

# Real Symmetric Matrices have Real Eigenvectors

**Notation.** For notational convenience, for any  $N \in \mathbb{N}$ , let  $\widetilde{N} = \{1, \dots, N\}$ .

In this document, we prove that for any  $M \times M$  real symmetric matrix,  $S_M$ , there exists for some eigenvalue  $\lambda$ , a corresponding **real** eigenvector  $\vec{v} \in \mathbb{R}^M$ . Prior to starting the main proof, we begin with a lemma.

**Lemma.** Suppose we have a  $M \times M$  real symmetric matrix with a some eigenvalue  $\lambda$ . If there we have a corresponding eigenvector  $v \in \mathbb{C}^M$ , then every entry of  $v$ , say  $v_i$  is equal to a **real** linear combination of the other entries  $v_j \mid j \neq i$ .

So, we will show that:

$$\forall i \in \widetilde{M} : v_i = \sum_{j \neq i} c_j v_j \quad (c_j \in \mathbb{R})$$

**Proof of Lemma.** Begin by taking a real symmetric matrix  $S_M$  for some  $M \in \mathbb{N}$ . Suppose we have an eigenvalue  $\lambda$ . Then, if we have some eigenvector  $v$ , we know that:

$$(1) : \forall i \in \widetilde{M} : a_1 v_1 + \dots + d_i v_i + \dots + a_{m-1} v_m = \lambda v_i \quad (a_j \in \mathbb{R})$$

We obtain (1) by expanding the equality  $Av = \lambda v$  and noticing that every row of  $Av$  is expressible as the sum of the non-diagonal entries multiplied by  $v_j \mid j \neq i$  plus  $d_i v_i$ . Note that since our matrix is symmetric, for some rows, some of the constants  $a_j$  are not distinct but this should not raise any issues. Next, we collect the terms:

$$\forall i \in \widetilde{M} : a_1 v_1 + \dots + a_{m-1} v_m = v_i (\lambda - d_i)$$

Since  $S_M$  is a real symmetric matrix, the  $a_j$  terms are real so we can say:

$$\forall i \in \widetilde{M} : v_i (\lambda - d_i) = \sum_{j \neq i} a_j v_j \quad (a_j \in \mathbb{R})$$

Finally, divide both sides by  $(\lambda - d_i)$ . Since  $S_M$  is a real symmetric matrix, we know  $\lambda \in \mathbb{R}$  then also  $(\lambda - d_i) \in \mathbb{R}$ . On the right hand side, the coefficients of the  $v_j$  become  $\frac{a_j}{(\lambda - d_i)}$ . Since  $a_j \in \mathbb{R}$ , then also  $\frac{a_j}{(\lambda - d_i)} \in \mathbb{R}$ . Letting  $c_j = \frac{a_j}{(\lambda - d_i)}$ , we obtain:

$$\forall i \in \widetilde{M} : v_i = \sum_{j \neq i} c_j v_j \quad (\forall j : c_j \in \mathbb{R})$$

Thus, for any  $M \in \mathbb{N}$ , a real symmetric matrix with eigenvalue  $\lambda$  must have a corresponding eigenvector  $v$  such that each of its entries is expressible as a real linear combination of the other entries.  $\square$

Now, we will prove the main theorem.

**Theorem.** Suppose we have a  $M \times M$  real symmetric matrix,  $S_M$ . Then, we will show that there exists for some eigenvalue  $\lambda$ , a corresponding **real** eigenvector  $\vec{v} \in \mathbb{R}^M$ .

**Proof.** For this proof we will induct on the dimension of the matrix,  $M$ . So let the inductive statement be

$$f(M) : S_M \text{ has a real eigenvector } v \text{ corresponding to an eigenvalue } \lambda$$

**Base Case.** Take the base case  $M = 2$ . Then by **Zoom Meeting 11.12**, we know  $f(2)$  is true.

**Inductive Step.** For our inductive step, we need to show that  $f(M) \Rightarrow f(M+1)$ . So, let us assume  $f(M)$ . This means that we can assume any real symmetric matrix  $S_M$  has a real eigenvector  $v \in \mathbb{R}^M$  corresponding to  $\lambda$ .

Next, we will write  $S_{M+1}$  as the matrix  $S_M$  augmented by some  $u \in \mathbb{R}^M$  as follows:

$$S_{M+1} = \left[ \begin{array}{c|c} S_M & u \\ \hline u^T & d_{M+1} \end{array} \right]$$

From our lemma, we use the fact that  $S_{M+1}$  is symmetric and our assumption of  $f(M)$  to obtain:

$$(1) : \forall i \in \{1, \dots, m+1\} : v_i = \sum_{j \neq i} c_j v_j \quad (c_j \in \mathbb{R})$$

$$(2) : \forall i \in \tilde{M} : v_i \in \mathbb{R}$$

In particular for (2), we know that  $v_i = \left( \sum_{j \neq i} \frac{a_j}{d_i - \lambda} v_j \right)$ .

From (1), we know that for row  $i = m+1$ :  $v_{m+1} = \sum_{j \neq m+1} c_j v_j \quad (c_j \in \mathbb{R})$  By (2), this is a linear combination of real entries  $v_i$ . Since  $v_{m+1} \in \mathbb{R}$ , it follows that:

$$\forall i \in \{1, \dots, m+1\} : v_i \in \mathbb{R}$$

So, we have established that  $f(m) \Rightarrow f(M+1)$ .

By the induction, the theorem is proved.  $\square$ .