Harmony and Pitch Groups

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Origins: The Integers

Integers

The integers are a set of numbers, defined as follows:

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$$

However, the set of numbers we are more interested in is the natural numbers.

$$\mathbb{N} = \{1, 2, 3, ...\}$$

Harmonics

One main element of music is pitch. Pitch is a resonance at a particular frequency, say f.

Example: $\lambda =$A4 $\Rightarrow f_\lambda = 440Hz$

Harmonics

In nature, there exists something called the Harmonic series. From physics, we know that a resonant body doesn't really ever resonate at a pure frequency. Instead, the frequency we perceive it as is called the fundamental frequency.

The harmonic series is defined as the product of some fundamental frequency and the series of natural numbers. Define the harmonic series of a given frequency f to be the set:

$$\mathsf{Harm}[f] = \{nf : n \in \mathbb{N}\}$$

Harmonics: Example

Let us dissect this notation. Let us take the same note as before, an octave lower. That is, let $\lambda =$ A3. So, $f_\lambda = 220Hz$. Now, recall that the natural numbers are defined as the set $\mathbb{N} = \{1, 2, 3, ...\}$. From our definition of the set of harmonics, we obtain:

$$\begin{aligned}\text{Harm}[f_\lambda] &= \{(1) \cdot 220Hz, (2) \cdot 220Hz, (3) \cdot 220Hz, ...\} \\ &= \{220Hz, 440Hz, 660Hz, ...\}\end{aligned}$$

Consonance and the Primes

The prime numbers is a subset of $\mathbb{N}$, and we denote it the set $\mathbb{P} \subset \mathbb{N}$.

We can start defining some actual musical objects now. Take the subset $A = \{1, 3, 5\} \subset \mathbb{N}$. Then the harmonic subset with respect to A can be defined as:

$$\begin{aligned} \mathsf{Harm}_A[f] &= \{af : a \in A\} \\ &= \{f, 3f, 5f\} \end{aligned}$$