Chapter 1

Dispersions

1.1 Introduction

In this section, we define the final spectral statistic studied in this chapter: eigenvalue dispersions. As the name suggests, these statistics are concerned with the distirbution of the spacings between the eigenvalues.

Oddly enough, this is almost as literal as it gets when we use the word "spectral". In physics and chemistry, atomic spectra are essentially differences between energy levels or quanta, so the translation is close.

In any case, we motivate a few definitions and formalisms in this section, then motivate the definition of a matrix's eigenvalue dispersion. To start off, we define an object useful for pairing our eigenvalues together, the pairing scheme.

1.1.1 Dispersion Metrics

Before we may even start to consider studying dispersions of eigenvalues, we must first formalize and make clear what version of spacing we are using. To do so, we motivate the dispersion metric.

Definition 1.1.1 (Dispersion Metric)**.** *A dispersion metric $\delta : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^+$ is defined as a function from the space of pairs of complex numbers to the positive reals. In simple terms, it is a way of measuring "space" between two complex numbers - our eigenvalues.*

Note, there is one special case where the domain of δ is not $\mathbb{R}^+$, that is when we are taking the simple difference of two complex numbers. In this case, the range of δ is also $\mathbb{C}$. We will denote this as the "identity difference" metric and call it δ_{id}.

Consider the following dispersion metrics below. Out of those 4 dispersion metrics, only the first one has a range of $\mathbb{C}$. The rest have a range of $\mathbb{R}^+$. Additionally, the second and third metrics are *symmetric* operations while the rest are not. The β-norm is only a symmetric operation when β is even.

1. The identity difference: $d_{id}(z, z') = z' - z$

2. The standard norm: $d_n(z, z') = |z' - z|$
3. The β-norm: $d_\beta(z, z') = |z' - z|^\beta$
4. The difference of absolutes: $d_{ad}(z, z') = |z'| - |z|$

1.1.2 Pairing Schema

You are a studious student, earnest to learn about eigenvalue spacings, for some reason. So you begin, and the first question pops into your head.

Student: For which eigenvalue pairs will we observe their dispersion?

Wigner's Ghost: The ones specified by the pairing scheme.

Unphased by the fact you are meeting a ghost from the 1990s - a wild spirit you are - you find it a good idea to interrogate the ghost physicist.

The student, irritated about having to learn yet another snippet of loaded notation complains and asks:

Student: Aha! Why don't we just look at all the eigenvalue pairs? Isn't more information always better?

Wigner's Ghost: Actually, no. We will later find out a few reason taking all pairs isn't best. For instance, some spacings are linear combination of others. Or if we have a dispersion metric that is symmetric, half of the information is going to be repeated!

1. The unique pair combinations are two related pair schema. By specifying $i > j$ or $i < j$, we are able to obtain all unique pairs of unique indices. The reason we call them upper and lower pair combinations is in reference to the indices of the upper and lower triangular matrices.

   (a) Let $\Pi_>$ be the set of unique (lower-pair) combinations of ordered eigenvalues. This will be the standard ordered pair scheme used in lieu of our dispersion metric argument orders (more later). In this pairing scheme, the eigenvalue with the lower rank is always listed first, and the higher rank second.
   $$\Pi_> = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i > j\}_{i=1}^{N-1}$$

   (b) For completeness, we will also define $\Pi_<$. This is the set of (upper-pair) unique combinations of ordered eigenvalues. Wont be used because we want bigger - smaller to make positive definite.
   $$\Pi_< = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i < j\}_{i=1}^{N-1}$$

   Benefits: Solves the issue of repeated pairs for symmetric metrics.

2. Let Π_C be the consecutive pairs of eigenvalues in a spectrum. This pairing scheme gives us the minimal information needed to express important bounds and spacings in terms of its elements.
   $$\Pi_C = \{\pi_j = (\lambda_{j+1}, \lambda_j)\}_{j=1}^{N-1}$$

Special pairs:

1. Let Π_1 be the largest pair of eigenvalues of a spectrum. Nice and simple.

$$\Pi_1 = \{(\lambda_2, \lambda_1)\}$$

[Plot showing difference when using different graphs]

1.1.3 Dispersions!

Finally, we are able to motivate the definition of a matrix dispersion! Suppose we have a $\mathcal{D}$-distributed random matrix $P \in \mathbb{F}^{N \times N}$ or a random matrix ensemble $\mathcal{E} = \{P_i \sim \mathcal{D} \mid P_i \in \mathbb{F}^{N \times N}\}$. Then we define their dispersion as follows.

