

Chapter 1

Spectral Statistics

1.1 Introduction

So, what are *spectral statistics*? Do they have to do with rainbows? Sceptres? No, they don't, but they're almost as colorful and regal. The word spectral is borrowed from the spectral-like patterns observed in statistical physics - whether it may be atomic spectra or other quantum mechanical phenomena. The borrowing is loose and not literal, but still somewhat well founded. In fact, the field of Random Matrix Theory was extensively developed in the 1930s by the nuclear physicist Eugene Wigner. He found connections between the deterministic properties of atomic nuclei and their random and stochastic behaviors. The link? Random matrices.

So in the context of this thesis, *spectral statistics* will be an umbrella term for random matrix statistics that somehow involve that matrix's eigenvalues and eigenvectors. That being said, if we fix a *random matrix*, we can study its features by studying its eigenvalues - fundamental numbers that tell us a lot about the matrix. The study of eigenvalues and eigenvectors primarily falls in the scope of Linear Algebra. They are quite important for many reasons. In statistical physics, many processes are represented by operators or matrices, and as such, their behaviours could be partially determined by the eigenvalues of their corresponding matrices. So, what are *eigenvalues* exactly?

1.1.1 The Quintessential Spectral Statistic: the Eigenvalue

Given any standard square matrix $P \in \mathbb{F}^{N \times N}$, its *eigenvalues* are simply the roots of the characteristic polynomial $\text{char}_P(\lambda) = \det(P - \lambda I)$. By the Fundamental Theorem of Algebra, we know that there is always have as many complex eigenvalues $\lambda \in \mathbb{C}$ as the dimension of the matrix.

That being said, when our random matrix has a specified distribution (say, standard normal), we can see patterns in the eigenvalue distributions. So, an eigenvalue is a **spectral statistic** of a random matrix! To talk about a matrix's eigenvalues in a more formal and concise manner, we motivate what is the eigenvalue spectrum.

Definition 1.1.1 (Spectrum). Suppose $P \in \mathbb{F}^{N \times N}$ is a square matrix of size N over \mathbb{F} . The (eigenvalue) spectrum of P is defined as the multiset of its eigenvalues and it is denoted $\sigma(P) = \{\lambda_i \in \mathbb{C}\}_{i=1}^N$. Note that it is important to specify that a spectrum is a multiset and not just a set; eigenvalues could be repeated due to algebraic multiplicity and we opt to always have N eigenvalues.

1.1.2 Interlude: Ensembles

While the spectrum of a matrix provides a good summary of the matrix, in random matrix theory, a matrix is considered a single point or observation. Additionally, simulating large matrices becomes harder and harder as $N \rightarrow \infty$. As such, to obtain more eigenvalue statistics efficiently, another dimension is introduced by motivating the *random matrix ensemble*.

Definition 1.1.2 (Random Matrix Ensemble). A \mathcal{D} -distributed random matrix ensemble \mathcal{E} over $\mathbb{F}^{N \times N}$ of size K is defined as a set of \mathcal{D} -distributed random matrices $\mathcal{E} = \{P_i \sim \mathcal{D} \mid P_i \in \mathbb{F}^{N \times N}\}_{i=1}^K$. In simple words, it is simply a collection of iterations of some specified class of random matrix.

Now that matrix ensembles are well defined, we can motivate a core object of our study - the spectrum of a random matrix ensemble. From its name, it is indeed what one might expect it to be.

Definition 1.1.3 (Ensemble Spectrum). If we have an ensemble \mathcal{E} , then we can naturally extend the definition of $\sigma(\mathcal{E})$. To take the spectrum of an ensemble, simply take the union of the spectra of each of its matrices. In other words, if $\mathcal{E} = \{P_i \sim \mathcal{D} \mid P_i \in \mathbb{F}^{N \times N}\}_{i=1}^K$, then $\sigma(\mathcal{E}) = \bigcup_{i=1}^K \sigma(P_i)$.

