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ASSIGNMENT-2 :

A travelling agent has to visit n cities, each of them $(n+1)$ times. In how many different ways can he do this if he is not allowed to start and finish in the same city?

Find the different ways for travelling if $n=4$ in the above problem.

Solution :

Since, we are given to solve this combinatorial problem in a generalised method (for n cities and $(n+1)$ visits to each one of them), it may get tougher at times to keep track of what we're doing as we progress in solving the problem.

In order to remove that ambiguity I'd like to design a step-by-step algorithm to encounter this problem and do necessary steps according to that algorithm.

So, let's begin.

Algorithm for solving this problem :-

1. Calculate no. of visits (Total number) possible before the start of the journey.

2. Find the number of ways to make those visits calculated in step-1.

3. Find the number of ways to make visits to n cities, each of them $(n+1)$ times, if the start and end is at ~~the~~ a particular city (say City-1 or C-1).

4. Find total number of such ways, calculated in step-3, for n number of cities.

5. Required Answer =

[Total number of ways to visit n cities, each one $(n+1)$ times (Calculated in Step-2)]

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[Number of ways to make visits to n cities, each one $(n+1)$ times, if the starting and ending ~~city~~ cities are ALWAYS THE SAME (Calculated in Step-4)]

Now, following our previously designed algorithm, we shall solve the problem step-by-step.

There are n cities and the agent has to visit each cities for $(n+1)$ times.

Hence, total ~~of~~ number of visits possible before the start of journey $= \underset{(\text{say})}{N} = n \times (n+1)$

$$\Rightarrow N = n(n+1)$$

Now, total number of ways to visit n cities, each of them $(n+1)$ times $= X$ (say)

$$\Rightarrow X = \frac{N!}{(n+1)! \cdot (n+1)! \cdot (n+1)! \cdots n \text{ times}}$$

$$\Rightarrow X = \frac{N!}{\{(n+1)!\}^n}$$

$$\Rightarrow X = \frac{\{n(n+1)\}!}{\{(n+1)!\}^n}$$

Now, let us assume that these n cities are $c_1, c_2, c_3, \ldots, c_n$ and the agent starts with c-1 as well as ends with c-1.

$\therefore$ ~~Total~~ Number of visits possible if the agent alway starts and ends at c-1 $= N'$ (say)

$$= n(n+1) - 2$$

$$\therefore N' = n(n+1) - 2$$

$\therefore$ Total number of ways to make these N' visits if the agent always starts and ends at any particular city $= U$ (say) $= \dfrac{N'!}{(n+1)!\cdot(n+1)!\cdots(n-1)\text{ times} \times [(n+1)-2]!}$

$$\Rightarrow U = \frac{N'!}{[(n+1)!]^{n-1}[(n+1)-2]!}$$

$$\Rightarrow U = \frac{[n(n+1)-2]!}{[(n+1)!]^{n-1}\cdot[(n+1)-2]!}$$

Obviously, the same argument applies for the number of visiting arrangements that

start and end ~~with~~ in C_2,
that start and end in C_3,
that start and end in C_4,
and so on
that, start and end in C_n

n-times (including C_1)

$\therefore$ Total number of ways to make visits to n cities, each one $(n+1)$ times, if the starting and ending cities are always the same

$= Y$ (say) $= n \cdot U$

$$\Rightarrow Y = n \cdot \frac{[n(n+1)-2]!}{[(n+1)!]^{n-1}\cdot[(n+1)-2]!}$$

Hence, Our Required Answer

= Total number of ways to make visits to n cities, each one $(n+1)$- times, if the starting and ending cities are NEVER THE SAME.

= (Total number of ways to visit n cities, each of them $(n+1)$ times) — (Total no. of ways to make visits to n cities, each one $(n+1)$-times, if the starting and ending cities are ALWAYS THE SAME)

$$= X - Y$$

$$= \frac{[n(n+1)]!}{[(n+1)!]^n} - n \cdot \frac{[n(n+1)-2]!}{[(n+1)!]^{n-1} \cdot [(n+1)-2]!}$$

Hence, the travelling agent can travel n cities, each of them $(n+1)$-times in $\frac{[n(n+1)]!}{[(n+1)!]^n} - n \cdot \frac{[n(n+1)-2]!}{[(n+1)!]^{n-1} \cdot [(n+1)-2]!}$ ways, if he is not allowed to start and finish in the same city. (Ans)

Find the number of different ways for travelling if $n=4$ in the above problem.

⇒ For $n=4$, we have 4 distinct cities and the travelling agent has to visit these 4 cities, each of them 5 times.

Total no. of visits he has to make $= 5 \times 4 = 20$

Total no. of ways that he can arrange these 20 visits $= \dfrac{20!}{5! \times 5! \times 5! \times 5!}$ [∵ 5! because he is to travell each cities for 5 times]

Now, say these 4 cities are A, B, C and D.

Let's assume that the traveller always

Starts at A and
Ends at A } 2 journeys / visits

∴ Number of independant visits left for him

$= 20 - 2 = 18$

He is free to choose the order in which he makes these remaining 18 visits.

As 3 of those visits will be to city A and 5 will be to each of the remaining three cities (B, C and D), this can be done in $\dfrac{18!}{5! \times 5! \times 5! \times 3!}$ ways.

Obviously, the same argument applies for the number of visiting arrangements that

start and end in B,
start and end in C,
start and end in D,

$\left.\right\}$ Total 4 arrangements. (including city-A)

∴ Number of ways to make visit to 4 cities, each one 5 times, if the starting and ending cities are 'Always the Same' $= 4 \times \dfrac{18!}{5! \times 5! \times 5! \times 3!}$

∴ Total Number of ways to make visit to 4 cities, each one 5 times, if the starting and ending cities are 'Never the Same'

= (Total number of ways to visit 4 cities, each 5 times)
− (Total number of ways to visit 4 cities, each 5 times if the starting and ending cities are Always Same)

$$= \frac{20!}{5! \times 5! \times 5! \times 5!} - \frac{4 \times 18!}{5! \times 5! \times 5! \times 3!}$$

= 9, 262, 693, 440 number of ways (Ans)

N.B. Even if we put $n=4$ in the formula we derived while solving the previous question of Assignment-2, we get,

$$\text{Number of ways} = \frac{[4 \times (4+1)]!}{[(4+1)!]^4} - 4 \times \frac{[4(4+1)-2]!}{[(4+1)!]^{4-1} \times [(4+1)-2]!}$$

$$= \frac{20!}{(5!)^4} - 4 \times \frac{18!}{(5!)^3 \times 3!} = 9,262,693,440$$

So we can see, we're getting the same result. This verifies our formula is correct.

REFERENCES

The resources I've used or took reference of while completing the assignment - 2 are listed below.

However, while ~~th~~ answering the question(s), I've written my answer in my own words.

1.) Class Notes provided by our respected professor Dr. Chintan Kumar Mondal.

2.) Alan Tucker : Applied Combinatorics.

3.) A Guide to Mathematics Olympiad for RMO and INMO 3rd Ed (Chapter 3 : Combinatorics).