Chapter 1

Dispersions

1.1 Introduction

In this section, we define the final spectral statistic studied in this chapter: eigenvalue dispersions. As the name suggests, these statistics are concerned with the distirbution of the spacings between the eigenvalues. Interestingly, this is almost as literal as it gets when we use the word "spectral". In physics and chemistry, atomic spectra are essentially differences between energy levels or quanta, so the translation is close.

In any case, we will begin this chapter by first motivating a few definitions and formalisms in this section. Then, once our setup is ready, we will motivate the definition of a matrix's eigenvalue dispersion and formalize it as a statistic. To outline the section, we will first define two things: the dispersion metric and the pairing scheme. In simple terms, we formalize **what** "eigenvalue spacings" are and **which** eigenvalue pairs' spacings to consider. Without further ado, we motivate the dispersion metric.

1.1.1 Dispersion Metrics

Before we may even start to consider studying dispersions of eigenvalues, we must first formalize and make clear what "metric" of spacing we are using. To do so, we motivate the dispersion metric.

Definition 1.1.1 (Dispersion Metric)**.** *A dispersion metric $\delta : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^+$ is defined as a function from the space of pairs of complex numbers to the positive reals. In simple terms, it is a way of measuring "space" between two complex numbers - our eigenvalues.*

Consider the following dispersion metrics below.

1. The standard norm: $\delta(z, z') = |z' - z|$
2. The β-norm: $\delta_\beta(z, z') = |z' - z|^\beta$
3. The difference of absolutes: $\delta_{\text{abs}}(z, z') = |z'| - |z|$

Remark 1.1.1 (Symmetric Metrics)**.** *Note that the standard norm is a symmetric operation. The β-norm can also be a symmetric operation when β is even. Otherwise, the difference of absolutes metric is not symmetric.*

While we have defined disperison metrics to be functions from $\mathbb{C}^2$, there is one special case where we can make an exception so that the domain of δ is not $\mathbb{R}^+$.

Remark 1.1.2 (Identity Difference Heuristic)**.** *Suppose we take the arithmetic difference of two complex numbers. Then, the range of δ is $\mathbb{C}$. For this reason, we won't consider the arithmetic difference a formal dispersion metric, but we will honor it as a dispersion huerestic. As such, we will denote this as the "identity difference" huerestic and call it δ_{id}. So, we define $\delta_{id} : (z, z') \mapsto z' - z$*

Common Dispersion Metrics

For the formula, assume the order of arguments is $\delta(z, z')$.

Table of Dispersion Metrics				
Metric*	Notation	Formula	Symmetric	Parameters
Standard Norm	δ	$\lvert z' - z \rvert$	True	-
β-Norm	δ_β	$\lvert z' - z \rvert^\beta$	β Even	$\beta \in \mathbb{N}$
Difference of Absolutes	δ_{abs}	$\lvert z' \rvert - \lvert z \rvert$	False	-
Identity Difference	δ_{id}	$z' - z$	False	-

*Note that the identity difference is a heurestic, and not a formal metric.

1.1.2 Pairing Schema

The next thing we need to motivate before talking about eigenvalue dispersions are pairing schema. In simple terms, pairing schema are templates (for some $N \in \mathbb{N}$) for which eigenvalue pairs to pick. There are many subtle reasons why this is important, which will be covered in detail later. Without further ado, we motivate the pairing schema.

Definition 1.1.2 (Pairing Scheme)**.** *Suppose P is any $N \times N$ matrix and $\sigma(P)$ is its corrosponding ordered spectrum. A pairing scheme for the matrix P is simply a subset of $\mathbb{N}_N \times \mathbb{N}_N$ - a subset of pairs of numbers from $\mathbb{N}_N = \{1, \ldots, N\}$. In other words, it is a subset of pair indices for N objects - in our case, eigenvalues. We denote a pairing scheme as some set $\Pi = \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{N}_N\} \subseteq \mathbb{N}_N \times \mathbb{N}_N$. To take the matrix's corrosponding eigenvalue pairs with respect to Π, we simply take the set of eigenvalue pairs with the corrosponding indices, $\tilde{\sigma}(P \mid \Pi) = \{(\lambda_\alpha, \lambda_\beta) \mid (\alpha, \beta) \in \Pi\}$.*

The reader might find this definition slightly obscure, and rightfully so. The definition is a mere formality, as we will usually only consider a few specific pairing schema. Seeing the explicit examples will hopefully make things more clear. With all the technical details aside, we can just say that a pairing scheme tells us which subset of eigenvalue pairs to consider. If one visualizes an array with N objects on two axes, we are simply choosing a subset of that plane. In fact, consider the following.

