Expressing a Polynomial as the Characteristic Polynomial of a Symmetric Matrix

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ABSTRACT

We show that given a polynomial, one can (without knowing the roots) construct a symmetric matrix whose characteristic polynomial is the given polynomial appropriately normed. For a real polynomial, the matrix can have purely imaginary off-diagonal entries. For a polynomial with only real distinct roots, the matrix can be chosen real.

1. INTRODUCTION

One of the best-known theorems from linear algebra states that the characteristic polynomial of a real symmetric matrix has all roots real. For many years, the author has thought about a formulation of a converse theorem, namely about an easy construction of a real symmetric matrix for which the given normed polynomial with real roots only would be the characteristic polynomial.

In this note, a simple solution to this problem using interpolation is given. It came out essentially as a consequence of the following more general theorem proved in [2] (slightly reworded):

THEOREM 1. *Let $b_1,\dots,b_n$ be distinct numbers, $n \geqslant 1$; let $v(x) = \prod_{k=1}^{n}(x - b_k)$. Suppose that $u(x)$ and $w(x)$ are polynomials of degree at most n*

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such that $u(b_k) \neq 0$ *for* $k = 1, \dots, n$. *Define n-by-n matrices* A, C, D *by*

$$A = \operatorname{diag}\left(-\frac{w(b_k)}{u(b_k)} \right),$$

$$C = (c_{ik}), \qquad i,k = 1,\dots,n,$$

where

$$c_{ik} = \frac{\dfrac{w(b_i)}{u(b_i)} - \dfrac{w(b_k)}{u(b_k)}}{b_i - b_k} \qquad \text{if} \quad i \neq k,$$

$$c_{ii} = \left(\frac{w(t)}{u(t)} \right)'_{t=b_i}, \qquad i,k = 1,\dots,n,$$

$$d = \operatorname{diag}(d_k),$$

where d_k *satisfies for a fixed* $\sigma \neq 0$

$$\sigma v'(b_k) d_k^2 - u(b_k) = 0, \qquad k = 1,\dots,n.$$

Then, for any t *for which* $v(t) \neq 0$, *the number* $-w(t)/v(t)$ *is an eigenvalue of the matrix*

$$\frac{u(t)}{v(t)} A - \sigma DCD$$

and $(d_k/(t - b_k))$ *is a corresponding eigenvector.*

REMARK. Theorem 1 is valid for numbers and polynomials over any field.

2. RESULTS

Observe that choosing t_0 as a root of $u(t)$ in Theorem 1, the matrix $-\sigma DCD$ will have an eigenvalue $-w(t_0)/v(t_0)$, and this number will be equal to t_0 if, for the monic polynomial $u(x)$, $w(x)$ is chosen as

$$w(x) = x[u(x) - v(x)].$$

This suggests the following theorem, a simple, direct proof of which will be supplied.

THEOREM 2. *Let $u(x)$ be a monic polynomial of degree $n \geqslant 1$; let $b_1, \ldots, b_n$ be distinct numbers such that $u(b_k) \neq 0$ for $k = 1, \ldots, n$. Set $v(x) = \prod_{k=1}^{n}(x - b_k)$, and define the n-by-n matrix $A = (a_{ik})$ by*

$$\begin{aligned} a_{ik} &= -\sigma d_i d_k \qquad \text{if} \quad i \neq k, \\ a_{kk} &= b_k - \sigma d_k^2, \qquad i,k = 1, \ldots, n, \end{aligned} \tag{1}$$

where σ is a fixed nonzero number and d_k is a root of

$$\sigma v'(b_k) d_k^2 - u(b_k) = 0. \tag{2}$$

Then $(-1)^n u(x)$ is the characteristic polynomial of the symmetric matrix A. If λ is an eigenvalue of A, then $(d_i/(\lambda - b_i))$ is a corresponding eigenvector.

If $u(x)$ is a polynomial with n real distinct roots, then choosing b_i as any real numbers interlacing the roots of A, one can choose σ as $+1$ or -1 in such a way that d_i are all real and A thus real symmetric.

Proof. Since A is the sum of the matrix $\operatorname{diag}(b_i)$ and a matrix of rank one,

$$\det(A - \lambda I) = \prod_{k=1}^{n} (b_k - \lambda) - \sigma \sum_{k=1}^{n} d_k^2 \prod_{\substack{j=1 \\ j \neq k}}^{n} (b_j - \lambda). \tag{3}$$

Write this polynomial as $(-1)^n R(\lambda)$. Then $R(\lambda)$ is monic and satisfies $R(b_k) = \sigma d_k^2 v'(b_k)$, which is, by (2), equal to $u(b_k)$. Thus $R(x) = u(x)$. From (3) and Theorem 1, the assertion about the eigenvector follows immediately.

