ODE Midterm

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Question 1. Solve the following first order differential equations:

(a) $\cos(x)dy = y(\sin(x) - y)dx$,

(b) $(2x + y - 3)dy = (x + 2y - 3)dx$.

Solution. (a) This D.E. turns into

$$y' - \tan(x)y = -\sec(x)y^2,$$

we recognize that it is a Bernoulli D.E. It follows that

$$y^{-2}y' - \tan(x)y^{-1} = -\sec(x),$$

substituting $v = y^{-1}$, we have $v' = -y^{-2}y'$ and so

$$v' + \tan(x)v = \sec(x)$$

which is a linear D.E.

Consider a function $\mu(x)$ such that $\mu'(x) = \mu(x)\tan(x)$

$$\mu(x)v' + \mu'(x)v = \mu(x)\sec(x).$$

Therefore, $\mu(x) = e^{\int \tan(x)\,dx}$ so that $\mu(x) = e^{-\ln|\cos(x)|} = \sec(x)$. Thus,

$$v(x) = \frac{\int \sec^2(x)\,dx + c}{\sec(x)} = \frac{\tan(x) + c}{\sec(x)} = \sin(x) + c\cos(x)$$

so that

$$y(x) = \frac{1}{\sin(x) + c\cos(x)}$$

where c is a constant.

(b) Let $u = x - 1$ and $v = y - 1$. Then

$$\frac{dv}{du} = \frac{u + 2v}{2u + v}$$

Assume that $z = \frac{v}{u}$, then $v' = z + uz'$; therefore

$$u\frac{dz}{du} = \frac{1 - z^2}{2 + z}.$$

It follows that

$$\int \frac{1}{1+z}\,dz - \int \frac{-1}{1-z}\,dz - \frac{1}{2}\int \frac{-2z}{1-z^2}\,dz = \int \frac{1}{u}\,du,$$

thus

$$ln|1 + z| - \ln|1 - z| - \frac{1}{2}\ln|1 - z^2| = \ln\left|\frac{1+z}{(1-z)^2}\right| = \ln|u| + C,$$

we get

$$\frac{1}{1-z} = Cu,$$

and finally

$$z = 1 - \frac{1}{Cu}.$$

Now, we rearrange the equation above

$$\frac{y-1}{x-1} = 1 - \frac{1}{C(x-1)}$$

to get

$$y(x) = x - \frac{1}{C}$$

where C is an arbitrary constant.

Question 2. Show that $y = 1-4x$ is a particular solution of $y' = y^2 + 8xy + 16x^2 - 5$, and find its general solution.

Solution. Call y, y_p; then $y_p = 1-4x$ and $y_p' = -4$. We check the validity of y_p as a solution:

$$y_p' = y_p^2 + 8xy_p + 16x^2 - 5 = (1-4x)^2 + 8x(1-4x) + 16x^2 - 5 = 1 - 8x + 16x^2 + 8x - 32x^2 + 16x^2 - 5 = -4,$$

therefore y_p is in fact a particular solution for our D.E.

Observe that $y' = y^2 + 8xy + 16x^2 - 5$ is a Riccati D.E. which can be turned into the form $(y+4x)^2 - 5$. So, let $u = y + 4x$. Since $u' = y' + 4$, we have $\frac{du}{dx} = u^2 - 1$ and so $\frac{du}{u^2-1} = dx$ which is a seperable equation.

It follows that

$$\int \frac{du}{u^2-1} = \frac{1}{2}\int \Big(\frac{1}{u-1} - \frac{1}{u+1}\Big)\, du = \frac{1}{2}\ln\Big|\frac{u-1}{u+1}\Big| = x + C,$$

so that

$$\frac{u-1}{u+1} = Ke^{2x}$$

where $K = e^{2C}$.

We have that

$$u - 1 = Ke^{2x}(u+1),$$

$$u(x) = \frac{1+K^{2x}}{1-K^{2x}},$$

thus

$$y(x) = \frac{1+K^{2x}}{1-K^{2x}} - 4x,$$

where $K \in \mathbb{R}$ is an arbitrary constant.