A common theme in this thesis will be that singleton matrices do not provide insightful information on their own. Rather, it is the collective behavior of a \mathcal{D} -distributed ensemble that tells us about how \mathcal{D} impacts our spectral statistics. So in a way, ensemble statistics are the engine of this research.

1.2 Eigenvalue Spectra

1.2.1 A Caveat: Eigenvalue Ordering

When we motivate the idea of matrix dispersion, we will consider order statistics of that matrix's eigenvalues in tandem with its dispersion. However, to do so presupposes that we have a sense of what "ordered" eigenvalues means. Suppose are given a matrix P which has an "unordered" spectrum $\sigma(P) = \{\lambda_j\}$. It is paramount to know what ordering scheme $\sigma(P)$ is using, because otherwise, the indices are meaningless! So, to delineate this, we add an index to σ . Often, the ordering context will be clear and the indexing will be omitted. Consider the two following *ordering schema*:

Classical definitions of an *ordered spectrum* follow the standard ordering in the reals; denote this as the ordering by "sign" scheme. Note that because well-ordering

is defined on the reals, we cannot use the sign scheme when $\sigma_P \subset \mathbb{C}^N$. Additionally, if we have a symmetric matrix (of real or complex entries), we have real eigenvalues. In that case, we could either this metric or the other. Without further ado, we write the *sign-ordered spectrum* as follows:

$$\sigma_S(P) = \{\lambda_j : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N\}_{j=1}^N$$

Alternatively, we can motivate a different scheme that properly handles complex eigenvalues. We could sort the spectrum by the norm of its entries; denote this as the ordering by "norm" scheme. Note that when we take the norms of the eigenvalues, we essentially ignore "rotational" features of the eigenvalues. Signs of eigenvalues indicate reflection or rotation, so when we take the norm, we essentially become more concerned with scaling. Without further ado, we write the *norm-ordered spectrum* as follows:

$$\sigma_N(P) = \{\lambda_j : |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N|\}_{j=1}^N$$

1.2.2 Order Statistics

Furthermore, we introduce a new equivalence class, the integer difference of the sorted eigenvalue ranks.

Definition 1.2.1 (Ranking Delta). *The ranking delta is a function $\delta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ which takes the index of two eigenvalues (from an **ordered** spectrum) and returns their difference. In other words, $\delta : (\lambda_i, \lambda_j) \mapsto (i - j)$.*

With the function δ , we may take the set of unique eigenvalue pairs $(i > j)$ and partition it into equivalence classes. To achieve this, we define the equivalence relation \sim_δ which says $(\lambda_m, \lambda_n) \sim_\delta (\lambda_p, \lambda_q) \iff (m - n) = (p - q)$. These equivalence classes then naturally correspond to pairs a set distance $\rho = i - j$ apart. So, for a $N \times N$ matrix, δ assumes a range $\rho \in \{1, \dots, N - 1\}$.

In summary, \sim_δ takes the set $\{(\lambda_i, \lambda_j) \mid \lambda_i, \lambda_j \in \sigma(P) \text{ and } i > j\}$ and surjectively partitions it onto the equivalence classes $[(\lambda_i, \lambda_j)]_\rho$ for $\rho \in \{1, \dots, N - 1\}$. Note that the sizes of each equivalence class are **never equal**. With this partition in mind, we consider the eigenvalue dispersions under each of those equivalence classes.

1.2.3 Analytical Results from Random Matrix Theory

1.3 Eigenvalue Dispersions

Another important spectral statistic is the spacings between the eigenvalues.

1.3.1 Pairing Schema

Next up, we introduce a new notation for a pairing scheme denoted Π . What are pairing schemes and why do they matter? Recall that our goal is to study the spacings

between eigenvalues. If we are studying spacing, then a priori, we are concerned with pairs of eigenvalues! Spacing, after all, is a binary relationship. So, with the definitions of spectra well motivated, a natural definition of pairing schema follows. Essentially, Π is just a subset of the Cartesian product of a spectrum with itself. In other words, if we denote $\mathbb{S} := \sigma(P)$, then we say that a pairing scheme is simply a subset $\Pi \subseteq \mathbb{S}^2$. There are a few special pairing schema to consider:

