Spectral Statistics of Random Matrices

Ali Taqi

Reed College

Spring 2021

Preamble

RMAT

This package was developed alongside this thesis. Two reasons: reproducibility and interactivity. Available on CRAN, GitHub, Code Appendix.

$\mathcal{D}$-distributions

$\mathcal{D}$-distributions

To simulate random matrices, we need to initialize their entries. But, how do we sample the random entries?

1. Random Variables: Sample from a probability distribution ($X \sim \mathcal{D}$)
2. Random Matrices: Define a new framework containing information about entry distributions: $\mathcal{D}$-distributions
   1. Explicit distributions: sample (some or all) of the entries as independent random variables
   2. Implicit distributions: use an algorithm to randomly generate the matrix.

Homogenous Explicit $\mathcal{D}$-distributions

Definition: Homogenous Explicit $\mathcal{D}$-distributions

Suppose $P \sim \mathcal{D}$ where $\mathcal{D}$ is a homogenous and explicit distribution. Additionally, let $\mathcal{D}^*$ denote the corresponding random variable analogue of $\mathcal{D}$. Then, every single entry of P is an i.i.d random variable with the corresponding distribution. That is,

$$P \sim \mathcal{D} \iff \forall i, j \mid p_{ij} \sim \mathcal{D}^*$$

Homogenous Explicit $\mathcal{D}$-distributions

Examples

Suppose $P \sim \mathcal{N}(0, 1)$ and that P is a 2×2 matrix. Then, $p_{11}, p_{12}, p_{21}, p_{22}$ are independent, identically distributed random variables with the standard normal distribution.

Non-Homogenous Explicit $\mathcal{D}$-distributions

Now, as opposed to having independent and identically distributed entries for the whole matrix, the random matrix distribution may be slightly more complex to define.

Fundamentally, we could characterize any matrix by identifying the distribution of every entry. But, we don't need to do so.

Non-Homogenous Explicit $\mathcal{D}$-distributions

Definition: Diagonal Bands

Suppose $P = (p_{ij})$ is an $N \times N$ matrix. Then, P may be partitioned into $2n - 1$ rows called diagonal bands. Each band is denoted $[\rho]_P$ where $[\rho]_P = \{p_{ij} \mid \rho = i - j\}$. We have $\rho \in \{-(N-1), \ldots, -1, 0, 1, \ldots, N-1\}$.

The Hermite β-Matrix

Definition: β-matrix

Suppose $P \sim \mathcal{H}(\beta)$ is an $N \times N$ matrix. Then, the main diagonal $[0]_P \sim \mathcal{N}(0, 2)$. Additionally, both the main off-digaonals are equal and they are given by $[1]_P = [-1]_P = \vec{X} = (X_k)_{k=1}^{N-1}$ where $X_k \sim \chi(\mathrm{df} = \beta k)$. As such, we obtain a Hermite-β distributed matrix. Note that this is a symmetric tridiagonal matrix.

The Hermite β-Matrix

$$H_\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_{\beta} \\ & & & \chi_{\beta} & N(0,2) \end{pmatrix}$$

Implicit Distributions

1. Stochastic Matrices: we will generate transition matrices representing a walk on a fully connected graph with **randomized weights**.
2. Erdos-Renyi p-Matrices: stochastic matrices with randomized weights, but parameterized probability p of observing severed edges (zero weight).

Random Matrices

Random Matrices

Random Matrices

Let $P \sim \mathcal{D}$ be an $N \times N$ matrix over $\mathbb{F}$. Then, the entries of P are elements in $\mathbb{F}$ completely determined by the $\mathcal{D}$-distribution, regardless of what type it is. Also, if $\mathcal{D}$ is an explicit distribution, $\mathcal{D}^{\dagger}$ represents the symmetric/hermitian version of $\mathcal{D}$.

