Image Denoising

by Silao_Xu and Cyang

According to the handout, the equation we are trying to minimize is as follows:

$$\underbrace{k\sum_i (x_i - x_{i_{right}})^2}_{A} + \underbrace{k\sum_j (x_j - x_{j_{up}})^2}_{B} + \sum_m (x_m - I_m)^2 \tag{1}$$

A and B above could be represented by the convolution form, so (1) is equivalent to

$$\underbrace{k(X \otimes K_1)^2}_{A^{Conv}} + \underbrace{k(X \otimes K_2)^2}_{B^{Conv}} + \sum_m (x_m - I_m)^2$$

where $K_1 = \begin{pmatrix} 1 & -1 & 0 & \cdots \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix}, K_2 = \begin{pmatrix} 1 & 0 & \cdots \\ -1 & 0 & \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix}$.

Now A^{Conv} and B^{Conv} could be expressed as using *Fourier* transform

$$k(F(X) \cdot F(K_1))^2 + k(F(X) \cdot F(K_1))^2 + \sum_k (x_k - I_k)^2$$

According to *Parseval's Theorm* — the sum of the squares of the magnitudes of x equals the sum of the squares of the magnitudes of its (1 or 2-dimensional) *Fourier* transform, we could tranform the term $\sum_k (x - I_k)^2$ into the *Fourier* transform, so we have

$$k(F(X) \cdot F(K_1))^2 + k(F(X) \cdot F(K_1))^2 + (F(X) - F(I))^2 \tag{2}$$

So taking the variable X, to calculate the derivative of (2) in order to minimize X, we have

$$\begin{aligned} 2kF(X) \cdot F(K_1)^2 + 2kF(X) \cdot F(K_2)^2 + 2F(X) - 2F(I) &= 0 \\ F(X) &= \frac{F(I)}{k(F(K_1)^2 + F(K_2)^2) + 1} \end{aligned}$$

Now using inverse *Fourier* transform we could get the X which could minimize the formula (1), that is

$$X = F^{-1}\left(\frac{F(I)}{k(F(K_1)^2 + F(K_2)^2) + 1}\right)$$