

## Chapter 2 Section 2

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The letter L can be represented by the vectors  $(0, 2)$  and  $(1, 0)$ .



The following problems ask for a linear transformation of the letter L. In the following problems, give the matrix of the transformation and plot the result.

**Problem 1.** Scale  $L$  by a factor of  $\frac{1}{2}$

**Solution.** The matrix of the transformation is

$$\begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}$$

*After the scaling, the  $L$  looks like this*



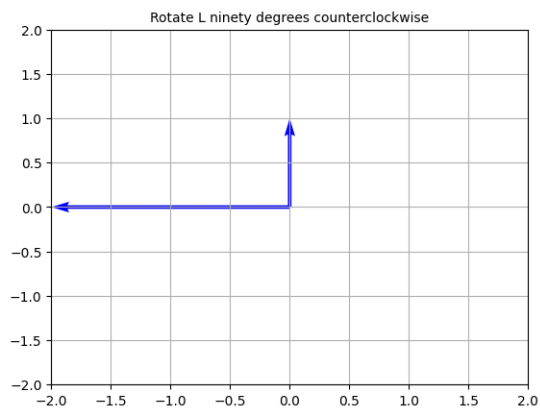
*Note that in creating this shape, we scaled both vectors that make up the L.*

**Problem 2.** *Rotate L ninety degrees counterclockwise*

**Solution.** *The matrix of the transformation is*

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

*After the rotation, the L looks like this*

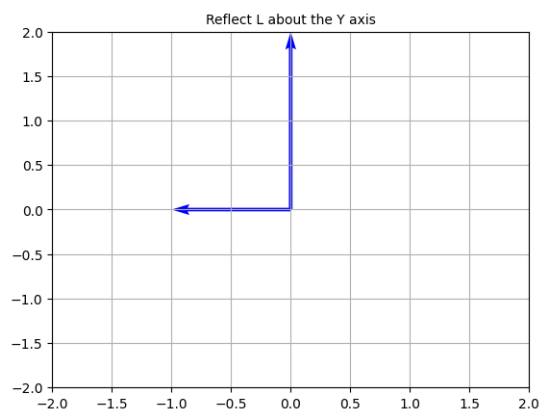


**Problem 3.** *Reflect L about the Y axis*

**Solution.** *The matrix of the transformation is*

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

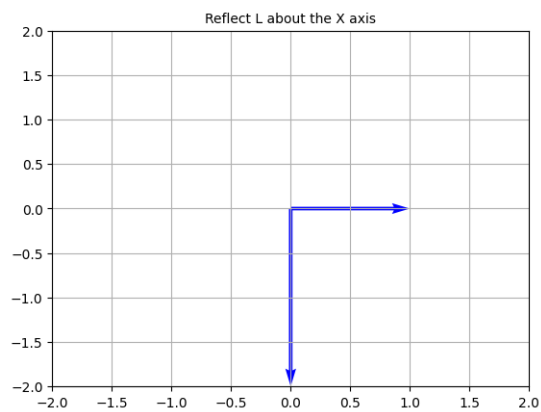
*The plot looks like this*



**Problem 4.** *Reflect  $L$  about the  $X$  axis*

**Solution.** *The matrix of the transformation is*

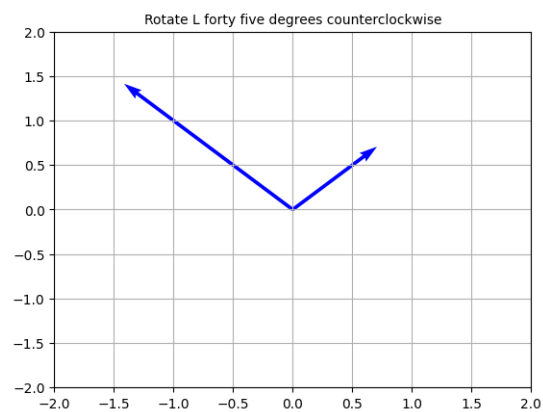
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



**Problem 5.** *Rotate  $L$  forty five degrees counterclockwise*

**Solution.** *The matrix of the transformation is*

$$\begin{bmatrix} \cos(\frac{\pi}{4}) & -1 * \sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix}$$



**Problem 6.** Find the orthogonal projection of  $L$  onto the  $x$ -axis

**Solution.** The matrix of the transformation is

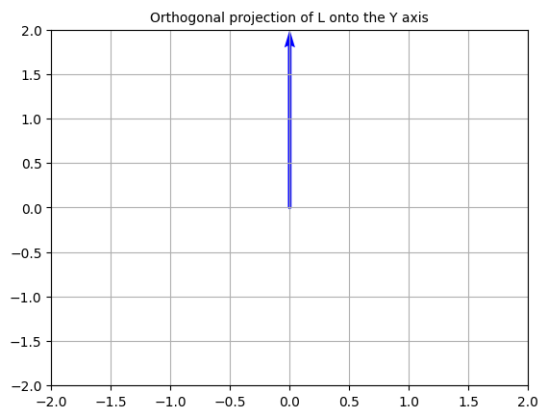
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