Random Matrix Ensembles

Definition: Random Matrix Ensembles

A $\mathcal{D}$-distributed ensemble $\mathcal{E}$ of $N \times N$ random matrices over $\mathbb{F}$ of size K is defined as a set of K iterations of that class of random matrix, and it is denoted:

$$\mathcal{E} = \bigcup_{i=1}^{K} P_i \text{ where } P_i \sim \mathcal{D} \text{ and } P_i \in \mathbb{F}^{N \times N}$$

Summary of $\mathcal{D}$-distributions

Table of Random Matrix Distributions			
Distribution	Notation ($\mathcal{D}$)	Parameters	Class
Normal	$\mathcal{N}(\mu, \sigma)$	$\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$	Explicit (H)
Uniform	Unif(a, b)	$a, b \in \mathbb{R}$	Explicit (H)
Hermite-β	$\mathcal{H}(\beta)$	$\beta \in \mathbb{N}$	Explicit (NH)
Stochastic	Stoch	-	Implicit
Erdos-p	ER(p)	$p \in [0, 1]$	Implicit

Checkpoint

Now, we have defined random matrices. We can simulate various random matrix ensembles, and are ready to analyze their spectral statistics!

Spectra

Spectra

Definition: Spectrum

Suppose $P \in \mathbb{F}^{N \times N}$ is a square matrix of size N over $\mathbb{F}$. Then, the (eigenvalue) spectrum of P is defined as the multiset of its eigenvalues and it is denoted

$$\sigma(P) = \{\lambda_i \in \mathbb{C} \mid \mathrm{char}_P(\lambda_i) = 0\}_{i=1}^{N}$$

Note that it is important to specify that a spectrum is a multiset and not just a set; eigenvalues could be repeated due to algebraic multiplicity and we opt to always have N eigenvalues.

Ensemble Spectrum

Definition: Ensemble Spectrum

Let $\mathcal{E} \sim \mathcal{D}$ be an ensemble of matrices $P_i \in \mathbb{F}^{n \times n}$. To take the spectrum of $\mathcal{E}$, simply take the union of the spectra of each of its matrices. In other words, if $\mathcal{E} = \{P_i \sim \mathcal{D}\}_{i=1}^{K}$, then we denote the spectrum of the ensemble

$$\sigma(\mathcal{E}) = \bigcup_{i=1}^{K} \sigma(P_i)$$

Spectrum Analysis

Order Statistics

One method of analyzing the spectrum of a matrix is to consider the framework of using order statistics.

Examples

Let's say you roll a die 3 times and obtain 5,6, and 2. Then, your order statistics take the form $X_1 = 6$, $X_2 = 5$, and $X_3 = 2$.

Order Statistics

That being said, consider the two order schema that we will use.

1. The **sign**-ordering scheme; works on $\mathbb{R}$.

$$\sigma_S(P) = \{\lambda_j : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N\}_{j=1}^{N}$$

2. The **norm**-ordering scheme; works on $\mathbb{R}$ and $\mathbb{C}$.

$$\sigma_N(P) = \{\lambda_j : |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N|\}_{j=1}^{N}$$

Order Statistics

Consider the following example:

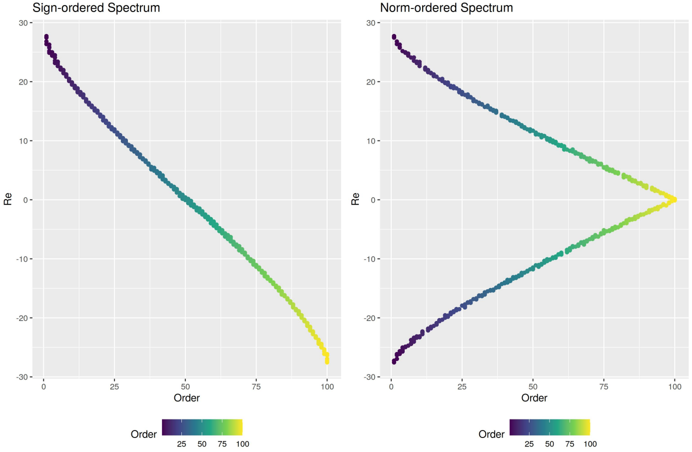


Figure: Spectrum of an Ensemble Using Two Different Ordering Schemes

Order Statistics

Expectation. $\mathbb{E}(\lambda_i \mid i)$ One useful summary statistic to consider when analyzing a spectrum is the expected norm or value (which we will just call quantity hereinafter) of the eigenvalue at the i^{th} rank.

