Primer parcial Señales y Sistemas

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1. La distancia media entre dos señales periodicas $x_1(t) \in \mathbb{R}\mathbb{C}$ y $x_2(t) \in \mathbb{R}\mathbb{C}$; Se puede expresar a partir de la potencia media de la diferencia entre ellas.

$$d^2(x_1, x_2) = \bar{P}_{x_1 - x_2} = \lim_{T \to \infty} \frac{1}{T} \int_T |x_1(t) - x_2(t)|^2 dt$$

Sea $x_1(t)$ y $x_2(t)$ dos señales definidas como:

$$x_1(t) = A e^{-jnW_0 t}$$

$$x_2(t) = B e^{jmw_0 t}$$

Con $W_0 = \frac{2\pi}{T}$; $T, A, B \in \mathbb{R}^+$ y n y $m \in \mathbb{Z}$

Determine la distancia entre las dos señales

Solucion:

$$\lim_{T\to\infty} \frac{1}{T}\int_T |x_1(t) - x_2(t)|^2 dt \qquad w_0 = \frac{2\pi}{T}$$

$$\overline{P}_{x_1} = \frac{1}{T}\int |x_1(t)|^2 dt$$

$$\overline{P}_x = \frac{1}{T}\int |A e^{-jnw_0t}|^2 dt$$

$$|A e^{-jnw_0t}|^2 = \frac{A}{\cancel{e^{jnw_0t}}} \cdot A \cancel{e^{jnw_0t}} = A^2$$

$$\overline{P}_{x_1} = \frac{1}{T}\int_T A^2 dt$$

$$\overline{P}_{x_1} = \frac{A^2}{T}\int_0^T dt$$

$$\overline{P}_{x_1} = \frac{A^2}{T} t \Big|_0^T$$

$$\overline{P}_{x_1} = \frac{A^2}{\cancel{T}} \cancel{T}$$

$$\overline{P}_{x_1} = A^2$$

$$\bar{P}_{x_2} = \frac{1}{T}\int_0^T |x_2(t)|^2 dt$$

$$\bar{P}_{x_2} = \frac{1}{T}\int_0^T |B e^{jm\omega_0 t}|^2 dt$$

$$\bar{P}_{x_2} = \frac{1}{T}\int_0^T B \cancel{e^{jm\omega_0 t}} \cdot \frac{B}{\cancel{e^{jm\omega_0 t}}} dt$$

$$\bar{P}_{x_2} = \frac{1}{T}\int_0^T B^2 dt$$

$$\bar{P}_{x_2} = \frac{B^2}{T} \cdot t \Big|_0^T$$

$$\bar{P}_{x_2} = \frac{B^2}{\cancel{T}} \cancel{T}$$

$$\bar{P}_{x_2} = B^2$$

$$\bar{P}_e = \bar{P}_{x_1} + \bar{P}_{x_2} - \frac{1}{T}\int_0^T (x_1 x_2^* + x_1^* x_2) dt$$

$$\bar{P}_e = A^2 + B^2 - \frac{1}{T}\int_0^T (A e^{-jn\omega_0 t} B e^{-jm\omega_0 t} + A e^{jn\omega_0 t} B e^{jm\omega_0 t}) dt$$

$$\bar{P}_e = A^2 + B^2 - \frac{1}{T}\int_0^T (AB e^{-j(n+m)\omega_0 t} + AB e^{j(n+m)\omega_0 t}) dt$$

$$\bar{P}_c = A^2 + B^2 - \frac{1}{T}\int_0^T \left(AB\, e^{-j(n+m)w_0 t} + AB\, e^{j(n+m)w_0 t}\right) dt$$

$$\bar{P}_c = A^2 + B^2 - \frac{AB}{T}\int_0^T \left(e^{-j(n+m)w_0 t} + e^{j(n+m)w_0 t}\right) dt$$

