Prelude in $\mathbb{Z}$

From Pythagorean Tuning to Negative Harmony

Cyrus Fidele

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If we have some frequency f, then if we multiply it by some $\iota \in \mathbb{Q}^+$, we obtain the ι-interval on f given by ιf

1.2 Definitions

Here are some relevant definitions.

Definition 1.2.1 — Harmonic Series. The harmonic series is defined as the product of some fundamental frequency τ and the series of natural numbers $\mathbb{N}$. Define the **harmonic series** of f to be the set:

$$\text{Harm}[f] = \{nf : n \in \mathbb{N}\}$$

Definition 1.2.2 — Harmonic Subset. Suppose $A \subseteq \mathbb{N}$. Then define the harmonic subset of f with respect to the subset A is the set defined as follows.

$$\text{Harm}_A[f] = \{af : a \in A\}$$

Now, we have the tools to start defining some basic musical objects. Take the subset $\Delta = \{1, 3, 5\}$. Then the harmonic subset with respect to Δ can be defined as:

$$\begin{aligned}\text{Harm}_\Delta[f] &= \{af : a \in \Delta\} \\ &= \{f, 3f, 5f\}\end{aligned}$$

This subset Δ is the holy grail of consonance, at least in the Western canon: **the major triad**. The connection between our modern day musical intervals and harmonics will be discussed later. There is also a caveat, which is that there exists no practical tuning system that perfectly captures tonic consonance using A for all elements of some non-trivial scale.

Definition 1.2.3 — Characteristic Harmonics. The prime numbers is a subset of $\mathbb{N}$, and we denote it the set $\mathbb{P} \subset \mathbb{N}$. Since $\mathbb{P}$ is a proper subset of a naturals, we may derive a unique set of harmonics, called the **characteristic harmonics** of a fundamental f. Namely,

$$\mathrm{Harm}_{\mathbb{P}}[f] = \{pf : p \in \mathbb{P}\}$$

To understand why this is a significant set, we will motivate the next definition, octave equivalency.

Definition 1.2.4 — Octave Equivalency. We may define an equivalence relation $\sim_{8\mathrm{ve}}$ on the space of fundamental frequencies, called **octave equivalency**, as follows:

$$f \sim_{8\mathrm{ve}} \hat{f} \iff \exists K \in \mathbb{Z} : \hat{f} = 2^K f$$

The beauty of this relation is that it is cross-cultural, agreed upon in the musical traditions of almost every culture if not all. The quick reasoning for this is because doubling some frequency τ is the same action as taking the first non-fundamental element from its harmonic series, $2\tau \in \mathrm{Harm}[\tau]$.

2. Intervals: The Rational Numbers $(\mathbb{Q})$

2.1 Musical Intervals and the Rational Numbers

If we have some frequency f, then if we multiply it by some $\iota \in \mathbb{Q}^+$, we obtain the ι-interval on f given by ιf.

Now that we have motivated octave equivalency, we may finally expand our musical horizons. Notice, that in the definition of octave equivalency, it suffices to find any $K \in \mathbb{Z}$ such that one frequency is 2^K multiples of the other for them to be equivalent.
Up until now, our definition of harmonics only utilized the multiplication of frequencies by positive whole numbers $(n \in \mathbb{N})$. However, as we can see, octave equivalency already requires us to use multiplicative inverses, or in other words, fractions.

Definition 2.1.1 — Justly Intonated Interval. Now, we may define an extremely powerful and versatile musical object. A **justly-intonatated interval** about a tonic frequency τ is any element of the set:

$$\mathbb{I}[\tau] = \{\iota \cdot \tau \mid \iota = \frac{p}{q} \in \mathbb{Q}^+\}$$

The set of intervals may be further partitioned into three categories as follows:

2.1.1 Interval Classification

- If $\iota \in (0,1)$, then $(\tau, \tau\iota)$ is *descending*.
- If $\iota = 1$, then it is the *identity/unison* interval.
- If $\iota \in (1,\infty)$, then it is *ascending*.