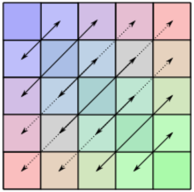
Variance. $\mathrm{Var}(\lambda_i \mid i)$ Similarly, the variance of the eigenvalue quantity at a given order i can tell us a lot about an ensemble.

Symmetric & Hermitian Matrices

Symmetric & Hermitian Matrices

Symmetric & Hermitian matrices are an important class of matrices in Linear Algebra. Suppose we have a matrix P.

1. P Symmetric $\iff p_{ij} = p_{ji}$
2. P Hermitian $\iff p_{ij} = \bar{p}_{ji}$



Symmetric & Hermitian Matrices

So, why do we care?

Theorem

Suppose P is Symmetric/Hermitian. Then, P has a set of real eigenvalues. That is,

$$P \textbf{ Symmetric/Hermitian} \implies \forall \lambda_i \in \sigma(P) \mid \lambda_i \in \mathbb{R}$$

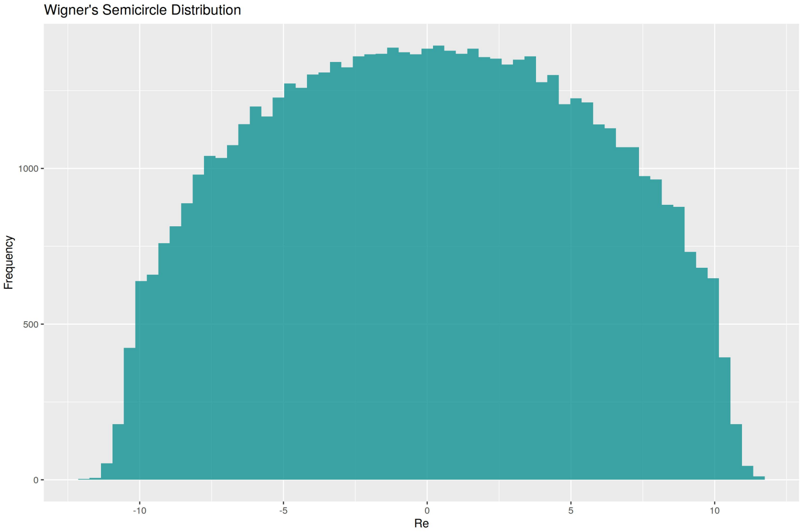
Wigner's Semicircle Distribution

Definition: Wigner's Semicircle Distribution

If a random variable X is semicircle distributed with radius $R \in \mathbb{R}^+$, then we say $X \sim \mathrm{SC}(R)$. X has the following probability density function:

$$\mathbf{P}(X = x) = \frac{2}{\pi R^2}\sqrt{R^2 - x^2} \text{ for } x \in [-R, R]$$

Wigner's Semicircle Distribution



Example: Real Normal Symmetric Matrices

In this section, we will be carefully analyzing an ensemble of symmetric matrices to showcase the special properties of symmetric and hermitian matrices. Namely, we will be considering an ensemble $\mathcal{E} \sim \mathcal{N}(0,1)^{\dagger}$ of 15×15 matrices over $\mathbb{R}$.

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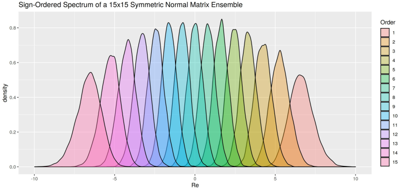
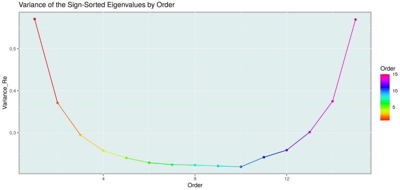
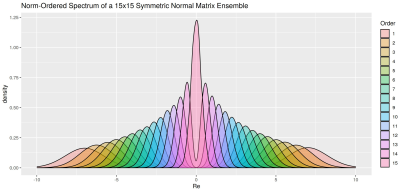
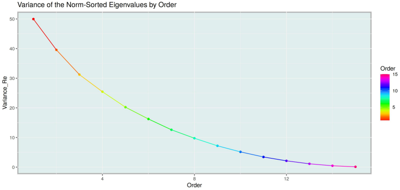
Example: Real Normal Symmetric Matrices