**Problem 7.** Find the orthogonal projection of  $L$  onto the  $y$ -axis

**Solution.** The matrix of the transformation is

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



**Problem 8.** Find the matrix  $P$  of the orthogonal projection onto the line  $L$  spanned by  $\vec{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

**Solution.**

$$\begin{aligned} P &= \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} \\ &= \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \\ &= \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix} \end{aligned}$$

**Problem 9.** Let  $V$  be the plane defined by  $2x_1 + x_2 - 2x_3 = 0$  and let  $\vec{x} = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$ . Find  $\text{ref}_V \vec{x}$ .

**Solution.** The vector  $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$  is perpendicular to the plane  $V$ .

We can get the unit vector  $\vec{u}$  perpendicular to the plane by

$$\begin{aligned} \vec{u} &= \frac{1}{\|\vec{v}\|} \vec{v} \\ &= \frac{1}{\sqrt{2^2 + 1^2 + (-2)^2}} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \end{aligned}$$

We can now use the formula

$$\begin{aligned}
 \text{ref}_V \vec{x} &= \text{proj}_V \vec{x} - \text{proj}_L \vec{x} \\
 &= \vec{x} - 2 \text{proj}_L \vec{x} \\
 &= \vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u} \\
 &= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - 2 \left( \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right) \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
 &= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
 &= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \\ -8 \end{bmatrix} \\
 &= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}
 \end{aligned}$$

The reflection of the vector  $\vec{x}$  over the plane  $V$  is the vector

$$\text{ref}_V \vec{x} = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$$

**Problem 10.** Give the matrix of a counterclockwise rotation through  $\frac{\pi}{6}$ .

**Solution.** The matrix of a counterclockwise rotation through  $\frac{\pi}{6}$  is

$$\begin{aligned}
 &\begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} \\
 &= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}
 \end{aligned}$$

**Problem 11.** Let  $T(\vec{x})$  be the linear transformation

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{x} \\ &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

How does the linear transformation  $T(\vec{x})$  affect the letter  $L$ ? Remember that our letter  $L$  is composed of vectors  $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

**Solution.** The vector  $\begin{bmatrix} a \\ b \end{bmatrix}$  can be written in polar coordinates as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \cos(\theta) \\ r \sin(\theta) \end{bmatrix}$$

We can substitute these polar coordinates into the linear transformation, and write it as

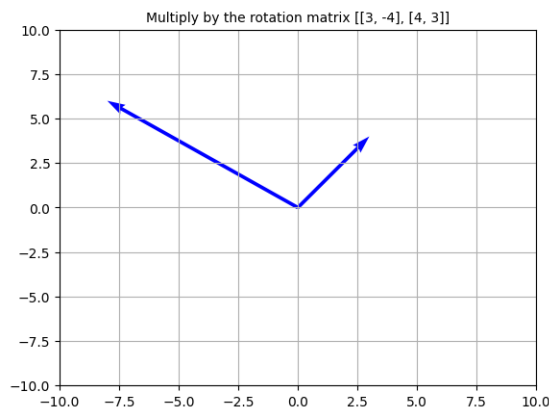
$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} r \cos(\theta) & -r \sin(\theta) \\ r \sin(\theta) & r \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

The linear transformation rotates the vector counterclockwise through an angle of  $\theta$  and scales the vector by a factor of  $r$ .

For example, let  $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ . Then we have

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &\approx 5 \begin{bmatrix} \cos(53.1^\circ) & -\sin(53.1^\circ) \\ \sin(53.1^\circ) & \cos(53.1^\circ) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

This rotates  $L$  approximately  $53.1^\circ$  and scales  $L$  by a factor of 5.



Thus we see that the linear transformation

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

rotates  $L$  counterclockwise through  $\theta$  degrees and scales  $L$  by a factor of  $r$ .

**Problem 12.** Sketch the image of the standard  $L$  under the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}$$

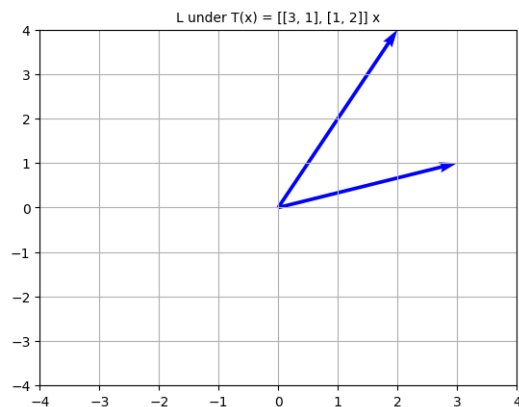
**Solution.** We can apply the linear transformation to both vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$  that make up our  $L$ .

