Chapter 2 Section 1

Andrew Taylor

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Problem 1. *Imagine yourself cruising in the Mediterranean as a crew member on a French coast goard boat, looking for evildoers. Periodically, your boat radios its position to headquarters in Marseille. You expect that communications will be intercepted. So, before you broadcast anything, you have to transform the actual position of the boat,*

$$\begin{bmatrix} x1 \\ x2 \end{bmatrix}$$

(x_1 for Eastern longitude, x_2 for Northern latitude), into an encoded position

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

You use the following code:

$$\begin{aligned} y_1 &= x_1 + 3x_2 \\ y_2 &= 2x_2 + 5x_2 \end{aligned}$$

For example, when the actual position of your boat is $5°E$, $42°N$, or

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 42 \end{bmatrix}$$

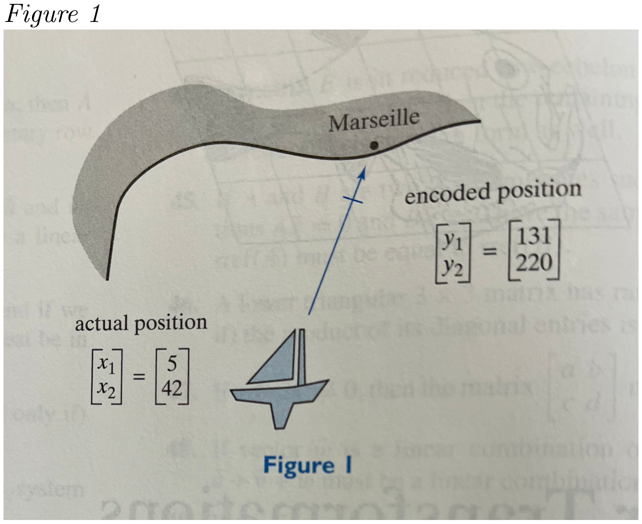
your encoded position will be

$$\begin{aligned} \vec{y} &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 + 5x_2 \end{bmatrix} \\ &= \begin{bmatrix} 5 + 3 * 42 \\ 2 * 5 + 5 * 42 \end{bmatrix} \\ &= \begin{bmatrix} 131 \\ 220 \end{bmatrix} \end{aligned}$$

The coding transformation can be represented as

$$\begin{aligned} \vec{y} &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Figure 1



As the ship reaches a new position, the sailor on duty at headquarters in Marseille receives the encoded message

$$\vec{b} = \begin{bmatrix} 133 \\ 223 \end{bmatrix}$$

He must determine the actual position of the boat. He will have to solve the linear system

$$A\vec{x} = \vec{b}$$

or, more explicitly,

$$\begin{vmatrix} x_1 + 3x_2 = 133 \\ 2x_1 + 5x_2 = 223 \end{vmatrix}$$

Here is his solution. Is it correct?

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 43 \end{bmatrix}$$

Solution. *We can use elementary row operations to solve the system. Remember, the elementary row operations are subtracting a multiple of a row, dividing a row by a scalar, and swapping rows.*

$$\begin{bmatrix} 1 & 3 & 133 \\ 2 & 5 & 223 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 & 133 \\ 0 & -1 & -43 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 & 133 \\ 0 & 1 & 43 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 43 \end{bmatrix}$$

Thus his solution is correct. The solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 43 \end{bmatrix}$$

What we really did is we applied the linear transformation (the code we were given) to the encoded message, to get the actual coordinates.

Consider the transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$. Which of these transformations are linear?

Problem 2.

$$\begin{aligned} y_1 &= 2x_2 \\ y_2 &= x_2 + 2 \\ y_3 &= 2x_2 \end{aligned}$$

Solution. *Let T be the linear transformation, and let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$.*

$$\begin{aligned} &T(\vec{v} + \vec{w}) \\ &= T((v_1, v_2, v_3) + (w_1, w_2, w_3)) \\ &= T((v_1 + w_1, v_2 + w_2, v_3 + w_3)) \\ &= (2(v_2 + w_2), v_2 + w_2 + 2, 2(v_2 + w_2)) \end{aligned}$$

$$\begin{aligned} &T(\vec{v}) + T(\vec{w}) \\ &= T((v_1, v_2, v_3)) + T((w_1, w_2, w_3)) \\ &= (2(v_2), v_2 + 2, 2(v_2)) + (2(w_2), w_2 + 2, 2(w_2)) \\ &= (2(v_2 + w_2), v_2 + w_2 + 2, 2(v_2 + w_2)) \end{aligned}$$

Thus

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$$

Let k be a scalar in $\mathbb{R}$.

$$\begin{aligned}
&T(k\vec{v}) \\
&= T((kv_1, kv_2, kv_3)) \\
&= (2kv_2, kv_2 + 2, 2kv_2)
\end{aligned}$$

$$\begin{aligned}
&kT(\vec{v}) \\
&= kT((v_1, v_2, v_3)) \\
&= k(2v_2, v_2 + 2, 2v_2) \\
&= (2kv_2, kv_2 + 2, 2kv_2)
\end{aligned}$$

Thus

$$T(k\vec{v}) = kT(\vec{v})$$

T satisfies the two properties of a linear transformation, so T is a linear transformation.

