CS4725/CS6705

Chapter 13: Quantifying Uncertainty

Logical agents

- We have looked at **logical agents**: agents that:
  - represent *facts* about their environment,
  - derive new information that follows logically from these facts, and
  - find plans that are *guaranteed* to work.
- Propositions are either true, false or unknown.
- In reality, an agent will very rarely have complete information about its environment.

Uncertainty

- In realistic situations, an agent should be able to represent the *likelihood* that sentences are true.
  - e.g., If the patient has a toothache, there is a *probability* of 0.8 that he has a cavity.
- The agent’s job is still to make the **rational decision**, but this now depends not only on the “goodness” (utility) of different outcomes, but also on their likelihood.

Utility Theory

- To evaluate the quality of a sequence of actions, we must know the agent's **preferences** between different possible **outcomes**.
- The **utility** of an outcome (or a state) is a real number that captures how useful or how good it is.
  – Agents will always prefer states with higher utility.

Decision Theory

- Decision theory = probability theory + utility theory
- *An agent is rational if and only if it chooses the action that yields the highest expected utility, averaged over all the possible outcomes of the action.* (Principle of **Maximum Expected Utility**)
- [Formula for expected utility and very simple example on the board]

Basic probability

- **Random variable:** some part of the world whose status is unknown (e.g., *Cavity*)
- **Domain:** all possible values for a random variable (e.g., <*true,false*>, <*win,loss,tie*>, <*sunny,rainy,cloudy,snow*>, the interval [0,1])
- **Propositions:** e.g., *Cavity = true*
- **Complex propositions:** e.g., *Cavity = true* ∧ *Toothache = false*

Basic probability (cont’d)

- **Atomic event:** an assignment of values to all variables making up the “world”
  - e.g., *cavity* $\wedge$ *toothache, cavity* $\wedge$ $\neg$*toothache,* $\neg$*cavity* $\wedge$ *toothache,* $\neg$*cavity* $\wedge$ $\neg$*toothache*

- Atomic events are *exhaustive* – at least one must be the case.
- Atomic events are *mutually exclusive* – at most one can be the case.

Prior probability

- Given a proposition *a*, the **prior probability** (or **unconditional probability**) of *a, P(a)*, is the degree of belief assigned to it *in the absence of any other information*.
  - e.g., *P(cavity)* = 0.05

- We can use a **vector** to represent the probabilities of **all** possible values of a random variable.
  - e.g., **P**(*Weather*) = <0.7, 0.2, 0.08, 0.02> might represent the probabilities that the weather will be sunny, rainy, cloudy, snowy, respectively.

Joint Probability Distributions

- Given a set of random variables, the **joint probability distribution** provides the probability of all possible combinations of values for the random variables: e.g.,

	Cavity = true	Cavity=false
Toothache = true	0.04	0.08
Toothache = false	0.01	0.87

Probability density functions

- For continuous random variables, we will talk about **probability density functions**.
  - e.g., The expected maximum temperature for tomorrow is uniformly distributed in the range [5,14].
  - e.g., Tim's expected mark on the exam is normally distributed, with mean = 80, standard deviation = 10.

Conditional probability

- Often, an agent will update the probabilities of different propositions based on **evidence** that it has observed.
- In this case, we refer to **conditional** or **posterior probabilities**.
- **Notation**: $P(a|b)$ is read as "the probability of a, given that all we know is b"
  - e.g., $P(\mathit{cavity}|\mathit{toothache}) = 0.8$

Conditional probability (cont'd)

- Note that conditional probabilities can be defined in terms of unconditional probabilities.

$$P(cavity \mid toothache) = \frac{P(cavity \wedge toothache)}{P(toothache)}$$

- or $P(cavity \wedge toothache) = P(toothache)\, P(cavity \mid toothache)$

Axioms of probability

- For any proposition a, $0 \leq P(a) \leq 1$
- $P(true) = 1$, $P(false) = 0$
- For any propositions a and b,
  $P(a \vee b) = P(a) + P(b) - P(a \wedge b)$

Inference using full joint distributions

- [Example on the board]

- Normalization: [example on the board]

Independence

- As mentioned before: in general (product rule),

$$P(a \land b) = P(a)\, P(b|a)$$

- In terms of variables,

$$\mathbf{P}(\mathrm{A,B}) = \mathbf{P}(\mathrm{A})\, \mathbf{P}(\mathrm{B|A})$$

- Sometimes, the probability of any value of B is not affected at all by the value of A. In this case, variables A and B are **independent**, and so

$$\mathbf{P}(\mathrm{A,B}) = \mathbf{P}(\mathrm{A})\, \mathbf{P}(\mathrm{B})$$

Independence

- If we have subsets of variables that are independent of each other, our full joint distribution can be *factored* into separate joint distributions on those subsets.
- Example:

  P(Toothache,Cavity,Catch,Weather) =
  P(Toothache,Cavity,Catch) **P**(Weather)

  [Instead of one table with 32 entries, we need one table with 8 entries and another with 4.]

Bayes' Rule

- Product rule:

$$P(a \land b) = P(a)\, P(b|a)$$

- Also:

$$P(a \land b) = P(b)\, P(a|b)$$

- Combining these two equations, we get

$$P(a)\, P(b|a) = P(b)\, P(a|b)$$

or

$$P(b\,|\,a) = \frac{P(a\,|\,b)P(b)}{P(a)}$$

Bayes' Rule

- Bayes' Rule is an essential equation for probabilistic inference in AI systems.
- In general, for multi-valued variables:

$$\mathbf{P}(Y \mid X) = \frac{\mathbf{P}(X \mid Y)\mathbf{P}(Y)}{\mathbf{P}(X)}$$

- [Example on the board]

Conditional independence

- Often, two variables are not completely independent of each other, but they might be independent **given** the value of some other variable.

- Example: *toothache* and *catch* are not independent, but **given** that you have a cavity (or given that you don't), they do not have any direct effect on each other.

- This is **conditional independence**.