2.2 Eulerian Consonance

Although consonance is a easy feature to describe quantiatively, there are no objective ways to say whether a ratio will sound consonant or not. Generally, if the denominator is "sufficiently small", we can expect the ratio to be consonant. So the closest standard of objectively capturing consonance is given to us by none other than Euler himself.

Corollary 2.2.1 — Euler's Consonance Formula. Let $E : \mathbb{Q}^+ \backslash \{0,1\} \to \mathbb{N}$ denote Euler's Consonance Formula. Define an interval as an ordered pair corresponding to its interval scalar as $\imath_i = (p_i, q_i)$. If $E(\imath_0) < E(\imath_1)$, Euler believes that this entails that $\imath_0$ is more **consonant** than $\imath_1$

3. Irrational Ratio Approximations: $(\mathbb{R})$

3.1 Irrational Approximations of Rational Numbers

$\log_2 \frac{3}{2} \approx \frac{7}{12}$. This is why 12TET is chosen, due to ε-tunedness to a consonant set of justly intoned intervals.

Pythagorean tuning; we know $(\frac{3}{2})^{12} \approx 2^7$. $\varphi^{12} \equiv (\varphi^0) \equiv (\varphi^0)^7 \equiv (\varphi^{12})^7$.

II Part Two: Tuning Systems

4. N-TET

4.1 Logarithms and Divisions of the Octave

The octave is 2^N. Let $k \in \mathbb{N}$, then we can divide the octave logarithmically by $2^{i/k}$ for $i \in \mathbb{N}_k$.

4.2 N-Tone Equal Temperament

Definition 4.2.1 — N-Tone Equal Temperament. . Given a positive whole number $N \in \mathbb{N}$, we may define a N-tone equal temperament tuning system (N-TET) about a tuning frequency τ to be the product of τ and the interval set $\mathbb{I}(N)$. The interval set is defined as:

$$\mathbb{I}(N) = \{2^{\frac{i}{N}} : i \in \mathbb{Z}/N\mathbb{Z}\}$$

Notation 4.1. *An N-TET system tuned to τ is denoted $\mathscr{E}_\tau^{(N)}$*

Definition 4.2.2 — Tuned Pitch Subset (Scale). Let $\mathscr{E}_{12} = \mathscr{E}(f_T, \mathbb{I}(12))$ be our 12-TET system tuned to f_T. Then $\mathscr{E}'_{12} \subseteq \mathscr{E}_{12}$ is a pitch subset of $\mathscr{E}_{12}$ if its interval set $\mathbb{I}_{\mathscr{E}'_{12}} \subseteq \mathbb{I}_{\mathscr{E}_{12}} = \mathbb{I}(12)$.

4.3 Case Study: $\mathscr{E}^{(12)}$ (12-TET)

Definition 4.3.1 — N-Tone Equal Temperament. . Given a positive whole number $N \in \mathbb{N}$, we may define a N-tone equal temperament tuning system (N-TET) about a tuning frequency τ to be the product of τ and the interval set $\mathbb{I}(N)$. The interval set is defined as:

$$\mathbb{I}(N) = \{2^{\frac{i}{N}} : i \in \mathbb{Z}/N\mathbb{Z}\}$$

4.3.1 Symmetric Scales in 12-TET

The symmetric pitch subsets of $\mathbb{Z}/12\mathbb{Z}$ are the tuning systems with interval sets $\mathbb{I}_K = \{2^{\frac{iK}{12}} : i \in \mathbb{Z}/12\mathbb{Z}\}$ with values of $K = \{1, 2, \ldots, 6\}$. They are as follows:

1. *The Chromatic Scale.* Corresponding to the interval set $\mathbb{I}_1$, the Chromatic Scale is actually equal to $\mathbb{I}(12)$.
2. *The Whole Tone Scales.* Corresponding to the interval set $\mathbb{I}_2$, the Whole Tone Scales divide the interval set of $\mathscr{E}_{12}$ into the two sets $\mathbb{I}_2(0)$ and $\mathbb{I}_2(1)$ where $\mathbb{I}_2(t) = \{2^{\frac{i}{6}+t} \mid i \in \mathbb{Z}/6\mathbb{Z}\}$. Note that we can also just say $t \in \mathbb{Z}/2\mathbb{Z}$.

