

# Chapter 1

## Dispersions

### 1.1 Introduction

In this section, we define the final spectral statistic studied in this chapter: eigenvalue dispersions. As the name suggests, these statistics are concerned with the distribution of the spacings between the eigenvalues. Interestingly, this is almost as literal as it gets when we use the word "spectral". In physics and chemistry, atomic spectra are essentially differences between energy levels or quanta, so the translation is close.

In any case, we will begin this chapter by first motivating a few definitions and formalisms in this section. Then, once our setup is ready, we will motivate the definition of a matrix's eigenvalue dispersion and formalize it as a statistic. To outline the section, we will first define two things: the dispersion metric and the pairing scheme. In simple terms, we formalize **what** "eigenvalue spacings" are and **which** eigenvalue pairs' spacings to consider. Without further ado, we motivate the dispersion metric.

#### 1.1.1 Dispersion Metrics

Before we may even start to consider studying dispersions of eigenvalues, we must first formalize and make clear what "metric" of spacing we are using. To do so, we motivate the dispersion metric.

**Definition 1.1.1** (Dispersion Metric). *A dispersion metric  $\delta : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}^+$  is defined as a function from the space of pairs of complex numbers to the positive reals. In simple terms, it is a way of measuring "space" between two complex numbers - our eigenvalues.*

Consider the following dispersion metrics below.

1. The standard norm:  $\delta(z, z') = |z' - z|$
2. The  $\beta$ -norm:  $\delta_\beta(z, z') = |z' - z|^\beta$
3. The difference of absolutes:  $\delta_{\text{abs}}(z, z') = |z'| - |z|$

**Remark 1.1.1** (Symmetric Metrics). *Note that the standard norm and the  $\beta$ -norm are symmetric operations. This means that switching the order of arguments will have no effect on the function's output. Otherwise, the difference of absolutes metric is not symmetric.*

While we have defined disperison metrics to be functions from  $\mathbb{C}^2$ , there is one special case where we can make an exception so that the domain of  $\delta$  is not  $\mathbb{R}^+$ .

**Remark 1.1.2** (Identity Difference Heuristic). *Suppose we take the arithmetic difference of two complex numbers. Then, the range of  $\delta$  is  $\mathbb{C}$ . For this reason, we won't consider the arithmetic difference a formal dispersion metric, but we will honor it as a dispersion huerestic. As such, we will denote this as the "identity difference" huerestic and call it  $\delta_{id}$ . So, we define  $\delta_{id} : (z, z') \mapsto z' - z$*

### Common Dispersion Metrics

For the formula, assume the order of arguments is  $\delta(z, z')$ .

Table of Dispersion Metrics				
Metric*	Notation	Formula	Symmetric	Parameters
Standard Norm	$\delta$	$ z' - z $	True	-
$\beta$ -Norm	$\delta_\beta$	$ z' - z ^\beta$	True	$\beta \in \mathbb{N}$
Difference of Absolutes	$\delta_{\text{abs}}$	$ z'  -  z $	False	-
Identity Difference	$\delta_{\text{id}}$	$z' - z$	False	-

\*Note that the identity difference is a heuristic, and not a formal metric.

### 1.1.2 Pairing Schema

The next thing we need to motivate before talking about eigenvalue dispersions are pairing schema. In simple terms, pairing schema are templates (for some  $N \in \mathbb{N}$ ) for which eigenvalue pairs to pick. There are many subtle reasons why this is important, which will be covered in detail later. Without further ado, we motivate the pairing schema.

Before we may begin talking about pairing scheme, we define an auxilliary object - the spectral pairs of a matrix, which we denote  $\sigma^{(2)}$ . Since we are now talking about eigenvalue pairs, it is helpful to define this object before proceeding.

