2-dimensional Symmetric Matrix

Take an arbitrary two dimensional symmetric matrix:

$$S_2 = \begin{pmatrix} d_1 & a \\ a & d_2 \end{pmatrix},$$

Notation. Let $\alpha_i(\lambda) = (d_i - \lambda)$.

We obtain the determinant of $(S_2 - \lambda I)$ to be:

$$\det(S_2 - \lambda I) = \alpha_1\alpha_2 - a^2$$

So to solve for the roots of our characteristic polynomial, we set $\det(S_2 - \lambda I) = 0$ and get the equation:

$$\alpha_1(\lambda) \cdot \alpha_2(\lambda) = a^2$$

If a were to be real, then we would expect $\alpha_1\alpha_2 \geq 0$, since otherwise, the square root would be imaginary. So we obtain some useful conditions by solving the inequality:

S_2 is real when any of the following is true:

(1) $\alpha_1, \alpha_2 \in \mathbb{R}^- \iff (d_1 > \lambda) \wedge (d_2 > \lambda)$
(2) $\alpha_1 = 0$ or $\alpha_2 = 0 \iff (d_1 = \lambda) \vee (d_2 = \lambda)$
(3) $\alpha_1, \alpha_2 \in \mathbb{R}^+ \iff (d_1 < \lambda) \wedge (d_2 < \lambda)$

3-dimensional Symmetric Matrix

Take an arbitrary three dimensional symmetric matrix:

$$S_3 = \begin{pmatrix} d_1 & a & b \\ a & d_2 & c \\ b & c & d_3 \end{pmatrix},$$

We obtain the determinant of $(S_3 - \lambda I)$ to be:

$$\det(S_2 - \lambda I) = \alpha_1\alpha_2\alpha_3 - \alpha_1 c^2 - \alpha_2 b^2 - \alpha_3 a^2 + 2abc$$

So solving for $\det(S_2 - \lambda I) = 0$ yields:

$$\alpha_1\alpha_2\alpha_3 - \alpha_1 c^2 - \alpha_2 b^2 - \alpha_3 a^2 + 2abc = 0$$

If we consolidate the left hand side into one function of our free parameters, we obtain:

$$f(d_1, d_2, d_3, a, b, c, \lambda) = 0$$

Generalization

Suppose we have an arbitrary m-dimensional symmetric matrix, call it S_m.

Then, we obtain that the solution to the roots of the characteristic polynomial is $\det(S_m - \lambda I) = 0$. We may find that the number of free parameters is m diagonal entries, and $T(m) = \frac{m(m+1)}{2}$ non-diagonal (triangular) entries and λ. So solving for the eigenvalues of a symmetric matrix yields:

$$f(d_1, \ldots, d_m, a_1, \ldots, a_{T(m)}, \lambda) = 0$$