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Sem - BCSE UG-2 Even Sem

Sub - Graph Theory Assignment-3

ASSIGNMENT-3 :

How many Hamiltonian Cycles does K_n have ?

⇒ To answer this question we must know the following things -

(1) What is a Hamiltonian Cycle ?

(2) What is K_n ?

(3) How to calculate number of Hamiltonian Cycles for a K_n ?

So, without any further delay let me answer and explain each of these one by one.

Firstly, what is a Hamiltonian Cycle ?

① Hamiltonian Cycle :

Def^n : A Hamiltonian cycle, also called a Hamiltonian circuit, Hamilton Cycle or Hamilton circuit is a graph-cycle (i.e., closed loop) through a graph that passes through / visits each node exactly once.

A graph possessing a Hamiltonian Cycle is said to be a Hamiltonian Graph.

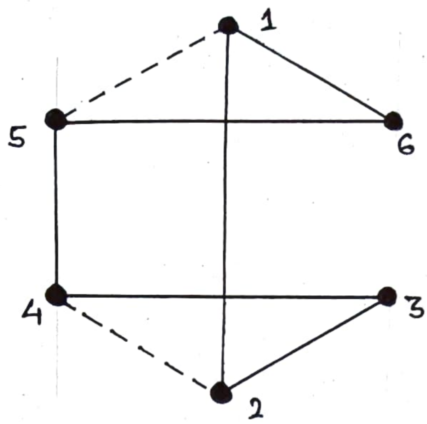

Fig: 3.1

Figure 3.1 has a Hamiltonian Cycle labeled by the solid line as shown.

>> Important theorems, conjectures and Remarks on Hamiltonian Cycle:-

⇨ Let G be a Simple Graph with at least 3 vertices. If every vertex of G has degree $\geq |V(G)|/2$, then G has a Hamiltonian Cycle.

⇨ Let G be a graph with at least 3 vertices ($n = |V(G)| \geq 3$) and vertex degrees $d_1 \leq d_2 \leq \dots \leq d_n$. If either $d_i > i$ or $d_{n-i} \geq n-i$ for every $1 \leq i \leq \frac{n}{2}$. then G is Hamiltonian.

⇨ If G is a k-connected graph with $|V(G)| \geq 3$ and $\alpha(G) \leq k$, then G is Hamiltonian.

⇨ If G is a d-regular graph where d is an odd number and $e \in E(G)$, then there are an even number of Hamiltonian cycles in G which pass through the edge e.

⇨ If G is a plane graph with a Hamiltonian Cycle C and G has f_i^{in} faces of length i inside C and f_i^{n} faces of length i outside C for every i, then $\sum (i-2)(f_i^{in} - f_i^{n}) = 0$

previously we have discussed generalised definition of a Hamiltonian Cycle. Now we will discuss formal Mathematical definition of a Hamiltonian Cycle.

Formal/Mathematical definition of Hamiltonian Cycle:-

Let $G = (V, E, w)$ be an edge-weighted, directed graph where $V = \{v_1, v_2, \dots, v_n\}$ is the set of $n = \# V$ vertices, $E \subseteq V \times V$ the set of (directed) edges, and $w : E \rightarrow \mathbb{R}^+$ a function assigning each edge $e \in E$ a weight $w(e)$.

A path in G is a list $(u_1, u_2, \dots, u_k)$ of vertices $v_i \in V$ $(i = 1, 2, \dots, k)$, such that any pair (u_i, u_{i+1}), $i = 1, 2, \dots, k-1$ is an edge in G. A cyclic path in G is a path for which the first and the last vertex coincide [i.e. $u_1 = u_k$ in the above notation].

A Hamiltonian Cycle in G is a cyclic path p in G that visits every vertex of G (except for its starting point) exactly once, i.e., $p = (u_1, u_2, \dots, u_n, u_1)$ is a Hamiltonian Cycle in G if and only if $n = \# V$ and $\{u_1, u_2, \dots u_n\} = V$.

Secondly, we shall discuss what is K_n ?

② <u>Discussion on K_n</u> :

→ In Graph theory, a complete graph ~~is denoted~~ with n vertices is denoted by the notation K_n.

→ <u>Def^n</u>: In the mathematical field of graph theory, a complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.

please note that, degree of each vertex will be $n-1$, where n is the order of the Graph.

So, we can say that a complete graph of order n is nothing but a $(n-1)$-regular graph of order-n.

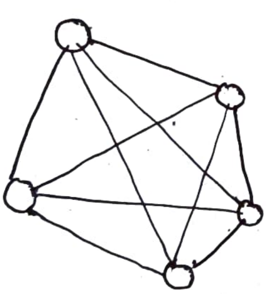

Fig 3.2 shows a complete graph of order 5 (K_5).

