CS157 Homework 8

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Problem 5

Duality:

1. Namely, recall from lecture that the dual to our linear program is: $\max_y b^T \cdot y$ subject to the constraints $A^T \cdot y = u^T$ and $y \geqslant 0$ (the unbounded variables x correspond to equality constraints under duality). Show that the coefficients $\{y_i\}$—that we found in our physics thought experiment above—make everything work out:

   a. show that $\{y_i\}$ is a feasible solution to the dual linear program

   Based on the fact from the physics intution, we have $y_1 A_1 + \ldots + y_m A_m = u$, there is the constraints such that $A^T \cdot y = u$. Based on the fact that $y_1, \ldots y_m$ are all non-negative, the constraint that in dual program—$y \geqslant 0$ could be satisfied. So, we can draw conclusion that $\{y_i\}$ is a feasible solution to the dual linear program.

   b. show that the objective value of $\{y_i\}$ in the dual linear program *equals* the optimal objective value of the original linear program, $u \cdot x^*$

   For the constraints that passes through x^*, we have $A_i \cdot x^* = b^i$. Suppose we have a coefficients $\{y_i^*\}$ such that $y_i^* > 0$, then left-multiplying y^{*T} in both side of $A \cdot x^* = b$, we get

$$\begin{aligned} y^{*T} \cdot A \cdot x^* &= y^{*T} \cdot b \\ A^T \cdot y^* \cdot x^* &= b^T \cdot y^* \\ u \cdot x^* &= b^T \cdot y^* \end{aligned} \tag{1}$$

   For any coefficients $\{y_i\}$ that is a feasible solution to the dual linear program which has $A_i \cdot x \geqslant b_i$, left-multiplying y^T in both side of the contraints of primal LP, we have

$$\begin{aligned} y^T \cdot A \cdot x &\geqslant y^T \cdot b \\ (y^T \cdot A \cdot x)^T &\geqslant (y^T \cdot b)^T \\ x^T \cdot (A^T \cdot y) &\geqslant b^T \cdot y \\ x^T \cdot u^T &\geqslant b^T \cdot y \\ u \cdot x &\geqslant b^T \cdot y \end{aligned} \tag{2}$$

   Plugging x^* into the left-hand side of (2), we have $u \cdot x^* \geqslant b^T \cdot y$, substituting $u \cdot x^*$ with $y^* \cdot b^T$ according to (1), we get the upper bound for the dual LP as follows

$$y^* \cdot b^T \geqslant y \cdot b^T \tag{3}$$

   Since $u \cdot x^*$ is the lower bound for the primal linear program, which has been already known, and $y^* \cdot b^T$ is the upper bound for the dual linear program and plus the equality (1), *Duality Theorem* proved here.

   c. summarize what you have found.

   For the primal LP, let $OPT =$ Maximize $C^T x$, *s.t.*, $Ax \leqslant b$, $\forall x_i \in x, x_i \geqslant 0$. For the dual LP, let $OPT^* =$ Minimize $b^T y$, *s.t.*, $A^T y \geqslant C$, $\forall y_i \in y, y_i \geqslant 0$.

When $OPT = +\infty$ and thus on its counterpart, the dual LP is infeasible; when primal LP is infeasible, $OPT^* = -\infty$ and *vice versa*.

When $OPT \in R$, OPT^* also is within R, their boundary coincide.

2. We consider randomized strategies for the players. Suppose that you, the row player, know that the column player picks rock with probability 0.5, paper with probability 0.2, and scissors with probability 0.3, then your *expected payoff* for each of your three strategies may be found by taking 0.5 times the first column of A, plus 0.2 times the second column of A, plus 0.3 times the third column of A, which equals $\begin{pmatrix} 0.1 & 0.2 & -0.3 \end{pmatrix}^T$, meaning that you could make expected profit of 0.1 by choosing rock all the time, 0.2 by choosing paper all the time, or -0.3 by choosing scissors all the time. To express this slightly more in the language of linear programming, the entry 0.1 guarantees you an expected profit of at least 0.1, regardless of the other entries of the matrix, etc. Given these choices, you will presumably choose paper, for an expected profit of 0.2. Your opponent wants to minimize your profit.

   a. (7 points) Assume the framework outlined above, where your opponent—the column player—declares a probability distribution on columns, and then you choose the best row to play. Write a linear program that your opponent could solve to choose the distribution on columns that would minimize your expected profit in this interaction. **Explain** how it works.

