1}{n}\right)$ for all $x \in \mathbb{R}$, for all $n \in \mathbb{N}$, then prove that $\{f_n\}$ is uniformly convergent on $\mathbb{R}$. 8+(4+2)+6

7. (a) Prove that the sum function of a uniformly convergent series $\sum\limits_n f_n$ of continuous functions defined on a set $D \subseteq \mathbb{R}$ is continuous on D.

(b) Examine whether $\sum\limits_{n=1}^{\infty}\left[n^2xe^{-n^2x^2} - (n-1)^2xe^{-(n-1)^2x^2}\right]$ is uniformly convergent on [0, 1].

(c) Using Abel's test prove that the series $\sum\limits_{n=1}^{\infty} a_n n^{-x}$ converges uniformly on [0, 1] if $\sum\limits_{n=1}^{\infty} a_n$ converges uniformly on [0, 1]. 8+8+4

8. (a) If a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ has radius of convergence $\rho \in (0, \infty)$ and $\sum\limits_{n=0}^{\infty} a_n \rho^n$ is convergent, prove that the power series is uniformly convergent on $[0, \rho]$.

(b) Find the largest interval in which the power series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n.10^{n-1}}$ is convergent.

(c) Prove or disprove : If a power series is neither nowhere convergent nor everywhere convergent, then its sum function is bounded. 8+8+4

2020

MATHEMATICS — HONOURS

Fifth Paper

(Module - X)

Full Marks : 50

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Candidates are required to give their answers in their own words as far as practicable.

Group - A

(Marks : 20)

Answer ***any one*** question.

1. (a) Let V and W be two vector spaces over a field F and $T : V \to W$ be a linear mapping. If $\mathrm{Ker}T = \{\theta\}$ and $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of V, prove that $\{T(\alpha_1), T(\alpha_2), ...T(\alpha_n)\}$ is a basis of $Im\ T$.

(b) Let $P_3(\mathbb{R})$ be the real vector space of all polynomials of degree atmost 3. Define $S : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ by $S(p(x)) = p(x + 1)$ for all $p(x) \in P_3(\mathbb{R})$. Find the matrix of S relative to the ordered basis $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$.

(c) If the matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the ordered basis $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$ is given by $\begin{pmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix}$, then find $\dim(Im\ T)$. 10+6+4

2. (a) Let V and W be two vector spaces of finite dimensions over a field F and let $T : V \to W$ be a linear mapping. Prove that T is invertible if and only if the matrix of T relative to any chosen pair of ordered bases of V and W is nonsingular.

(b) A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T(x, y, z) = (y + z, z + x, x + y, x + y + z)$, $(x, y, z) \in \mathbb{R}^3$. Find ImT and dimension of ImT. Also find nullity T. 10+4+4+2

3. (a) If a subgroup H of a group G is defined to be normal if $a\,\mathrm{H}a^{-} \subseteq \mathrm{H}$ for all $a \in \mathrm{G}$, prove that H is a normal subgroup of G if $a\,\mathrm{H}a^{-1} = \mathrm{H}$ for all a in G.

(b) Is there any group G of order 6 with the quotient group $G/Z(G)$ of order 3, where $Z(G) = \{a \in G;\ ax = xa \text{ for all } x \in G\}$? Justify your answer.

(c) Prove that any two infinite cyclic groups are isomorphic. Is this true for finite cyclic groups? Justify your answer. 6+4+(6+4)

4. (a) Let G and G' be two groups and $\phi : G \to G'$ be an onto homomorphism. Let $H = \text{Ker}\phi$. Show that $G/H \simeq G'$. Hence show that there does not exist any homomorphism from Z_9 onto Z_6.

(b) Let $a, b \in \mathbb{R}$ and a mapping $T_{ab} : \mathbb{R} \to \mathbb{R}$ be defined by $T_{ab}(x) = ax + b,\ x \in \mathbb{R}$. Let $G = \{T_{ab} : a \neq 0\}$. Assume that $(G, *)$ is a group where $*$ is the composition of mappings. If $H = \{T_{ab} : a = 1\}$, prove that H is a normal subgroup of $(G, *)$. (6+4)+10

Group - B

(Marks : 15)

Answer ***any one*** question.

5. If (A_1, A_2) is a covariant vector in Cartesian coordinates x^1, x^2 where $A_1 = \frac{x^1}{x^2}$ and $A_2 = \frac{x^2}{x^1}$, find its components in polar coordinates. 15

6. (a) Show that angle between two vectors at a point in a Riemannian space is an invariant under coordinate transformation.

(b) Prove that the covariant derivative of the metric tensor g_{ij} vanishes. 15

7. Prove that if A_i and B_i be two covariant vectors, then $A_iB_j - A_jB_i$ is a skew symmetric tensor. 15

8. Calculate the quantities (g^{ij}) and (g_{ij}) where the metric is given by $ds^2 = (dx_1)^2 + {x_1}^2(dx_2)^2 + {x_1}^2 \sin^2 x_2 (dx_3)^2$. Also calculate $\left\{ \begin{matrix} & 2 & \\ 2 & & 1 \end{matrix} \right\}$. 15

9. Prove that $A^{jk}\left[ij, k\right] = \frac{1}{2} A^{jk} \frac{\partial g_{jk}}{\partial x^i}$, where A^{ij} are components of a symmetric contravariant tensor of rank 2. 15

Answer either **Group - C** *or* **Group - D**

Group - C

(Marks : 15)

Answer ***any one*** question.

10. (a) (i) Show that $L(t^n) = \frac{n!}{p^{n+1}}$, where n is a positive integer and $t > 0$.

(ii) Find the Laplace transform of $f(t)$ defined by

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < 2 \\ 3 & \text{if } t > 2 \end{cases}$$

(b) Using Laplace transform solve $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+10y=0$, when $y(0)=1$, $y\left(\frac{\pi}{4}\right)=\sqrt{2}$.

(c) Find the power series solution of $\frac{d^2y}{dx^2}+(x-3)\frac{dy}{dx}+y=0$ near $x=2$. 15

11. (a) Evaluate $L^{-1}\left(\frac{1}{(p-3)^2}\cdot\frac{1}{p+4}\right)$

(b) Using Laplace transform, solve $y''(t)+4y'(t)+4y(t)=4e^{-2t}$, $y(0)=-1$ and $y'(0)=4$.

(c) State a set of sufficient conditions for the existence of Laplace transform. Show that Laplace transform is a linear operator. If f is the Laplace transform of F, determine the Laplace transform of G defined by $G(t)=\begin{cases}0 & 0<t<a\\ F(t-a) & t>a\end{cases}$ 15

Group - D

(Marks : 15)

Answer ***any two*** questions.

12. State and prove the necessary and sufficient condition for a connected graph to be an Euler graph. 7½

13. If n, e and f are respectively the number of vertices, number of edges and number of faces of a planar graph, then show that $n-e+f=2$. 7½

14. If $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of the vertices of a graph with n vertices and e edges then prove that $\delta(G)\leqslant\frac{2e}{n}\leqslant\Delta(G)$. 7½

15. (a) Define complement of a graph. Show that the complement of P_4(a path on 4 vertices) is again P_4.

(b) In a tree T prove that there exists one and only one path between every pair of vertices. 7½

16. Construct the minimum spanning tree for the given graph using Prim's Algorithm. 7½