<u>Fig: 3.2</u>

Properties of a Complete Graph :

>> A complete graph K_n has $^nC_2 = n(n-1)/2$ number of edges and is a regular graph of degree $(n-1)$.

>> All complete graphs are their own maximal cliques.

[A clique, C, in an undirected graph $G=(V, E)$ is a subset of the vertices, $C \subseteq V$, such that every two distinct vertices are adjacent.

A maximal clique is a clique that cannot be extended by including one more adjacent.]

>> If the edges of a complete graph are each given an orientation, the resulting directed-graph is called a Tournament.

>> K_n can be decomposed into n trees T_i such that T_i has i vertices.

(3) Since, we've pretty much covered everything needed for our final discussion so without a further delay let us dive into ~~the~~ our final goal which is to calculate number of ~~K_n~~ Hamiltonian Cycles for a Complete Graph K_n.

→ Let, us understand the scenario first with a very simple example.

We are taking $\underline{K_3}$ into consideration here—

Let's call its vertices A, B an C. then do we consider $A \to B \to C \to A$ same as $B \to C \to A \to B$? If YES, $A \to B \to C \to A$ is same as $B \to C \to A \to B$, which is the same as $C \to A \to B \to C$ in K_3. Then we will have <u>two</u> Hamiltonian Cycles $A \to B \to C \to A$ and $A \to C \to B \to A$.

Moreover, if we consider $A \to B \to C \to A$ and $A \to C \to B \to A$ being the same because the second one is obtained by reversing the direction of the first one, then we have only <u>one</u> Hamiltonian Cycle, in K_3.

For general K_n (vertices numbered as $1, 2, 3, \ldots, n$), it's the same. $1 \to 2 \to \ldots \to n \to 1$ is the same as $n \to 1 \to 2 \to \cdots \to n$ is the same as $2 \to \cdots \to n \to 1 \to 2$.

And the $1 \to 2 \to \cdots \to n \to 1$ and $1 \to n \to \cdots \to 2 \to 1$ being the same because the second one is obtained by reversing direction of the first one.

Now, let us generalise our thoughts ~~in~~ using Mathematical intuition and conjecture.

In a complete graph, every vertex is adjacent to every other vertex.

Therefore, if we were to take all the vertices in a complete graph in any order, there will ~~x~~ be a path (defined previously), through those vertices in any order.

Joining either end of that path gives us a Hamiltonian Cycle.

Using, Cardinality of Set of Bijections, there are $n!$ different ways of picking the vertices of G in some order.

> Cardinality of Set of Bijections :-
>
> ⇒ Let S and T be sets such that $|S| = |T| = n$. then there exists $n!$ bijections from S to T.

Hence, there are $n!$ ways of building such a Hamiltonian Cycle.

Not all these are different, though.

On any such cycle, there are :-

→ n different places we can start

→ 2 different directions we can travel.

So, any one of these $n!$ cycles is in a set of $2n$ cycles which all contain the same set of edges.

Hence, keeping in mind both of possible cases -

Total (non-distinct) Hamilton Circuits in the complete graph K_n is $= \frac{n!}{n}$

$$= (n-1)!$$

This follows from the fact that starting from any vertex we have (n-1) edges to choose from first vertex, (n-2) edges to choose from second vertex, (n-3) to choose from the third and so on. These being independent choices, we get (n-1)! possible number of choices.

Total number of distinct not edge disjoint Hamiltonian Circuits in complete graph is $= \frac{n!}{2n}$

$$= \frac{(n-1)!}{2}$$

Above number (number of non-distinct H.C.-s) is divided by 2, because each Hamiltonian circuit ~~have~~ has been counted twice (in reverse direction of each other like these: $A \to B \to C \to A$ and $A \to C \to B \to A$)

Conclusion:

Let's summarise our whole discussion in not more than two sentences only.

- Number of non-distinct Hamiltonian Circuits in the Complete Graph K_n is $(n-1)!$
- Number of distinct Hamiltonian Circuits in the Complete Graph K_n is $(n-1)!/2$.

REFERENCES

All the resources I've used or took reference of during solving this assignment -3 are mentioned below. However, while answering the question(s) I've written my answer in my own words.

1.) F. Harary : Graph Theory,

2.) N. Deo : Graph Theory with Applications to Engineering and Computer Science.

3.) Douglas B. West : Introduction to Graph Theory.

4.) Internet (Wikipedia and YouTube mainly)