CS4725/CS6705 - Fall 2017

Assignment # 3

SAMPLE SOLUTIONS

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1. **(11 marks)** Consider the following full joint distribution, showing probabilities associated with whether a hockey team practiced the day before their game, whether the players stayed up all night, and whether the team won the game.

	up-all-night		¬up-all-night	
	practice	¬practice	practice	¬practice
win	.036	.016	.336	.162
¬win	.204	.144	.084	.018

Calculate the following:

(a) (2 marks) $P(\text{win})$

$$P(\texttt{win}) = .036 + .016 + .336 + .162 = .550$$

(b) (3 marks) $\mathbf{P}(\text{Win} \mid \neg\text{up-all-night})$
[Reminder of notation: With the bold **P** and the capitalized `Win`, this means that you are being asked to provide a vector $< P(\text{win} \mid \neg\text{up-all-night}),\ P(\neg\text{win} \mid \neg\text{up-all-night}) >$]

$$P(\texttt{win} \mid \neg\texttt{up-all-nt}) = \frac{P(\texttt{win}\wedge\neg\texttt{up-all-night})}{P(\neg\texttt{up-all-night})} = \frac{.336+.162}{.336+.162+.084+.018} = \frac{.498}{.600} = 0.83$$

$$P(\neg\texttt{win} \mid \neg\texttt{up-all-nt}) = \frac{P(\neg\texttt{win}\wedge\neg\texttt{up-all-night})}{P(\neg\texttt{up-all-night})} = \frac{.084+.018}{.600} = \frac{.102}{.600} = 0.17$$

$$\mathbf{P}(\texttt{Win} \mid \texttt{up-all-night}) = < 0.83, 0.17 >$$

(c) (3 marks) $P(\text{practice} \mid \text{win})$
(How does this compare to $P(\text{practice})$?)

$$P(\texttt{practice} \mid \texttt{win}) = \frac{P(\texttt{practice}\wedge\texttt{win})}{P(\texttt{win})} = \frac{.036+.336}{.036+.016+.336+.162} = \frac{.372}{.550} = 0.676$$

$P(\texttt{practice})$ `is` $0.036 + 0.204 + 0.336 + 0.084 = 0.660$`, so the probability that they practiced given that they won is only slightly higher than the prior probability that they practiced.`

(d) (3 marks) $P(\text{up-all-night} \mid \neg\text{win})$
(How does this compare to $P(\text{up-all-night})$?)

$$P(\texttt{up-all-nt} \mid \neg\texttt{win}) = \frac{P(\texttt{up-all-night}\wedge\neg\texttt{win})}{P(\neg\texttt{win})} = \frac{.204+.144}{.204+.144+.084+.018} = \frac{.348}{.450} = 0.773$$

$P(\texttt{up-all-night}) = .036 + .016 + .204 + .144 = .400$`, so the probability that they were up all night given that they did not win is significantly higher than the prior probability that they were up all night.`

2. **(5 marks)** Suppose that a student writes 60% of her exams during the daytime and 40% in the evening. She passes 80% of her daytime exams, but only 50% of her evening exams.

   Given that she failed her CS1000 exam, what is the probability that it was written in the evening?

   Hints:

   - Use Bayes' Rule.
   - As part of your work, you will want to calculate the prior probability that she would fail any given exam. To do this, you should consider the probability of an exam being written in the daytime and being failed, as well as the probability of an exam being written in the evening and being failed.
     Reminder: $P(\text{daytime} \wedge \neg\text{pass}) = P(\text{daytime})P(\neg\text{pass} \mid \text{daytime})$.

We are looking for the probability that the exam was written in the evening (e), given that the student failed the exam (f).

Bayes' Rule tells us that $P(e|f) = \frac{P(f|e)P(e)}{P(f)}$.

From the description above, $P(f|e) = 0.5$ and $P(e) = 0.4$. Calculating $P(f)$ takes a bit more work. She could have failed if the exam was written in the daytime and she could have failed if the exam was written in the evening.