Problem 3.

$$\begin{aligned}
y_1 &= 2x_2 \\
y_2 &= 3x_3 \\
y_3 &= x_1
\end{aligned}$$

Solution. *Let T be the linear transformation, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$.*

$$\begin{aligned}
&T(\vec{v} + \vec{w}) \\
&= T((v_1, v_2, v_3) + (w_1, w_2, w_3)) \\
&= T((v_1 + w_1, v_2 + w_2, v_3 + w_3)) \\
&= (2(v_2 + w_2), 3(v_3 + w_3), v_1 + w_1)
\end{aligned}$$

$$\begin{aligned}
&T(\vec{v}) + T(\vec{w}) \\
&= (2v_2, 3v_3, v_1) + (2w_2, 3w_3, w_1) \\
&= (2(v_2 + w_2), 3(v_3 + w_3), v_1 + w_1)
\end{aligned}$$

Thus

$$T(\vec{v}) + T(\vec{w}) = T(\vec{v} + \vec{w})$$

Let k be a scalar in the reals.

$$\begin{aligned}
&T(k\vec{v}) \\
&= T((kv_1, kv_2, kv_3) \\
&= (2kv_2, 3kv_3, kv_1)
\end{aligned}$$

$$\begin{aligned}
&kT(\vec{v}) \\
&= k(2v_2, 3v_3, v_1) \\
&= (2kv_2, 3kv_3, kv_1)
\end{aligned}$$

Thus

$$T(k\vec{v}) = kT(\vec{v})$$

Therefore T is a linear transformation, because it satisfies both requirements of a linear transformation.

Problem 4.

$$\begin{aligned}
y_1 &= x_2 - x_3 \\
y_2 &= x_1 x_3 \\
y_3 &= x_1 - x_2
\end{aligned}$$

Solution. *Let T be the linear transformation and let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$.*

$$\begin{aligned}
&T(\vec{v} + \vec{w}) \\
&= T((v_1, v_2, v_3) + (w_1, w_2, w_3)) \\
&= T((v_1 + w_1, v_2 + w_2, v_3 + w_3)) \\
&= (v_2 + w_2 - v_3 - w_3, (v_1 + w_1) * (v_3 + w_3), v_1 + w_1 - v_2 - w_2)
\end{aligned}$$

$$\begin{aligned}
&T(\vec{v}) + T(\vec{w}) \\
&= (v_2 - v_3, v_1 * v_3, v_1 - v_2) + (w_2 - w_3, w_1 * w_3, w_1 - w_2) \\
&= (v_2 - v_3 + w_2 - w_3, v_1 * v_3 + w_1 * w_3, v_1 - v_2 + w_1 - w_2)
\end{aligned}$$

The two vectors are not equal, therefore T is not a linear transformation. In other words,

$$T(\vec{v}) + T(\vec{w}) \neq T(\vec{v} + \vec{w})$$

so T is not a linear transformation.

Problem 5. *Find the matrix of the linear transformation*

$$\begin{aligned} y_1 &= 9x_1 + 3x_2 - 3x_3 \\ y_2 &= 2x_1 - 9x_2 + x_3 \\ y_3 &= 4x_1 - 9x_2 - 2x_3 \\ y_4 &= 5x_1 + x_2 + 5x_3 \end{aligned}$$

Solution.

$$\begin{aligned} \vec{y} &= A\vec{x} \\ &= \begin{pmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{pmatrix} \vec{x} \\ &= \begin{pmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

We see that the matrix has four rows and three columns, transforming a three-dimensional vector into a four-dimensional vector.

Problem 6. *Consider the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ with*

$$\begin{aligned} T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \\ T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 6 \\ 9 \end{bmatrix}, \\ T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} -13 \\ 17 \end{bmatrix} \end{aligned}$$

Find the matrix A of T.

