Izvestiya: *Mathematics* **79**:1 *Izvestiya RAN*: *Ser. Mat.* **79**:1 217–224

DOI

Localization of eigenvalues and estimation of the spread for complex matrices

J. Wu, D. Wang, and J. Zhao

Abstract. For a given complex matrix we describe new methods for localizing the eigenvalues and new upper bounds for the spread. These methods and upper bounds are sharper than those previously known.

Keywords: eigenvalues, localization, spread, upper bounds, new methods.

§ 1. Introduction

Throughout the paper, $\mathbb{C}^{n\times n}$ stands for the set of complex $n\times n$ matrices. We denote the trace of any matrix $A\in\mathbb{C}^{n\times n}$ by $\operatorname{tr} A$, the Euclidean norm by $\|A\|$, and the conjugate transpose by A^*. The *spread* of a matrix $A\in\mathbb{C}^{n\times n}$ was defined by Mirsky [1] in 1956 by the formula

$$s(A)=\max_{i,j}|\lambda_i-\lambda_j|,$$

where λ_i, $i=1,2,\dots,n$, are the eigenvalues of A. Mirsky [1], [2] was the first to study the properties of spread. He obtained many interesting results, including upper and lower bounds. For example, he showed that

$$s(A)\leqslant\biggl\{2\|A\|^2-\frac{2}{n}|\operatorname{tr} A|^2\biggr\}^{1/2}.$$

Various bounds (upper and lower) for the spread of complex matrices of any order were later obtained in [3]–[5]. These results prompted research into the spread of complex matrices.

Here we take this matter a little further. In § 2 we combine algebraic and geometric methods to obtain upper bounds for $s(A)$. Our governing thought is that if all the eigenvalues of a given matrix lie in a specific disc or rectangle in the complex plane, then the spread of the matrix does not exceed the diameter of the disc or the diagonal of the rectangle. Thus one can use methods for localizing eigenvalues to obtain upper bounds for $s(A)$. We shall exhibit various new forms of upper bounds for $s(A)$ which are sharper than those previously known.

This paper was written with the support of the Chinese National Foundation for Natural Sciences (grant no. 70872123).

AMS 2010 *Mathematics Subject Classification.* 15A18, 65F15.

For brevity we write $[A,B]=AB-BA$. Every matrix $M\in\mathbb{C}^{n\times n}$ can be written in the form

$$M=\begin{bmatrix} A_{k\times k} & B_{k\times(n-k)}\\ C_{(n-k)\times k} & D_{(n-k)\times(n-k)}\end{bmatrix}.$$

We associate with it the following four matrix-valued functions:

$$M(x)=\begin{bmatrix} A_{k\times k} & xB_{k\times(n-k)}\\ x^{-1}C_{(n-k)\times k} & D_{(n-k)\times(n-k)}\end{bmatrix},$$

$$\Delta_M(k,x)=\|M\|^2-\big((1-x^2)\|B_{k\times(n-k)}\|^2+(1-x^{-2})\|C_{(n-k)\times k}\|^2\big)-\frac{|\operatorname{tr}M|^2}{n},$$

$$f_M(k,x)=\left(\big(\Delta_M(k,x)\big)^2-\frac{1}{2}\|[M(x),M(x)^*]\|^2\right)^{1/2}+\frac{|\operatorname{tr}M|^2}{n},$$

$$g_M(k,x)=\Delta_M(k,x)-\big(|x|\,\|B_{k\times(n-k)}\|-|x^{-1}|\,\|C_{(n-k)\times k}\|\big)^2+\frac{|\operatorname{tr}M|^2}{n},$$

where $A_{k\times k}$ is the principal $(k\times k)$-submatrix of M for $1\leqslant k\leqslant n-1$ and x is any *non-zero* real number.