Question 3. Show that $y_1 = 3x^2 - 1$ is a solution. Find the general solution of

$$(1-x^2)y'' - 2xy' + 6y = 0. \, (*)$$

Solution. Note that $y_1' = 6x$ and $y_1'' = 6$. It follows that

$$(1-x^2)6 - 2x \cdot 6x + 6(3x^2 - 1) = 6 - 6x^2 - 12x^2 + 18x^2 - 6 = 0.$$

We can rewrite $(*)$ as

$$y'' - \frac{2x}{1-x^2}y' + \frac{6}{1-x^2}y = 0.$$

Now, we look for a second solution $y_2(x)$ such that

$$y_2(x) = v(x)y_1(x)$$

where $v(x)$ is a first-order D.E. Compute

$$y_2' = v'y_1 + vy_1', \qquad y_2'' = v''y_1 + 2(v'y_1') + vy_1'';$$

therefore

$$\big[v''y_1 + 2(v'y_1') + vy_1''\big] - \frac{2x}{1-x^2}\big[v'y_1 + vy_1' + 6xv\big] + \frac{6}{1-x^2}\big[vy_1\big] = 0.$$

Since y_1 itself satisfies $(*)$, all the terms proportional to v cancel out so letting $P(x) = \frac{2x}{1-x^2}$, we are left with

$$\big[v''y_1 + 2(v'y_1')\big] - P\big[v'y_1\big] = 0.$$

Set $w(x) = v'(x)$. The equation above turns into

$$w'y_1 + (y_1' - Py_1)w = 0 \implies \frac{dw}{dx} + \Big(\frac{y_1'}{y_1} - P\Big)w = 0$$

which is a first order linear D.E.

Question 4. Find an integrating factor in terms of $u = x^2 + y^2$ for the differential equation and then solve it:

$$(x^3 + xy^2 - y)dx + xdy = 0.$$

Solution. Note that $\frac{du}{dx} = 2x$ and $\frac{du}{dy} = 2y$. The equation is not exact since $\frac{\partial M}{\partial y} = 2xy - 1$ and $\frac{\partial N}{\partial x} = 1$.

$$\begin{aligned}
\frac{\mu'(u)}{\mu(u)} &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{2yM - 2xN} \\
&= \frac{1 - (2xy - 1)}{2y(x^3 + xy^2 - y) - 2x^2} \\
&= \frac{2(1 - xy)}{2(xy - 1)(x^2 + y^2)} \\
&= \frac{-1}{x^2 + y^2},
\end{aligned}$$

thus

$$\mu(u) = e^{-\int \frac{1}{u}\, du} = e^{-\ln|u|} = \frac{1}{u} = \frac{1}{x^2 + y^2}.$$

Multiplying by the integrating factor, the equation becomes:

$$\left(x - \frac{y}{x^2 + y^2}\right) dx + \frac{x}{x^2 + y^2} dy = 0.$$

Now, $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} = -\frac{x^2 - y^2}{(x^2+y^2)^2}$, confirming exactness. Integrate M' with respect to x to get

$$\frac{x^2}{2} - \arctan\left(\frac{x}{y}\right) + h(y).$$

Differentiate with respect to y and solve for $h(y)$ to find $h(y)$ is constant. Thus, the solution is:

$$\frac{x^2}{2} - \arctan\left(\frac{x}{y}\right) = C.$$

Question 5. Find the orthogonal curves of the family $y = -x - 1 + ce^x$ $(*)$.

Solution. We have $\frac{dy}{dx} = ce^x - 1$ $(\cdot)$, adding $-(*)$ and $(\cdot)$ we get

$$\frac{dy}{dx} = x + y,$$

$$\left.\frac{dy}{dx}\right|_{\text{orth}} = -\frac{1}{\frac{dy}{dx}} = -\frac{1}{x + y},$$

thus

$$\frac{dx}{dy} = -x - y$$

is orthogonal to the curves defined by $(*)$.

