Group Theory and the Lydian Chromatic Concept

Ali Taqi

Introduction

This short text is intended to be a group-theoretic analysis of some of the results in George Russell's Lydian Chromatic Concept.

N-TET

(Definition) N-Tone Equal Temperament. Given a positive whole number $N \in \mathbb{N}$, we may define a N-tone equal temperament tuning system (N-TET) about a tuning frequency f_T to be the product of f_T and the interval set $\mathbb{I}(N)$. The interval set is defined as:

$$\mathbb{I}(N) = \{2^{\frac{i}{N}} : i \in \mathbb{Z}/N\mathbb{Z}\}$$

(Notation) N-TET. An N-TET system tuned to f_T is denoted $\mathcal{E}(f_T, \mathbb{I}(N))$.

(Example) 12-TET. A 12-TET system tuned to f_T is denoted $\mathcal{E}(f_T, \mathbb{I}(12))$ where $\mathbb{I}(12) = \{2^{\frac{i}{N}} : i \in \mathbb{Z}/12\mathbb{Z}\}$

Pitch Groups

(Definition) Pitch Subset. Let $\mathcal{E}_{12} = \mathcal{E}(f_T, \mathbb{I}(12))$ be our 12-TET system tuned to f_T. Then $\mathcal{E}'_{12} \subseteq \mathcal{E}_{12}$ is a pitch subset of $\mathcal{E}_{12}$ if its interval set $\mathbb{I}_{\mathcal{E}'_{12}} \subseteq \mathbb{I}_{\mathcal{E}_{12}} = \mathbb{I}(12)$.

(Example) Symmetric Pitch Subsets of Z/12Z. The symmetric pitch subsets of $\mathbb{Z}/12\mathbb{Z}$ are the tuning systems with interval sets $\mathbb{I}_K = \{2^{\frac{iK}{12}} : i \in \mathbb{Z}/12\mathbb{Z}\}$ with values of $K = \{1, 2, \ldots, 6\}$. They are as follows:

1. *The Chromatic Scale.* Corresponding to the interval set $\mathbb{I}_1$, the Chromatic Scale is actually equal to $\mathbb{I}(12)$.

2. *The Whole Tone Scales.* Corrosponding to the interval set $\mathbb{I}_2$, the Whole Tone Scales divide the interval set of $\mathcal{E}_{12}$ into the two sets $\mathbb{I}_2(0)$ and $\mathbb{I}_2(1)$ where $\mathbb{I}_2(t) = \{2^{\frac{i}{6}+t} : i \in \mathbb{Z}/6\mathbb{Z}\}$. Note that we can also just say $t \in \mathbb{Z}/2\mathbb{Z}$.

   Generally, we can say that given $t = K$ symmetric pitch subsets $\mathbb{I}_K(t)$ of size N/K, we have $t \in \mathbb{Z}/K\mathbb{Z}$.

3. *The Diminished Scales.* Corresponding to the interval set $\mathbb{I}_3$, the Diminished Scales divide the interval set of $\mathcal{E}_{12}$ into the three sets:

$$\mathbb{I}_3(t) = \{2^{\frac{i}{4}+t} : i \in \mathbb{Z}/4\mathbb{Z}\} \,|\, t \in \mathbb{Z}/3\mathbb{Z}$$

4. *The Augmented Scales.* Corresponding to the interval set $\mathbb{I}_4$, the Augmented Scales divide the interval set of $\mathcal{E}_{12}$ into the four sets:

$$\mathbb{I}_4(t) = \{2^{\frac{i}{3}+t} : i \in \mathbb{Z}/3\mathbb{Z}\} \,|\, t \in \mathbb{Z}/4\mathbb{Z}$$

5. *The ϕ-Sequence.* Corrosponding to the interval set $\mathbb{I}_5$, the coprimality of 5 and 12 leads to the ϕ-Sequence being a unique reordering of $\mathbb{I}(12)$:

$$\mathbb{I}_\phi = \{2^{\frac{5i}{12}} : i \in \mathbb{Z}/12\mathbb{Z}\}$$

6. *The Tritones.* Corresponding to the interval set $\mathbb{I}_6$, the Tritones divide the interval set of $\mathcal{E}_{12}$ into the six sets:

$$\mathbb{I}_6(t) = \{2^{\frac{i}{2}+t} : i \in \mathbb{Z}/2\mathbb{Z}\} \,|\, t \in \mathbb{Z}/6\mathbb{Z}$$

Various Results

(Lemma) Tritone Existence. Suppose we have an N-TET system $\mathcal{E}_N$ tuned to f_T where N is an even number ($N \bmod 2 \equiv 0$). Then $\sqrt{2} \in \mathbb{I}_{\mathcal{E}_N}$. That is:

$$\exists K \in \mathbb{N} : \mathcal{E}_N = \mathcal{E}(2K) \iff \iota_\delta \in \mathbb{I}_{\mathcal{E}_1} \text{ where } \iota_\delta = \sqrt{2}$$

. Proof: If N is even, then $N/2 \in \mathbb{Z}/N\mathbb{Z}$. This means $\iota = 2^{\frac{N/2}{N}} = 2^{\frac{1}{2}} \in \mathbb{I}(N)$.

Question. What if we did a fifth equivalence relation and have a (N, p) TET system where the interval set not only partitions the octave equally but also the perfect fifth. So:

$$\mathbb{I}(N, p) = \{2^{i/N} 3^{i/N} : N \in \mathbb{Z}/N\mathbb{Z}\}$$