§ 2. Localization of eigenvalues and estimation of the spread for complex matrices

In [6], Gu proposed a method for localizing all the eigenvalues of a complex $n\times n$ matrix in one closed disc. He proved that all the eigenvalues of any complex $n\times n$ matrix A lie in the disc

$$\left|\lambda-\frac{\operatorname{tr}A}{n}\right|\leqslant\left(\frac{n-1}{n}\left(\|A\|^2-\frac{|\operatorname{tr}A|^2}{n}\right)\right)^{1/2}. \tag{2.1}$$

It follows easily that

$$s(A)\leqslant 2\left(\frac{n-1}{n}\left(\|A\|^2-\frac{1}{n}|\operatorname{tr}A|^2\right)\right)^{1/2}. \tag{2.2}$$

We note that Zou and Jiang [7] improved this result by showing that all the eigenvalues of a complex $n\times n$ matrix M lie in the disc

$$\left|\lambda-\frac{\operatorname{tr}M}{n}\right|\leqslant\min_{1\leqslant k\leqslant n-1}\left(\frac{n-1}{n}\right)^{1/2}\times\left(\|M\|^2-\frac{|\operatorname{tr}M|^2}{n}-\big(\|B_{k\times(n-k)}\|-\|C_{(n-k)\times k}\|\big)^2\right)^{1/2}. \tag{2.3}$$

This result also yields the estimate

$$s(M)\leqslant\min_{1\leqslant k\leqslant n-1}2\left(\frac{n-1}{n}\right)^{1/2}\left(\|M\|^2-\frac{|\operatorname{tr}M|^2}{n}-\big(\|B_{k\times(n-k)}\|-\|C_{(n-k)\times k}\|\big)^2\right)^{1/2}. \tag{2.4}$$

In what follows we sharpen these results. We shall find much smaller discs that contain all the eigenvalues of a given complex $n\times n$ matrix. We shall also use rectangles containing all the eigenvalues of a given complex $n \times n$ matrix to deduce various upper bounds for the spread of such matrices.

The following lemma was proved in [8].

Lemma 2.1. *Let $z_1, z_2, \ldots, z_n$ be complex numbers. Then for every i, $1 \leqslant i \leqslant n$, we have*

$$\left| z_i - \frac{1}{n} \sum_{j=1}^{n} z_j \right|^2 \leqslant \frac{n-1}{n} \left(\sum_{j=1}^{n} |z_j|^2 - \frac{1}{n} \left| \sum_{j=1}^{n} z_j \right|^2 \right).$$

Lemma 2.1 shows that any n complex numbers $z_1, z_2, \ldots, z_n$ are contained in the disc

$$\left| z - \frac{1}{n} \sum_{j=1}^{n} z_j \right| \leqslant \left(\frac{n-1}{n} \right)^{1/2} \left(\sum_{j=1}^{n} |z_j|^2 - \frac{1}{n} \left| \sum_{j=1}^{n} z_j \right|^2 \right)^{1/2}.$$

Lemma 2.2. *Let $M \in \mathbb{C}^{n \times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqslant \min(f_M, g_M), \tag{2.5}$$

where $f_M = \min_{x \neq 0} \min_{1 \leqslant k \leqslant n-1} f_M(k,x)$ and $g_M = \min_{x \neq 0} \min_{1 \leqslant k \leqslant n-1} g_M(k,x)$.

Proof. Since

$$\begin{aligned} M(x) &= \begin{bmatrix} A_{k \times k} & x B_{k \times (n-k)} \\ x^{-1} C_{(n-k) \times k} & D_{(n-k) \times (n-k)} \end{bmatrix} \\ &= \begin{bmatrix} x I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \begin{bmatrix} A_{k \times k} & B_{k \times (n-k)} \\ C_{(n-k) \times k} & D_{(n-k) \times (n-k)} \end{bmatrix} \begin{bmatrix} x^{-1} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix}, \end{aligned} \tag{2.6}$$

where I_k is the $k \times k$ identity matrix, we see that $M(x)$ is similar to M and, therefore, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $M(x)$. By Kress' inequality [9] we have

