

## Chapter 3 Section 4

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**Problem 1.** Let  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ . Let  $V = \text{span}(\vec{v}_1, \vec{v}_2)$ . Is the vector

$\vec{w} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$  on the plane  $V$ ?

**Solution.** If the vector  $\vec{w}$  is on the plane  $V$ , then there exist some  $x_1, x_2 \in \mathbb{R}$  such that  $\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2$ . This gives us the equations

$$x_1 + x_2 = 5$$

$$x_1 + 2x_2 = 7$$

$$x_1 + 3x_2 = 9$$

We can solve these equations using a matrix.

$$\begin{bmatrix} 1 & 1 & | & 5 \\ 1 & 2 & | & 7 \\ 1 & 3 & | & 9 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & | & 5 \\ 0 & 1 & | & 2 \\ 0 & 2 & | & 4 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & | & 5 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$$

This gives us  $x_1 = 3$  and  $x_2 = 2$ .

Thus  $\vec{w}$  is on the plane  $V$  because  $\vec{w} = 3\vec{v}_1 + 2\vec{v}_2$ .

**Problem 2.** Consider the basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  consisting of the vectors  $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ .

- If  $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$  find  $\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}$
- If  $\begin{bmatrix} \vec{y} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$  find  $\vec{y}$

**Solution.** We can find the coordinates of  $\vec{x}$  with respect to  $\mathfrak{B}$  by means of an equation.

$$\begin{aligned}\vec{x} &= c_1 \vec{v}_1 + c_2 \vec{v}_2 \\ \begin{bmatrix} 10 \\ 10 \end{bmatrix} &= c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 10 \\ 10 \end{bmatrix} &= \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\end{aligned}$$

We can solve this equation using elementary row operations.

$$\begin{aligned}&\left( \begin{array}{cc|c} 3 & -1 & 10 \\ 1 & 3 & 10 \end{array} \right) \\&\left( \begin{array}{cc|c} 0 & -10 & -20 \\ 1 & 3 & 10 \end{array} \right) \\&\left( \begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 3 & 10 \end{array} \right) \\&\left( \begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 4 \end{array} \right) \\&\left( \begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array} \right)\end{aligned}$$

By reducing the matrix, we find that  $c_1 = 4$  and  $c_2 = 2$ . Thus  $\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ .

We can also use an equation to solve for  $\vec{y}$ .

$$\begin{aligned}
\vec{y} &= 2\vec{v}_1 - \vec{v}_2 \\
&= 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 6 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 7 \\ -1 \end{bmatrix}
\end{aligned}$$

**Problem 3.** Let  $\vec{v}_1$  and  $\vec{v}_2$  be perpendicular unit vectors in  $\mathbb{R}^3$ . Let  $\vec{v}_3$  be the cross product of  $\vec{v}_1$  and  $\vec{v}_2$ , that is,  $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$ . We know from the properties of the cross product that  $\vec{v}_3$  is perpendicular to  $\vec{v}_1$  and  $\vec{v}_2$ . Thus the three vectors are linearly independent. The three vectors form a basis for  $\mathbb{R}^3$ .

1. What is  $\vec{v}_1 \times \vec{v}_3$ ?
2. Find the  $\mathfrak{B}$ -matrix of the linear transformation  $T(x) = \vec{v}_1 \times \vec{x}$ .

**Solution.**  $\vec{v}_1 \times \vec{v}_3 = -\vec{v}_2$ .

The  $\mathfrak{B}$ -matrix is the matrix  $B$  such that

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = B \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}$$

We can find the coordinates of  $\vec{x}$  with respect to  $\mathfrak{B}$  by means of an equation.

$$\begin{aligned}
\vec{x} &= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \\
&= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
T(x) &= \vec{v}_1 \times \vec{x} \\
&= \vec{v}_1 \times (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \\
&= c_1(\vec{v}_1 \times \vec{v}_1) + c_2(\vec{v}_1 \times \vec{v}_2) + c_3(\vec{v}_1 \times \vec{v}_3) \\
&= c_2\vec{v}_3 - c_3\vec{v}_2 \\
&= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix}
\end{aligned}$$

Thus

$$\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

and

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix}$$

Now let's find the matrix  $B$  such that

$$\begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix} = B \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

By inspection, we see that

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}$$

and the  $\mathfrak{B}$ -matrix of  $T$  is

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

**Problem 4.** Let  $T$  be the linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  that projects any vector onto the line  $L$  spanned by the vector  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . Earlier we found that the  $\mathfrak{B}$ -matrix of  $T$  with respect to the basis  $\mathfrak{B} = \left( \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right)$  is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

What is the relationship between  $B$  and the standard matrix  $A$  of  $T$  (such that  $T(x) = Ax$ )?

**Solution.**

$$\begin{aligned}
 \text{proj}_L(\vec{x}) &= \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} \\
 &= \frac{1}{3^2 + 1^2} \begin{bmatrix} 3^2 & 3 * 1 \\ 3 * 1 & 1^2 \end{bmatrix} \\
 &= \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{9}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix} \\
 &= \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}
 \end{aligned}$$

Thus the standard matrix of  $T$  is

$$A = \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}$$

and

$$\begin{aligned}
 T(\vec{x}) &= A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
 &= \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
 \end{aligned}$$

Now we're going to find the relationship between  $A$  and  $B$ .

$$\text{Let } S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

Let's write  $T(\vec{x})$  in terms of  $A$  and  $S$ .

$$\begin{aligned}
 \vec{x} &= S \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} \\
 T(\vec{x}) &= A\vec{x} \\
 T(\vec{x}) &= AS \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}
 \end{aligned}$$

Now let's write  $T(\vec{x})$  in terms of  $B$  and  $S$ .

$$\begin{aligned}
\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} &= B \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} \\
T(\vec{x}) &= S \begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} \\
T(\vec{x}) &= SB \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}
\end{aligned}$$

The above equations show that  $AS = SB$  and  $A = SBS^{-1}$ .

The equation  $A = SBS^{-1}$  gives us another way of finding  $A$  (since we know  $S$ , we know  $B$ , and we can calculate  $S^{-1}$ ).

**Definition 1.** Two  $n \times n$  matrices  $A$  and  $B$  are similar if there exists an invertible matrix  $S$  such that

$$AS = SB, \text{ or } B = S^{-1}AS$$