$$e^{-j(n+m)w_0 t} = \cos((n+m)w_0 t) - j\,\text{Sen}((n+m)w_0 t)$$

$$e^{j(n+m)w_0 t} = \cos((n+m)w_0 t) + \hat{j}\,\text{Sen}((n+m)w_0 t)$$

$$\cos((n+m)w_0 t) \cancel{- j\,\text{Sen}((n+m)w_0 t)} + \cos((n+m)w_0 t) \cancel{+ j\,\text{Sen}((n+m)w_0 t)}$$

$$\bar{P}_c = A^2 + B^2 - \frac{AB}{T}\int_0^T 2\cos((n+m)w_0 t)\, dt$$

Cuando $n+m \neq 0$

$$\bar{P}_c = A^2 + B^2 - \frac{2AB}{T}\,\frac{1}{(n+m)w_0}\left[\text{sen}\,(n+m)\frac{2\pi}{T}t\right]\Bigg|_0^T$$

$\text{Sen}(k2\pi) = 0$

$$\bar{P}_c = A^2 + B^2 - \overbrace{\frac{2AB}{\cancel{T}}\,\frac{1}{(n+m)\frac{2\pi}{\cancel{T}}}}^{0}\left[\text{sen}\left(\cancel{(n+m)\frac{2\pi T}{T}}\right)^{0} - \text{sen}\left(\cancel{(n+m)\frac{2\pi}{T}\cdot 0}\right)^{0}\right]$$

$$\boxed{\bar{P}_c = A^2 + B^2}$$

Cuando $n+m=0$

$$\bar{P}_c = A^2 + B^2 - \frac{2AB}{T}\int_0^T \cos\left(\cancel{(0)w_0 t}\right) dt$$

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$$\bar{P}_c = A^2 + B^2 - \frac{2AB}{T}\int_0^T dt$$

$$\bar{P}_c = A^2 + B^2 - \frac{2AB}{T}\left[t\right]\Big|_0^T$$

$$\bar{P}_c = A^2 + B^2 - \frac{2AB}{\cancel{T}}\cancel{T}$$

$$\bar{P}_c = A^2 + B^2 - 2AB$$

$$\boxed{\bar{P}_c = (A-B)^2}$$

Conclusion:

$$\bar{P}_c = \begin{cases} n+m \neq 0 & A^2 + B^2 \\ n+m = 0 & (A-B)^2 \end{cases}$$

$$d^2(x_1, x_2) = \begin{cases} n+m \neq 0 & \sqrt{A^2 + B^2} \\ n+m = 0 & |A-B| \end{cases}$$

$$d^2(x_1, x_2) = \begin{cases} n+m \neq 0 & \sqrt{A^2+B^2} \\ n+m = 0 & |A-B| \end{cases}$$

② Encuentre la señal en el tiempo discreto al utilizar un conversor analogo dijital con frecuencia de muestreo 5 kHz y 4 bits de capacidad de representacion, aplicada a la senal continua

$$x(t) = 3\cos(1000\pi t) + 5\sin(3000\pi t) + 10\cos(11000\pi t)$$

Solucion:

$F_s = 5\,kHz$

$t = nT_s \qquad T_s = \frac{1}{F_s} \Rightarrow T_s = \frac{1}{5000}$

$$t = n\left(\frac{1}{5000}\right)$$

$$t = \frac{n}{5000}$$

$$F = \frac{w}{2\pi}$$

$$f_1 = \frac{1000\,\pi}{2\pi} = 500\ Hz$$

$$f_2 = \frac{3000\,\pi}{2\pi} = 1500\ Hz$$

$$f_3 = \frac{11000\,\pi}{2\pi} = 5500\ Hz$$

No cumple el teorema de Nyquist

Ya que $F_s \geq 2F_3$

$$5000 \geq 11000 \rightarrow \text{Falso}$$

Discretisamos asi el teorema no se cumpla

$$X[n] = X(nT_s)$$

$$X[n] = 3\cos\left(\frac{1000\pi\,n}{5000}\right) + 5\,\text{Sen}\left(\frac{3000\pi . n}{5000}\right) +$$

$$10\cos\left(\frac{11000\pi\,n}{5000}\right)$$

$$x[n] = 3\cos\left(\frac{\pi n}{5}\right) + 5\,\text{Sen}\left(\frac{3\pi n}{5}\right) + 10\cos\left(\frac{11\pi n}{5}\right)$$