$P(f) = P(e)P(f|e) + P(\neg e)P(f|\neg e) = (0.4)(0.5) + (0.6)(0.2) = 0.32$.

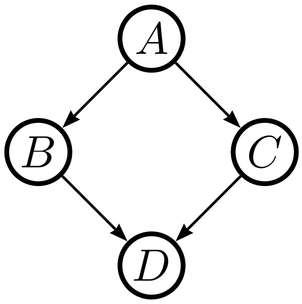
Finally, we can compute $P(e|f) = \frac{P(f|e)P(e)}{P(f)} = \frac{(0.5)(0.4)}{0.32} = 0.625$.

Given that she failed the exam, the probability that it was an evening exam is 0.625. (Note that this is significantly higher than the prior probability of an exam being written in the evening (0.4).)

[Note that this question can also be done by constructing the joint distribution table and calculating from those values.]

3. **(12 marks)**

Suppose that you have been given the following Bayesian network, containing four variables. Three of the variables are Boolean (true or false), but variable C has three possible values: c_{high}, c_{med} and c_{low}.



The information from the conditional probability tables is the following:

- $P(a) = 0.7$
- $P(b \mid a) = 0.6$, $P(b \mid \neg a) = 0.1$
- $P(c_{high} \mid a) = 0.1$, $P(c_{med} \mid a) = 0.5$, $P(c_{low} \mid a) = 0.4$, $P(c_{high} \mid \neg a) = 0.6$, $P(c_{med} \mid \neg a) = 0.3$, $P(c_{low} \mid \neg a) = 0.1$
- $P(d \mid b, c_{high}) = 0.9$, $P(d \mid b, c_{med}) = 0.6$, $P(d \mid b, c_{low}) = 0.3$, $P(d \mid \neg b, c_{high}) = 0.8$, $P(d \mid \neg b, c_{med}) = 0.5$, $P(d \mid \neg b, c_{low}) = 0.2$

Use the method of **exact inference by enumeration** discussed in class (and in Section 14.4 of the textbook) to determine $\mathbf{P}(C \mid \neg d)$.

Note that this requires you to calculate $P(c_{high} \mid \neg d)$, $P(c_{med} \mid \neg d)$, and $P(c_{low} \mid \neg d)$. **You must show your work.**

$$\begin{aligned}
&\mathbf{P}(c_{high} \mid \neg d)\\
&= \alpha \textstyle\sum_A \sum_B P(A, B, c_{high}, \neg d)\\
&= \alpha \textstyle\sum_A \sum_B P(A)\ P(B \mid A)\ P(c_{high} \mid A)\ P(\neg d \mid B, c_{high})\\
&= \alpha \textstyle\sum_A P(A)\ P(c_{high} \mid A) \sum_B P(B \mid A)\ P(\neg d \mid B, c_{high})\\
&= \alpha[P(a)\ P(c_{high} \mid a)(P(b \mid a)\ P(\neg d \mid b, c_{high}) + P(\neg b \mid a)\ P(\neg d \mid \neg b, c_{high}))\\
&\quad\quad + P(\neg a)\ P(c_{high} \mid \neg a)(P(b \mid \neg a)\ P(\neg d \mid b, c_{high}) + P(\neg b \mid \neg a)\ P(\neg d \mid \neg b, c_{high}))]\\
&= \alpha[(0.7)(0.1)((0.6)(1-0.9) + (1-0.6)(1-0.8))\\
&\quad\quad +(1-0.7)(0.6)((0.1)(1-0.9) + (1-0.1)(1-0.8))]\\
&= \alpha[(0.7)(0.1)((0.6)(0.1) + (0.4)(0.2)) + (0.3)(0.6)((0.1)(0.1) + (0.9)(0.2))]\\
&= \alpha[(0.7)(0.1)(0.06 + 0.08) + (0.3)(0.6)(0.01 + 0.18))]\\
&= 0.044\alpha
\end{aligned}$$