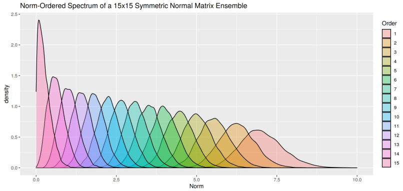


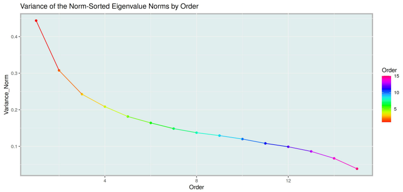
Example: Real Normal Symmetric Matrices





Example: Real Normal Symmetric Matrices





A Survey Of Spectra

Uniform Ensemble Spectra

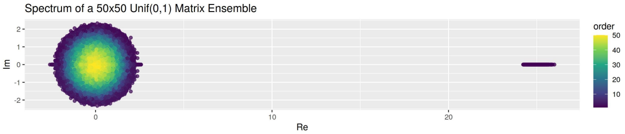


Figure: Spectrum of a Uniform(0,1) Matrix ensemble

Stochastic Matrix Ensemble Spectra

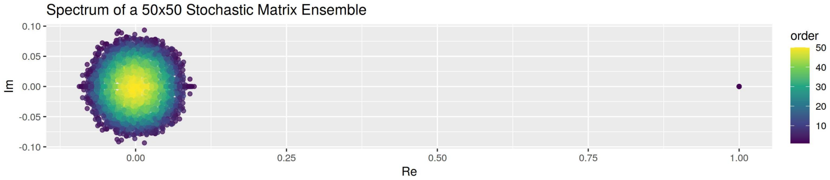


Figure: Spectrum of a Stochastic Matrix ensemble

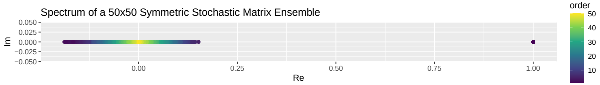
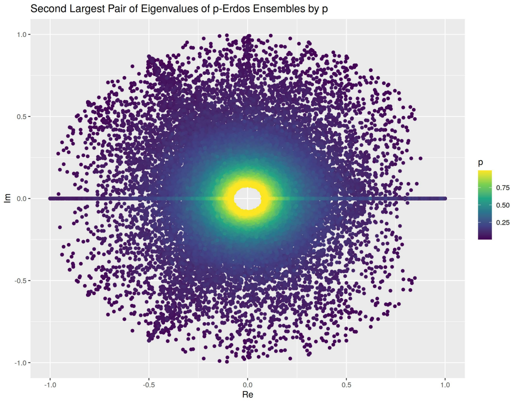


Figure: Spectrum of a Symmetric Stochastic Matrix ensemble

Erdos-Renyi Ensemble Spectra



Dispersions

Introduction

We will study the spacings between the eigenvalues. We will denote that as a dispersion of a random matrix (ensemble). However, we must formalized what a spacing **is** and **which** eigenvalue pairs to consider. To do so, we formalize the notions of dispersion metric and pairing schema respectively.

Eigenvalue Pairs

Eigenvalue Pair

Suppose P is a matrix and $\sigma(P)$ is its ordered spectrum. Then, an eigenvalue pair with respect to this ordered spectrum is denoted π_{ij}. It is defined as the ordered pair $\pi_{ij} = (\lambda_i, \lambda_j)$.

Consecutive Pair

Suppose P is a matrix and $\sigma(P)$ is its ordered spectrum. Then, let $\tilde{\pi}_j$ denote a pair of consecutive eigenvalues, the largest of the two being the j^{th} largest eigenvalue. So, $\tilde{\pi}_j = (\lambda_{j-1}, \lambda_j)$.

