# Homework 0

## Solution Key

### Problem 1

**Problem:** Let the function $f(n)$, for positive integer $n$, be defined by the following sum

$$f(n) = \sum_{i=1}^{n} 2^i(i+1)$$

Show that the function $f(n)$ has the closed form $n \cdot 2^{n+1}$ by using mathematical induction.

**Solution:** Let $f(n) = \sum_{i=1}^{n} 2^i(i+1)$, where $n$ is a positive integer, as defined in the problem. We will prove the claim that $f(n) = n \cdot 2^{n+1}$, for all integers $n \geq 1$ by induction.

Base Case: For $n = 1$,

$$\begin{aligned} f(1) &= \sum_{i=1}^{1} 2^i(i+1) \\ &= 2^1(1+1) \\ &= 4 \end{aligned}$$

Then, if we substitute $n = 1$ into the closed form, we get

$$n \cdot 2^{n+1} = 1 \cdot 2^{1+1} = 4$$

And this is trivially equal to the value we got for $f(1)$. So our base case holds, and $f(n) = n \cdot 2^{n+1}$ is true for $n = 1$.

Inductive Step: Let $P(k)$ be $f(n) = n \cdot 2^{n+1}$ for any $n \leq k$. We must show that $P(k) \implies P(k+1)$. Assume $P(k)$. Then

$$\begin{aligned} f(k) &= k \cdot 2^{k+1} \\ f(k) + 2^{k+1}(k+2) &= k \cdot 2^{k+1} + 2^{k+1}(k+2) \\ \sum_{i=1}^{k} 2^i(i+1) + 2^{k+1}(k+2) &= 2^{k+1}(2k+2) \\ \sum_{i=1}^{k+1} 2^i(i+1) &= 2^{k+1}(2)(k+1) \\ f(k+1) &= (k+1)2^{k+2}, \end{aligned}$$

and thus $P(k+1)$. We have shown that the base case holds and that $P(k) \implies P(k+1)$, thus by induction $P(k)$ is true $\forall k \geq 1$. Therefore, by induction, $f(n) = n \cdot 2^{n+1}$, for all integers $n \geq 1$.

## Problem 2

The final ordering is $e^{\log\log n}$, $n$, $6n\log n$, $4n^2$, $7n^3-10n$, $n^{8621909}$, $n^{\log n}$, $3^n$, $n!$.

Consider the function $e^{\log\log n}$ (**f**). Because $e^{\log a}=a$, we have that

$$e^{(\log\log n)}=\log n.$$

Since $n$ (**c**) grows faster than $\log n$, **c** grows faster than **f**.

Next, $6n\log n$ (**h**) grows faster than $n$ (**c**) due to the extra factor of $\log n$ in **h**.

Consider the functions $6n\log n$ (**h**) and $4n^2$ (**b**). Removing the common factor of $n$ from both and noting that $n$ grows faster than $\log n$, we see that **b** must grow faster than **h**.

The polynomial function $7n^3-10n$ (**a**) grows faster than $4n^2$ (**b**) because the degree of **a** is greater than the degree of **b**. By the same argument, **d** must grow faster than **a**.

Consider the functions $n^{8621909}$ (**d**) and $n^{\log n}$ (**g**). Observe that for $n>e^{8261909}$, $\log n>8261909$, and thus **g** must grow faster than **d**.

One way to argue that $3^n$ (**e**) grows faster than $n^{\log n}$ (**g**) is to compare the value after taking the logarithm of both functions. The function in **e** becomes $\log 3^n=(\log 3)\cdot n$ and the function in **g** becomes $\log n^{\log n}=(\log n)(\log n)$, so **e** grows faster than **g**.

Finally, the claim that $n!$ (**i**) grows faster than $3^n$ (**e**) relies on the fact that $n!>3^n$ for integers $n\geq 7$. This could easily be proven with induction using the facts that $(n+1)!=(n+1)\cdot n!$ and $3^{n+1}=3\cdot 3^n$ and noting that thus for $n\geq 3$, the change in **i** is greater than the change in **e**.

## Problem 3

1. Both $n!$ and $n^n$ are the products of $n$ terms. Let $t_k$ be any of the terms in $n!$, and let $t_m$ be any of the terms in $n^n$. We know that $t_k\leq t_m$, so we know that $n!\leq n^n$.

2. Using a similar approach, we know that $n!$ is the product of all integers between 1 and $n$. We note that there are at least $n/2$ terms in the product which are at least $n/2$. Since the number of such terms and the terms themselves are integers, there are at least $\lceil n/2\rceil$ terms each at least $\lceil n/2\rceil$. This means when we multiply all the terms in $n!$ together, we get a product that must necessarily be larger than $\lceil n/2\rceil$.

