

## Chapter 3 Section 1

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**Problem 1.** Find the kernel of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$$

from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .

**Solution.** Let's solve the linear system  $T(\vec{x}) = 0$  to get the kernel of  $T$ .

$$\begin{array}{l} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array} \right) \\ \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \\ \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \end{array}$$

This tells us that  $x_1 = x_3$  and  $x_2 = -2x_3$ .

Let  $t = x_3$  be an arbitrary real number. Then the solutions to the linear system are

$$\ker(T) = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel of  $T$  is the line spanned by the vector  $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .

**Problem 2.** Find the kernel of the linear transformation

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from  $\mathbb{R}^5$  to  $\mathbb{R}^4$ .

**Solution.** Let's solve the linear system  $T(\vec{x}) = Ax = 0$ .

We can solve this linear system by creating the augmented matrix  $[A \mid \vec{0}]$  and calculating  $\text{rref } [A \mid \vec{0}]$ .

$$\begin{pmatrix} 1 & 2 & 2 & -5 & 6 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 4 & 8 & 5 & -8 & 9 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & -4 & 5 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 4 & 8 & 5 & -8 & 9 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Adding row2 to row1}$$

$$\begin{pmatrix} 0 & 0 & 1 & -4 & 5 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Adding row2 to row1}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Subtracting row3 from row1}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & -2 & 8 & -10 & | & 0 \end{pmatrix} \text{ Adding row2 to row4}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Adding row2 to row4}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & 0 & -3 & 4 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Adding row3 to row2}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & -4 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Swapping rows and multiplying row1 by -1}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_4 - 4x_5 &= 0 \\ x_3 - 4x_4 + 5x_5 &= 0 \end{aligned}$$

$$x_1 = -2x_2 - 3x_4 + 4x_5$$

$$x_3 = 4x_4 - 5x_5$$

Let  $r = x_2$ , let  $s = x_4$ , let  $t = x_5$ .

Then the solutions to the linear system (the kernel) are of the form:

$$\begin{aligned} \ker(T) &= \begin{bmatrix} -2r - 3s + 4t \\ r \\ 4s - 5t \\ s \\ t \end{bmatrix} \\ &= \begin{bmatrix} -2r \\ r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3s \\ 0 \\ 4s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 4t \\ 0 \\ -5t \\ 0 \\ t \end{bmatrix} \\ &= r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \\ &= \text{span} \left( \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

**Problem 3.** For an invertible  $n \times n$  matrix find  $\ker A$ .

**Solution.** Let  $T(\vec{x}) = A\vec{x}$  be a linear transformation with an invertible  $n \times n$  matrix  $A$ . We know that  $T$  is invertible because  $A$  is invertible. Thus there can only be one unique solution to the equation  $A\vec{x} = \vec{0}$ . Since  $\vec{x} = \vec{0}$  is the unique solution, we know that  $\ker A = \{\vec{0}\}$ .

**Problem 4.** For which  $n \times m$  matrices is  $\ker A = \{\vec{0}\}$ . Give your answer in terms of the rank of  $A$ .

**Solution.** Let  $A$  be a  $n \times m$  matrix. When  $\text{rank } A = m$ , we get the unique solution  $\vec{x} = \vec{0}$ . Thus  $\ker A = \{\vec{0}\}$  when  $\text{rank } A = m$ .

In the following problems, find vectors that span the kernel of  $A$  and the image of  $A$ .

**Problem 5.**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

**Solution.** Let  $T(\vec{x}) = A\vec{x}$ . Let's solve the equation  $A\vec{x} = \vec{0}$ .

$$\begin{array}{ll} \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array} \right] & \text{Augmented matrix } \left[ A \mid \vec{0} \right] \\ \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \end{array} \right] & \text{Subtract 3 times row1 from row2} \\ \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -2 & 0 \end{array} \right] & \text{Add row2 to row1} \\ \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] & \text{Divide row2 by -2} \end{array}$$

The solution set is  $\vec{x} = \vec{0}$ , thus  $\ker A = \{0\}$ .

The kernel of  $A$  is spanned by the zero vector. The image of  $A$  is the span of the column vectors  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ .

In other words,  $T(\vec{x}) = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ .

**Problem 6.**

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

**Solution.** Let's solve the equation

$$x_1 + 2x_2 + 3x_3 = 0$$

From inspection, we can find two nonparallel vectors in the solution set, the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

There is no constant  $c$  such that  $\vec{v}_2 = c\vec{v}_1$ , thus the two vectors are linearly independent and form a basis for the kernel.

Thus the kernel of  $A$  is the plane through the origin spanned by the vectors  $\vec{v}_1$  and  $\vec{v}_2$ .

$$\ker A = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right)$$

The image of  $A$  is  $\mathbb{R}$ , since for any  $y \in \mathbb{R}$ , we have the solution  $\vec{x} = \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix}$ .

**Problem 7.**

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**Solution.** The kernel of  $A$  is  $\mathbb{R}^2$  and the image of  $A$  is  $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . We can write the kernel of  $A$  as the span of the unit vectors  $e_1$  and  $e_2$ .

$$\ker A = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

**Problem 8.**

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$$

**Solution.** The image is the span of the column vectors. Since the second column vector is redundant, the image of  $A$  is the line spanned by the vector  $\begin{bmatrix} 2 & 3 \end{bmatrix}$ .

The kernel of  $A$  is the solution set of the equation  $2x_1 + 3x_2 = 0$ .

$$x_1 = -\frac{3}{2}x_2$$

This equation describes a line that passes through the origin.

