Chapter 3 Section 4

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Problem 1. *Let* $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. *Let* $V = \mathrm{span}(\mathrm{v}_1, \mathrm{v}_2)$. *Is the vector* $\vec{w} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$ *on the plane V?*

Solution. *If the vector* $\vec{w}$ *is on the plane V, then there exist some* $x_1, x_2 \in \mathbb{R}$ *such that* $\vec{w} = x_1\vec{v_1} + x_2\vec{v_2}$. *This gives us the equations*

$$\begin{aligned} x_1 + x_2 &= 5 \\ x_1 + 2x_2 &= 7 \\ x_1 + 3x_2 &= 9 \end{aligned}$$

We can solve these equations using a matrix.

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

This gives us $x_1 = 3$ *and* $x_2 = 2$.

Thus $\vec{w}$ *is on the plane V because* $\vec{w} = 3\vec{v_1} + 2\vec{v_2}$.

Problem 2. *Consider the basis* $\mathfrak{B}$ *of* $\mathbb{R}^2$ *consisting of the vectors* $\vec{v_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ *and* $\vec{v_2} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

- *If* $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ *find* $\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}$
- *If* $\begin{bmatrix} \vec{y} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ *find* $\vec{y}$

Solution. *We can find the coordinates of* $\vec{x}$ *with respect to* $\mathfrak{B}$ *by means of an equation.*

$$\begin{aligned} \vec{x} &= c_1\vec{v_1} + c_2\vec{v_2} \\ \begin{bmatrix} 10 \\ 10 \end{bmatrix} &= c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 10 \\ 10 \end{bmatrix} &= \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

We can solve this equation using elementary row operations.

$$\left(\begin{array}{cc|c} 3 & -1 & 10 \\ 1 & 3 & 10 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 0 & -10 & -20 \\ 1 & 3 & 10 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 3 & 10 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 4 \end{array}\right)$$
$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right)$$

By reducing the matrix, we find that $c_1 = 4$ *and* $c_2 = 2$. *Thus* $\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

We can also use an equation to solve for $\vec{y}$.

$$
\begin{aligned}
\vec{y} &= 2\vec{v_1} - \vec{v_2} \\
&= 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 6 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 7 \\ -1 \end{bmatrix}
\end{aligned}
$$

Problem 3. *Let $\vec{v_1}$ and $\vec{v_2}$ be perpendicular unit vectors in $\mathbb{R}^3$. Let $\vec{v_3}$ be the cross product of $\vec{v_1}$ and $\vec{v_2}$, that is, $\vec{v_3} = \vec{v_1} \times \vec{v_2}$. We know from the properties of the cross product that $\vec{v_3}$ is perpendicular to $\vec{v_1}$ and $\vec{v_2}$. Thus the three vectors are linearly independent. The three vectors form a basis for $\mathbb{R}^3$.*

1. *What is $\vec{v_1} \times \vec{v_3}$?*
2. *Find the $\mathfrak{B}$-matrix of the linear transformation $T(x) = \vec{v_1} \times \vec{x}$.*

Solution. $\vec{v_1} \times \vec{v_3} = -\vec{v_2}$.

The $\mathfrak{B}$-matrix is the matrix B such that

$$
\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = B \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}
$$

We can find the coordinates of $\vec{x}$ with respect to $\mathfrak{B}$ by means of an equation.

$$
\begin{aligned}
\vec{x} &= c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} \\
&= \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
\end{aligned}
$$

Likewise, we have

$$
\begin{aligned}
T(x) &= \vec{v_1} \times \vec{x} \\
&= \vec{v_1} \times (c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}) \\
&= c_1(\vec{v_1} \times \vec{v_1}) + c_2(\vec{v_1} \times \vec{v_2}) + c_3(\vec{v_1} \times \vec{v_3}) \\
&= c_2 v_3 - c_3\vec{v_2} \\
&= \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix} \begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix}
\end{aligned}
$$

Thus

$$\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

and

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix}$$

Now let's find the matrix B such that

$$\begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix} = B \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

By inspection, we see that

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}$$

and the $\mathfrak{B}$-matrix of T is

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$