   (Generally, we can say that given $t = K$ symmetric pitch subsets $\mathbb{I}_K(t)$ of size N/K, we have $t \in \mathbb{Z}/K\mathbb{Z}$.)
3. *The Diminished Scales.* Corresponding to the interval set $\mathbb{I}_3$, the Diminished Scales divide the interval set of $\mathscr{E}_{12}$ into the three sets:
$$\mathbb{I}_3(t) = \{2^{\frac{i}{4}+t} : i \in \mathbb{Z}/4\mathbb{Z}\} \mid t \in \mathbb{Z}/3\mathbb{Z}$$
4. *The Augmented Scales.* Corresponding to the interval set $\mathbb{I}_4$, the Augmented Scales divide the interval set of $\mathscr{E}_{12}$ into the four sets:
$$\mathbb{I}_4(t) = \{2^{\frac{i}{3}+t} : i \in \mathbb{Z}/3\mathbb{Z}\} \mid t \in \mathbb{Z}/4\mathbb{Z}$$
5. *The φ-Sequence (Circle of Fourths).* Corresponding to the interval set $\mathbb{I}_5$, the coprimality of 5 and 12 leads to the φ-Sequence being a unique reordering of $\mathbb{I}(12)$:
$$\mathbb{I}_\varphi = \{2^{\frac{5i}{12}} : i \in \mathbb{Z}/12\mathbb{Z}\}$$
6. *The Tritones.* Corresponding to the interval set $\mathbb{I}_6$, the Tritones divide the interval set of $\mathscr{E}_{12}$ into the six sets:
$$\mathbb{I}_6(t) = \{2^{\frac{i}{2}+t} : i \in \mathbb{Z}/2\mathbb{Z}\} \mid t \in \mathbb{Z}/6\mathbb{Z}$$

4.4 Beyond 12-TET

Theorem 4.4.1 — Tritone Existence. Suppose we have an N-TET system tuned to τ, $\mathscr{T} = \mathscr{E}_{\tau}^{(N)}$. Then, $\tau\sqrt{2} \in \mathscr{T} \iff$ N is even. In other words:

$$\iota_\delta \in \mathbb{I}(\mathscr{E}_{\tau}^{(N)}) \text{ where } \iota_\delta = \sqrt{2} \iff N \text{ is even} \tag{4.1}$$

$$\tag{4.2}$$

The proof of the theorem is straightforward. If N is even, then $N/2 \in \mathbb{Z}/N\mathbb{Z}$. This means $\iota = 2^{\frac{N/2}{N}} = 2^{\frac{1}{2}} \in \mathbb{I}(N)$.

4.5 Beyond Octave Equivalency

Proposition 4.5.1 — Generalized Harmonic Equivalency. What if we did a fifth equivalence relation and have a (N, p) TET system where the interval set not only partitions the octave equally but also the perfect fifth. So, what type of system would we get if we said the following?

$$\mathbb{I}(N, p) = \{2^{i/N}3^{i/N} : N \in \mathbb{Z}/N\mathbb{Z}\}$$

5. P-JET

5.1 Pythagorean Principles

Use justly-intonated intervals to make consonant tuning systems. Specifically, consider the generator $\varphi = \frac{3}{2}$. Note that $\varphi \setminus \tau \sim \mathbb{H}_\tau(3)$.

5.2 p-limit Justly-Intonated Tuning.

This is an example of a definition. A definition could be mathematical or it could define a concept.