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{aligned}$$

We can now draw these vectors on the Cartesian plane.





**Problem 13.** Find the matrix of a rotation through an angle of  $60^\circ$  in the counterclockwise direction.

**Solution.** The matrix of the rotation is

$$\begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

**Problem 14.** Consider a linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^3$ . Use  $T(\vec{e}_1)$  and  $T(\vec{e}_2)$  to describe the image of the unit square geometrically.

**Solution.** Let  $\vec{x}$  be a vector in the unit square. We can write the vector  $\vec{x}$  as

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$$

where  $0 \leq x_1, x_2 \leq 1$  and  $\vec{e}_1, \vec{e}_2$  are the unit vectors. Thus

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + x_2\vec{e}_2) \\ &= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) \\ &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) \end{aligned}$$

The vectors  $T(\vec{e}_1)$  and  $T(\vec{e}_2)$  describe a parallelogram in  $\mathbb{R}^3$ .

Thus  $T(\vec{x}) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2)$  is a vector in that parallelogram in  $\mathbb{R}^3$ .

Note that we got our equation for  $T(\vec{x})$  using the properties of linear transformations.

$T(a + b) = T(a) + T(b)$  and  $T(ka) = kT(a)$  for vectors  $a, b$  and scalar  $k$ .

**Problem 15.** Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$

**Solution.** The matrix of the transformation is a rotation matrix combined with a scaling. Since a rotation matrix has the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

We see that  $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The vector  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  forms a  $-45^\circ$  angle with the  $X$  axis. The vector has magnitude  $\sqrt{2}$ .

Thus the linear transformation  $T$  applies a rotation of  $45^\circ$  in the clockwise direction and a scaling of  $\sqrt{2}$ .

**Problem 16.** The matrix

$$\begin{bmatrix} -0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

represents a rotation. Find the angle of rotation (in radians).

**Solution.** We have  $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix}$  and  $a^2 + b^2 = 1$ .

Thus  $\cos(\theta) = -0.8$  and  $\sin(\theta) = 0.6$  where  $\theta$  is the angle of rotation.

We can solve for  $\theta$ .

$$\begin{aligned} \cos(\theta) &= -0.8 \\ \theta &= \cos^{-1}(-0.8) = 2.49809... \end{aligned}$$

Thus  $\theta \approx 2.49809$  radians.

**Problem 17.** Let  $L$  be the line in  $\mathbb{R}^3$  that consists of all scalar multiples of the vector  $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ . Find the orthogonal projection of the vector  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  onto  $L$ .

**Solution.** Let  $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ .

$$\begin{aligned} \text{proj}_L \vec{x} &= \left( \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w} \\ &= \left( \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}} \right) \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\ &= \left( \frac{5}{9} \right) \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix} \end{aligned}$$

Since the magnitude of  $\vec{x}$  is  $\sqrt{3}$ , it's fine that the first and third components of  $\text{proj}_L \vec{x}$  exceed 1.

The vector  $\text{proj}_L \vec{x}$  represents the component of  $\vec{x}$  that is parallel to  $\vec{w}$ .

Let  $V$  be the plane perpendicular to  $L$ .

$$\begin{aligned} \vec{x} &= \text{proj}_L \vec{x} + \text{proj}_V \vec{x} \\ &= \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix} + \begin{bmatrix} \frac{-1}{9} \\ \frac{4}{9} \\ \frac{-1}{9} \end{bmatrix} \end{aligned}$$

**Problem 18.** Let  $L$  be the line in  $\mathbb{R}^3$  that consists of all scalar multiples of the vector  $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ . Find the reflection of the vector  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  onto  $L$ .

**Solution.**

$$\begin{aligned}
 \text{ref}_L \vec{x} &= 2 \text{proj}_L \vec{x} - \vec{x} \\
 &= 2 \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{20}{9} \\ \frac{10}{9} \\ \frac{20}{9} \end{bmatrix} - \begin{bmatrix} \frac{9}{9} \\ \frac{9}{9} \\ \frac{9}{9} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{11}{9} \\ \frac{1}{9} \\ \frac{11}{9} \end{bmatrix}
 \end{aligned}$$

**Problem 19.** Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \vec{x}$$

**Solution.** The matrix of the transformation represents a reflection. Recall that a reflection matrix is of the form

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

Here we have  $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

In the linear transformation  $T(\vec{x})$ , the vector  $\vec{x}$  is reflected about the line spanned by the vector  $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$