Distribution of Entry Ratios in Sequences of Powers of Transition Matrices

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Notation

Random Batches

Let $\mathbb{F}$ be a field, and fix some $M \in \mathbb{N}$. Let $\mathcal{B}_\lambda \subset \mathbb{F}^M$ be a uniformly random batch of points in the M-hypercube of length λ. That is,

$$\mathcal{B}_\lambda = \{\vec{x} \mid x_i \sim \text{Unif}(-\lambda, \lambda) \text{ for } i = 1, \dots, M\}$$

Note: If $\mathbb{F} = \mathbb{C}$, then take $\vec{x} \in \mathcal{B}_\lambda$ to mean $\vec{x} = a + bi$ where $a, b \sim \text{Unif}(-\lambda, \lambda)$.

Finite Evolution Sequences

Suppose we sample a random point from $\mathcal{B}_\lambda$, emulating a random point $\vec{v} \in \mathbb{F}^M$. Additionally, let $Q \in \mathbb{F}^{M \times M}$ be a transition matrix over $\mathbb{F}$. Fixing a maximum power ('time') $T \in \mathbb{N}$, define the evolution sequence of $\vec{v}$ as follows:

$$\mathcal{S}(v, Q, T) = (\alpha_n)_{n=1}^T \text{ where } \alpha_k = vQ^k$$

If we do not impose a finiteness constraint on the sequence, we consider powers for $n \in \mathbb{N}$ or $t = \infty$

Consecutive Ratio Sequences

Accordingly, define the consecutive ratio sequence (CST) of $\vec{v}$ as follows:

$$\mathcal{R}(v, Q, T) = (r_n)_{n=2}^T \text{ where } (r_n)_j = \frac{(\alpha_n)_j}{(\alpha_{n-1})_j} \text{ for } j = 1, \dots, M$$

In other words, the consecutive ratio sequence of v can be obtained by performing **component-wise division** on consecutive elements of the evolution sequence of v.

Near Convergence

Because these sequences may never truly converge to eigenvectors of the matrix, we formalize a notion of "near convergence". As a prelimenary, we first define ε-equivalence. Let $\mathbb{F}$ be a field, and fix $\varepsilon \in \mathbb{R}^+$. Suppose we have vectors $v, v' \in \mathbb{F}^M$. Then, $v \sim_\varepsilon v'$ if $||v - v'|| < \varepsilon$ where $|| \cdot ||$ is the norm on $\mathbb{F}$.

Let $\varepsilon \in \mathbb{R}^+$, and suppose we have an evolution sequence $(a[\vec{v}])_n$. Then, a_n ε-converges at $N \in \mathbb{N}$ if:

$$\forall n \geq N \mid a_N \sim_\varepsilon a_n$$

Questions

1. How are the entries of the CRS distributed? Are they normal, and if so, what is its mean?
2. Are the entries of the CRS i.i.d as $t \to \infty$?
3. For an Erdos-Renyi matrix, is the mixing time t dependent on the parameter p?
4. What impact does the running time parameter T have on σ (the variance of the distribution of the CRS entries)?

Initial Findings

It seems to be the case that the **log-transformed** entries of the CRS are Cauchy distributed about $\log \lambda_1$ where $\lambda_1 = \max(\sigma(Q))$, the largest eigenvalue of Q. That is,

$$r_i \sim \text{Cauchy}(\ln \lambda_1) \text{ for } i = 1, \ldots, M$$