Definition 1.1.2 (Dispersion)**.** *The dispersion of a matrix $P \in \mathbb{F}^{N \times N}$ with respect to a dispersion metric $d : \mathbb{C} \times \mathbb{C} \to \mathbb{F}$ and pairing scheme Π, call it $\Delta_d(P, \Pi)$, is defined as follows. Suppose $\sigma(P) := \mathbb{S}$ is the ordered spectrum of P where $\sigma(P) = \{\lambda_1, \ldots, \lambda_N\}$. Then, let $\Pi = \{\pi_{ij} = (\lambda_i, \lambda_j)\} \subseteq \mathbb{S}^2$ be a subset of eigenvalue ordered pairs. Then, the dispersion of P with respect to d is simply the set $\Delta_d(P, \Pi) = \{\delta_{ij} = d(\pi_{ij}) \mid \pi_{ij} = (\lambda_i, \lambda_j) \in \Pi\}$.*

As we usually do, we define the dispersion of an ensemble in a similar fashion.

Definition 1.1.3 (Ensemble Dispersion)**.** *If we have an ensemble $\mathcal{E}$, then we can naturally extend the definition of $\Delta_d(\mathcal{E}, \Pi)$. To take the dispersion of an ensemble, simply take the union of the dispersions of each of its matrices. In other words, if $\mathcal{E} = \{P_i \sim \mathcal{D} \mid P_i \in \mathbb{F}^{N \times N}\}_{i=1}^K$, then $\Delta_d(\mathcal{E}, \Pi) = \bigcup_{i=1}^K \Delta_d(P_i, \Pi)$.*

With our spectral statistics defined, we are prepared to discuss prominent results in Random Matrix Theory alongside our new findings from the simulations.

1.2 Order Statistics

1.2.1 Introduction

With eigenvalue dispersions and eigenvalue orderings well-defined, we may proceed to start talking about their order statistics.

In addition to these simple order statistics, we introduce a new variant statistic called the **ranking difference class**. Instead of observing a single eigenvalue at a given rank, we will now observe a pair of eigenvalues at a time. To standardize the process, we introduce a new eqivalence class called the *ranking difference*. As suggested, it is precisely the integer difference of the eigenvalue ranks.

Definition 1.2.1 (Ranking Difference)**.** *The ranking difference is a function $\delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which takes the index of two eigenvalues (from an ordered spectrum) and returns their difference. In other words, $\delta : (\lambda_i, \lambda_j) \mapsto (i - j)$.*

With the function δ, we may take the set of unique eigenvalue pairs $(i > j)$ and partition it into equivalence classes. To do so, we define the equivalence relation $\sim_\delta$ which says $(\lambda_a, \lambda_b) \sim_\delta (\lambda_c, \lambda_d) \iff (a-b) = (c-d)$. These equivalence classes then naturally corrospond to pairs a set distance $\rho = i - j$ apart. So, for an $N \times N$ matrix, δ assumes a range $\rho \in \{1, \ldots, N-1\}$.

In summary, $\sim_\delta$ takes the set $\{(\lambda_i, \lambda_j) \mid \lambda_i, \lambda_j \in \sigma(P) \text{ and } i > j\}$ and surjectively partitions it onto the equivalence classes $[(\lambda_i, \lambda_j)]_\rho$ for $\rho \in \{1, \ldots, N-1\}$. For example, if we consider $\rho = 1$, then we are considering all pairs of eigenvalue neighbors.

Note that the sizes of each equivalence class are **never equal**. With this partition in mind, we can consider various statistics conditioning on the value of ρ. Conditioning on ρ will be especially useful in the cases where we are considering matrices like the Hermite-β matrices; the eigenvalues of those matrices tend to *repel*, so to speak, and we can observe these patterns using ρ.

1.2.2 Conditional Statistics

We will considering the conditional statistics $\mathbb{E}(\delta_{ij} \mid \rho)$ and $\mathrm{Var}(\delta_{ij} \mid \rho)$.

1.3 Analytical Results

1.3.1 Case Study: Wigmer's Surmise

Limiting distribution of the eigenvalue spacings of symmetric matrices.

[Plot]

1.4 Findings