**Definition 1.1.2** (Spectral Pairs). *Suppose  $P$  is an  $N \times N$  random matrix. Then, taking the spectral pair of  $P$ , denoted  $\sigma^{(2)}(P)$  is equivalent to taking the Cartesian product of its ordered spectrum  $\sigma(P)$ . That is,  $\sigma^{(2)}(P) = \sigma(P) \times \sigma(P) = \{(\lambda_i, \lambda_j) \mid i, j \in \mathbb{N}_N\}$ .*

Now, to select eigenvalue pairs, we motivate the pairing scheme - this is what tells us which indices to select.

**Definition 1.1.3** (Pairing Scheme). *Suppose  $P$  is any  $N \times N$  matrix and  $\sigma^{(2)}(P)$  are its spectral pairs. A pairing scheme for the matrix  $P$  is a subset of  $\mathbb{N}_N \times \mathbb{N}_N$  - a subset of pairs of numbers from  $\mathbb{N}_N = \{1, \dots, N\}$ . In other words, it is a subset of pair indices for  $N$  objects - in our case, eigenvalues. We denote a pairing scheme as a set  $\Pi = \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{N}_N\} \subseteq \mathbb{N}_N \times \mathbb{N}_N$ . To take a matrix's spectral pairs with respect to  $\Pi$ , we simply take the set of eigenvalue pairs with the matching indices,  $\sigma^{(2)}(P \mid \Pi) = \{(\lambda_\alpha, \lambda_\beta) \mid (\alpha, \beta) \in \Pi\}$ .*

The reader might find this definition slightly obscure, and rightfully so. The definition is a mere formality, as we will usually only consider a few specific pairing schema. Seeing the explicit examples will hopefully make things more clear. With all the technical details aside, we can just say that a pairing scheme tells us which subset of eigenvalue pairs to consider. If one visualizes an array with  $N$  objects on two axes, we are simply choosing a subset of that plane. In fact, consider the following.

**Remark 1.1.3** (Proper Subset). *If  $\sigma^{(2)}(P)$  is the spectral pairs of the matrix  $P$ , then for any pairing scheme  $\Pi$ , we will find that  $\sigma^{(2)}(P \mid \Pi) \subseteq \sigma^{(2)}(P)$ . By definition, taking a spectral pair with respect to some pairing scheme constricts which pairs to select - meaning it is a proper subset of the general spectral pairs.*

### Common Pairing Schema

Suppose  $P$  in an  $N \times N$  square matrix, and  $\sigma^{(2)}(P)$  are its spectral pairs.

1. The unique pair combinations schema are two complementary pair schema. By specifying **either**  $i > j$  or  $i < j$ , we characterize this scheme to entail all unique pair combinations of eigenvalues without repeats. The reason we call them upper and lower pair combinations is an allusion to the indices of the upper and lower triangular matrices.

- (a) Let  $\Pi_{>}$  be the lower-pair combinations of ordered eigenvalues. This will be the standard unique pair combination scheme used in lieu of the argument orders of our dispersion metrics (more later). In this pairing scheme, the eigenvalue with the lower rank is always listed first, and the higher rank second.

$$\sigma^{(2)}(P \mid \Pi_{>}) = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i > j\}_{i=1}^{N-1}$$

- (b) For completeness, we will also define the upper-pair scheme.  $\Pi_{<}$  is the set of upper-pair unique combinations of ordered eigenvalues. Wont be used because we want bigger - smaller to make positive definite.

$$\sigma^{(2)}(P \mid \Pi_{<}) = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i < j\}_{i=1}^{N-1}$$

**Benefits:** Solves the issue of repeated pairs for symmetric dispersion metrics.

2. Let  $\Pi_C$  be the consecutive pairs of eigenvalues in a spectrum.

$$\sigma^{(2)}(P \mid \Pi_C) = \{\pi_j = (\lambda_{j+1}, \lambda_j)\}_{j=1}^{N-1}$$

**Benefits:** This pairing scheme gives us the minimal information needed to express important bounds and spacings in terms of its elements.

