Chapter 3 Section 4

Andrew Taylor

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Problem 1. *Let* $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. *Let* $V = \mathrm{span}(\mathrm{v}_1, \mathrm{v}_2)$. *Is the vector* $\vec{w} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$ *on the plane V?*

Solution. *If the vector* $\vec{w}$ *is on the plane V, then there exist some* $x_1, x_2 \in \mathbb{R}$ *such that* $\vec{w} = x_1\vec{v_1} + x_2\vec{v_2}$. *This gives us the equations*

$$\begin{aligned} x_1 + x_2 &= 5 \\ x_1 + 2x_2 &= 7 \\ x_1 + 3x_2 &= 9 \end{aligned}$$

We can solve these equations using a matrix.

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

This gives us $x_1 = 3$ *and* $x_2 = 2$.

Thus $\vec{w}$ *is on the plane V because* $\vec{w} = 3\vec{v_1} + 2\vec{v_2}$.

Problem 2. *Consider the basis* $\mathfrak{B}$ *of* $\mathbb{R}^2$ *consisting of the vectors* $\vec{v_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ *and* $\vec{v_2} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

- *If* $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ *find* $[\vec{x}]_{\mathfrak{B}}$
- *If* $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ *find* $\vec{y}$

Solution. *We can find the coordinates of* $\vec{x}$ *with respect to* $\mathfrak{B}$ *by means of an equation.*

$$\vec{x} = c_1\vec{v_1} + c_2\vec{v_2}$$
$$\begin{bmatrix} 10 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

We can solve this equation using elementary row operations.

$$\left(\begin{array}{cc|c} 3 & -1 & 10 \\ 1 & 3 & 10 \end{array} \right)$$
$$\left(\begin{array}{cc|c} 0 & -10 & -20 \\ 1 & 3 & 10 \end{array} \right)$$
$$\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 3 & 10 \end{array} \right)$$
$$\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 4 \end{array} \right)$$
$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array} \right)$$

By reducing the matrix, we find that $c_1 = 4$ *and* $c_2 = 2$. *Thus* $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

We can also use an equation to solve for $\vec{y}$.

$$
\begin{aligned}
\vec{y} &= 2\vec{v_1} - \vec{v_2} \\
&= 2\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}-1\\3\end{bmatrix} \\
&= \begin{bmatrix}6\\2\end{bmatrix} - \begin{bmatrix}-1\\3\end{bmatrix} \\
&= \begin{bmatrix}7\\-1\end{bmatrix}
\end{aligned}
$$