Frecuencias en discreta

$$\Omega_1 = \frac{\pi}{5} \in [-\pi, \pi]$$

$$\Omega_2 = \frac{3\pi}{5} \in [-\pi, \pi]$$

$$\Omega_3 = \frac{11\pi}{5} \notin [-\pi, \pi]$$

Por lo tanto Ω_3 es una copia (Aliasing)

procedemos a restar vueltas completas

$$\Omega_3 = \frac{11\pi}{5} - 2\pi = \frac{1}{5}\pi \in [-\pi, \pi]$$

Una vez correjida la señal discreta es

$$x[n] = 3\cos\left(\frac{\pi n}{5}\right) + 5\,\text{Sen}\left(\frac{3\pi n}{5}\right) + 10\cos\left(\frac{\pi n}{5}\right)$$

- Ahora vamos a comprobar la discretización hallando las frecuencias originales

$$\Omega_{org} = 2\pi \frac{f_{org}}{F_s} \; ; \; f_{org} = \frac{\Omega_{org} \cdot F_s}{2\pi}$$

$$F_{1\,org} \; \frac{\frac{\pi}{5} \cdot 5000\,Hz}{2\pi} = 500\,Hz$$

$$F_{2\,org} \; \frac{\frac{3\pi}{5}\, 5000\,Hz}{2\pi} = 1500\,Hz$$

$$F_{3\,org} \; \frac{\frac{\pi}{5}\, 5000\,Hz}{} = 500\,Hz$$

Se propone una frecuencia de muestreo Nueva que sea mayor o igual al doble de la f_{max} de la señal original

Para el casa $F_s = 11000$

Así queda la señal en discreta

$$x[n] = 3\cos\left(\frac{1000\pi n}{11000}\right) + 5\,\mathrm{Sen}\left(\frac{3000\pi n}{11000}\right) + \cos\left(\frac{1100\pi n}{11000}\right)$$

$$x[n] = 3\cos\left(\frac{\pi n}{11}\right) + 5\,\mathrm{Sen}\left(\frac{3\pi n}{11}\right) + 10\cos\left(\pi n\right)$$

Buscando nuevamente frecuencias de la nueva señal discreta:

$$\Omega_1 = \frac{\pi}{11} \in [-\pi, \pi]$$

$$\Omega_2 = \frac{3\pi}{11} \in [-\pi, \pi]$$

$$\Omega_3 = \pi \in [-\pi, \pi]$$

Buscando Frecuencias Originales

$$F_1 = \frac{\Omega_1 F_s}{2\pi} = \frac{\frac{\pi}{11} 1100}{2\pi} = 500 \text{ Hz}$$

$$F_2 = \frac{\Omega_2 F_s}{2\pi} = \frac{\frac{3\pi}{11} 1100}{2\pi} = 1500 \text{ Hz}$$

$$F_2 = \frac{\Omega_3 F_s}{2\pi} = \frac{\pi \, 1100}{2\pi} = 5500 \text{ Hz}$$

Entonces F_s es apropiado ya que las frecuencias en continuo son iguales con $\Omega_1, \Omega_2, \Omega_3$

Por lo tanto

$$x[n] = 3\cos\left(\frac{\pi n}{11}\right) + 5\,\mathrm{Sen}\left(\frac{3\pi n}{11}\right) + 10\cos(\pi n)$$

Probamos si las señales son cuasiperiodicas para poder simular

$W_1 = 1000$; $W_2 = 3000$; $W_3 = 11000$

$$\frac{w_1}{w_2} = \frac{1000}{3000} = \frac{1}{3} \in \mathbb{Q}$$

$$\frac{w_1}{w_3} = \frac{1000}{11000} = \frac{1}{11} \in \mathbb{Q}$$

$$\frac{w_2}{w_3} = \frac{3000}{11000} = \frac{3}{11} \in \mathbb{Q}$$

- Como $\frac{w_1}{w_2}$; $\frac{w_1}{w_3}$; $\frac{w_2}{w_3} \in \mathbb{Q}$ entonces