3. Combining the previous parts, we obtain that $(\lceil n/2\rceil)^{\lceil n/2\rceil}\leq n!\leq n^n$. By taking the logarithm of the functions in the inequalities, we get

   $$\lceil n/2\rceil\log\lceil n/2\rceil\leq\log n!\leq n\log n$$

   Since $\log n!$ is bounded above and below by functions in $\Theta(n\log n)$, $\log n!$ itself must also be in $\Theta(n\log n)$.

4. We cannot use the logic from part 3 because $(\lceil n/2\rceil)^{\lceil n/2\rceil}$ is not $\Theta(n^n)$.

   Beginning a proof by contradiction, assume that $n!=\Theta(n^n)$. In order for this to be true, there must exist some positive constant $c$ and some positive integer $n_0$ such that for all $n>n_0$, $cn^n\leq n!$. This means that

   $$c\leq\frac{n!}{n^n}=\frac{1}{n}\cdot\frac{2}{n}\cdot\ldots\cdot\frac{n}{n}$$

Each term in the product is at most 1, so the product is less than or equal to any individual term of the product. Thus we have

$$c \leq \frac{1}{n}$$

Since this holds for all $n > n_0$, this implies that $c = 0$, which is a contradiction. Therefore, $n!$ is not bounded below by $n^n$, and thus $n! = \Theta(n^n)$ is false.

## Problem 4

1. Motivation: First, notice that $T(0) = 0$, $T(1) = 1$, $T(2) = 3$, $T(3) = 7$ ...suggesting $T(n) = 2^n - 1$.

   Proof: We will prove this by induction.

   Base Case: Consider the base case $n = 0$.

   $$T(0) = 0 = 2^0 - 1.$$

   Inductive Step: Let $P(k)$ be $T(n) = 2^n - 1 \; \forall n \leq k$. We must show that $P(k) \implies P(k+1)$. Assume $P(k)$. Then

   $$\begin{aligned} &T(k+1) = 2 * T(k) + 1 \\ &T(k+1) = 2(2^k - 1) + 1 \text{ (by the inductive hypothesis)} \\ &T(k+1) = 2^{k+1} - 1, \end{aligned}$$

   and thus $P(k+1)$. We have shown that the base case holds and that $P(k) \implies P(k+1)$, thus by induction $P(k)$ is true $\forall k \geq 0$.

2. Motivation: Expanding the first few terms yields

   $$\begin{aligned} T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\ &= 4 \cdot T\left(\frac{n}{4}\right) + 2n \\ &= 8 \cdot T\left(\frac{n}{8}\right) + 3n, \end{aligned}$$

   suggesting an alternative formula of $T(n) = 2^k \cdot T(\frac{n}{2^k}) + kn$ for any $k \geq 1$.

   Proof: We will prove this by induction.

   Base Case: When $k = 1$, we get $2^1 \cdot T(\frac{n}{2^1}) + 1 \cdot n$, the original relation.

Inductive Step: Let $P(k)$ be $T(n) = 2^l \cdot T(\frac{n}{2^l}) + ln$ for any $l \leq k$. We must show that $P(k-1) \implies P(k)$. Assume $P(k-1)$. Then

$$\begin{aligned}
T(n) &= 2^{k-1} \cdot T\left(\frac{n}{2^{k-1}}\right) + (k-1)n \\
&= 2^{k-1}\left[2 \cdot T\left(\frac{n}{2^{k-1} \cdot 2}\right) + \frac{n}{2^{k-1}}\right] + (k-1)n \qquad \text{(by inductive hypothesis)} \\
&= 2^{k-1}\left[2 \cdot T\left(\frac{n}{2^{k}}\right) + \frac{n}{2^{k-1}}\right] + (k-1)n \\
&= 2^{k} \cdot T\left(\frac{n}{2^{k}}\right) + n + (k-1)n \\
&= 2^{k} \cdot T\left(\frac{n}{2^{k}}\right) + kn,
\end{aligned}$$

and thus $P(k)$. We have shown that the base case holds and that $P(k-1) \implies P(k)$, thus by induction $P(k)$ is true $\forall k \geq 1$. Using the alternative form we have proven, when $k = \log_2 n$,

$$\begin{aligned}
T(n) &= 2^{\log_2 n} \cdot T\left(\frac{n}{2^{\log_2 n}}\right) + n\log_2 n \\
&= n \cdot T(1) + n\log_2 n \\
&= \Theta(n \log n).
\end{aligned}$$

