Chapter 3 Section 4

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Problem 1. *Let* $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. *Let* $V = \mathrm{span}(\mathrm{v}_1, \mathrm{v}_2)$. *Is the vector* $\vec{w} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$ *on the plane V?*

Solution. *If the vector* $\vec{w}$ *is on the plane V, then there exist some* $x_1, x_2 \in \mathbb{R}$ *such that* $\vec{w} = x_1\vec{v_1} + x_2\vec{v_2}$. *This gives us the equations*

$$\begin{aligned} x_1 + x_2 &= 5 \\ x_1 + 2x_2 &= 7 \\ x_1 + 3x_2 &= 9 \end{aligned}$$

We can solve these equations using a matrix.

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

This gives us $x_1 = 3$ *and* $x_2 = 2$.

Thus $\vec{w}$ *is on the plane V because* $\vec{w} = 3\vec{v_1} + 2\vec{v_2}$.

Problem 2. *Consider the basis* $\mathfrak{B}$ *of* $\mathbb{R}^2$ *consisting of the vectors* $\vec{v_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ *and* $\vec{v_2} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

- *If* $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ *find* $\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}$
- *If* $\begin{bmatrix} \vec{y} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ *find* $\vec{y}$

Solution. *We can find the coordinates of* $\vec{x}$ *with respect to* $\mathfrak{B}$ *by means of an equation.*

$$\vec{x} = c_1\vec{v_1} + c_2\vec{v_2}$$
$$\begin{bmatrix} 10 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

We can solve this equation using elementary row operations.

$$\left(\begin{array}{cc|c} 3 & -1 & 10 \\ 1 & 3 & 10 \end{array} \right)$$
$$\left(\begin{array}{cc|c} 0 & -10 & -20 \\ 1 & 3 & 10 \end{array} \right)$$
$$\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 3 & 10 \end{array} \right)$$
$$\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 4 \end{array} \right)$$
$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array} \right)$$

By reducing the matrix, we find that $c_1 = 4$ *and* $c_2 = 2$. *Thus* $\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

We can also use an equation to solve for $\vec{y}$.

$$
\begin{aligned}
\vec{y} &= 2\vec{v_1} - \vec{v_2} \\
&= 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 6 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 7 \\ -1 \end{bmatrix}
\end{aligned}
$$

Problem 3. *Let $\vec{v_1}$ and $\vec{v_2}$ be perpendicular unit vectors in $\mathbb{R}^3$. Let $\vec{v_3}$ be the cross product of $\vec{v_1}$ and $\vec{v_2}$, that is, $\vec{v_3} = \vec{v_1} \times \vec{v_2}$. We know from the properties of the cross product that $\vec{v_3}$ is perpendicular to $\vec{v_1}$ and $\vec{v_2}$. Thus the three vectors are linearly independent. The three vectors form a basis for $\mathbb{R}^3$.*

1. *What is $\vec{v_1} \times \vec{v_3}$?*
2. *Find the $\mathfrak{B}$-matrix of the linear transformation $T(x) = \vec{v_1} \times \vec{x}$.*

Solution. $\vec{v_1} \times \vec{v_3} = -\vec{v_2}$.

The $\mathfrak{B}$-matrix is the matrix B such that

$$
\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = B \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}
$$

We can find the coordinates of $\vec{x}$ with respect to $\mathfrak{B}$ by means of an equation.

$$
\begin{aligned}
\vec{x} &= c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} \\
&= \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
\end{aligned}
$$

Likewise, we have

$$
\begin{aligned}
T(x) &= \vec{v_1} \times \vec{x} \\
&= \vec{v_1} \times (c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}) \\
&= c_1(\vec{v_1} \times \vec{v_1}) + c_2(\vec{v_1} \times \vec{v_2}) + c_3(\vec{v_1} \times \vec{v_3}) \\
&= c_2 v_3 - c_3\vec{v_2} \\
&= \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix} \begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix}
\end{aligned}
$$

Thus

$$\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

and

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix}$$

Now let's find the matrix B such that

$$\begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix} = B \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

By inspection, we see that

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}$$

and the $\mathfrak{B}$-matrix of T is

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 4. *Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that projects any vector onto the line L spanned by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Earlier we found that the $\mathfrak{B}$-matrix of T with respect to the basis $\mathfrak{B} = \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right)$ is*

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

What is the relationship between B and the standard matrix A of T (such that $T(x) = Ax$)?