3. Let  $\Pi_0$  be all the pairs in a spectrum. For completeness, we define this pairing scheme as a standard.

$$\sigma^{(2)}(P \mid \Pi_0) = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i, j \in \mathbb{N}_N\}$$

**Remark 1.1.4** (Order Statistics). *Since the pairing scheme assumes the eigenvalues are given in an ordered spectrum, our analysis will be mostly leveraging this fact by using the indices. However, as notated above, some pairing schemes are indexed with  $i$  and  $j$ , and some like the consecutive pairs only with  $j$ . These indices are interchangeable with “order statistic”, so in the next section, we will devise a synthetic order statistic that uses two indices.*

### Special pairs

There are a few pairs that are so special, it may be worth highlighting them explicitly.

1. Let  $\tilde{\pi}_j$  denote a pair of consecutive eigenvalues, the largest of the two being the  $j^{\text{th}}$  largest eigenvalue.

$$\tilde{\pi}_j = (\lambda_{j+1}, \lambda_j)$$

Since the consecutive pair scheme can be sufficiently indexed by one index ( $j$ ), we will use the convention of omitting the index of the smaller eigenvalue to be more concise and idiomatic. So, whenever one sees the notation  $\tilde{\pi}_j$ , think the  $j^{\text{th}}$  largest eigenvalue and its smaller neighbour.

2. So, for example, let  $\tilde{\pi}_1$  be the pair of the two largest eigenvalues in an ordered spectrum.

$$\tilde{\pi}_1 = (\lambda_2, \lambda_1)$$

### Common Pairing Schema

Suppose  $P$  is an  $N \times N$  matrix.

Table of Pairing Schema		
Scheme	Notation	Formula
Lower	$\Pi_<$	$\{(i, j) \mid i < j \text{ for } i, j \in \mathbb{N}_N\}$
Upper	$\Pi_>$	$\{(i, j) \mid i > j \text{ for } i, j \in \mathbb{N}_N\}$
Consecutive	$\Pi_C$	$\{(i, j) \mid i = j + 1 \text{ for } i, j \in \mathbb{N}_N\}$
All	$\Pi_0$	$\{(i, j) \mid i, j \in \mathbb{N}_N\}$

Another way we can define the pairing scheme is defining an auxilliary object: the eigenvalue (pair) matrix.

### An Alternative Reperesetation: Eigenvalue Matrix

**Definition 1.1.4** (Eigenvalue Matrix). *Suppose  $P$  is an  $N \times N$  square matrix. Then, the eigenvalue matrix of  $P$ , given by  $\Lambda(P)$  is the matrix with entries  $\pi_{ij} = (\lambda_i, \lambda_j)$ . It is assumed that the eigenvalues,  $\lambda_i$  come from some **ordered** spectrum  $\sigma(P)$ .*

Using the eigenvalue matrix, we have an alternative, potentially more intrinsic, equivalent representation of pairing schemes. For example, the entire matrix represents all pairs. The upper and lower schemes represent the upper-triangular matrix and the lower-triangular respectively. Lastly, the consecutive pairs are simply the nearest off-diagonal in the lower-triangle of the matrix.

### Pairing Scheme Diagrams for a $5 \times 5$ Matrix

All Pairs

$$\begin{bmatrix} O & O & O & O & O \\ O & O & O & O & O \\ O & O & O & O & O \\ O & O & O & O & O \\ O & O & O & O & O \end{bmatrix}$$

The Lower Pair Combinations

$$\begin{bmatrix} - & - & - & - & - \\ O & - & - & - & - \\ O & O & - & - & - \\ O & O & O & - & - \\ O & O & O & O & - \end{bmatrix}$$

The Upper Pair Combinations

$$\begin{bmatrix} - & O & O & O & O \\ - & - & O & O & O \\ - & - & - & O & O \\ - & - & - & - & O \\ - & - & - & - & - \end{bmatrix}$$

The Consecutive Pairs

$$\begin{bmatrix} - & - & - & - & - \\ O & - & - & - & - \\ - & O & - & - & - \\ - & - & O & - & - \\ - & - & - & O & - \end{bmatrix}$$

### 1.1.3 Dispersions

Alas, with dispersion metrics and pairing schemes defined, we are finally able to motivate the definition of a matrix dispersion for both singleton matrices and their ensemble counterparts.

