Harmony and Pitch Groups

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Reed Student Colloquium

Origins: The Integers (500BC)

Integers

"God created the integers; all else is the work of man." -Leopold Kronecker

The integers are a set of numbers, defined as follows:

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$$

However, the set of numbers we are more interested in is the natural numbers.

$$\mathbb{N} = \{1, 2, 3, ...\}$$

Harmonics

One of the primary elements of music is pitch. A pitch, measured in Hz(s^{-1}), is a resonance at a particular frequency, say f.
Example: $\lambda = \text{A4} \Rightarrow f_\lambda = 440\text{Hz}$

Harmonics

In nature, there exists something called the Harmonic Series. From physics, we know that a resonant body doesn't really ever resonate at a pure frequency. Instead, the frequency we perceive it as is called the fundamental frequency. The harmonic series is defined as the product of some fundamental frequency f and the series of natural numbers $\mathbb{N}$. Define the **harmonic series** of f to be the set:

$$\text{Harm}[f] = \{nf : n \in \mathbb{N}\}$$

Harmonics: Example

Let us dissect this notation. Let us take the same note as before, an octave lower. That is, let $\lambda = \text{A3}$. So, $f_\lambda = 220\text{Hz}$. Now, recall that the natural numbers are defined as the set $\mathbb{N} = \{1, 2, 3, ...\}$. From our definition of the set of harmonics, we obtain:

$$\begin{aligned} \text{Harm}[f_\lambda] &= \{1 \cdot 220\text{Hz}, 2 \cdot 220\text{Hz}, 3 \cdot 220\text{Hz}, ...\} \\ &= \{220\text{Hz}, 440\text{Hz}, 660\text{Hz}, ...\} \end{aligned}$$

Consonance: The Major Triad

Now, we have the tools to start defining some basic musical objects. Take the subset $A = \{1, 3, 5\} \subset \mathbb{N}$. Then the harmonic subset with respect to A can be defined as:

$$\begin{aligned} \mathsf{Harm}_A[f] &= \{af : a \in A\} \\ &= \{f, 3f, 5f\} \end{aligned}$$

The subset A is precisely the holy grail of consonance, at least in the Western canon: the major triad. The connection between our modern day musical intervals and harmonics will be discussed later. There is also a caveat, which is that there exists no practical tuning system that perfectly captures tonic consonance using A for all elements of some non-trivial scale.

Harmonics: Primes

The prime numbers is a subset of $\mathbb{N}$, and we denote it the set $\mathbb{P} \subset \mathbb{N}$. Since $\mathbb{P}$ is a proper subset of a naturals, we may derive a unique set of harmonics, called the **characteristic harmonics** of a fundamental f. Namely,

$$\mathsf{Harm}_{\mathbb{P}}[f] = \{pf : p \in \mathbb{P}\}$$

To understand why this is a significant set, we will motivate the next definition, octave equivalency.

Harmonics: Octave Equivalency

We may define an equivalence relation $\sim_{\text{8ve}}$ on the space of fundamental frequencies, called **octave equivalency**, as follows:

$$f \sim_{\text{8ve}} \hat{f} \iff \exists K \in \mathbb{Z} : \hat{f} = 2^K f$$

The beauty of this relation is that it is cross-cultural, agreed upon in the musical traditions of almost every culture if not all. The quick reasoning for this is because doubling some frequency f is the same action as taking the first non-fundamental element from its harmonic series, $2f \in \text{Harm}[f]$.

Now that we have motivated octave equivalency, we may finally expand our musical horizons. Notice, that in the definition of octave equivalency, it suffices to find any $K \in \mathbb{Z}$ such that one frequency is 2^K multiples of the other for them to be equivalent. Up until now, our definition of harmonics only utilized the multiplication of frequencies by positive whole numbers $(n \in \mathbb{N})$. However, as we can see, octave equivalency already requires us to use multiplicative inverses, or in other words, fractions.

Intervals and the Rational Numbers

Now, we may define an extremely powerful and versatile musical object. A **justly-intonatated interval**, about a tonic frequency f, is any element of the set:

$$\mathbb{I}[f] = \left(\frac{p}{q} \cdot f : \frac{p}{q} \in \mathbb{Q}^{+} \backslash \{0, 1\} \right)$$

The set of intervals may be further partitioned into two categories as follows: an interval is *descending* if its interval scalar $pq^{-1} \in (0, 1)$, and *ascending* if $pq^{-1} \in (1, \infty)$.

Euler's Consonance Formula

$$E : \mathbb{Q}^+ \backslash \{0, 1\} \to \mathbb{N}$$

Define an interval as an ordered pair corresponding to its interval scalar as $\iota_i = (p_i, q_i)$. If $E(\iota_0) < E(\iota_1)$, Euler believes that this entails that ι_0 is more **consonant** than ι_1

Beauty of Standardization: 12-TET (1700AD)

New horizons: N-TET

To solve the problems of just intonation, mathematicians utilized the freshly discovered idea of a logarithm to define a new tuning system. Define a N-TET ($N \in \mathbb{N}$) tuning system about a fundamental f to be the set of frequencies:

$$[\text{N-TET}]_f = \{2^{\frac{i}{N}} f : i \in \mathbb{N} \,|\, 0 \leq i \leq N-1\}$$

It is worthwhile to note that this set is a cyclic group isomorphic to $\mathbb{Z}/N\mathbb{Z}$! This is even more powerful when we drop the restriction on the domain of i. That is, if we allow $i \in \mathbb{Z}$, and attach an octave equivalence $\sim_{\text{8ve}}$, we are able to access much more frequencies!

Fun fact: Our current system is a N-TET system, with $N = 12$, and it is tuned about $f = 440$Hz, the note A4.