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqslant \left(\|M\|^4 - \frac{1}{2} \|[M, M^*]\|^2 \right)^{1/2}. \tag{2.7}$$

Applying (2.7) to the matrix $N = M(x) - \frac{\operatorname{tr} M}{n} I$, we get

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqslant \left(\|N\|^4 - \frac{1}{2} \|[N, N^*]\|^2 \right)^{1/2} + \frac{|\operatorname{tr} M|^2}{n}. \tag{2.8}$$

Noting that

$$\begin{aligned} \|N\|^4 &= \left(\operatorname{tr} \left(\left(M(x) - \frac{\operatorname{tr} M}{n} I \right) \left(M(x) - \frac{\operatorname{tr} M}{n} I \right)^* \right) \right)^2 \\ &= \left(\|M\|^2 - \left((1 - x^2) \|B_{k \times (n-k)}\|^2 + (1 - x^{-2}) \|C_{(n-k) \times k}\|^2 \right) - \frac{|\operatorname{tr} M|^2}{n} \right)^2 \\ &= \left(\Delta_M(k,x) \right)^2, \end{aligned}$$

$$[N, N^*] = \left[M(x) - \frac{\operatorname{tr} M}{n} I, M(x)^* - \frac{\overline{\operatorname{tr} M}}{n} I \right] = [M(x), M(x)^*],$$

we deduce that

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqslant f_M. \tag{2.9}$$

On the other hand, by Tu's inequality [10] we have

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqslant \|M\|^2 - (\|B_{k\times(n-k)}\| - \|C_{(n-k)\times k}\|)^2. \tag{2.10}$$

Applying (2.10) to the matrix $N = M(x) - \frac{\operatorname{tr} M}{n} I$, we get

$$\sum_{j=1}^{n} \left|\lambda_j - \frac{\operatorname{tr} M}{n}\right|^2 \leqslant \left\|M(x) - \frac{\operatorname{tr} M}{n}\right\|^2 - (|x|\,\|B_{k\times(n-k)}\| - |x^{-1}|\,\|C_{(n-k)\times k}\|)^2.$$

Thus we have

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqslant g_M. \tag{2.11}$$

Combining (2.9) and (2.11), we obtain (2.5). □

Remark 2.1. Since

$$\min(f_M, g_M) \leqslant \min\bigl(f_M(k,1), g_M(k,1)\bigr),$$

the inequality (2.5) is stronger than (2.8) and (2.10).

Theorem 2.1. *Let $M \in \mathbb{C}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then all the eigenvalues of M are contained in the disc*

$$\left|\lambda - \frac{\operatorname{tr} M}{n}\right| \leqslant \left(\frac{n-1}{n}\right)^{1/2} \left(\min(f_M, g_M) - \frac{|\operatorname{tr} M|^2}{n}\right)^{1/2}. \tag{2.12}$$

Proof. Taking $z_j = \lambda_j$, $j = 1, 2, \dots, n$, in Lemma 2.1, we have the following inequality for every p, $1 \leqslant p \leqslant n$:

$$\left|\lambda_p - \frac{1}{n}\sum_{j=1}^{n}\lambda_j\right|^2 \leqslant \frac{n-1}{n}\left(\sum_{j=1}^{n}|\lambda_j|^2 - \frac{1}{n}\left|\sum_{j=1}^{n}\lambda_j\right|^2\right) = \frac{n-1}{n}\left(\sum_{j=1}^{n}|\lambda_j|^2 - \frac{|\operatorname{tr} M|^2}{n}\right).$$

By Lemma 2.2 we obtain (2.12). □

One can deduce from Theorem 2.1 that

$$s(M) \leqslant 2\left(\frac{n-1}{n}\right)^{1/2}\left(\min(f_M, g_M) - \frac{|\operatorname{tr} M|^2}{n}\right)^{1/2}. \tag{2.13}$$