**Definition 1.1.5** (Dispersion). *Suppose  $P$  is an  $N \times N$  matrix, and  $\sigma^{(2)}(P)$  are its spectral pairs. The dispersion of  $P$  with respect to the pairing scheme  $\Pi$  and dispersion metric  $\delta$  is denoted by  $\Delta_\delta(P | \Pi)$  and it is given by the following:*

$$\Delta_\delta(P | \Pi) = \{\delta(\pi_{ij}) \mid \pi_{ij} \in \sigma^{(2)}(P | \Pi)\}$$

We extend the definition of dispersion for an ensemble as we usually do.

**Definition 1.1.6** (Ensemble Dispersion). *If we have an ensemble  $\mathcal{E}$ , then we can naturally extend the definition of  $\Delta_\delta(\mathcal{E} | \Pi)$ . To take the dispersion of an ensemble, simply take the union of the dispersions of each of its matrices. In other words, if  $\mathcal{E} = \{P_i \sim \mathcal{D}\}_{i=1}^K$ , then its dispersion is given by:*

$$\Delta_d(\mathcal{E} | \Pi) = \bigcup_{i=1}^K \Delta_\delta(P_i | \Pi)$$

With our spectral statistics defined, we are prepared to discuss prominent results in Random Matrix Theory alongside our new findings from the simulations.

## 1.2 More Order Statistics

### 1.2.1 Introduction

With eigenvalue dispersions and eigenvalue orderings well-defined, we may proceed to start talking about their order statistics.

In **Chapter 2**, we observed simple order statistics regarding the eigenvalues. This time around, we have two eigenvalues being compared at once, so there needs to be further setup with regards to dispersion order statistics.

However, prior to introducing any new concepts, there is one scenario where using one index is sufficient in terms of discussing dispersion order statistics.

**Remark 1.2.1** (Consecutive Pair Dispersions). *When we consider the dispersion with respect to the consecutive pairs, things are much simpler since our pairs are intrinsically defined by one index ( $j$ ). As such, we will consider dispersion statistics in the form of  $\mathbb{E}(\tilde{\pi}_j | j)$  and  $\text{Var}(\tilde{\pi}_j | j)$ .*

Otherwise, we introduce a new synthetic order statistic called the **ranking difference class**. Since we are no longer observing a single eigenvalue at a given rank, we will need a way to standardize observing a pair of eigenvalues at a time. To do so, we introduce a new **equivalence relation** (see A.x) called the **ranking difference**. As the name suggests, it is precisely the integer difference of the eigenvalue orders.

**Definition 1.2.1** (Ranking Difference). *The ranking difference is a function  $\rho : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  which takes the index of two ordered eigenvalues and returns their difference. In other words, it is the function  $\rho : (\lambda_i, \lambda_j) \mapsto (i - j)$ .*

With the ranking difference, we may now take the spectral pairs of some random matrix with respect to either the lower ( $\Pi_{<}$ ) or upper ( $\Pi_{>}$ ) pairing scheme and partition the eigenvalues into distinct equivalence classes.

**Remark 1.2.2** (Equivalence Relation). *Formally speaking, we may define  $\sim_\rho$  as an equivalence relation. It satisfies all three properties of reflexivity, symmetry, and transitivity. More precisely, it is an equivalence class which partitions  $\mathbb{N}_N \times \mathbb{N}_N$  into  $N - 1$  equivalence classes given by  $[r]_\rho$  for  $r \in \{1, \dots, N - 1\}$ . We would define the equivalence relation  $\sim_\rho$  as follows:*

$$(\lambda_a, \lambda_b) \sim_\rho (\lambda_c, \lambda_d) \iff (a - b) = (c - d)$$