Then,

$$\frac{dx}{dy} + x = -y$$

which is a first order linear D.E., it follows that the integrating factor

$$\mu(y) = e^{\int 1\, dy} = e^y,$$

multiply all terms by $\mu(y)$

$$e^y \frac{dx}{dy} + e^y x = \frac{d}{dy}(xe^y) = -e^y y.$$

Integrate with respect to y

$$xe^y = -\int ye^y\, dy + K = (1 - y)\, e^y + K$$

so that

$$x = 1 - y + Ke^{-y}$$

is a family of curves orthogonal to the family $(*)$.

Question 6. Let $y_1, y_2, \dots, y_n$ be n solutions of the homogeneous linear n-th order differential equation on an interval I. Then the set of solutions is linearly independent on I if and only if $W(y_1, y_2, \dots, y_n) \neq 0$ for every x in the interval.

Solution. Suppose, to the contrary, that $W(y_1, y_2, \dots, y_n)(x_0) \neq 0$ at some point $x_0 \in I$ but the y_i are linearly dependent. Then there exist constants $c_1, \dots, c_n$, not all zero, so that

$$c_1 y_1(x) + \cdots + c_n y_n(x) = 0$$

on I.

Differentiating $n-1$ times gives a homogeneous linear system

$$\begin{bmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & \cdots & y_n'(x_0) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0.$$

Call the matrices on the left-hand side W and c respectively. A nontrivial solution c exists only if W has determinant zero, since if W had nonzero determinant, then it would be invertible, so that $W^{-1}Wc = c = 0$. But its determinant is exactly $W(y_1, y_2, \dots, y_n)(x_0)$, which we assumed nonzero, a contradiction! Thus, $W(y_1, y_2, \dots, y_n)(x) \neq 0$ implies that the set of solutions is linearly independent on I.

Now, assume that the set of solutions is linearly independent on I. It follows that there exist a_i corresponding to each y_i such that

$$\sum_{i=1}^{n} a_i y_i = 0,$$

differentiating one times we get

$$\sum_{i=1}^{n} a_i y_i' = 0$$

and so on. We thus obtain that all columns are linearly independent, meaning that the rank of the matrix is n, since row and column ranks are equal, row rank is also n which implies that the determinant of this matrix must be nonzero. The result follows.

Question 7. Find the general solution of the following differential equations:

(a) $y^{(4)} + 8y'' + 16y = 0$,

(b) $9y'' + 6y' + 5y = 0$.

Solution. (a) Assume a solution of the form $y = e^{rx}$, then $y^{(i)} = r^i e^{rx}$ for $i \in \mathbb{N}$. Substituting, our equation becomes

$$r^4 e^{rx} + 8r^2 e^{rx} + 16 e^{rx} = 0,$$

since $e^{rx} \neq 0$, we get

$$r^4 + 8r^2 + 16 = 0.$$

Now, let $u = r^2$. The equation becomes

$$u^2 + 8u + 16 = (u+4)^2 = 0.$$

So, $u = -4$ is a repeated root of multiplicity 2, and so $r = \pm\sqrt{-4} = \pm 2i$, following that $r = +2i$ and $r = -2i$ are both repeated roots of multiplicity 2. The repeated roots $r = \pm 2i$ (each with multiplicity 2) lead to the general solution:

$$y(x) = (C_1 + C_2 x)\cos(2x) + (C_3 + C_4 x)\sin(2x),$$

where C_1, C_2, C_3, C_4 are constants.

(b) For the differential equation $9y''+6y'+5y = 0$, we solve the characteristic equation $9r^2+6r+5 = 0$. Using the quadratic formula:

$$r = \frac{-6 \pm \sqrt{36 - 180}}{18} = \frac{-6 \pm 12i}{18} = -\frac{1}{3} \pm \frac{2}{3}i.$$

The general solution with complex roots $\alpha \pm \beta i$ is:

$$(x) = e^{-\frac{x}{3}} \left(C_1 \cos\left(\frac{2x}{3}\right) + C_2 \sin\left(\frac{2x}{3}\right) \right),$$

where C_1, C_2 are constants.