

Chapter 2 Section 3

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Problem 1. *Calculate the matrix product*

$$\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Solution.

$$\begin{aligned} \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} &= \begin{bmatrix} 6 * 1 + 7 * 3 & 6 * 2 + 7 * 5 \\ 8 * 1 + 9 * 3 & 8 * 2 + 9 * 5 \end{bmatrix} \\ &= \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix} \end{aligned}$$

Problem 2. *Compute the products BA and AB for*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Interpret your answers geometrically, as composites of linear transformation.

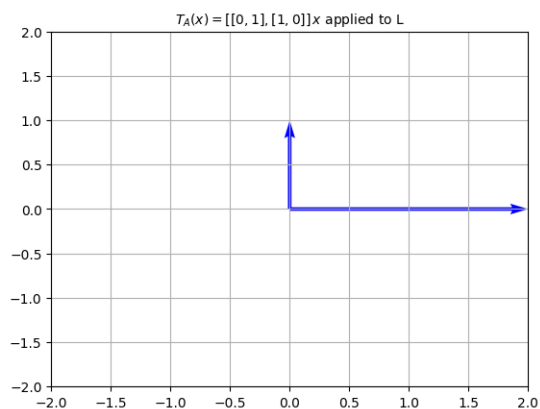
Solution. Let $T_A(\vec{x}) = A\vec{x}$ and $T_B(\vec{y}) = B\vec{y}$. We can write

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [T_A(e_1) \quad T_A(e_2)] \\ B &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = [T_B(e_1) \quad T_B(e_2)] \end{aligned}$$

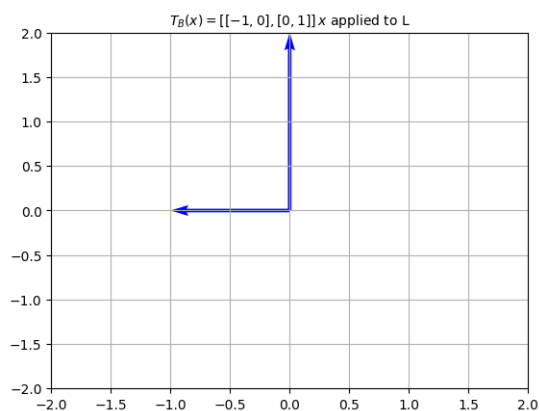
Thus

$$\begin{aligned} T_A(\vec{e}_1) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, T_A(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T_B(\vec{e}_1) &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, T_B(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

We see that T_A is a reflection about the line $y = x$. In other words, T_A is a reflection about the line spanned by the vector $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



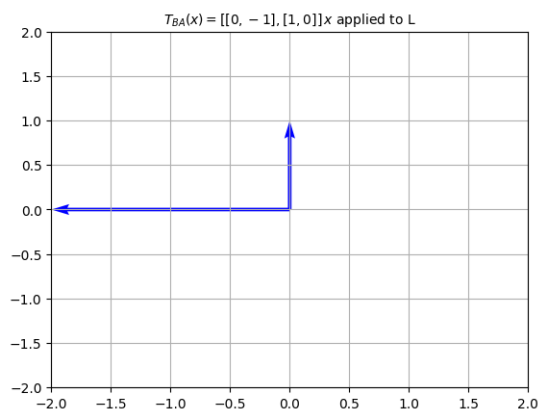
We see that T_B is a reflection about the line $x = 0$. In other words, T_B is a reflection about the line spanned by the vector $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



Now let's compute the products BA and AB .

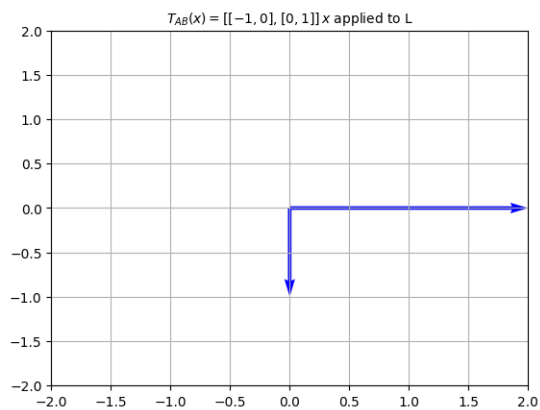
$$\begin{aligned} BA &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

The product BA is a rotation matrix that rotates a vector ninety degrees counterclockwise.



$$\begin{aligned}
 AB &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
 \end{aligned}$$

The product AB is a rotation matrix that rotates a vector ninety degrees clockwise.