Solution.

$$\begin{aligned}
\operatorname{proj}_L(\vec{x}) &= \frac{1}{w_1^2 + w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix} \\
&= \frac{1}{3^2 + 1^2}\begin{bmatrix} 3^2 & 3*1 \\ 3*1 & 1^2 \end{bmatrix} \\
&= \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{9}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix} \\
&= \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}
\end{aligned}$$

Thus the standard matrix of T is

$$A = \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}$$

and

$$\begin{aligned}
T(\vec{x}) &= A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}$$

Now we're going to find the relationship between A and B.

Let $S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$

Let's write $T(\vec{x})$ in terms of A and S.

$$\begin{aligned}
\vec{x} &= S\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} \\
T(\vec{x}) &= A\vec{x} \\
T(\vec{x}) &= AS\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}
\end{aligned}$$

Now let's write $T(\vec{x})$ in terms of B and S.

$$\left[T(\vec{x})\right]_{\mathfrak{B}} = B\left[\vec{x}\right]_{\mathfrak{B}}$$
$$T(\vec{x}) = S\left[T(\vec{x})\right]_{\mathfrak{B}}$$
$$T(\vec{x}) = SB\left[\vec{x}\right]_{\mathfrak{B}}$$

The above equations show that $AS = SB$ and $A = SBS^{-1}$.

The equation $A = SBS^{-1}$ gives us another way of finding A (since we know S, we know B, and we can calculate S^{-1}).

Definition 1. *Two $n \times n$ matrices A and B are similar if there exists an invertible matrix S such that*

$$AS = SB, \text{ or } B = S^{-1}AS$$

Problem 5. *Is matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ similar to $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$?*

Solution. *We're looking for a matrix $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that*

$$\begin{bmatrix} x + 2z & y + 2t \\ 4x + 3z & 4y + 3t \end{bmatrix} = \begin{bmatrix} 5x & -y \\ 5z & -t \end{bmatrix}$$

By inspection we see that $z = 2x$ and $t = -y$. Therefore

$$S = \begin{bmatrix} x & y \\ 2x & -y \end{bmatrix}$$

Now let's look at the determinant of S.

$$det(S) = -3xy$$

The matrix S is invertible when $x \neq 0$ and $y \neq 0$. Thus

$$S = \begin{bmatrix} x & y \\ 2x & -y \end{bmatrix}$$

where $x \neq 0$ and $y \neq 0$.

We have found invertible matrices S such that $AS = SB$, so we know that matrix A is similar to matrix B.

Theorem 1. *If matrix A is similar to matrix B, then its power A^t is similar to B^t for all positive integers t.*

Proof. Let matrix A be similar to matrix B. Then there exists an invertible matrix S such that

$$\begin{aligned} AS &= SB \\ B &= S^{-1}AS \end{aligned}$$

When we simplify the expression for B^t, most of the S^{-1} and S terms cancel.

$$\begin{aligned} B^t &= (S^{-1}AS)^t \\ &= (S^{-1}AS)(S^{-1}AS)\cdots(S^{-1}AS) \\ &= S^{-1}A^tS \end{aligned}$$

Arriving at the equation $B^t = S^{-1}A^tS$, we have proven that B^t is similar to A^t for all positive integers t.

□

Definition 2. *A diagonal matrix has the form*

$$\begin{bmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & c_n \end{bmatrix}$$

where $c_i \neq 0$ for all $1 \leq i \leq n$

Problem 6. *Let T be a linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $\mathfrak{B} = (\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n})$ be a basis for $\mathbb{R}^n$. When is the $\mathfrak{B}$-matrix B of T diagonal?*

Solution. *The $\mathfrak{B}$-matrix of T is diagonal when*

$$T(v_i) = c_i v_i$$

for some constant $c_i \neq 0$ for all $1 \leq i \leq n$.

This follows from the fact that

$$\begin{aligned} \begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} &= B \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} \\ &= \left(\begin{bmatrix} T(\vec{v_1}) \end{bmatrix}_{\mathfrak{B}} \begin{bmatrix} T(\vec{v_2}) \end{bmatrix}_{\mathfrak{B}} \begin{bmatrix} T(\vec{v_3}) \end{bmatrix}_{\mathfrak{B}} \cdots \begin{bmatrix} T(\vec{v_n}) \end{bmatrix}_{\mathfrak{B}} \right) \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} \end{aligned}$$

When $T(v_i) = c_i v_i$ *the matrix* B *is diagonal.*

$$\left[T(\vec{x})\right]_{\mathfrak{B}} = \begin{bmatrix} c_1 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & c_n \end{bmatrix} \left[\vec{x}\right]_{\mathfrak{B}}$$