CS4725/CS6705

Chapter 16: Making Simple Decisions

Decision Theory

- Decision-theoretic agent: one that makes rational decisions based on what it believes and what it wants.
  - Again, decision theory = probability theory + utility theory

Maximum Expected Utility

- A rational agent will choose the action that maximizes expected utility.
  - Suppose *a* is an action with many possible results.
  - Suppose **e** is a set of evidence observations.
  - Then, the expected utility EU(a | **e**) is:

$$EU(a \mid \mathrm{e}) = \sum_{s'} P(\mathrm{Result}(a) = s' \mid a, \mathrm{e}) U(s')$$

where *U(s')* is the utility of state *s'*

Preferences

- $A \succ B$: A is preferred over B
- $A \sim B$: the agent is indifferent between A and B
- $A \succsim B$: the agent prefers A over B or is indifferent

- Define a **lottery** L as a probability distribution over possible outcomes $C_1, C_2, \ldots, C_n$ with probabilities $p_1, p_2, \ldots, p_n$

$$L = [p_1, C_1;\ p_2, C_2;\ \ldots\ ;\ p_n, C_n]$$

Utility functions

- A **utility function** *U* assigns a real value to every possible state such that:
  - $U(A) > U(B) \Leftrightarrow A \succ B$
  - $U(A) = U(B) \Leftrightarrow A \sim B$

- The utility of a lottery is the sum of the probability of each outcome times its utility.

$$U([p_1, C_1; \ldots; p_n, C_n]) = \sum_i p_i U(C_i)$$

Axioms of utility theory

- Maximizing expected utility is rational if the following axioms hold:
  - **Orderability**: $(A \succ B) \vee (B \succ A) \vee (A \sim B)$
  - **Transitivity**: $(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$
  - **Continuity**: $A \succ B \succ C \Rightarrow \exists p\ [p, A; 1\text{-}p, C] \sim B$
  - **Substitutability**: $A \sim B \Rightarrow [p, A; 1\text{-}p, C] \sim [p, B; 1\text{-}p, C]$
  - **Monotonicity**:

    $$A \succ B \Rightarrow (p \geq q \Leftrightarrow [p, A; 1\text{-}p, B] \succsim [q, A; 1\text{-}q, B])$$
  - **Decomposability**: $[p, A; 1\text{-}p, [q, B; 1\text{-}q, C]] \sim$

    $$[p, A; (1\text{-}p)q, B; (1\text{-}p)(1\text{-}q), C]$$

Utility functions

- Strictly speaking, a utility function just maps every state to a real number.
- In theory, this **could** be any arbitrary function
  - e.g., U(getting 100% on the test) = 0.75

    U(80%) = 0.42

    U(60%) = 0.98

    U(40%) = 0.63

Utility functions

- However, normally, a person's preferences are more systematic and we can represent them with a utility function like

  - U(grade) = grade * 0.01

  or

  - $U(grade) = \begin{cases} 0 & \text{if grade} < 50 \\ (grade - 50) * 0.02 & \text{if grade} \geq 50 \end{cases}$

  or

  - $U(grade) = \begin{cases} 0 & \text{if grade} < 50 \\ \log_2(1+(grade - 50)/50) & \text{if grade} \geq 50 \end{cases}$

Utility of money

- Utility theory has its roots in economics, and people's preferences over different amounts of money have been studied extensively.

Attitudes toward risk

- Consider the following choice:
  - a guarantee of receiving \$500,000, or
  - a **lottery** with:
    - a 50% chance of receiving \$1,000,000
    - a 50% chance of receiving nothing
- Expected monetary value and utility of sure thing and lottery (on board)

Risk neutrality

- If a decision-maker is **risk-neutral**, then he/she would be indifferent between accepting the sure thing or taking a chance on the lottery. (Both options have the same expected monetary value.)
- Risk neutrality = linear utility function

Risk aversion

- If a decision-maker is **risk-averse**, then he/she would prefer to accept a sure thing with value equal to the expected monetary value of a lottery.

