Chapter 3 Section 2

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Problem 1. *Is* $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 \colon x \geq 0 \text{ and } y \geq 0 \right\}$ *a subspace of* $\mathbb{R}^2$*?*

Solution. *W contains the zero vector and is closed under addition. But W is not closed under scalar multiplication. Therefore W is not a subspace of $\mathbb{R}^2$.*

Problem 2. *Show that the only subspaces of $\mathbb{R}^2$ are $\mathbb{R}^2$ itself, the set $\{\vec{0}\}$, and any of the lines through the origin.*

Solution. *Let W be a subspace of $\mathbb{R}^2$ that is neither a line through the origin nor the set $\{\vec{0}\}$. Then we can choose two nonzero nonparallel vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ from our subspace W. Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. We will show that we can write $\vec{u}$ as a linear combination of $\vec{v}$ and $\vec{w}$.*

If $\vec{u}$ can be written as a linear combination of $\vec{v}$ and $\vec{w}$, then there are solutions to the equation

$$x_1\vec{v} + x_2\vec{w} = \vec{u}$$

where x_1 and x_2 are real numbers. We can write this equation in matrix form

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This equation has solutions when $A = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ is invertible. We know that A is invertible when $\det A$ is nonzero.

There exist real numbers c_1 and c_2 such that $c_1v_1 = w_1$ and $c_2v_2 = w_2$. We know that $c_1 \neq c_2$ since the two vectors are not parallel (or else one vector would be a scalar multiple of the other). We can substitute these expressions when we calculate the determinant of A.

$$\begin{aligned}\det A &= v_1 w_2 - v_2 w_1 \\ &= v_1(c_2 v_2) - v_2(c_1 v_1) \\ &= c_2 v_1 v_2 - c_1 v_1 v_2 \\ &= v_1 v_2 (c_2 - c_1)\end{aligned}$$

The only case where the equations $v_1 v_2 = 0$ *and* $w_1 w_2 = 0$ *both hold true is when* $\vec{v}$ *and* $\vec{w}$ *are scalar multiples of the unit vectors* e_1 *and* e_2*, respectively. In this case, it's easy to show that* $\vec{u}$ *can be written as a linear combination of* $\vec{v}$ *and* $\vec{w}$*.*

Suppose $\vec{v}$ *and* $\vec{w}$ *are not scalar multiples of the unit vectors. Then either* $\vec{v}$ *or* $\vec{w}$ *has two nonzero components. Let* $\vec{v}$ *be the vector with two nonzero components. Then the determinant of* A *is not zero, since* $c_1 \neq c_2$*. Therefore* A *is invertible, and the equation*

$$x_1 \vec{v} + x_2 \vec{w} = \vec{u}$$

has solutions for x_1 *and* x_2*.*

Since W *is closed under linear combinations, the vector* $\vec{u}$ *has to be in the subspace* W*. This means that every real number is contained in* W*, so* $W = \mathbb{R}^2$*.*