On Spectra and Dispersions

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Definitions

Definition 0.1 (Spectrum)**.** *Given a matrix P, the spectrum of P is defined as the ordered multiset of its eigenvalues, denoted by $\sigma(P) = \{\lambda_i\}_{i=1}^n$ and $\lambda_1 \leq \cdots \leq \lambda_n$.*

Definition 0.2 (Hermite-Gaussian β-ensemble)**.** *The Hermite β-ensemble is the ensemble of random matrices whose eigenvalues have the joint probability density function:*

$$f_\beta(\lambda) = c_H^\beta \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-1/2 \sum_i \lambda_i^2}$$

where the normalization constant c_H^β is given by:

$$c_H^\beta = (2\pi)^{-n/2} \prod_{j=1}^{n} \frac{\Gamma(1+\beta/2)}{\Gamma(1+\beta j/2)}$$

They represent a matrix whose entries have β real number components.

Definition 0.3 (Ranking Delta)**.** *The ranking delta is a function $\delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which takes the index of two eigenvalues (from an **ordered** spectrum) and returns their difference. In other words, $\delta : (\lambda_i, \lambda_j) \mapsto (i-j)$.*

With the function δ, we may take the set of unique eigenvalue pairs $(i > j)$ and partition it into equivalence classes. To achieve this, we define the equivalence relation $\sim_\delta$ which says $(\lambda_m, \lambda_n) \sim_\delta (\lambda_p, \lambda_q) \iff (m-n) = (p-q)$. These equivalence classes then naturally corrospond to pairs a set distance $\rho = i - j$ apart. So, for a $N \times N$ matrix, δ assumes a range $\rho \in \{1, \ldots, N-1\}$.

In summary, $\sim_\delta$ takes the set $\{(\lambda_i, \lambda_j) \mid \lambda_i, \lambda_j \in \sigma(P) \text{ and } i > j\}$ and surjectively partitions it onto the equivalence classes $[(\lambda_i, \lambda_j)]_\rho$ for $\rho \in \{1, \ldots, N-1\}$. Note that the sizes of each equivalence class are **never equal**. With this partition in mind, we consider the eigenvalue dispersions under each of those equivalence classes.

Definition 0.4 (Dispersion)**.** *Define the dispersion of a pair of two eigenvalues λ_i, λ_j to be the output of a function that maps their difference to $\mathbb{R}$ using some norm (metric of distance) $|\cdot|^*$. More concisely, a dispersion metric is some function $\Delta : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ such that $\Delta : (\lambda_i, \lambda_j) \mapsto |\lambda_j - \lambda_i|^*$.*

One usual metric is the standard norm $|\cdot|$. As seen in the density of the Hermite-Gaussian β ensembles, another norm worth worth noting is the β-norm: $|\cdot|^\beta$ for $\beta \in \mathbb{N}$.

Spectral Statistics

We consider the following spectral statistics:

1. $\mathbb{E}(\Delta \mid \rho)$
2. $\text{Var}(\Delta \mid \rho)$.