Dispersion Metric

Definition: Dispersion Metric

A dispersion metric $\delta : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^+$ is defined as a function from the space of pairs of complex numbers to the positive reals. In simple terms, it is a way of measuring “space” between two complex numbers - our eigenvalues.

Selected Dispersion Metrics

Table of Dispersion Metrics				
Metric*	Notation	Formula	Symmetric	Parameters
Standard Norm	δ_n	$\lvert z' - z \rvert$	True	-
β-Norm	δ_β	$\lvert z' - z \rvert^\beta$	True	$\beta \in \mathbb{N}$
Diff. of Absolutes	δ_{abs}	$\lvert z' \rvert - \lvert z \rvert$	False	-
Identity Difference	δ_{id}	$z' - z$	False	-

Pairing Scheme

Definition: Pairing Scheme

Suppose P is any $N \times N$ matrix and $\sigma^{(2)}(P)$ are its spectral pairs. Then, a pairing scheme is a subset of indices $\Pi = \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{N}_N\}$ such that taking the spectral pairs of P returns a specified subset of its eigenvalue pairs. It is denoted $\sigma^{(2)}(P \mid \Pi) = \{(\lambda_\alpha, \lambda_\beta) \mid (\alpha, \beta) \in \Pi\}$.

Selected Pairing Schema

1. Let Π_C be the consecutive pairs of eigenvalues in a spectrum.

$$\sigma^{(2)}(P \mid \Pi_C) = \{\tilde{\pi}_j = (\lambda_{j+1}, \lambda_j)\}_{j=1}^{N-1}$$

2. Let $\Pi_>$ be the lower-pair combinations of ordered eigenvalues.

$$\sigma^{(2)}(P \mid \Pi_>) = \{\pi_{ij} = (\lambda_i, \lambda_j) \mid i > j\}_{i=1}^{N-1}$$

Dispersions

Definition: Dispersion

Suppose P is an $N \times N$ matrix, and $\sigma^{(2)}(P)$ are its spectral pairs. The dispersion of P with respect to the pairing scheme Π and dispersion metric δ_M is denoted by $\Delta_M(P \mid \Pi)$ and it is given by the following:

$$\Delta_M(P \mid \Pi) = \{\delta_M(\pi_{ij}) \mid \pi_{ij} \in \sigma^{(2)}(P \mid \Pi)\}$$

Dispersions

Definition: Ensemble Dispersion

If we have an ensemble $\mathcal{E}$, then we can naturally extend the definition of $\Delta_M(\mathcal{E} \mid \Pi)$. To take the dispersion of an ensemble, simply take the union of the dispersions of each of its matrices. In other words, if $\mathcal{E} = \{P_i \sim \mathcal{D}\}_{i=1}^{K}$, then its dispersion is given by:

$$\Delta_M(\mathcal{E} \mid \Pi) = \bigcup_{i=1}^{K} \Delta_M(P_i \mid \Pi)$$

Wigner's Surmise

Wigner's Surmise

Wigner's surmise is a result found by Eugene Wigner regarding the limiting distribution of eigenvalue spacings of for symmetric matrices. To start talking about this, we must talk about normalized spacings, which are the precise items considered in the distribution. Before, we can talk about the normalized spacing, we define the mean spacing.

Definition: Mean Spacing

Suppose P is an $N \times N$ symmetric matrix, and $\sigma(P)$ are its real, sign-ordered eigenvalues. Then, the mean (eigenvalue) spacing, denoted $\langle s \rangle$ is the average distance between two consecutive eigenvalues. That is,

$$\langle s \rangle = \mathbb{E}[\Delta_\delta(P \mid \Pi_C)] = \mathbb{E}[\delta(\tilde{\pi}_j)]_{j=1}^{N-1}$$

Wigner's Surmise

So, with the mean spacing defined, we now define the normalized spacing between a pair of consecutive eigenvalues below.

