Chapter 3 Section 2

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Problem 1. *Is* $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 \text{: } x \geq 0 \text{ and } y \geq 0 \right\}$ *a subspace of* $\mathbb{R}^2$*?*

Solution. *W contains the zero vector and is closed under addition. But W is not closed under scalar multiplication. Therefore W is not a subspace of $\mathbb{R}^2$.*

Problem 2. *Show that the only subspaces of $\mathbb{R}^2$ are $\mathbb{R}^2$ itself, the set $\{\vec{0}\}$, and any of the lines through the origin.*

Solution. *Let W be a subspace of $\mathbb{R}^2$ that is neither a line through the origin nor the set $\{\vec{0}\}$. Then we can choose two nonzero nonparallel vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ from our subspace W. Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. We will show that we can write $\vec{u}$ as a linear combination of $\vec{v}$ and $\vec{w}$.*

If $\vec{u}$ can be written as a linear combination of $\vec{v}$ and $\vec{w}$, then there are solutions to the equation

$$x_1\vec{v} + x_2\vec{w} = \vec{u}$$

where x_1 and x_2 are real numbers. We can write this equation in matrix form

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This equation has solutions when $A = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ is invertible. We know that A is invertible when $\det A$ is nonzero.

The components v_1, v_2, w_1, w_2 can either be zero or nonzero. There is a small number of possible cases, since both vectors are not the zero vector, and since the two vectors are not parallel.

Case 1: $v_1 = 0, v_2 \neq 0, w_1 \neq 0, w_2 = 0$
Case 2: $v_1 \neq 0, v_2 = 0, w_1 = 0, w_2 \neq 0$

In both of these cases, the two vectors are scalar multiples of the unit vectors, and it easy to write any vector $\vec{u}$ as a linear combination of $\vec{v}$ and $\vec{w}$.

Case 3: *At least one of the vectors ($\vec{v}$ and $\vec{w}$) has two nonzero components.*

Let $\vec{v}$ be the vector with two nonzero components.

There exist real numbers c_1 and c_2 such that $c_1 v_1 = w_1$ and $c_2 v_2 = w_2$. We know that $c_1 \neq c_2$ since the two vectors are not scalar multiples of each other. We can substitute these expressions when we calculate the determinant of A.

$$\begin{aligned} \det A &= v_1 w_2 - v_2 w_1 \\ &= v_1(c_2 v_2) - v_2(c_1 v_1) \\ &= c_2 v_1 v_2 - c_1 v_1 v_2 \\ &= v_1 v_2 (c_2 - c_1) \end{aligned}$$

Since $v_1 \neq 0$, $v_2 \neq 0$ and $c_2 \neq c_1$, the determinant of A is nonzero. Thus the matrix A is invertible, and the equation

$$x_1 \vec{v} + x_2 \vec{w} = \vec{u}$$

has solutions for x_1 and x_2.

Since W is closed under linear combinations, the vector $\vec{u}$ is in the subspace W. This means that W contains every real number, so $W = \mathbb{R}^2$.

We can also express this using a linear transformation. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation.

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

We have shown that the matrix $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ is invertible. This means that T is invertible, and that T is a bijection, and that the image of T is $\mathbb{R}^2$.

This is equivalent to saying any vector $\vec{u}$ in $\mathbb{R}^2$ can be written as a linear combination of $\vec{v}$ and $\vec{w}$.