

Chapter 3 Section 1

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Problem 1. Find the kernel of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$$

from \mathbb{R}^3 to \mathbb{R}^2 .

Solution. Let's solve the linear system $T(\vec{x}) = 0$ to get the kernel of T .

$$\begin{array}{l} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array} \right) \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \\ \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \end{array}$$

This tells us that $x_1 = x_3$ and $x_2 = -2x_3$.

Let $t = x_3$ be an arbitrary real number. Then the solutions to the linear system are

$$\ker(T) = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel of T is the line spanned by the vector $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Problem 2. Find the kernel of the linear transformation

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from \mathbb{R}^5 to \mathbb{R}^4 .

Solution. Let's solve the linear system $T(\vec{x}) = Ax = 0$.

We can solve this linear system by creating the augmented matrix $[A \mid \vec{0}]$ and calculating $\text{rref } [A \mid \vec{0}]$.

$$\begin{pmatrix} 1 & 2 & 2 & -5 & 6 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 4 & 8 & 5 & -8 & 9 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & -4 & 5 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 4 & 8 & 5 & -8 & 9 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Adding row2 to row1}$$

$$\begin{pmatrix} 0 & 0 & 1 & -4 & 5 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Adding row2 to row1}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Subtracting row3 from row1}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & -2 & 8 & -10 & | & 0 \end{pmatrix} \text{ Adding row2 to row4}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Adding row2 to row4}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & 0 & -3 & 4 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Adding row3 to row2}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & -4 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Swapping rows and multiplying row1 by -1}$$

$$\begin{aligned}
x_1 + 2x_2 + 3x_4 - 4x_5 &= 0 \\
x_3 - 4x_4 + 5x_5 &= 0
\end{aligned}$$

$$x_1 = -2x_2 - 3x_4 + 4x_5$$

$$x_3 = 4x_4 - 5x_5$$

Let $r = x_2$, let $s = x_4$, let $t = x_5$.

Then the solutions to the linear system (the kernel) are of the form:

$$\begin{aligned} \ker(T) &= \begin{bmatrix} -2r - 3s + 4t \\ r \\ 4s - 5t \\ s \\ t \end{bmatrix} \\ &= \begin{bmatrix} -2r \\ r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3s \\ 0 \\ 4s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 4t \\ 0 \\ -5t \\ 0 \\ t \end{bmatrix} \\ &= r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \\ &= \text{span} \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Problem 3. For an invertible $n \times n$ matrix find $\ker A$.

Solution. Let $T(\vec{x}) = A\vec{x}$ be a linear transformation with an invertible $n \times n$ matrix A . We know that T is invertible because A is invertible. Thus there can only be one unique solution to the equation $A\vec{x} = \vec{0}$. Since $\vec{x} = \vec{0}$ is the unique solution, we know that $\ker A = \{\vec{0}\}$.

Problem 4. For which $n \times m$ matrices is $\ker A = \{\vec{0}\}$. Give your answer in terms of the rank of A .

Solution. Let A be a $n \times m$ matrix. When $\text{rank } A = m$, we get the unique solution $\vec{x} = \vec{0}$. Thus $\ker A = \{\vec{0}\}$ when $\text{rank } A = m$.

In the following problems, find vectors that span the kernel of A and the image of A .

Problem 5.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution. Let $T(\vec{x}) = A\vec{x}$. Let's solve the equation $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array} \right] \quad \text{Augmented matrix } \left[A \mid \vec{0} \right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \end{array} \right] \quad \text{Subtract 3 times row1 from row2}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -2 & 0 \end{array} \right] \quad \text{Add row2 to row1}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] \quad \text{Divide row2 by -2}$$

The solution set is $\vec{x} = \vec{0}$, thus $\ker A = \{0\}$.

The kernel of A is spanned by the zero vector. The image of A is the span of the column vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

In other words, $T(\vec{x}) = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Problem 6.

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$