Problem 3. Multiply the block matrices

$$\left(\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \end{array} \right) \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array} \right)$$

Afterwards, compute the product without a partition and see if you get the same result.

Solution.

$$\begin{aligned}
& \left(\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \end{array} \right) \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array} \right) \\
&= \left(\begin{array}{cc|c} \left[\begin{array}{cc} 0 & 1 \end{array} \right] \left[\begin{array}{cc} 1 & 2 \end{array} \right] + \left[\begin{array}{c} -1 \\ 1 \end{array} \right] \left[\begin{array}{cc} 7 & 8 \end{array} \right] & \left[\begin{array}{cc} 0 & 1 \end{array} \right] \left[\begin{array}{c} 3 \\ 6 \end{array} \right] + \left[\begin{array}{c} -1 \\ 1 \end{array} \right] \left[\begin{array}{c} 9 \end{array} \right] \\ \hline \end{array} \right) \\
&= \left(\begin{array}{cc|c} \left[\begin{array}{cc} 4 & 5 \end{array} \right] + \left[\begin{array}{cc} -7 & -8 \end{array} \right] & \left[\begin{array}{c} 6 \\ 3 \end{array} \right] + \left[\begin{array}{c} -9 \\ 9 \end{array} \right] \\ \hline \end{array} \right) \\
&= \left(\begin{array}{cc|c} \left[\begin{array}{cc} -3 & -3 \end{array} \right] & \left[\begin{array}{c} -3 \\ 12 \end{array} \right] \\ \hline \end{array} \right) \\
&= \left(\begin{array}{cc|c} -3 & -3 & -3 \\ 8 & 10 & 12 \end{array} \right)
\end{aligned}$$

Now let's compute the product without a partition.

$$\begin{aligned}
& \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\
&= \begin{bmatrix} 4-7 & 5-8 & 6-9 \\ 1+7 & 2+8 & 3+9 \end{bmatrix} \\
&= \begin{bmatrix} -3 & -3 & -3 \\ 8 & 10 & 12 \end{bmatrix}
\end{aligned}$$

We get the same result.

In the following problems, multiply the matrices and describe the linear transformations geometrically.

Problem 4.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution. The first matrix is a horizontal shear with $k = 1$.

$$\begin{aligned}
& \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}
\end{aligned}$$

Problem 5.

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

Solution. Let A be a n by p matrix and let B be a q by m matrix. The product of a matrix AB is defined if and only if $p = q$.

In the matrices below, $p = q$, so the product of the matrices is defined.

$$\begin{aligned} & \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3-1 & 2-0 \\ 2 & 0 \\ 6+1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix} \end{aligned}$$

Problem 6.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution. The matrix product is not defined, since the number of columns in the first matrix (3) does not equal the number of rows in the second matrix (2)

Problem 7.

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix}$$

Solution.

$$\begin{aligned} & \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7-3 & 5-1 \\ -14+6 & -10+2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 4 \\ -8 & -8 \end{bmatrix} \end{aligned}$$

Problem 8.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution.

$$\begin{aligned} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Problem 9.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 10.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \\ = \begin{bmatrix} 1-2 & 2-1 & 3-3 \\ 3+2 & 2+1 & 1+3 \\ 1-3-4 & 2-2-2 & 3-1-6 \end{bmatrix} \\ = \begin{bmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{bmatrix}$$

Problem 11.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Solution.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

The matrix product represents a scaling by a factor of $ad - bc$.

Problem 12.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -6 & 8 \\ 3 & -4 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -6 & 8 \\ 3 & -4 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 13.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \quad 2 \quad 3]$$

Solution. *Multiplying the column vector by the row vector, we get*

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \quad 2 \quad 3] \\ = 1 + 4 + 9 \\ = 14$$

Problem 14.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [3 \quad 2 \quad 1]$$

Solution.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [3 \quad 2 \quad 1] \\ = 3 + 4 + 3 \\ = 10$$

Problem 15.

$$[1 \quad 0 \quad -1] \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution.

$$[1 \quad 0 \quad -1] \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \\ = [0 \quad 1]$$

Problem 16.

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Solution.

$$\begin{aligned} & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} g & h & k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= h \end{aligned}$$