Solution. *We can write the equation*

$$\begin{pmatrix}7\\1\end{pmatrix} = A\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}7\\1\end{pmatrix} = \begin{pmatrix}c_1 & c_2 & c_3\\d_1 & d_2 & d_3\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

So $c_1 = 7$ *and* $d_1 = 1$

$$\begin{pmatrix}6\\9\end{pmatrix} = A\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}6\\9\end{pmatrix} = \begin{pmatrix}c_1 & c_2 & c_3\\d_1 & d_2 & d_3\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

So $c_2 = 6$ *and* $d_2 = 9$.

$$\begin{pmatrix}-13\\17\end{pmatrix} = A\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}-13\\17\end{pmatrix} = \begin{pmatrix}c_1 & c_2 & c_3\\d_1 & d_2 & d_3\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

So $c_3 = -13$ *and* $d_3 = 17$. *Thus*

$$A = \begin{pmatrix}7 & 6 & -13\\1 & 9 & 17\end{pmatrix}$$

and

$$T(\vec{x}) = \begin{pmatrix}7 & 6 & -13\\1 & 9 & 17\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

Problem 7. *Consider the transformation* T *from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$ *given by*

$$T\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix}$$

Is this transformation linear? If so, find its matrix.

Solution.

$$\begin{aligned} T(\vec{x}+\vec{y}) &= T((x_1+y_1, x_2+y_2) \\ &= (x_1+y_1)\begin{bmatrix}1\\2\\3\end{bmatrix} + (x_2+y_2)\begin{bmatrix}4\\5\\6\end{bmatrix} \end{aligned}$$

$$\begin{aligned} T(\vec{x})+T(\vec{y}) &= x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix} + y_1\begin{bmatrix}1\\2\\3\end{bmatrix} + y_2\begin{bmatrix}4\\5\\6\end{bmatrix} \\ &= (x_1+y_1)\begin{bmatrix}1\\2\\3\end{bmatrix} + (x_2+y_2)\begin{bmatrix}4\\5\\6\end{bmatrix} \end{aligned}$$

$$\begin{aligned} T(k\vec{x}) &= T((kx_1, kx_2)) \\ &= kx_1\begin{bmatrix}1\\2\\3\end{bmatrix} + kx_2\begin{bmatrix}4\\5\\6\end{bmatrix} \end{aligned}$$

$$\begin{aligned} kT(\vec{x}) &= kT(x_1, x_2) \\ &= k\left(x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix}\right) \\ &= kx_1\begin{bmatrix}1\\2\\3\end{bmatrix} + kx_2\begin{bmatrix}4\\5\\6\end{bmatrix} \end{aligned}$$

T satisfies both properties of a linear transformation, thus T is a linear transformation.

$$\begin{aligned} T(\vec{x}) &= T((x_1, x_2)) \\ &= x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix} \\ &= \begin{bmatrix}x_1+4x_2\\2x_1+5x_2\\3x_1+6x_2\end{bmatrix} \\ &= \begin{bmatrix}1 & 4\\2 & 5\\3 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \end{aligned}$$

The matrix of the transformation is

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Thus we can write

$$\begin{aligned} T(\vec{x}) &= A\vec{x} \\ &= \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Problem 8. *Suppose $\vec{v_1}$, $\vec{v_2}$, ..., $\vec{v_m}$ are arbitrary vectors in $\mathbb{R}^n$. Consider the transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ given by*

$$T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_m v_m$$

Is this transformation linear? If so, find its matrix A in terms of the vectors $\vec{v_1}$, $\vec{v_2}$, $\ldots$, $\vec{v_m}$

Solution.

$$\begin{aligned} &T(\vec{x} + \vec{y}) \\ &= T \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \\ &= (x_1 + y_1)\vec{v_1} + (x_2 + y_2)\vec{v_2} + \cdots + (x_n + y_n)\vec{v_m} \end{aligned}$$

$$\begin{aligned} &T(\vec{x}) + T(\vec{y}) \\ &= T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + T \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= (x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_m}) + (y_1\vec{v_1} + y_2\vec{v_2} + \cdots + y_n\vec{v_m}) \\ &= (x_1 + y_1)\vec{v_1} + (x_2 + y_2)\vec{v_2} + \cdots + (x_n + y_n)\vec{v_m} \end{aligned}$$

$$
\begin{aligned}
&kT(\vec{x}) \\
&= kT \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= k(x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_m})
\end{aligned}
$$

$$
\begin{aligned}
&T(k\vec{x}) \\
&= T \begin{bmatrix} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{bmatrix} \\
&= (kx_1\vec{v_1} + kx_2\vec{v_2} + \cdots + kx_n\vec{v_m}) \\
&= k(x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_m})
\end{aligned}
$$

Thus T is a linear transformation. The matrix of the transformation is

$$
A = \begin{bmatrix} \vec{v_{11}} & \vec{v_{21}} & \cdots & \vec{v_{m1}} \\ \vec{v_{12}} & \vec{v_{22}} & \cdots & \vec{v_{m2}} \\ \vdots & & & \\ \vec{v_{1n}} & \vec{v_{2n}} & \cdots & \vec{v_{mn}} \end{bmatrix}
$$

The vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_m}$ are column vectors in the matrix A.