Suppose now that $u(x)$ has n real distinct roots and that $b_1, \ldots, b_n$ interlace the roots of u (i.e., in every open interval between two consecutive roots of u is exactly one number b_i). Then, for an appropriate choice of $\sigma = 1$ or -1,

$$\sigma u(b_k)v'(b_k) > 0, \qquad k = 1,\dots,n.$$

Therefore, d_k is real and A is real. ■

REMARK. If u is real and all b_k's are real, then each d_k is either real or purely imaginary. The matrix A can thus be brought by a simultaneous permutation of rows and columns into the block form

$$A = \begin{pmatrix} A_{11} & iA_{12} \\ iA_{12}^T & A_{22} \end{pmatrix},$$

where all the matrices A_{ik} are real (and A_{12} has rank one). This matrix is, of course, similar to the real matrix

$$\tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ -A_{12}^T & A_{22} \end{pmatrix}.$$

One can obtain an even simpler solution to the mentioned problem by letting one of the numbers b_k, say b_n, tend to infinity. The result is:

THEOREM 3. *Let $u(x)$ be a polynomial of degree n, $n \geqslant 1$, of the form*

$$u(x) = x^n + px^{n-1} + r(x),$$

$r(x)$ of degree less than $n-1$ or zero. Let $b_1,\dots,b_{n-1}$ be distinct numbers such that $u(b_k) \neq 0$, $k = 1,\dots,n-1$. Define $v(x) = \prod_{k=1}^{n-1}(x - b_k)$, $B =$ diag(b_i). *Let $c = (c_k)$ be a column vector with c_k satisfying*

$$v'(b_k)c_k^2 + u(b_k) = 0, \qquad k = 1,\dots,n-1. \tag{4}$$

Then the symmetric matrix

$$A = \begin{pmatrix} B & c \\ c^T & d \end{pmatrix}, \tag{5}$$

where

$$d = -p - \sum_{k=1}^{n-1} b_k, \tag{6}$$

has characteristic polynomial $(-1)^n u(x)$.

If λ *is an eigenvalue of* A, *then* $(q_1, \ldots, q_n)^T$ *is a corresponding eigenvector, where*

$$q_k = \frac{c_k}{\lambda - b_k}, \qquad k = 1, \ldots, n-1,$$

$$q_n = 1.$$

In the case that $u(x)$ ***has only real simple roots and the*** b_k***'s interlace these roots, the matrix*** A ***is real.***

Proof. By Schur's formula,

$$\det(A - \lambda I) = \det(B - \lambda I)\left[d - \lambda - c^T(B - \lambda I)^{-1}c\right]$$

unless $v(\lambda) = 0$. Denoting the polynomial on the left-hand side by $(-1)^n R(\lambda)$, it follows that (now identically)

$$R(\lambda) = (\lambda - d)v(\lambda) - \sum_{k=1}^{n-1} c_k^2 \prod_{\substack{j=1 \\ j \neq k}}^{n-1} (\lambda - b_j).$$

The polynomial $R(\lambda)$ has thus the form

$$\lambda^n - \left(d + \sum_{k=1}^{n-1} b_k\right)\lambda^{n-1} + \cdots,$$

which is, by (6),

$$\lambda^n + p\lambda^{n-1} + \cdots$$

and satisfies

$$R(b_k) = -c_k^2 v'(b_k), \qquad k = 1, \dots, n-1.$$

Since this is $u(b_k)$ by (4), $R(x) = u(x)$. The rest is easy. ■

REMARK. As in the previous remark, if $u(x)$ is real, we can find a real (not necessarily symmetric) matrix with the characteristic polynomial $(-1)^n u(x)$ for any choice of real distinct numbers $b_1, \dots, b_{n-1}$.

3. COMMENTS

Theorems 2 and 3 can serve (a) for using matrix software to compute the roots of an algebraic equation, (b) for obtaining estimates of these roots. A general reference in this respect is the book [3]. It seems that the matrices in (1) or (5) are in many cases preferable to the usual companion matrix. This is also supported by the form of the eigenvectors. One can hope to obtain, by some sophisticated special choice of the numbers b_k, stable or even universal algorithms for solving algebraic equations (using e.g. the algorithm from [1]).

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