Theorem 2.2. *Let $M \in \mathbb{C}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then we have the following upper bound for the spread of M:*

$$s(M) \leqslant \sqrt{2}\left(\min(f_M, g_M) - \frac{|\operatorname{tr} M|^2}{n}\right)^{1/2}. \tag{2.14}$$

Proof. We first consider the case when $\operatorname{tr} M = 0$. For all $k \neq l$, $1 \leqslant k, l \leqslant n$, we write

$$\begin{aligned}\sum_{j=1}^{n} |\lambda_j|^2 &= \sum_{j=1}^{n} \left|\lambda_j - \frac{\lambda_k + \lambda_l}{2} + \frac{\lambda_k + \lambda_l}{2}\right|^2 \\ &= \frac{|\lambda_k - \lambda_l|^2}{2} + \sum_{j \neq k,l} \left|\lambda_j - \frac{\lambda_k + \lambda_l}{2}\right|^2 - n\left|\frac{\lambda_k + \lambda_l}{2}\right|^2.\end{aligned}$$

Noting that $\sum_{j\neq k,l}\left|\lambda_j - \frac{\lambda_k+\lambda_l}{2}\right|^2 \geqslant n\left|\frac{\lambda_k+\lambda_l}{2}\right|^2$, we have

$$\sum_{j=1}^{n} |\lambda_j|^2 \geqslant \frac{|\lambda_k - \lambda_l|^2}{2}. \tag{2.15}$$

If $\operatorname{tr} M \neq 0$, then inequality (2.15) for the matrix $M(x) - \frac{\operatorname{tr} M}{n} I$ yields that

$$\sum_{j=1}^{n} \left|\lambda_j - \frac{\operatorname{tr} M}{n}\right|^2 \geqslant \frac{|\lambda_k - \lambda_l|^2}{2}. \tag{2.16}$$

It follows from (2.15) and (2.16) that (2.16) holds for all matrices M, whether or not $\operatorname{tr} M = 0$. Thus,

$$s(M) = \max_{k\neq l} |\lambda_k - \lambda_l| \leqslant \sqrt{2}\sqrt{\sum_{j=1}^{n} |\lambda_j|^2 - \frac{|\operatorname{tr} M|^2}{n}}. \tag{2.17}$$

By Lemma 2.2 we get the desired result. $\square$

Note that Sharma and Kumar [11] gave the following upper bound for the spread of any matrix $M \in \mathbb{C}^{n\times n}$:

$$s(M) \leqslant \sqrt{2}\left[\left(\|M\|^2 - \frac{|\operatorname{tr} M|^2}{n}\right)^2 - \frac{1}{2}\|[M, M^*]\|^2\right]^{1/4}. \tag{2.18}$$

Furthermore, if M is partitioned into blocks of the form

$$M = \begin{bmatrix} A_{k\times k} & B_{k\times(n-k)} \\ C_{(n-k)\times k} & D_{(n-k)\times(n-k)} \end{bmatrix},$$

then

$$s(M) \leqslant \sqrt{2}\left[\|M\|^2 - \frac{|\operatorname{tr} M|^2}{n} - (\|B_{k\times(n-k)}\| - \|C_{(n-k)\times k}\|)^2\right]^{1/2}. \tag{2.19}$$

Remark 2.2. It follows from Remark 2.1 that (2.14) is stronger than (2.18) and (2.19).

Remark 2.3. If $n \geqslant 3$, then $2\left(\frac{n-1}{n}\right)^{1/2} > \sqrt{2}$. Therefore (2.14) is stronger than (2.13).