Thus

$$
T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vec{v_{11}} & \vec{v_{21}} & \cdots & \vec{v_{m1}} \\ \vec{v_{12}} & \vec{v_{22}} & \cdots & \vec{v_{m2}} \\ \vdots & & & \\ \vec{v_{1n}} & \vec{v_{2n}} & \cdots & \vec{v_{mn}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
$$

Problem 9. *Find the inverse of the linear transformation*

$$
\begin{aligned}
y_1 &= x_1 + 7x_2 \\
y_2 &= 3x_1 + 20x_2
\end{aligned}
$$

Solution.

$$\begin{bmatrix} x_1 + 7x_2 = y_1 \\ 3x_1 + 20x_2 = y_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1 + 7x_2 = y_1 \\ -x_2 = y_2 - 3y_1 \end{bmatrix}$$
$$\begin{bmatrix} x_1 + 7x_2 = y_1 \\ -x_2 = y_2 - 3y_1 \end{bmatrix}$$
$$\begin{bmatrix} x_1 + 7x_2 = y_1 \\ x_2 = 3y_1 - y_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1 = y_1 - 21y_1 + 7y_2 \\ x_2 = 3y_1 - y_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1 = -20y_1 + 7y_2 \\ x_2 = 3y_1 - y_2 \end{bmatrix}$$

Thus the inverse of the transformation is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -20 & 7 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

In the next four exercises, decide whether the given matrix is invertible, and find the inverse if it exists.

Problem 10.

$$\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$$

Solution. *We can create the linear system*

$$\begin{bmatrix} 2x_1 + 3x_2 = y_1 \\ 6x_1 + 9x_2 = y_2 \end{bmatrix}$$

and we can use row ops to find the inverse of the coefficient matrix.

$$\begin{bmatrix} 2x_1 + 3x_2 = y_1 \\ 6x_1 + 9x_2 = y_2 \end{bmatrix}$$
$$\begin{bmatrix} 2x_1 + 3x_2 = y_1 \\ 0 = y_2 - 3y_1 \end{bmatrix}$$

We know that

$$3y_1 = y_2$$
$$y_1 = \frac{y_2}{3}$$

This is a reason why we use determinants. I started doing row ops before I calculated the determinant of the matrix. The matrix has a determinant of 0 so the matrix does not have an inverse. In other words, the matrix is not invertible. For a matrix to be invertible, its determinant has to be nonzero. In fact, there are algorithms for calculating the inverse of a matrix that divide every number in the matrix by the determinant. If the determinant were zero, this division would not be possible.

Problem 11.

$$\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$$

Solution. *The determinant is* $1 * 9 - 4 * 2 = 1$. *Thus the matrix is invertible. We can create the linear system*

$$\begin{bmatrix} x_1 + 2x_2 = y_1 \\ 4x_1 + 9x_2 = y_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1 + 2x_2 = y_1 \\ x_2 = y_2 - 4y_1 \end{bmatrix}$$
$$\begin{bmatrix} x_1 = y_1 - 2y_2 + 8y_1 \\ x_2 = y_2 - 4y_1 \end{bmatrix}$$
$$\begin{bmatrix} x_1 = 9y_1 - 2y_2 \\ x_2 = -4y_1 + y_2 \end{bmatrix}$$

Thus the inverse of the matrix is

$$\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$$

Let's check

$$\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The inverse of the linear system is

$$\begin{bmatrix} x_1 = 9y_1 - 2y_2 \\ x_2 = -4y_1 + y_2 \end{bmatrix}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Problem 12.

$$\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$$

Solution. *The determinant is $1*9-3*2=3$, so the matrix is invertible. We can create the augmented matrix*

$$\begin{bmatrix} 1 & 2 & y_1 \\ 3 & 9 & .y_2 \end{bmatrix}$$

Now we can do row operations to find the inverse of the coefficient matrix.

$$\begin{bmatrix} 1 & 2 & y_1 \\ 3 & 9 & .y_2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 & y_1 \\ 0 & 3 & .y_2-3y_1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 & y_1 \\ 0 & 1 & .\dfrac{y_2}{3}-y_1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & y_1-\frac{2y_2}{3}+2y_1 \\ 0 & 1 & \frac{y_2}{3}-y_1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 3y_1-\frac{2y_2}{3} \\ 0 & 1 & -y_1+\frac{y_2}{3} \end{bmatrix}$$

Thus we get the linear system

$$x_1 = 3y_1 - \frac{2y_2}{3}$$
$$x_2 = -y_1 + \frac{y_2}{3}$$

The inverse of the matrix is

$$\begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}$$

Let's check

$$\begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$