Remark 1.1.3 (Spectrum Subset)**.** *If we denote $\mathbb{S} = \sigma(P)$ as the **ordered** spectrum of some matrix P, then for any valid pairing scheme Π, we will find that $\tilde{\sigma}(P \mid \Pi) \subseteq \mathbb{S}^2$ by definition. Note the emphasis on ordered - otherwise, the indices are meaningless.*

Common Pairing Schema

Suppose P in an $N \times N$ square matrix, and $\sigma(P)$ is its ordered spectrum.

1. The unique pair combinations schema are two complementary pair schema. By specifying **either** $i > j$ or $i < j$, we characterize this scheme to entail all unique pair combinations of eigenvalue without repeats. The reason we call them upper and lower pair combinations is an allusion to the indices of the upper and lower triangular matrices.

   (a) Let $\Pi_>$ be the lower-pair combinations of ordered eigenvalues. This will be the standard unique pair combination scheme used in lieu of the argument orders of our dispersion metrics (more later). In this pairing scheme, the eigenvalue with the lower rank is always listed first, and the higher rank second.
   $$\sigma(P \mid \Pi_>) = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i > j\}_{i=1}^{N-1}$$

   (b) For completeness, we will also define the upper-pair scheme. $\Pi_<$ is the set of upper-pair unique combinations of ordered eigenvalues. Wont be used because we want bigger - smaller to make positive definite.
   $$\sigma(P \mid \Pi_<) = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i < j\}_{i=1}^{N-1}$$

   Benefits: Solves the issue of repeated pairs for symmetric dispersion metrics.

2. Let Π_C be the consecutive pairs of eigenvalues in a spectrum.
   $$\sigma(P \mid \Pi_C) = \{\pi_j = (\lambda_{j+1}, \lambda_j)\}_{j=1}^{N-1}$$

   Benefits: This pairing scheme gives us the minimal information needed to express important bounds and spacings in terms of its elements.

3. Let Π_0 be all the pairs in a spectrum. For completeness, we define this pairing scheme as a standard.
   $$\sigma(P \mid \Pi_0) = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i, j \in \mathbb{N}_N\}$$

Remark 1.1.4 (Order Statistics)**.** *Since the pairing scheme assumes the eigenvalues are given in an ordered spectrum, our analysis will be mostly leveraging this fact by using the indices. However, as notated above, some pairing schemes are indexed with i and j, and some like the consecutive pairs only with j. These indices are interchangable with "order statistic", so in the next section, we will devise a synthetic order statistic that uses two indices.*

So, if a pair π_j has only one index j, the pair will be a consecutive pair denoted by $\pi_j = (\lambda_{j+1}, \lambda_j)$. Otherwise, π_{ij} will be given by (λ_i, λ_j). The ranking difference class will group pairs into equivalence classes given by an integer $\rho = i - j$.

Special pairs

There are a few pairs that are so special, it may be worth highlighting them explicity.

1. Let π_j be the pair of consecutive eigenvalues, the largest of the two being the j^{th} largest eigenvalue.
$$\pi_j = (\lambda_j, \lambda_{j+1})$$
Since the consecutive pair scheme can be sufficiently indexed by one index (j), we will use the convention of omitting the index of the smaller eigenvalue to be more concise and idiomatic. So, whenever one sees the notation π_j, think the j^{th} largest eigenvalue and its smaller neighbour.

2. So, for example, let π_1 be the pair of the two largest eigenvalues in an ordered spectrum.
$$\pi_1 = (\lambda_2, \lambda_1)$$

Another way we can define the pairing scheme is defining an auxilliary object: the eigenvalue (pair) matrix.