The kernel of  $A$  is the line spanned by the vector  $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ .

**Problem 9.**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

**Solution.** Let's solve the linear system  $A\vec{x} = 0$  to get the kernel of  $A$ .

$$\begin{aligned}
& \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{array} \right] \\
& \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 3 & 5 & 0 \end{array} \right] \\
& \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array} \right] \\
& \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \\
& \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]
\end{aligned}$$

This gives us the equations

$$\begin{aligned}
x_1 &= x_3 \\
x_2 &= -2x_3
\end{aligned}$$

Let  $t$  be an arbitrary real number. Then we can write the kernel of  $A$  as

$$\begin{aligned}
\ker A &= \left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \right\} \\
&= \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \\
&= \text{span} \left( \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)
\end{aligned}$$

Thus the kernel of  $A$  is the line spanned by the vector  $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .

In our matrix  $A$ , the third column vector is redundant. Thus

$$\text{im } A = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$$

The image of  $A$  is a plane spanned by the above vectors that passes through the origin.

**Problem 10.**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

**Solution.** Let's solve the linear system  $A\vec{x} = 0$ .

$$\begin{array}{l} \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array} \right] \\ \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \\ \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \end{array}$$

This gives us the equations

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \end{aligned}$$

Let  $t$  be an arbitrary real number. Then we can write the kernel of  $A$  as

$$\begin{aligned} \ker A &= \left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \right\} \\ &= \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \\ &= \text{span} \left( \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Thus the kernel of  $A$  is the line spanned by the vector  $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .

In our matrix  $A$ , the third column vector is redundant. Thus

$$\text{im } A = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

The image of  $A$  is a plane that passes through the origin.



**Problem 11.**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

**Solution.** *Let's solve the linear system  $A\vec{x} = \vec{0}$ .*

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 0 \\ 3 & 2 & 1 & 0 \end{array} \right] \\ & \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 \end{array} \right] \\ & \left[ \begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 \end{array} \right] \\ & \left[ \begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -14 & 0 \end{array} \right] \\ & \left[ \begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -7 & 0 \end{array} \right] \\ & \left[ \begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -6 & 0 \end{array} \right] \\ & \left[ \begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \\ & \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \end{aligned}$$

Since  $\text{rref } A = I_3$ ,  $\ker A = \{\vec{0}\}$ .

**Problem 12.**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

**Solution.** *We can solve the linear system  $A\vec{x} = \vec{0}$ .*

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This gives us the equation  $x_1 + x_2 + x_3 = 0$ .

This equation describes a plane passing through the origin. Two solutions to the equation are

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel of  $A$  has to be a subspace of  $\mathbb{R}^3$ . Possible subspaces are  $\mathbb{R}^3$ , a plane passing through the origin, a line passing through the origin, and  $\{\vec{0}\}$ . Since the kernel of  $A$  is not the domain, and since it contains two nonparallel vectors, it has to be a plane passing through the origin.

The kernel of  $A$  is the span of these vectors.

$$\ker A = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$$

**Problem 13.**

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

**Solution.**

$$\text{rref } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This gives us the equations  $x_1 = 0$  and  $x_2 = 0$ .

Thus  $\ker A = \{\vec{0}\}$ .

**Problem 14.**

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

**Solution.** We can solve the linear system  $A\vec{x} = \vec{0}$  with elementary row operations on an augmented matrix.

$$\begin{aligned} & \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{array} \right) \\ & \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{array} \right) \\ & \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right) \\ & \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \\ & \left( \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \end{aligned}$$

We get the equations  $x_1 = 0$  and  $x_2 = 0$ . Thus

$$\ker A = \{\vec{0}\}$$

**Problem 15.**

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix}$$

**Solution.** We can solve the linear system  $A\vec{x} = \vec{0}$ .



*This gives us the equations*

$$\begin{aligned}x_3 &= -x_4 \\x_5 &= 0\end{aligned}$$

*Let  $r, s$  be real numbers. Then the solutions are*

$$\ker A = \begin{bmatrix} s - 2t \\ s \\ -t \\ t \\ 0 \end{bmatrix}$$

**Problem 16.**

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}$$

**Solution.** *Let's solve the linear system  $A\vec{x} = \vec{0}$ .*

$$\left( \begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & 0 \end{array} \right)$$

$$\left( \begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array} \right)$$

$$\left( \begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

$$\left( \begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 3 & 4 & -6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

$$\left( \begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 4 & -12 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

$$\left( \begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

$$\left( \begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

*This gives us the equations*

$$x_1 = -2x_3$$

$$x_2 = 3x_3$$

$$x_4 = 0$$

*Thus the solutions are*

$$\begin{aligned}
\ker A &= \begin{bmatrix} -2t \\ 3t \\ t \\ 0 \end{bmatrix} \\
&= t \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \\
&= \text{span} \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

The kernel (above) can be interpreted geometrically as a line.

**Problem 17.**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Solution.** Let's solve the linear system  $A\vec{x} = \vec{0}$ .

$$\begin{aligned}
&\left( \begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right) \\
&\left( \begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right) \\
&\left( \begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right)
\end{aligned}$$

This gives us the equations

$$\begin{aligned}
x_1 &= x_3 \\
x_2 &= -2x_3 \\
x_4 &= 0
\end{aligned}$$

Thus the solutions are

$$\begin{aligned}
\ker A &= \begin{bmatrix} t \\ -2t \\ t \\ 0 \end{bmatrix} \\
&= t \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \\
&= \text{span} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

*The kernel of  $A$  is a line that passes through the origin.*