CS157 Homework 8

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Problem 3

Some courses are rewarding to take; other courses you take only because they are prerequisites for rewarding courses later on.

Suppose a university offers n courses, and each course i has a set of prerequisites $p_i \subset [n]$, all of which you must take if you plan on taking course i. (You may assume that there are no cyclic dependencies in the prerequisite list.) In addition, for each course i, you also know how rewarding it will be to take, a number r_i, in arbitrary units, which could be positive or negative! Your goal is to find a set C of courses to take, subject to the prerequisite constraints (for each course i and each prerequisite $j \in p_i$, if i is in C then j is in C), in order to maximize $\sum_{i \in C} r_i$.

1. (5 points) Express this problem as an integer program, where each course corresponds to a variable that has value 0 if the course is not taken, and 1 if it is. Each constraint should enforce a single relation of the form "course j is a prerequisite for course i".

   The variables we define are as follows:

   $$x_i = \text{whether course } i \text{ is taken or not, 1 represents it will be taken, 0 as opposite}$$

   Our objective is to maximize the overall rewards. The reward of course-i would be taken into account when $x_i = 1$. The objective function is shown in (1).

   $$\text{Maximize} \sum_{i \in [n]} r_i x_i \tag{1}$$

   The constraint for each course-i we could derive is represented in (2), which means if course-i would be taken, all of its corresponding prerequisite courses should be taken for sure and thus the right-hand side (the sum of all prerequisite) has to be no less than $|p_i|$ times x_i no matter whether course-i was taken or not.

   $$|p_i| \cdot x_i \leqslant \sum_{j \in p_i} x_j \tag{2}$$

   We define *IsPre* as a function denoting the preceding relationship between each pair of courses, its formal representation is as follows:

   $$\mathit{IsPre}(i,j) = \begin{cases} -1 & \text{course-}j \text{ is course-}i\text{'s correspoinding prerequisite} \\ 0 & \text{course-}j \text{ isn't } i\text{'s prerequisite} \end{cases}$$

   Now (2) could be represented in the form $A \cdot x \leqslant b$, where x is the $n \times 1$ input vector representing the corresponding variable for each course, b is a $n \times 1$ zeros vector. A is a $n \times n$ matrix shown as follows in detail:

   $$A_{n \times n} = \begin{pmatrix} |p_1| & \mathit{IsPre}(1,2) & \mathit{IsPre}(1,3) & \cdots & \mathit{IsPre}(1,n) \\ \mathit{IsPre}(2,1) & |p_2| & \mathit{IsPre}(2,3) & \cdots & \mathit{IsPre}(2,n) \\ \vdots & & & \ddots & \vdots \\ \mathit{IsPre}(n,1) & \mathit{IsPre}(n,2) & \mathit{IsPre}(n,3) & \cdots & |p_n| \end{pmatrix}$$

To sum these up, the problem represented in the integer program form is as below:

$$\text{Maximize } \sum_{i\in[n]} r_i x_i, \text{ such that } A \cdot x \leqslant b, \text{ where } x_i \in \{0,1\} \tag{3}$$

2. (10 points) Show that this problem can in fact be solved in polynomial time by linear programming: taking the above integer program and ignoring the integrality constraints, show that the resulting linear program has an optimal solution all of whose coordinates are integers.

   (**Hint**: Two possible ways to prove this include 1) consider a hypothetical optimum solution to the linear program, take all its non-integer coordinates, and figure out how to modify them to a solution that is at least as good but which has more of its coordinates equal to 0 or 1; or 2) make use of the fact that the set of optima of a linear program always includes a vertex of the constraint polytope, and show that such a vertex must have 0-1 coordinates.)

   We know that the feasible region of a linear program is specied by a set of inequalities and is therefore the intersection of the corresponding half-spaces, a convex polyhedron.

   Based on the face that the set of optima of a linear program always includes a vertex of the constraint polytope, and there are n variables in (1), we need at least n linear equality functions to uniquely determine a feasible solution. Given the integer program in (3), the vertex we are striving to find is supposed to be satisfied with these n linear equality constraints such that

$$|p_i| x_i - \sum_{j\in p_i} x_j = 0 \tag{4}$$

   From (4), we could infer that x_i and each of its corresponding courses' coefficient x_j all would be either 1 or 0 and thus every dimension of the vertex must have either coordinate 0 or 1. So the vertex must have 0-1 coordinates. This also implies that the resulting linear program has an optimal solution all of whose coordinates are integers.

   Now we can draw a conclusion that the problem can be solved in polynomial time by linear programming.