## Problem 5

1. The sum of matrices
$$\begin{bmatrix} 1 & 0 & 5 \\ 2 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 6 & 2 \\ 1 & 7 & 4 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 7 \\ 3 & 9 & 5 \\ 0 & 3 & 4 \end{bmatrix}$$

2. The product of matrices

   (a) $\begin{bmatrix} 2 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 7 & 6 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 7 & 4 & 2 & 6 \\ 2 & 1 & 1 & 1 & 9 \\ 100 & 99 & 98 & 97 & 96 \\ 0 & 10 & 5 & 2 & 1 \\ 10 & 1 & 19 & 28 & 33 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 97 \\ 2 \\ 28 \end{bmatrix}$

3. The transpose of the matrix
$$\begin{bmatrix} 1 & 2 \\ 0 & 6 \\ 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 6 & 2 \end{bmatrix}$$

4. Find the inverse of the following matrices where possible. If it is impossible, say why it is impossible.

   (a) $\begin{bmatrix} 1 & 3 & 5 \\ 8 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -0.1 & 0.1 & 0.3 \\ -0.3 & -0.2 & 1.9 \\ 0.4 & 0.1 & -1.2 \end{bmatrix}$

(b) This is impossible because the determinant of the matrix is 0.

5. (a) $\begin{bmatrix} 7 & 0 \end{bmatrix} = 3 \cdot \begin{bmatrix} 3 & 1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 2 & 3 \end{bmatrix}$

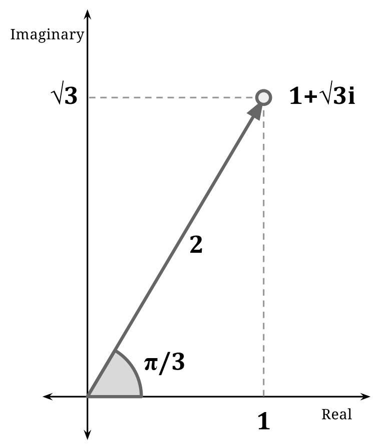
   (b) $\left\{ \begin{bmatrix} 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \end{bmatrix} \right\}$ is a basis of a vector space which contains the vector $\begin{bmatrix} 7 & 0 \end{bmatrix}$. We can represent $\begin{bmatrix} 7 & 0 \end{bmatrix}$ in the space as $3 \cdot \begin{bmatrix} 3 & 1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 2 & 3 \end{bmatrix}$.

   (c) The vector representing the coefficients of the linear combinations of $\begin{bmatrix} 7 & 0 \end{bmatrix}$ in terms of $\begin{bmatrix} 3 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \end{bmatrix}$ is

$$\begin{bmatrix} 7 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \end{bmatrix}$$

## Problem 6

Figure 1: $1 + \sqrt{3}$ in polar coordinates

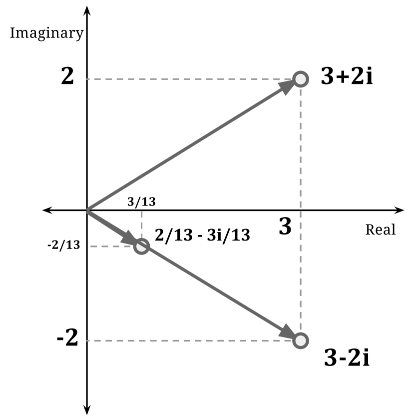



1. Magnitude of $1 + \sqrt{3}i$ is $\sqrt{1^2 + \sqrt{3}^2} = 2$ and the angle is $\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$.

2. The complex conjugate of $3 + 2i$ is $3 - 2i$, by definition.

3. The reciprocal of $3+2i$ is $\dfrac{1}{3+2i} \cdot \dfrac{3-2i}{3-2i} = \dfrac{3-2i}{3^2+2^2} = \dfrac{3}{13} - \dfrac{2i}{13}$

Figure 2: $3+2i$ and its reciprocal and conjugate




4. Given two complex numbers $(r_1, \theta_1)$ and $(r_2, \theta_2)$, the product of both numbers is $(r_1 \cdot r_2, \theta_1 \cdot \theta_2)$.
5. The $k$-th power of the complex number $(r, \theta)$ is $(r^k, k\theta)$.
6. Solving for $z^6 = 1$: suppose that $z = (r, \theta)$, then

$$(r^6, 6\theta) = (1, 0 \mod 2\pi)$$

That is, $r^6 = 1$ and $6\theta = 0 \mod 2\pi$. Therefore, $r = 1$ and $\theta = \frac{2\pi k}{6} = \frac{\pi k}{3}$ for $k = 0, 1, \ldots, 5$. Therefore, all complex numbers whose 6th power is 1 are $z = (r, \pi k/3)$ for $k = 0, 1, \ldots, 5$.