*So, two eigenvalue pairs would belong to the same class  $[r]_\rho$  if their ranking difference is the same. That is,*

$$[r]_\rho = \{(\lambda_\alpha, \lambda_\beta) \mid \alpha - \beta = r\}$$

**Remark 1.2.3** (Matrix Analogy). *Using the eigenvalue matrix analogy,  $\sim_\rho$  partitions the matrix into diagonal bands. The diagonal represents  $[0]_\rho$ , the first off-diagonal represents  $[1]_\rho$ , and so on... Notice that this means every equivalence class has a different size.*

With this partition in mind, we can consider various statistics conditioning on the value of  $r$ . Conditioning on  $r$  will be especially useful in the cases where we are considering matrices like the Hermite- $\beta$  matrices; the eigenvalues of those matrices tend to *repel*, so to speak, and we can observe these patterns using  $r$ .

Consider the following example.

**Example 1.2.1** (Ranking Differences for a  $4 \times 4$  Matrix). *Suppose  $P$  is a  $4 \times 4$  random matrix. Then, if we take the spectral pairs with respect to the lower pairs  $\sigma^{(2)}(P \mid \Pi_{<})$  and partition it by  $\rho$ , we get the following classes:*

1.  $[1]_\rho = \{\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3\}$
2.  $[2]_\rho = \{\pi_{31}, \pi_{42}\}$
3.  $[3]_\rho = \{\pi_{41}\}$

## 1.2.2 Conditional Statistics

We will consider the conditional statistics  $\mathbb{E}(\rho_{ij} \mid \rho)$  and  $\text{Var}(\rho_{ij} \mid \rho)$ .



## 1.3 Analytical Results

### 1.3.1 Case Study: Wigner's Surmise

Wigner's surmise is a result found by Richard Wigner regarding the limiting distribution of eigenvalue spacings of for symmetric matrices. To start talking about this, we must talk about normalized spacings, which are the precise items considered in the distribution. Before, we can talk about the normalized spacing, we define the mean spacing. Recall that the mean spacing showed up in the normalization of the largest eigenvalue in the *Tracy-Widom distribution*. This is precisely the same notion.

**Definition 1.3.1** (Mean Spacing). *Suppose  $P$  is an  $N \times N$  symmetric matrix. Then, the mean (eigenvalue) spacing, denoted  $\langle s \rangle = \langle \lambda_j - \lambda_i \rangle$  is given by taking the consecutive pairs of the sign-ordered spectrum of  $P$  and taking the mean of their difference. That is,  $\langle s \rangle = \mathbb{E}(\lambda_j - \lambda_{j-1})$  for  $j = 1, \dots, N - 1$ .*

So, with the mean spacing defined, we now define the normalized spacing between a pair of consecutive eigenvalues below.

**Definition 1.3.2** (Normalized Spacing). *Suppose  $P$  is an  $N \times N$  symmetric matrix, and  $\sigma(P)$  are its real, sign-ordered eigenvalues. Then, the normalized spacing of the  $j^{\text{th}}$  pair of eigenvalues, denoted  $s_j$  is given by the following formula. Note that in the alternate notation  $\delta = \delta_{id}$ .*

$$s_j = \frac{(\lambda_j - \lambda_{j+1})}{\langle s \rangle} = \frac{\delta(\tilde{\pi}_j)}{\mathbb{E}[\delta(\tilde{\pi}_j)]}$$

The analytical results for the Hermite  $\beta$ -ensembles for  $\beta = 1, 2, 4$ , denoted by  $w_\beta$ , are stated below.

$$\begin{aligned} w_1(s) &= \frac{\pi}{2} \exp\left(-\frac{\pi}{4}s^2\right) \\ w_2(s) &= \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi}s^2\right) \\ w_4(s) &= \frac{2^{18}}{3^6 \pi^3} s^4 \exp\left(-\frac{64}{9\pi}s^2\right) \end{aligned}$$

[Plot]

## 1.4 Findings