Definition 1.1.3 (Eigenvalue Matrix)**.** *Suppose P is an $N \times N$ square matrix. Then, the eigenvalue matrix of P, given by $\Lambda(P)$ is the matrix with entries $\pi_{ij} = (\lambda_i, \lambda_j)$. It is assumed that the eigenvalues, λ_i come from some* ***ordered*** *spectrum $\sigma(P)$.*

Common Pairing Schema

Suppose P is an $N \times N$ matrix.

Table of Pairing Schema		
Scheme	Notation	Formula
Lower	$\Pi_<$	$\{(i,j) \mid i < j \text{ for } i,j \in \mathbb{N}_N\}$
Upper	$\Pi_>$	$\{(i,j) \mid i > j \text{ for } i,j \in \mathbb{N}_N\}$
Consecutive	Π_C	$\{(i,j) \mid i = j+1 \text{ for } i,j \in \mathbb{N}_N\}$
All	Π_0	$\{(i,j) \mid i,j \in \mathbb{N}_N\}$

Pairing Scheme Diagrams for a 5×5 Matrix

All Pairs

$$\begin{bmatrix} O & O & O & O & O \\ O & O & O & O & O \\ O & O & O & O & O \\ O & O & O & O & O \\ O & O & O & O & O \end{bmatrix}$$

The Lower Pair Combinations

$$\begin{bmatrix} - & - & - & - & - \\ O & - & - & - & - \\ O & O & - & - & - \\ O & O & O & - & - \\ O & O & O & O & - \end{bmatrix}$$

The Upper Pair Combinations

$$\begin{bmatrix} - & O & O & O & O \\ - & - & O & O & O \\ - & - & - & O & O \\ - & - & - & - & O \\ - & - & - & - & - \end{bmatrix}$$

The Consecutive Pairs

$$\begin{bmatrix} - & - & - & - & - \\ O & - & - & - & - \\ - & O & - & - & - \\ - & - & O & - & - \\ - & - & - & O & - \end{bmatrix}$$

1.1.3 Dispersions

Finally, we are able to motivate the definition of a matrix dispersion! Suppose we have a $\mathcal{D}$-distributed random matrix $P \in \mathbb{F}^{N\times N}$ or a random matrix ensemble $\mathcal{E} = \{P_i \sim \mathcal{D} \mid P_i \in \mathbb{F}^{N\times N}\}$. Then we define their dispersion as follows.

Definition 1.1.4 (Dispersion)**.** *The dispersion of a matrix $P \in \mathbb{F}^{N\times N}$ with respect to a dispersion metric $d : \mathbb{C}\times\mathbb{C} \to \mathbb{F}$ and pairing scheme Π, call it $\Delta_d(P, \Pi)$, is defined as follows. Suppose $\sigma(P) := \mathbb{S}$ is the ordered spectrum of P where $\sigma(P) = \{\lambda_1, \ldots, \lambda_N\}$. Then, let $\Pi = \{\pi_{ij} = (\lambda_i, \lambda_j)\} \subseteq \mathbb{S}^2$ be a subset of eigenvalue ordered pairs. Then, the dispersion of P with respect to d is simply the set $\Delta_d(P, \Pi) = \{\delta_{ij} = d(\pi_{ij}) \mid \pi_{ij} = (\lambda_i, \lambda_j) \in \Pi\}$.*

As we usually do, we define the dispersion of an ensemble in a similar fashion.

Definition 1.1.5 (Ensemble Dispersion)**.** *If we have an ensemble $\mathcal{E}$, then we can naturally extend the definition of $\Delta_d(\mathcal{E}, \Pi)$. To take the dispersion of an ensemble, simply take the union of the dispersions of each of its matrices. In other words, if $\mathcal{E} = \{P_i \sim \mathcal{D} \mid P_i \in \mathbb{F}^{N\times N}\}_{i=1}^K$, then $\Delta_d(\mathcal{E}, \Pi) = \bigcup_{i=1}^K \Delta_d(P_i, \Pi)$.*

With our spectral statistics defined, we are prepared to discuss prominent results in Random Matrix Theory alongside our new findings from the simulations.

1.2 Order Statistics

1.2.1 Introduction

With eigenvalue dispersions and eigenvalue orderings well-defined, we may proceed to start talking about their order statistics.