Theorem 2.3. *Let $M \in \mathbb{C}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then all the eigenvalues of M lie in the disc*

$$\left|\lambda - \frac{\operatorname{tr} M}{n}\right| \leqslant \left(\frac{n-1}{2n}\right)^{1/2}\left(\min(f_M, g_M) - \frac{|\operatorname{tr} M|^2}{n} + \left|\operatorname{tr} M^2 - \frac{\operatorname{tr}^2 M}{n}\right|\right)^{1/2}. \tag{2.20}$$

Proof. We first consider the case when $\operatorname{tr} M = 0$. Put $z_j = \operatorname{Re}(e^{i\theta}\lambda_j)$, where $\theta = \arg \overline{\lambda}_p$ and $j, p = 1, 2, \ldots, n$. By Lemma 2.1, for every p, $1 \leqslant p \leqslant n$, we have

$$|\lambda_p| = |z_p| = \left| z_p - \frac{1}{n}\sum_{j=1}^{n} \operatorname{Re}(e^{i\theta}\lambda_j) \right| \leqslant \left(\frac{n-1}{n}\right)^{1/2} \left(\sum_{j=1}^{n} \bigl(\operatorname{Re}(e^{i\theta}\lambda_j)\bigr)^2 \right)^{1/2}. \tag{2.21}$$

The following equalities are easily verified:

$$\begin{aligned}\sum_{j=1}^{n} \bigl(\operatorname{Re}(e^{i\theta}\lambda_j)\bigr)^2 &= \frac{1}{2}\sum_{j=1}^{n} \bigl(\sqrt{2}\operatorname{Re}\lambda_j \cos\theta - \sqrt{2}\operatorname{Im}\lambda_j \sin\theta\bigr)^2 \\ &= \frac{1}{2}\sum_{j=1}^{n} \bigl(|\lambda_j|^2 + \operatorname{Re}(e^{2i\theta}\lambda_j^2)\bigr) = \frac{1}{2}\sum_{j=1}^{n} |\lambda_j|^2 + \frac{1}{2}\operatorname{Re}(e^{2i\theta}\operatorname{tr} M^2).\end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22), we get

$$|\lambda_p|^2 \leqslant \left(\frac{n-1}{2n}\right)\left(\sum_{j=1}^{n} |\lambda_j|^2 + |\operatorname{tr} M^2| \right). \tag{2.23}$$

If $\operatorname{tr} M \neq 0$, then inequality (2.23) for the matrix $M(x) - \frac{\operatorname{tr} M}{n} I$ yields that

$$\left| \lambda_p - \frac{\operatorname{tr} M}{n} \right|^2 \leqslant \left(\frac{n-1}{2n}\right)\left(\sum_{j=1}^{n} |\lambda_j|^2 - \frac{|\operatorname{tr} M|^2}{n} + \left| \operatorname{tr} M^2 - \frac{|\operatorname{tr} M|^2}{n} \right| \right). \tag{2.24}$$

It follows from (2.23) and (2.24) that (2.24) holds for all matrices M, whether or not $\operatorname{tr} M = 0$. Applying Lemma 2.2, we complete the proof of the theorem. □

Note that the following inequality holds:

$$\begin{aligned}\left| \operatorname{tr} M^2 - \frac{\operatorname{tr}^2 M}{n} \right| &= \left| \sum_{j=1}^{n} \lambda_j^2 - \frac{\operatorname{tr}^2 M}{n} \right| = \left| \sum_{j=1}^{n} \left(\lambda_j - \frac{\operatorname{tr} M}{n} \right)^2 \right| \\ &\leqslant \sum_{j=1}^{n} \left| \lambda_j - \frac{\operatorname{tr} M}{n} \right|^2 = \sum_{j=1}^{n} |\lambda_j|^2 - \frac{|\operatorname{tr} M|^2}{n}.\end{aligned}$$

Therefore (2.20) is stronger than (2.12).