$x(t)$ es una señal cuasiperiodica

Buscar(T) para simular

$$T = \frac{2\pi}{w} = \frac{1}{f}$$

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{1000\pi} = \frac{1}{500} \; [s]$$

$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{3000\pi} = \frac{1}{1500} \; [s]$$

$$T_3 = \frac{2\pi}{\omega_3} = \frac{2\pi}{11000\pi} = \frac{1}{5500} \; [s]$$

$$T = \pm k T_1 = \pm r T_2 = \pm L T_3$$

$$T = \frac{k}{500} = \frac{r}{1500} = \frac{L}{5500}$$

$$\text{mcm}(500, 1500, 5500) = 16500$$

$$16500\, T = \frac{k}{500} = \frac{r}{1500} = \frac{L}{5500} \times 16500$$

$$16500\, T = 33k = 11r = 3L$$

$$\begin{array}{ccc|c} 33 & 11 & 3 & 11 \\ 3 & 1 & & 3 \\ 1 & & 1 & \end{array}$$

$$16500\, T = 33$$

$$T = \frac{33}{16500}$$

$$T = \frac{1}{}$$

500

El periodo de muestreo es $\frac{1}{500}$

3 Sea $x''(t)$ la segunda derivada de la señal $x(t)$, donde $t \in [t_i, t_F]$. Demuestre que los coeficientes de la serie exponencial de Fourier se pueden calcular segun:

$$C_n = \frac{1}{(t_i - t_F)n^2 w_0^2} \int_{t_i}^{t_F} x''(t) e^{-jnw_0 t} dt \; ; n \in Z$$

→ Como se pueden calcular los coeficientes a_n y b_n desde $x''(t)$ en la serie trigonometrica de Fourier?

Solucion

$$x''(t) = \frac{d^2}{dt^2} x(t) \; ; \; t \in [t_i, t_F]$$

$$C_n = \frac{1}{(t_i - t_F) n^2 w_0^2} \int_{t_1}^{t_F} x''(t) e^{-jnw_0 t} \, dt \; ; n \in Z$$

$n \in \mathbb{Z}$

$$x(t) = \sum_n C_n e^{jnw_0 t}$$

$$\frac{d}{dt}\{x(t)\} = x'(t) = \frac{d}{dt}\left\{\sum_n C_n e^{jnw_0 t}\right\}$$

$$x'(t) = \sum_n C_n \frac{d}{dt}\left\{e^{jnw_0 t}\right\}$$

$$x''(t) = \frac{d}{dt}\{x'(t)\} = \frac{d}{dt}\left\{\sum_n C_n \frac{d}{dt}\left\{e^{jnw_0 t}\right\}\right\}$$

$$x''(t) = \sum_n C_n \frac{d^2}{dt^2}\left\{e^{jnw_0 t}\right\}$$

$$\frac{d}{dt}\left\{e^{jnw_0 t}\right\} = jnw_0 e^{jnw_0 t}$$

$$\frac{d}{dt}\left\{jnw_0 e^{jnw_0 t}\right\} = \frac{d^2}{dt^2}\left\{e^{jnw_0 t}\right\}$$

$$= j n w_0 \, j n w_0 \, e^{jnw_0 t}$$

$$= j^2 n^2 w_0^2 \, e^{jnw_0 t}$$

$$= (-1) \, n^2 w_0^2 \, e^{jnw_0 t}$$

$$\rightarrow x''(t) = \sum_n \underbrace{C_n (-1) n^2 w_0^2}\, e^{jnw_0 t}$$

$$\tilde{C}_n = -C_n n^2 w_0^2$$

$$\rightarrow x''(t) = \sum_n \tilde{C}_n \, e^{jnw_0 t}$$

$$C_n = \frac{1}{T} \int_{T_i}^{T_F} x(t) \, e^{-jnw_0 t} \, dt \rightarrow x(t) = \sum_n C_n \, e^{jnw_0 t}$$

