Chapter 3 Section 1

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Problem 1. *Find the kernel of the linear transformation*

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$$

from $\mathbb{R}^3$ *to* $\mathbb{R}^2$.

Solution. *Let's solve the linear system* $T(\vec{x}) = 0$ *to get the kernel of* T.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

This tells us that $x_1 = x_3$ *and* $x_2 = -2x_3$.

Let $t = x_3$ *be an arbitrary real number. Then the solutions to the linear system are*

$$\ker(T) = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel of T *is the line spanned by the vector* $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Problem 2. *Find the kernel of the linear transformation*

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from $\mathbb{R}^5$ *to* $\mathbb{R}^4$.

Solution. *Let's solve the linear system* $T(\vec{x}) = Ax = 0$.

We can solve this linear system by creating the augmented matrix $\begin{bmatrix} A & | & \vec{0} \end{bmatrix}$ *and calculating* $\operatorname{rref} \begin{bmatrix} A & | & \vec{0} \end{bmatrix}$.

$$\left(\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 4 & 8 & 5 & -8 & 9 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & -4 & 5 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 4 & 8 & 5 & -8 & 9 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right) \textit{Adding row2 to row1}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & -4 & 5 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right) \textit{Adding row2 to row1}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right) \textit{Subtracting row3 from row1}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & -2 & 8 & -10 & 0 \end{array}\right) \textit{Adding row2 to row4}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \textit{Adding row2 to row4}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & -3 & 4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \textit{Adding row3 to row2}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \textit{Swapping rows and multiplying row1 by -1}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_4 - 4x_5 &= 0 \\ x_3 - 4x_4 + 5x_5 &= 0 \end{aligned}$$

$$x_1 = -2x_2 - 3x_4 + 4x_5$$
$$x_3 = 4x_4 - 5x_5$$

Let $r = x_2$*, let* $s = x_4$*, let* $t = x_5$*.*

Then the solutions to the linear system (the kernel) are of the form:

$$\begin{aligned}
\ker(T) &= \begin{bmatrix} -2r - 3s + 4t \\ r \\ 4s - 5t \\ s \\ t \end{bmatrix} \\
&= \begin{bmatrix} -2r \\ r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3s \\ 0 \\ 4s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 4t \\ 0 \\ -5t \\ 0 \\ t \end{bmatrix} \\
&= r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \\
&= \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right)
\end{aligned}$$

Problem 3. *For an invertible* $n \times n$ *matrix find* $\ker A$*.*

Solution. *Let* $T(\vec{x}) = A\vec{x}$ *be a linear transformation with an invertible* $n \times n$ *matrix* A*. We know that* T *is invertible because* A *is invertible. Thus there can only be one unique solution to the equation* $A\vec{x} = 0$*. Since* $\vec{x} = \vec{0}$ *is the unique solution, we know that* $\ker A = \{\vec{0}\}$*.*

Problem 4. *For which* $n \times m$ *matrices is* $\ker A = \{\vec{0}\}$*. Give your answer in terms of the rank of* A*.*

Solution. *Let* A *be a* $n \times m$ *matrix. When* $\operatorname{rank} A = m$*, we get the unique solution* $\vec{x} = \vec{0}$*. Thus* $\ker A = \{\vec{0}\}$ *when* $\operatorname{rank} A = m$*.*