We deduce from Theorem 2.3 that

$$s(M) \leqslant \min_{x \neq 0} \min_{1 \leqslant k \leqslant n-1} 2\left(\frac{n-1}{2n}\right)^{1/2} \left(\min(f_M, g_M) - \frac{|\operatorname{tr} M|^2}{n} + \left| \operatorname{tr} M^2 - \frac{\operatorname{tr}^2 M}{n} \right| \right)^{1/2}. \tag{2.25}$$

Corollary 2.1. *Let $M \in \mathbb{C}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the spectral radius $\rho(M)$ satisfies the inequality*

$$\rho(M) \leqslant \left(\frac{n-1}{2n}\right)^{1/2} \left(\min(f_M, g_M) - \frac{|\operatorname{tr} M|^2}{n} + \left| \operatorname{tr} M^2 - \frac{\operatorname{tr}^2 M}{n} \right| \right)^{1/2} + \frac{|\operatorname{tr} M|}{n}.$$

Theorem 2.4. *Let $M \in \mathbb{C}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then all the eigenvalues of M lie in the rectangle*

$$\left[\frac{\operatorname{Re}(\operatorname{tr} M)}{n} - \alpha, \frac{\operatorname{Re}(\operatorname{tr} M)}{n} + \alpha\right] \times \left[\frac{\operatorname{Im}(\operatorname{tr} M)}{n} - \beta, \frac{\operatorname{Im}(\operatorname{tr} M)}{n} + \beta\right],$$

where

$$\alpha = \left(\frac{n-1}{n}\right)^{1/2}\left(\frac{1}{2}(\min(f_M, g_M) + \operatorname{Re}(\operatorname{tr} M^2)) - \frac{(\operatorname{Re}(\operatorname{tr} M))^2}{n}\right)^{1/2},$$

$$\beta = \left(\frac{n-1}{n}\right)^{1/2}\left(\frac{1}{2}(\min(f_M, g_M) - \operatorname{Re}(\operatorname{tr} M^2)) - \frac{(\operatorname{Im}(\operatorname{tr} M))^2}{n}\right)^{1/2}.$$

Proof. Putting $z_j = \operatorname{Re}\lambda_j$ and then $z_j = \operatorname{Im}\lambda_j$, $j = 1, 2, \ldots, n$, in Lemma 2.1, we have

$$\left|\operatorname{Re}\lambda_j - \frac{\operatorname{Re}(\operatorname{tr} M)}{n}\right| \leqslant \left(\frac{n-1}{n}\right)^{1/2}\left(\sum_{j=1}^{n}(\operatorname{Re}\lambda_j)^2 - \frac{(\operatorname{Re}(\operatorname{tr} M))^2}{n}\right)^{1/2},$$

$$\left|\operatorname{Im}\lambda_j - \frac{\operatorname{Im}(\operatorname{tr} M)}{n}\right| \leqslant \left(\frac{n-1}{n}\right)^{1/2}\left(\sum_{j=1}^{n}(\operatorname{Im}\lambda_j)^2 - \frac{(\operatorname{Im}(\operatorname{tr} M))^2}{n}\right)^{1/2}.$$

Noting that

$$\sum_{j=1}^{n}(\operatorname{Re}\lambda_j)^2 = \frac{1}{2}\left(\sum_{j=1}^{n}|\lambda_j|^2 + \operatorname{Re}(\operatorname{tr} M^2)\right),$$

$$\sum_{j=1}^{n}(\operatorname{Im}\lambda_j)^2 = \frac{1}{2}\left(\sum_{j=1}^{n}|\lambda_j|^2 - \operatorname{Re}(\operatorname{tr} M^2)\right),$$

we deduce that

$$\left|\operatorname{Re}\lambda_j - \frac{\operatorname{Re}(\operatorname{tr} M)}{n}\right| \leqslant \left(\frac{n-1}{n}\right)^{1/2} \times \left(\frac{1}{2}\left(\sum_{j=1}^{n}|\lambda_j|^2 + \operatorname{Re}(\operatorname{tr} M^2)\right) - \frac{(\operatorname{Re}(\operatorname{tr} M))^2}{n}\right)^{1/2},$$

$$\left|\operatorname{Im}\lambda_j - \frac{\operatorname{Im}(\operatorname{tr} M)}{n}\right| \leqslant \left(\frac{n-1}{n}\right)^{1/2} \times \left(\frac{1}{2}\left(\sum_{j=1}^{n}|\lambda_j|^2 - \operatorname{Re}(\operatorname{tr} M^2)\right) - \frac{(\operatorname{Im}(\operatorname{tr} M))^2}{n}\right)^{1/2}.$$

Using Lemma 2.2, we complete the proof. □

It follows from Theorem 2.4 that

$$s(M) \leqslant 2(\alpha^2 + \beta^2)^{1/2}. \tag{2.26}$$

The authors are grateful to the anonymous referees for their valuable comments and suggestions for improving the paper.

