Chapter 3 Section 1

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Problem 1. *Find the kernel of the linear transformation*

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$$

from $\mathbb{R}^3$ *to* $\mathbb{R}^2$.

Solution. *Let's solve the linear system* $T(\vec{x}) = 0$ *to get the kernel of* T.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

This tells us that $x_1 = x_3$ *and* $x_2 = -2x_3$.

Let $t = x_3$ *be an arbitrary real number. Then the solutions to the linear system are*

$$\ker(T) = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel of T *is the line spanned by the vector* $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Problem 2. *Find the kernel of the linear transformation*

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from $\mathbb{R}^5$ *to* $\mathbb{R}^4$.

Solution. *Let's solve the linear system* $T(\vec{x}) = Ax = 0$.

We can solve this linear system by creating the augmented matrix $\begin{bmatrix} A & | & \vec{0} \end{bmatrix}$ *and calculating* $\operatorname{rref}\begin{bmatrix} A & | & \vec{0} \end{bmatrix}$.

$$\left(\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 4 & 8 & 5 & -8 & 9 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & -4 & 5 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 4 & 8 & 5 & -8 & 9 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right) \textit{Adding row2 to row1}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & -4 & 5 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right) \textit{Adding row2 to row1}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right) \textit{Subtracting row3 from row1}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & -2 & 8 & -10 & 0 \end{array}\right) \textit{Adding row2 to row4}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \textit{Adding row2 to row4}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & -3 & 4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \textit{Adding row3 to row2}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \textit{Swapping rows and multiplying row1 by -1}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_4 - 4x_5 &= 0 \\ x_3 - 4x_4 + 5x_5 &= 0 \end{aligned}$$

$$x_1 = -2x_2 - 3x_4 + 4x_5$$
$$x_3 = 4x_4 - 5x_5$$

Let $r = x_2$, *let* $s = x_4$, *let* $t = x_5$.

Then the solutions to the linear system (the kernel) are of the form:

$$\begin{aligned}
\ker(T) &= \begin{bmatrix} -2r - 3s + 4t \\ r \\ 4s - 5t \\ s \\ t \end{bmatrix} \\
&= \begin{bmatrix} -2r \\ r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3s \\ 0 \\ 4s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 4t \\ 0 \\ -5t \\ 0 \\ t \end{bmatrix} \\
&= r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \\
&= \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right)
\end{aligned}$$

Problem 3. *For an invertible* $n \times n$ *matrix find* $\ker A$.

Solution. *Let* $T(\vec{x}) = A\vec{x}$ *be a linear transformation with an invertible* $n \times n$ *matrix* A. *We know that* T *is invertible because* A *is invertible. Thus there can only be one unique solution to the equation* $A\vec{x} = 0$. *Since* $\vec{x} = \vec{0}$ *is the unique solution, we know that* $\ker A = \{\vec{0}\}$.

Problem 4. *For which* $n \times m$ *matrices is* $\ker A = \{\vec{0}\}$. *Give your answer in terms of the rank of* A.

Solution. *Let* A *be a* $n \times m$ *matrix. When* $\operatorname{rank} A = m$, *we get the unique solution* $\vec{x} = \vec{0}$. *Thus* $\ker A = \{\vec{0}\}$ *when* $\operatorname{rank} A = m$.

In the following problems, find vectors that span the kernel of A and the image of A.

Problem 5.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution. *Let $T(\vec{x}) = A\vec{x}$. Let's solve the equation $A\vec{x} = \vec{0}$.*

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right] \qquad \textit{Augmented matrix } \begin{bmatrix} A & | & \vec{0} \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \end{array}\right] \qquad \textit{Subtract 3 times row1 from row2}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -2 & 0 \end{array}\right] \qquad \textit{Add row2 to row1}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \qquad \textit{Divide row2 by -2}$$

The solution set is $\vec{x} = \vec{0}$, thus $\ker A = \{0\}$.

The kernel of A is spanned by the zero vector. The image of A is the span of the column vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

In other words, $T(\vec{x}) = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Problem 6.

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Solution. *Let's solve the equation*

$$x_1 + 2x_2 + 3x_3 = 0$$

From inspection, we can find two nonparallel vectors in the solution set, the vectors

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \vec{v_2} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

There is no constant c such that $\vec{v_2} = c\vec{v_1}$, thus the two vectors are linearly independent and form a basis for the kernel.

Thus the kernel of A is the plane through the origin spanned by the vectors $\vec{v_1}$ and $\vec{v_2}$.

$$\ker A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}\right)$$

The image of A is $\mathbb{R}$, since for any $y \in \mathbb{R}$, we have the solution $\vec{x} = \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix}$.

Problem 7.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution. *The kernel of A is $\mathbb{R}^2$ and the image of A is* $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. *We can write the kernel of A as the span of the unit vectors e_1 and e_2.*

$$\ker A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Problem 8.

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$$

Solution. *The image is the span of the column vectors. Since the second column vector is redundant, the image of A is the line spanned by the vector* $\begin{bmatrix} 2 & 3 \end{bmatrix}$.

The kernel of A is the solution set of the equation $2x_1 + 3x_2 = 0$.

$$x_1 = -\frac{3}{2}x_2$$

This equation describes a line that passes through the origin.

The kernel of A is the line spanned by the vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.