$$\begin{aligned}
&\mathbf{P}(c_{med} \mid \neg d)\\
&= \alpha \textstyle\sum_A \sum_B P(A, B, c_{med}, \neg d)\\
&= \alpha \textstyle\sum_A \sum_B P(A)\ P(B \mid A)\ P(c_{med} \mid A)\ P(\neg d \mid B, c_{med})\\
&= \alpha \textstyle\sum_A P(A)\ P(c_{med} \mid A) \sum_B P(B \mid A)\ P(\neg d \mid B, c_{med})\\
&= \alpha[P(a)\ P(c_{med} \mid a)(P(b \mid a)\ P(\neg d \mid b, c_{med}) + P(\neg b \mid a)\ P(\neg d \mid \neg b, c_{med}))\\
&\quad\quad + P(\neg a)\ P(c_{med} \mid \neg a)(P(b \mid \neg a)\ P(\neg d \mid b, c_{med}) + P(\neg b \mid \neg a)\ P(\neg d \mid \neg b, c_{med}))]\\
&= \alpha[(0.7)(0.5)((0.6)(1-0.6) + (1-0.6)(1-0.5)) + (1-0.7)(0.3)((0.1)(1-0.6) + (1-0.1)(1-0.5))]\\
&= \alpha[(0.7)(0.5)((0.6)(0.4) + (0.4)(0.5)) + (0.3)(0.3)((0.1)(0.4) + (0.9)(0.5))]\\
&= \alpha[(0.7)(0.5)(0.24 + 0.20) + (0.3)(0.3)(0.04 + 0.45))]\\
&= 0.1981\alpha
\end{aligned}$$

$$
\begin{aligned}
&\mathbf{P}(c_{low} \mid \neg d)\\
&\quad = \alpha \textstyle\sum_A \sum_B P(A, B, c_{low}, \neg d)\\
&\quad = \alpha \textstyle\sum_A \sum_B P(A)\ P(B \mid A)\ P(c_{low} \mid A)\ P(\neg d \mid B, c_{low})\\
&\quad = \alpha \textstyle\sum_A P(A)\ P(c_{low} \mid A) \sum_B P(B \mid A)\ P(\neg d \mid B, c_{low})\\
&\quad = \alpha[P(a)\ P(c_{low} \mid a)(P(b \mid a)\ P(\neg d \mid b, c_{low}) + P(\neg b \mid a)\ P(\neg d \mid \neg b, c_{low}))\\
&\qquad\quad + P(\neg a)\ P(c_{low} \mid \neg a)(P(b \mid \neg a)\ P(\neg d \mid b, c_{low}) + P(\neg b \mid \neg a)\ P(\neg d \mid \neg b, c_{low}))]\\
&\quad = \alpha[(0.7)(0.4)((0.6)(1-0.3) + (1-0.6)(1-0.2)) + (1-0.7)(0.1)((0.1)(1-0.3) + (1-0.1)(1-0.2))]\\
&\quad = \alpha[(0.7)(0.4)((0.6)(0.7) + (0.4)(0.8)) + (0.3)(0.1)((0.1)(0.7) + (0.9)(0.8))]\\
&\quad = \alpha[(0.7)(0.4)(0.42 + 0.32) + (0.3)(0.1)(0.07 + 0.72))]\\
&\quad = 0.2309\alpha
\end{aligned}
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Because $0.044\alpha + 0.1981\alpha + 0.2309\alpha = 1$,
the normalization constant α is $\frac{1}{0.044+0.1981+0.2309} = \frac{1}{0.473}$.

$$
\begin{aligned}
&\mathbf{P}(C \mid \neg d)\\
&\quad = < 0.044\alpha, 0.1981\alpha, 0.2309\alpha >\\
&\quad = < 0.0930, 0.4188, 0.4882 > \quad \text{(high, medium, low)}
\end{aligned}
$$

4. **(10 marks)** For this question, we will assume that the value of money is linear, even though we discussed in class that this is rarely the case for people. Use the utility function $U(\text{winning } \$x) = x$. In this case, the expected utility of an event (or lottery) will be exactly the same as the expected monetary value and it will be in the range $[0, 100000]$.
Suppose that you have the choice of entering one of two contests.