Definition: Normalized Spacing

Suppose P is an $N \times N$ symmetric matrix, and $\sigma(P)$ are its real, sign-ordered eigenvalues. Then, the normalized spacing of the j^{th} pair of eigenvalues, denoted s_j is given by the following formula.

$$s_j = \frac{(\lambda_j - \lambda_{j+1})}{\langle s \rangle} = \frac{\delta(\tilde{\pi}_j)}{\langle s \rangle}$$

Wigner Dispersion

Finally, we define the Wigner dispersion, which we may reconstruct using our notation. This way, we can formalize Wigner's Surmise as an observation of the Wigner dispersion for symmetric matrices.

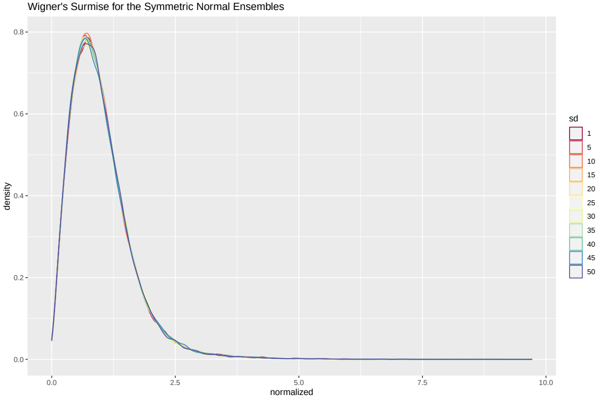
Definition: Wigner Dispersion

Suppose P is an $N \times N$ symmetric matrix, and $\sigma(P)$ are its real, sign-ordered eigenvalues. Then, the Wigner dispersion denoted $\Delta_W(P)$ is given by the set of normalized consecutive eigenvalues of P. That is,

$$\Delta_W(P) = \left\{ \frac{\delta_n(\pi)}{\langle s \rangle} \mid \pi \in \sigma^{(2)}(P \mid \Pi_C) \right\}$$

The extension for ensembles is trivial; it inherits the same notation for matrices and the definition is extended similar to how we did so for the spectrum and dispersion of an ensemble.

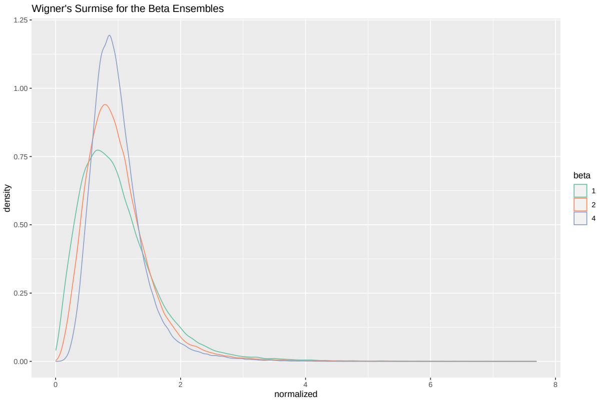
Wigner Dispersion: Symmetric Matrices



Wigner Dispersion: Symmetric Matrices

How can we vary the distributions? σ has no impact... Answer: β-ensembles.

Wigner Dispersion: β-ensembles



β-ensembles

β-ensembles

Definition: β-ensembles

A (Hermite) β-ensemble is an ensemble of random matrices parameterized by β, which determines the joint eigenvalue p.d.f that characterizes it. So, given an observed set of eigenvalues $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$. Then, the joint p.d.f. of Λ is as follows:

$$f_\beta(\Lambda) = C_\beta \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{1}{2}\sum_{i=1}^N \lambda_i^2}$$

where the normalization constant C_β is given by:

$$C_\beta = (2\pi)^{-n/2} \prod_{j=1}^{n} \frac{\Gamma(1+\frac{\beta}{2})}{\Gamma(1+\frac{\beta}{2}j)}$$