Definition 5.2.1 — p-limit Justly-Intonated Temperament. . Let $\mathbb{P}_p = \{p_j \in \mathbb{P} : p_j \leq p\}$. Given a prime number $p \in \mathbb{P}$, we may define a p-limit justly tempered tuning system (p-JET) about a tuning frequency f_T to be the product of f_T and the interval set $\mathbb{I}(p)$. Let $P = \{a_1, a_2, \ldots, a_k\}$ be the set of the first k prime numbers such that $\max(P) = a_k = p$. In other words:

$$\mathbb{P}_5 = \{p_j \in \mathbb{P} : p_j \leq 5\} = \{2,3,5\} \tag{5.1}$$

$$\mathbb{I}(p) = \{\prod_i p_i{}^{n_i} : p_i \in P, n_i \in \mathbb{Z}\} \tag{5.2}$$

Notation 5.1. *An p-JET system tuned to f_T is denoted $\mathscr{J}(f_T, \mathbb{I}(p))$.*

■ **Example 5.1** Let $\mathscr{T}$ be the 5-limit justly-intonated tuning system tuned about τ. Then, $\mathscr{T} = \mathscr{J}_\tau^{(5)}$ where:

$$\mathbb{I}(\mathscr{T}) = \{2^a 3^b 5^c : a,b,c \in \mathbb{Z}\}$$

■

III

Fidelian Analysis

6. Microtonality: Generalized Tuning Theory

6.1 Pitch Spaces and Tuning Systems

Definition 6.1.1 — Pitch Continuum. A pitch continuum Ψ is defined as the set of frequencies bounded below by τ_L and bounded above by τ_U. In other words,

$$\Psi = \{f \mid f \in [\tau_L, \tau_U]\}$$

Definition 6.1.2 — Tuning System. Let $\mathbb{I} \subseteq \mathbb{R}$ be some set of real-valued interval multipliers. Then, we obtain the tuning system $\mathscr{T}(\mathbb{I})$ by taking the quotient space of $\mathbb{I}$ under octave equivalency ($\sim_{8\text{ve}}$). So, the tuning system is the set $\mathscr{T} = \{\aleph_i\}_{i=1}^{|\mathbb{I}|}$ where $\aleph_i = [\iota_i \in \mathbb{I}]_{\sim_{8\text{ve}}}$ is the equivalence class of the corrosponding element $\iota_i \in \mathbb{I}$ and $|\mathbb{I}|$ is the number of equivalence classes.

Definition 6.1.3 — Tuned Pitch Space. Let Ψ be a pitch continuum. Also, let $\mathscr{T}$ be a tuning system with the generating pitches $\mathscr{T} = \{\aleph_i\}_{i=1}^{|\mathbb{I}|}$. Then the corrosponding tuned pitch space Ψ' given by the pair $(\Psi, \mathscr{T})$ is given by the set:

$$\Psi' = \{2^t \aleph_i \mid \aleph_i \in \mathscr{T}, t \in \mathbb{Z}\} \cap \Psi$$

6.2 Analysis of Tuning Systems

Definition 6.2.1 — Tuning Matrix. Let $\mathscr{T}$ be a tuning system with the generating pitches $\mathscr{T} = \{\aleph_i\}_{i=1}^{|\mathbb{I}|}$. Then, the corrosponding tuning matrix $\mathbf{T}$ is defined over every pair $(\aleph_i, \aleph_j)$ with $\aleph_i, \aleph_j \in \mathscr{T}$ as follows:

$$\mathbf{T} = (\iota_{ij}) \text{ where } \iota_{ij} = \aleph_j / \aleph_i$$

Definition 6.2.2 — ε-equivalent. Let Ψ' be a tuned pitch space $(\Psi, \mathscr{T})$ with the generating pitches $\mathscr{T} = \{\aleph_i\}_{i=1}^{|\mathbb{I}|}$. Then, $\aleph_i$ and $\aleph_k$ are said to be ε-equivalent in Ψ' iff $|\aleph_k - \aleph_i| < \varepsilon$. So:

$$\aleph_i \sim_\varepsilon \aleph_k \iff |\aleph_k - \aleph_i| < \varepsilon$$

Definition 6.2.3 — ε-tuned. Let Ψ' be a tuned pitch space $(\Psi, \mathscr{T})$ with the generating pitches $\mathscr{T} = \{\aleph_i\}_{i=1}^{|\mathbb{I}|}$. Then, $\mathscr{T}$ is said to be ε-tuned in Ψ' with respect to some tuning matrix $\mathbf{T}$ if every pair in $\mathscr{T}$ is elementwise ε-equivalent to the corrosponding pitch in Ψ'. So:

$$\forall \aleph_i, \aleph_j \in \mathscr{T} \mid \aleph_j \sim_\varepsilon \aleph_i \iota_{ij}$$

7. Negative Harmony: Collierian Group Theory

Collierian Group Theory will be restricted to the 12-TET tuning system, $\mathscr{T} = \mathscr{E}^{(12)}$. One standard procedure is to assume that the Lydian scale is the generating scale by stacking the generator $\varphi \in \mathscr{T}$ a total of 6 times above the root.