1. Let $\Pi_{>}$ be the set of unique upper-pair (i) combinations of ordered eigenvalues. This will be the standard ordered pair scheme used in lieu of our dispersion metric argument orders (more later).

$$\Pi_{>} = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i > j\}_{i=1}^{N-1}$$

1. Let $\Pi_{<}$ be the set of unique lower-pair (j) combinations of ordered eigenvalues.

$$\Pi_{<} = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i < j\}_{i=1}^{N-1}$$

1. Let Π_1 be the largest pair of eigenvalues of a spectrum. Nice and sweet.

$$\Pi_1 = \{(\lambda_2, \lambda_1)\}$$

1. Let Π_C be the consecutive pairs of eigenvalues in a spectrum. This pairing scheme gives us the minimal information needed to express important bounds and spacings in terms of its elements.

$$\Pi_C = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i = j + 1\}_{i=1}^{N-1}$$

1.3.2 Dispersion Metrics

When we define the dispersion of a matrix, we will see that there is a free argument d called the dispersion metric. This function d is a general function whose domain is always two eigenvalues. In set notation, this is the set $\mathbb{C} \times \mathbb{C}$ - a pair of two complex numbers. Its range will often be the positive reals \mathbb{R}^+ ; this is because the dispersion metric often will be substitutable with distance metric. Sometimes, the range will be \mathbb{C} . So, the dispersion metric will take the following form:

$$d : \mathbb{C} \times \mathbb{C} \rightarrow \{\mathbb{R}^+, \mathbb{C}\}$$

Consider the following dispersion metrics below. Out of those 4 dispersion metrics, only the first one has a range of \mathbb{C} . The rest have a range of \mathbb{R}^+ . Additionally, the second and third metrics are ****symmetric**** operations while the rest are not. The β -norm is only a symmetric operation when β is even.

1. The identity difference: $d_{id}(z, z') = z' - z$
2. The standard norm: $d_n(z, z') = |z' - z|$
3. The β -norm: $d_\beta(z, z') = |z' - z|^\beta$
4. The difference of absolutes: $d_{ad}(z, z') = |z'| - |z|$

Finally, we are able to motivate the definition of a matrix dispersion! Suppose we have a \mathcal{D} -distributed random matrix $P \in \mathbb{F}^{N \times N}$ or a random matrix ensemble $\mathcal{E} = \{P_i \sim \mathcal{D} \mid P_i \in \mathbb{F}^{N \times N}\}$. Then we define their dispersion as follows.

1.3.3 Matrix and Ensemble Dispersions

Definition 1.3.1 (Dispersion). *The dispersion of a matrix $P \in \mathbb{F}^{N \times N}$ with respect to a dispersion metric $d : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{F}$ and pairing scheme Π , call it $\Delta_d(P, \Pi)$, is defined as follows. Suppose $\sigma(P) := \mathbb{S}$ is the ordered spectrum of P where $\sigma(P) = \{\lambda_1, \dots, \lambda_N\}$. Then, let $\Pi = \{\pi_{ij} = (\lambda_i, \lambda_j)\} \subseteq \mathbb{S}^2$ be a subset of eigenvalue ordered pairs. Then, the dispersion of P with respect to d is simply the set $\Delta_d(P, \Pi) = \{\delta_{ij} = d(\pi_{ij}) \mid \pi_{ij} = (\lambda_i, \lambda_j) \in \Pi\}$.*

Definition 1.3.2 (Ensemble Dispersion). *If we have an ensemble \mathcal{E} , then we can naturally extend the definition of $\Delta_d(\mathcal{E}, \Pi)$. To take the dispersion of an ensemble, simply take the union of the dispersions of each of its matrices. In other words, if $\mathcal{E} = \{P_i \sim \mathcal{D} \mid P_i \in \mathbb{F}^{N \times N}\}_{i=1}^K$, then $\Delta_d(\mathcal{E}, \Pi) = \bigcup_{i=1}^K \Delta_d(P_i, \Pi)$.*

1.3.4 Analytical Results from Random Matrix Theory