CS157 Homework 8

BY SILAO_XU AND TC28

Problem 1

Many industries have to solve some version of the following "hiring problem." Suppose you run a business that needs 8000 hours of labor in May, 9000 hours in June, 7000 in July, 10,000 in August, 9000 in September, and and 11,000 in October. Also, suppose that currently, as May starts, you have 60 experienced employees working for you. Each month, each experienced employee can either work up to 150 hours for in that month or work up to 50 hours that month while also training one new hire who will then be ready to work the following month (and will be called an "experienced" worker the following month). At the end of each month, 10% of your experienced employees quit. It costs \$3400 a month to employ an experienced worker, and \$1800 a month for each trainee.

1. (10 points) Write a linear program that represents this problem, and describe what corresponds to what, and why it accurately represents the problem. (Note, do not worry about integrality of the solution; the optimum of the linear program may include things that mean, for example, "hire 27.3 people in June".)

   The variables we define are as follows:

$$\begin{aligned} w_i &= \text{number of workers for month } i \\ t_i &= \text{number of experienced workers participate in the training for month } i \\ c_i &= \text{the cost for employing in month } i \end{aligned}$$

   The constraints we could infer are as follows:

$$\begin{aligned} 150 \cdot (w_{May} - t_{May}) + 50 \cdot t_{May} &\geqslant 8000 \\ 150 \cdot (w_{Jun} - t_{Jun}) + 50 \cdot t_{Jun} &\geqslant 9000 \\ 150 \cdot (w_{Jul} - t_{Jul}) + 50 \cdot t_{Jul} &\geqslant 7000 \\ 150 \cdot \left(w_{Aug} - t_{Aug}\right) + 50 \cdot t_{Aug} &\geqslant 10000 \\ 150 \cdot \left(w_{Sep} - t_{Sep}\right) + 50 \cdot t_{Sep} &\geqslant 9000 \\ 150 \cdot (w_{Oct} - t_{Oct}) + 50 \cdot t_{Oct} &\geqslant 11000 \\ w_{May} &= 60 \\ w_{Jun} &= 0.9 \cdot w_{May} + t_{May} \\ w_{Jul} &= 0.9 \cdot w_{Jun} + t_{Jun} \\ w_{Aug} &= 0.9 \cdot w_{Jul} + t_{Jul} \\ w_{Sep} &= 0.9 \cdot w_{Aug} + t_{Aug} \\ w_{Oct} &= 0.9 \cdot w_{Sep} + t_{Sep} \\ w_{May}, w_{Jun}, w_{Jul}, w_{Aug}, w_{Sep}, w_{Oct} &\geqslant 0 \\ t_{May}, t_{Jun}, t_{Jul}, t_{Aug}, t_{Sep}, t_{Oct} &\geqslant 0 \end{aligned} \tag{1}$$

   For each c_i, it could be interpretted as

$$c_i = 3400 \cdot w_i - 1600 \cdot t_i$$

Now we have 12-variable vector $[w_{May}, t_{May}, ..., w_{Oct}, t_{Oct}]$. The matrix form of representing the constraints in (1) should be interpreted as inequality constraints $A \cdot X \leqslant b$ together with equality constraints $A' \cdot x = b'$, where $b = (\ -8000\ \ -9000\ \ -7000\ \ -10000\ \ -9000\ \ -11000\ )^T$, $b' = (\ 60\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ )^T$, $X = (\ w_{May}\ \ t_{May}\ \ \cdots\ \ w_{Oct}\ \ t_{Oct}\ )^T$. In A and A', each two columns represents a pair w_i and t_i, each row-i represents the negation of the total hours all workers would contribute in month i.

And thus our objective function could be represented as follows:

$$\text{Minimize } 3400\sum_i w_i - 1600\sum_i t_i, \text{ such that } A\cdot x \leqslant b, \forall x_j \in X, x_j \geqslant 0$$

2. (5 points) Solve the linear program via Matlab's **linprog** routine, and describe the optimal hiring strategy, along with the details of how you set up the problem in Matlab. (Type help linprog for details.)

We have expressed the linear program in the matrix form $A \cdot x \leqslant b$ and $A' \cdot x = b'$. Matrix A and A' are expressed as follows respectively:

$$A_{6\times 12} = \begin{pmatrix} -150 & 100 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -150 & 100 & & & \vdots \\ \vdots & & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -150 & 100 \end{pmatrix}$$

$$A'_{6\times 12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -0.9 & -1 & 1 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9 & -1 & 1 & & & & & \vdots \\ \vdots & & & & & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & & -0.9 & -1 & 1 & 0 \end{pmatrix}$$

In A, each two columns represents a pair w_i and t_i, each row-i represents the neation of the total hours all workers contributed in that month. In A', each three column represents w_{i-1}, t_i and w_i for the reason that each month's total workers number consists of last month's 90 percent of last month's experienced workers and last month new workers who got trained.

The resulting strategy computed by the linear programming is as follows:

$$X = \begin{pmatrix} 60.00 \\ 7.81 \\ 61.80 \\ 2.71 \\ 58.34 \\ 17.51 \\ 70.02 \\ 5.03 \\ 68.06 \\ 12.08 \\ 73.33 \\ 0.00 \end{pmatrix}$$

which means we could arrange 7.81, 2.71, 17.51, 5.03, 12.08 and 0 experienced works for training from May to Octorber respectively. And the overall minimal cost for employing in these 6 months is 1259100.