   Firstly, we lable the column player's probabilities of choosing rock, pocket and scissor to be r_c, p_c, s_c and thus the *expected payoff* for column player's three strategies could be computed as $A \cdot \begin{pmatrix} r_c \\ p_c \\ s_c \end{pmatrix}$, which is equivalent to $\begin{pmatrix} -p_c + s_c \\ r_c - s_c \\ -r_c + p_c \end{pmatrix}$, we name it as matrix B. For row player, his/her strategy is choossing the maximum *expected payoff* from row of B. As oppose to the row player, we know the opponent could solve to choose the distribution on columns that would minimize row player's expected profit in this interaction and therefore the objective function could be written as

$$\text{Minimize } \; w \;=\; \max(-p_c + s_c, r_c - s_c, -r_c + p_c) \tag{1}$$

   We let the maximum-*expected payoff* choice of row player to be w and now the objecive function becomes minimizing $C \cdot X$, where $X = \begin{pmatrix} r_c & p_c & s_c & w \end{pmatrix}^T$ and $C = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ and the constraints could be interpreted as follows

$$\begin{aligned} -p_c + s_c - w &\geqslant 0 \\ r_c - s_c - w &\geqslant 0 \\ -r_c + p_c - w &\geqslant 0 \\ r_c + p_c + s_c &= 1 \\ r_c, p_c, s_c, w &\geqslant 0 \end{aligned} \tag{2}$$

   The matrix form for representing the constraints in (2) could be interpretted as inequality constraints $B \cdot X \leqslant b$ together with equality constraints $B' \cdot X = b'$, where $b = [0, 0, 0]^T$, $b' = [1]$, $B = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & -1 \end{pmatrix}$, $B' = \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}$, where $r_c, p_c, s_c \geqslant 0$.

   So, the linear problem is represented as

$$\begin{aligned} \text{Minimize } \; C \cdot X, \; s.t.& \\ B \cdot X &\geqslant b \\ B' \cdot X &= b' \qquad \forall x_i \in X, x_i \geqslant 0 \end{aligned}$$

   b. (3 points) Take the dual of this linear program, and explain how it solves the same problem as above but with the roles of the two players swapped.

We define B_r as $\left(\begin{smallmatrix} B \\ B' \end{smallmatrix}\right)^T$, multiplying B_r by new variables vector $Y = (\begin{smallmatrix} r_r & p_r & s_r & w_r \end{smallmatrix})^T$, $\forall y_i \in Y$, $y_i \geqslant 0$, we know that each row of the resulting matrix forming the coefficients for the original X in objective function, C, which should be no bigger than C. So we have the following new constraints

$$\begin{aligned} p_r - s_r &\leqslant w_r \\ -r_r + s_r &\leqslant w_r \\ r_r - p_r &\leqslant w_r \\ -r_r - p_r - s_r &\leqslant 1 \end{aligned} \tag{3}$$

Multiplying b^T by Y, we get the new objective function as follows, which strives to find the best possible lower bound

$$\text{Maximize } \ w_r$$

As the role of the two players swapped, the problem becomes

$$\begin{aligned} \text{Maximize } \ b^T \cdot Y,\ s.t. & \\ B_r \cdot Y \leqslant C & \qquad \forall y_i \in Y, y_i \geqslant 0 \end{aligned}$$

The left-hand side of (3) is equivalent to $A \cdot (\begin{smallmatrix} r_r & p_r & s_r \end{smallmatrix})^T$, and w_r is interpreted as $\max(p_r - s_r, -r_r + s_r, r_r - p_r)$ and thus the problem is interpreted as choosing the best strategy which would maximize *expected payoff* for row player himself/herself.

c. (5 points) Because these two linear programs are dual, the linear programming duality theorem says that they have the *same* optimal objective value. Consider the following process: you solve the linear program from part b) and choose a row according to that probability distribution, and your opponent solves the linear program from part a) and chooses a column according to that probability distribution. Conclude that both players would be happy to play like this, that neither can make more profit by changing her strategy.

(This pair of probabilistic strategies is called a *Nash equilibrium*, and we have shown its existence here, and found it, by the power of linear programming duality.)

Based on *Duality Theorem*, problems in a) and b) are dual problem to each other and thus they have the same optimum *expected payoff*.

From column player's perspective, the strategy guarantees that the row player's *expected payoff* would be at most E. On the other hand, from row player's perspective, his/her strategy guarantees that his/her own *expected payoff* would be at least E. So they are happy to play like this and E would be 0, the possibilities of these 3 choices would be $\frac{1}{3}$ equally. From here, we could draw a conclusion that neither can make more profit by changing his/her strategy.