β-ensembles: Breaking it Down

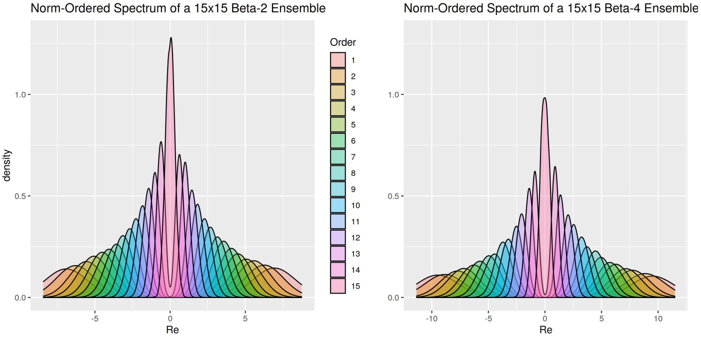
$$f_\beta(\Lambda) = C_\beta \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{1}{2}\sum_{i=1}^N \lambda_i^2}$$

β-ensembles: Breaking it Down

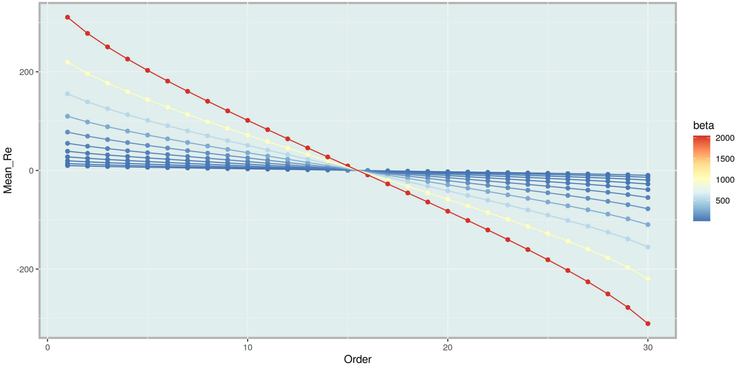
Altogether, here is what we can say about the β-ensemble joint eigenvalue p.d.f just from observing the terms.

1. When λ_i is large, then $\mathbf{P}(\Lambda)$ is small.
2. When $\delta(\lambda_i, \lambda_j)$ is small, then $\mathbf{P}(\Lambda)$ is small.

Spectra: Standard Beta Ensembles



Spectra: Extending the Beta Ensembles



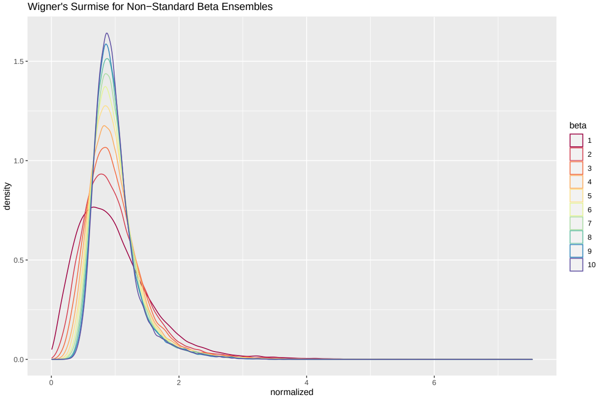
Charged Particle Model

Alongside these great algebraic properies, the β-ensembles also have a intepretation as a physical model. This is because as mentioned previously, these ensembles show up frequently in statistical physics. So, we will cover one physical model that the β-ensemble represents.
Suppose that $P \sim \mathcal{H}(\beta)$ is an $N \times N$ matrix. Then, the eigenvalues of P have a representation as a model of charged point particles.

Charged Particle Model

Low Replsuion, High Temperature. As $\beta \to 0$, the temperature of the system $T \to \infty$. At these values, the model starts to behave like an ideal gas. This returns a **fully stochastic** model of particles trying to align themselves along the field potential.
High Repulsion, Low Temperature. As $\beta \to \infty$, the temperature of the system $T \to 0$. At these values, the model loses its stochastic properties and starts to become **deterministic**. This returns a fully deterministic* model of particles that align themselves equidistantly.

Wigner's Surmise



Conclusion

In the end, we simulated and verified some results, found a few interesting patterns and phenomena.