Bibliography

[1] L. Mirsky, “The spread of a matrix”, *Mathematika* **3**:2 (1956), 127–130.

[2] L. Mirsky, “Inequalities for normal and Hermitian matrices”, *Duke Math. J.* **24**:4 (1957), 591–599.

[3] E. Deutsch, “On the spread of matrices and polynomials”, *Linear Algebra Appl.* **22** (1978), 49–55.

[4] P. R. Beesack, “The spread of matrices and polynomials”, *Linear Algebra Appl.* **31** (1980), 145–149.

[5] Bo Xun Tu, “On the spread of a matrix”, *Fudan Xuebao* **23**:4 (1984), 435–442. (Chinese)

[6] Yi Xi Gu, “The distribution of eigenvalues of a matrix”, *Acta Math. Appl. Sinica* **17**:4 (1994), 501–511.

[7] Limin Zou and Youyi Jiang, “Estimation of the eigenvalues and the smallest singular value of matrices”, *Linear Algebra Appl.* **433**:6 (2010), 1203–1211.

[8] Ó. Rojo, R. L. Soto, H. Rojo, and Tin Yau Tam, “Eigenvalue localization for complex matrices”, *Proyecciones* **11**:1 (1992), 11–19.

[9] R. Kress, H. L. de Vries, and R. Wegmann, “On nonnormal matrices”, *Linear Algebra Appl.* **8**:2 (1974), 109–120.

[10] Bo Xun Tu, “A lower bound for the rank of a matrix and sufficient conditions for nonsingularity of a matrix. I”, *Fudan Xuebao* **21**:4 (1982), 416–422. (Chinese)

[11] R. Sharma and R. Kumar, “Remark on upper bounds for the spread of a matrix”, *Linear Algebra Appl.* **438**:11 (2013), 4359–4362.

[12] H. Wolkowicz and G. P. H. Styan, “Bounds for eigenvalues using traces”, *Linear Algebra Appl.* **29** (1980), 471–506.

[13] H. Wolkowicz and G. P. H. Styan, “More bounds for eigenvalues using traces”, *Linear Algebra Appl.* **31** (1980), 1–17.

[14] J. L. Wu, “Distribution and estimation for eigenvalues of real quaternion matrices”, *Comput. Math. Appl.* **55**:9 (2008), 1998–2004.

[15] Junliang Wu, Limin Zou, Xiangping Chen, and Shengjie Li, “The estimation of eigenvalues of sum, difference, and tensor product of matrices over quaternion division algebra”, *Linear Algebra Appl.* **428**:11-12 (2008), 3023–3033.

[16] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge Univ. Press, Cambridge 1985; Russian transl., Mir, Moscow 1989.

[17] R. A. Horn and C. R. Johnson, *Topics in matrix analysis*, Cambridge Univ. Press, Cambridge 1991.

[18] R. S. Varga, *Geršgorin and his circles*, Springer Ser. Comput. Math., vol. 36, Springer-Verlag, Berlin 2004.

Junliang Wu
College of Mathematics and Statistics,
Chongqing University, China
E-mail: `jlwu678@163.com`

Received 26/JUL/13
12/FEB/14
Edited by A. V. DOMRIN

Dafei Wang
College of Mathematics and Statistics,
Chongqing University, China

Jianguo Zhao
College of Mathematics and Statistics,
Chongqing University, China