In addition to these simple order statistics, we introduce a new variant statistic called the **ranking difference class**. Instead of observing a single eigenvalue at a given rank, we will now observe a pair of eigenvalues at a time. To standardize the process, we introduce a new eqivalence class called the *ranking difference*. As suggested, it is precisely the integer difference of the eigenvalue ranks.

Definition 1.2.1 (Ranking Difference)**.** *The ranking difference is a function $\delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which takes the index of two eigenvalues (from an ordered spectrum) and returns their difference. In other words, $\delta : (\lambda_i, \lambda_j) \mapsto (i - j)$.*

With the function δ, we may take the set of unique eigenvalue pairs $(i > j)$ and partition it into equivalence classes. To do so, we define the equivalence relation $\sim_\delta$ which says $(\lambda_a, \lambda_b) \sim_\delta (\lambda_c, \lambda_d) \iff (a-b) = (c-d)$. These equivalence classes then naturally corrospond to pairs a set distance $\rho = i - j$ apart. So, for an $N \times N$ matrix, δ assumes a range $\rho \in \{1, \ldots, N-1\}$.

In summary, $\sim_\delta$ takes the set $\{(\lambda_i, \lambda_j) \mid \lambda_i, \lambda_j \in \sigma(P) \text{ and } i > j\}$ and surjectively partitions it onto the equivalence classes $[(\lambda_i, \lambda_j)]_\rho$ for $\rho \in \{1, \ldots, N-1\}$. For example, if we consider $\rho = 1$, then we are considering all pairs of eigenvalue neighbors.

Note that the sizes of each equivalence class are **never equal**. With this partition in mind, we can consider various statistics conditioning on the value of ρ. Conditioning on ρ will be especially useful in the cases where we are considering matrices like the Hermite-β matrices; the eigenvalues of those matrices tend to *repel*, so to speak, and we can observe these patterns using ρ.

1.2.2 Conditional Statistics

We will considering the conditional statistics $\mathbb{E}(\delta_{ij} \mid \rho)$ and $\mathrm{Var}(\delta_{ij} \mid \rho)$.

1.3 Analytical Results

1.3.1 Case Study: Wigner's Surmise

Wigner's surmise is a result found by Richard Wigner regarding the limiting distribution of eigenvalue spacings of for symmetric matrices. To start talking about this, we must talk about normalized spacings, which are the precise items considered in the distribution. Before, we can talk about the normalized spacing, we define the mean spacing. Recall that the mean spacing showed up in the normalization of the largest eigenvalue in the *Tracy-Widom distribution*. This is precisely the same notion.

Definition 1.3.1 (Mean Spacing)**.** *Suppose P is an $N \times N$ symmetric matrix. Then, the mean (eigenvalue) spacing, denoted $\langle s \rangle = \langle \lambda_j - \lambda_i \rangle$ is given by taking the consecutive pairs of the sign-ordered spectrum of P and taking the mean of their difference. That is, $\langle s \rangle = \mathbb{E}(\lambda_j - \lambda_{j-1})$ for $j = 1, \ldots, N-1$).*

So, with the mean spacing defined, we now define the normalized spacing between a pair of consecutive eigenvalues below.

Definition 1.3.2 (Normalized Spacing)**.** *Suppose P is an $N \times N$ symmetric matrix, and $\sigma(P)$ are its real, sign-ordered eigenvalues. Then, the normalized spacing of the j^{th} pair of eigenvalues, denoted s_j is given by the following formula. Note that in the alternate notation $\delta = \delta_{id}$.*

$$s_j = \frac{(\lambda_j - \lambda_{j+1})}{\langle s \rangle} = \frac{\delta(\pi_j)}{\mathbb{E}[\delta(\pi_j)]}$$

The analytical results for the Hermite β-ensembles for $\beta = 1, 2, 4$, denoted by w_β, are stated below.

$$w_1(s) = \frac{\pi}{2} \exp\left(-\frac{\pi}{4} s^2\right)$$

$$w_2(s) = \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi} s^2\right)$$

$$w_4(s) = \frac{2^{18}}{3^6 \pi^3} s^4 \exp\left(-\frac{64}{9\pi} s^2\right)$$

[Plot]

1.4 Findings