7.1 Primary Triad Construction

The two consonant triads in $\mathbb{Z}/12\mathbb{Z}$ are the major (Δ) and minor (∇) triads. They maximally partition the perfect fifth φ and can be seen as the arithmetic inversions of each other:

$$\Delta[\rho] \sim \{0,4,7\} = \{\rho, \rho+4, \rho+(4+3)\}$$
$$\nabla[\rho] \sim \{0,3,7\} = \{\rho, \rho+3, \rho+(3+4)\}$$

Definition 7.1.1 — Primary Triad Construction. To construct the Ion diatonic scale using primary triad construction (PTC), we take the major triads on the root ρ and the elements one fifth above ($\varphi(\rho)$) and below the root ($\varphi^{-1}(\rho)$). In other words:

$$\mathrm{Ion}[\rho] = \{\Delta[\varphi^k(\rho)] \mid k = -1, 0, 1\}$$

We construct the Aeo diatonic scale by taking the minor triads on the root ρ and the elements a fifth above and below $\varphi^{\pm 1}[\rho]$.

$$\mathrm{Aeo}[\rho] = \{\nabla[\varphi^k(\rho)] \mid k = -1, 0, 1\}$$

We can construct Dorian as follows:

$$\mathrm{Dor}[\rho] = \{\Delta[\varphi^{-1}(p)]\} \cup \{\nabla[\varphi^k(\rho)] \mid k = 0, 1\}$$

And Mixolydian as follows:

$$\mathrm{Mixo}[\rho] = \{\nabla[\varphi^{-1}(p)] \mid k = 0, 1\} \cup \{\nabla[\varphi^1(\rho)]\}$$

7.1.1 Alternative Notation

Notation 7.1. *Let ρ be the identity/root tone. Take the dot product of the primary triad qualities, Λ and the primary triad basis β_ρ to get the corresponding* ***primary triad scale*** *given by $\Sigma_p = \Lambda \cdot \beta_\rho$. The primary triad basis of ρ is always the same, and it is given by:*

$$\beta_\rho = [\varphi^{-1}[\rho], \varphi^{0}[\rho], \varphi^{1}[\rho]]$$

Mixolydian $\flat 6$. We can construct the Mixolydian $\flat 6$ scale on ρ, call it $(\text{Mixo } \flat 6)[\rho]$, as follows. Denote the scale $\Sigma_{M\flat 6} = \Lambda_{M\flat 6} \cdot \beta_\rho$. Then, $\Lambda_{M\flat 6} = [\nabla, \Delta, \nabla]$.

7.2 Triadic Lattice Theory

Definition 7.2.1 — Triadic Lattice. The Triadic lattice is the product group given by the set $\mathbf{L} = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Each axis corrosponds to each type of third, minor (3) and major (4) respectively. As such, we obtain the fifth on the diagonal and its corresponding triadic third on the latitudinal line.

Theorem 7.2.1 — Triadic Lattice Isomorphism. The Triadic Lattice $\mathbf{L}$ is isomorphic to the tuning system $\mathscr{T} = \mathscr{E}^{(12)}$.

$$\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \tag{7.1}$$

$$\mathscr{T} \cong \mathbf{L} \tag{7.2}$$

IV

References

8. Reference Tables

8.1 Semitones in 12-TET Intervals

Every element of the interval set $\mathbb{I}(12)$ has a unique label. They are as follows:

Semitones	Interval
0	Unison
1	m2
2	M2
3	m3
4	M3
5	P4
6	A4/d5
7	P5
8	m6
9	M6
10	m7
11	M7
12	Octave

Table 8.1: Number of Semitones for 12-TET musical intervals

Bibliography