- Risk aversion = concave utility function

Risk seeking

- If a decision-maker is **risk-seeking**, then he/she would prefer to take a chance on a lottery that has the same expected monetary value as some guaranteed payoff.

- Risk seeking = convex utility function

Another example: St. Petersburg Paradox

- Suppose you are offered a chance to play the following game:
  - Flip a coin until it comes up *heads*.
  - If *heads* occurs on the first toss, you win \$2.
  - If *heads* first occurs on the second toss: \$4
  - Third toss: \$8

    :
  - n^{th} toss: $\$2^n$

- How much would you pay for a chance to play this game?

St. Petersburg Paradox

- Expected monetary value of playing the game (on board)
- Looking only at expected *monetary values*, you should be willing to pay *any* finite amount of money to play.
- This doesn't seem rational.

Utility of money

- Bernoulli (1738) proposed a theory that the utility of money is logarithmic:

$$U(\text{having } n \text{ dollars}) = a \log_2 n + b$$

- Expected utility of St. Petersburg game then works out to be finite (depends on the amount of money someone has before playing the game)

- It is generally believed that logarithmic functions are a good fit for people's utility functions for money.

Multi-attribute utility functions

- Most real-world decision problems involve evaluating different possible outcomes over a number of different factors, or *attributes*.
- Example:
  - Utility of <red, 2002, Honda Accord, 90000 km>?
  - Utility of <black, 2002, Honda Accord, 90000 km>?
  - Utility of <red, 2002, Toyota Camry, 75000 km>?
  - Utility of <red, 2004, Honda Accord, 90000 km>?
  - etc.

Multi-attribute utility functions

- How do we evaluate the different possible outcomes, when some are better according to some attributes, but worse according to others?
- We need a function that takes into account the decision maker’s preferences for each attribute and the relative importance of each.

Special cases

- In general, multi-attribute utility functions can be very complex, but we will consider two special cases.
  - **Additive utility functions**
  - **Multiplicative utility functions**

Additive value functions

- If the attributes involved (X_1, X_2, ..., X_n) have a special property called **mutual preferential independence**, then we can use an **additive value function:**

  $$V(x_1, x_2, ..., x_n) = V_1(x_1) + V_2(x_2) + ... + V_n(x_n)$$

  where each V_i is a value function referring only to the attribute X_i

- This is an appropriate model for some real problems, and it can be a useful approximation even when mutual preferential independence does not hold.

Multiplicative utility functions

- When uncertainty is involved (lotteries), we can extend the notion of preference independence to the idea of **utility independence**. If attributes are **mutually utility-independent**, we can use a **multiplicative utility function**:
- [Three-attribute case]

$$U = k_1U_1 + k_2U_2 + k_3U_3 + k_1k_2U_1U_2 + k_1k_3U_1U_3 + k_2k_3U_2U_3 + k_1k_2k_3U_1U_2U_3$$

where the U_i are utility functions for each attribute and the k_i are constants (weights)

Multi-attribute examples

- [Examples on the board]

Decision networks

- Decision networks = Bayesian networks + additional nodes for actions and utilities
  – Chance nodes (ovals): random variables, just like in Bayesian networks
  – Decision nodes (rectangles)
  – Utility nodes (diamonds)

- [Small example on the board]

Evaluating decision networks

1. Set the evidence variables for the current state.
2. For each possible value of the decision node(s):
   a) Set the decision nodes to that value.
   b) Calculate the posterior probabilities for the parent nodes of the utility node, using a standard probabilistic inference algorithm.
   c) Calculate the resulting utility for the action.
3. Return the action with the highest utility.

Value of information

- One of the most important parts of decision making is knowing what questions to ask.
- **Information value theory** can help with deciding what information to acquire in a decision problem.
- [Simple example on the board]

Value of information

- It turns out that the value of information is **always nonnegative.**
- However, there is usually a **cost** involved in gathering more information.
- An agent might consider many different information-gathering actions and choose the one that maximizes the difference between the expected benefit and the expected cost.