$$\tilde{C}_n = \frac{1}{T} \int_{T_i}^{T_F} x''(t) \, e^{-jnw_0 t} \, dt \rightarrow x''(t) = \sum_n \tilde{C}_n \, e^{jnw_0 t}$$

$$-C_n n^2 w_0^2 = \frac{1}{T} \int_{T_i}^{T_F} x''(t) \, e^{-jnw_0 t} \, dt$$

$$T = t_F - t_i$$

$$C_n = \frac{1}{-(t_F - t_i) n^2 w_0^2} \int_{T_i}^{T_F} x''(t) \, e^{-jnw_0 t} \, dt$$

$$\boxed{C_n = \frac{1}{} \int^{T_F} x''(t) \, e^{-jnw_0 t} \, dt}$$

$$\boxed{C_n = \frac{1}{(t_i - t_f)n^2 W_0^2} \int_{T_i}^{T_2} x''(t) e^{-jnw_0 t} \, dt}$$

→ Sea la serie trigonometrica de fourier de $x(t)$:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nw_0 t) + b_n \sin(nw_0 t) \right]$$

$$W_0 = \frac{2\pi}{T}, \quad T = t_f - t_i$$

Primera derivada

$$x'(t) = \sum_{n=1}^{\infty} \left[-n W_0 a_n \sin(nw_0 t) + n W_0 b_n \cos(nw_0 t) \right]$$

Segunda derivada

$$x''(t) = \sum_{n=1}^{\infty} \left[-(nw_0)^2 a_n \cos(nw_0 t) - (nW_0)^2 b_n \sin(nw_0 t) \right]$$

Entonces:

$$x''(t) = -\sum_{n=1}^{\infty} (nW_0)^2 \left[a_n \cos(nw_0 t) + b_n \sin(nw_0 t) \right]$$

Calculo de a_n

Multiplico por $\cos(m\omega_0 t)$ e integro en $[t_i, t_f]$:

$$\int_{t_i}^{t_f} x''(t)\cos(m\omega_0 t)\,dt$$

$$= -\sum_{n=1}^{\infty}(n\omega_0)^2 a_n \int_{t_i}^{t_f} \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt$$

Por ortogonalidad

$$\int_{t_i}^{t_f} \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt = \begin{cases} T/2, & n=m \\ 0, & n\neq m \end{cases}$$

Entonces

$$\int_{t_i}^{t_f} x''(t)\cos(m\omega_0 t)\,dt = \frac{T}{2} m^2\omega_0^2 a_m$$

Despejando a_m

$$\boxed{a_m = -\frac{2}{Tm^2\omega_0^2}\int_{t_i}^{t_f} x''(t)\cos(m\omega_0 t)\,dt,\quad m\geq 1}$$

Calculo de b_n

Calculo de bn

Multiplico por $\sin(m w_0 t)$ e integro:

$$\int_{t_i}^{t_F} x''(t)\sin(m w_0 t)\,dt$$

$$= -\sum_{n=1}^{\infty} (n w_0)^2 b_n \int_{t_i}^{t_F} \sin(n w_0 t)\sin(m w_0 t)\,dt$$

Por ortogonalidad

$$\int_{t_i}^{t_F} \sin(n w_0 t)\sin(m w_0 t)\,dt = \begin{cases} T/2, & n=m \\ 0, & n\neq m \end{cases}$$

así

$$\int_{t_i}^{t_F} x''(t)\sin(m w_0 t)\,dt = -\frac{T}{2} m^2 w_0^2 b_m$$

Despejando bm

$$\boxed{b_m = -\frac{2}{T m^2 w_0^2}\int_{T_i}^{t_F} x''(t)\sin(m w_0 t)\,dt, \quad m \geq 1}$$