In the first contest, you will be given a ticket with a 4-digit number (between 0000 and 9999). A random 4-digit number will then be drawn. You could win money, based on the following rules.

- If all four digits of your ticket number match the winning number, then you win $100 000.
- If the last three digits match (but not all four), you win $10 000.
- If the last two digits match (but not the last three), you win $1000.
- If the last digit matches (but not the last two), you win $100.
- Otherwise, you do not win any money.

In the second contest, a fair coin will be flipped four times. If the result of the flip is *heads* all four times, you win $640. Otherwise, you do not win any money.

(a) Which of the two contests should you choose to enter, and what is the expected utility of your decision?

In the first contest:

- There is only one ticket out of 10000 that would match the winning number exactly, so the probability of matching all four digits is 1/10000.
- There are 10 tickets that would match the winning number on the last three digits, but subtracting the one that matches on all four means that there are 9, and so the probability is 9/10000.
- There are 100 tickets that would match the winning number on the last two digits, but subtracting the 10 that have already been counted above means that there are 90, and so the probability is 90/10000.
- There are 1000 tickets that would match the winning number on the last digit, but subtracting the 100 that have already been counted above means that there are 900, and so the probability is 900/10000.
- The remaining 9000 tickets will not satisfy any of the winning conditions, and so the probability is 9000/10000.

In the second contest:

- There are sixteen different (equally likely) coin-flipping sequences: HHHH, HHHT, HHTH, *etc.* Only one of these (HHHH) would allow us to win, and so the probability of winning $640 is 1/16.

A decision tree for this problem can be found in a separate PDF file on Desire2Learn.

Contest 2 has a higher expected utility (40 vs. 37), so we should choose to enter Contest 2. Our expected utility is 40.

(b) Now, suppose that you have a chance to find out one of the following two pieces of information (but not both) **before you have to choose your contest**.

- the last digit of the winning ticket number for Contest 1 (Assume that you already know your own ticket number.)
- the outcomes of the first three coin flips for Contest 2

Which piece of information should you request? Show your work and explain your answer.

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The work is shown in a separate PDF file on Desire2Learn.  It is
slightly more valuable for us to learn the last digit of the winning
ticket for Contest 1, so this is the piece of information we should
request.

Not surprisingly, our strategy will be:
  * If we learn that the last digit matches ours, enter Contest 1.
  * Otherwise, enter Contest 2.
  * The expected utility of this strategy will be 73.
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5. **(7 marks)**

[Note that this question is taken from Exercise 16.15 in the textbook, with some minor changes made.]
Consider a student who has the choice to buy or not buy a textbook for a course. We'll model this as a decision problem with one Boolean decision node, B, indicating whether the agent chooses to buy the book, and two Boolean chance nodes, M, indicating whether the student has mastered the material in the book, and P, indicating whether the student passes the course. Of course, there is also a utility node, U.
A certain student, Sam, has the following utility function: 1.00 for passing the course and not buying the book, 0.95 for passing the course and buying the book, 0.05 for failing the course and not buying the book, and 0.00 for failing the course and buying the book.
Sam's conditional probability estimates are as follows:

$$P(p \mid b, m) = 0.9 \qquad P(m \mid b) = 0.9$$
$$P(p \mid b, \neg m) = 0.5 \qquad P(m \mid \neg b) = 0.7$$
$$P(p \mid \neg b, m) = 0.8$$
$$P(p \mid \neg b, \neg m) = 0.3$$

You might think that P would be independent of B given M, but this course has an open-book final – so having the book helps.

(a) Draw the decision network for this problem.

(b) Compute the expected utility of buying the book and of not buying it.

(c) What should Sam do?

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The answers are shown in a separate PDF file on Desire2Learn.
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