Chapter 3 Section 3

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Problem 1. *Find bases of the image and kernel of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}$$

Solution. *The second column vector is redundant because it is a scalar multiple of the first. The third column vector is redundant because it's the zero vector. The fifth column vector is redundant because it's the sum of the first column vector and the fourth column vector. This leaves us with the first and fourth column vectors, which form a basis for the image of A.*

$$\operatorname{im} A = \operatorname{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right)$$

We know from the rank-nullity theorem that

$$\dim \ker A + \dim \operatorname{im} A = m$$

The matrix A is a 4×5 matrix, which means that $m = 5$. Furthermore, we know that $\dim \operatorname{im} A = 2$. This gives us

$$\dim \ker A = 3$$

Thus the basis of the kernel must have three linearly independent vectors. Let's find three linearly independent vectors in the kernel of A.

We'll proceed by finding relations among the redundant column vectors in A. Let's refer to the column vectors in A as $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_5}$

$$v_2 = 2v_1 \implies -2v_1 + v_2 = 0$$
$$v_3 = 0$$
$$v_5 = v_1 + v_4 \implies v_1 + v_4 - v_5 = 0$$

This gives us the vectors

$$\vec{w_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{w_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{w_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

These three vectors are linearly independent, and they form a basis for the kernel of A.

$$\ker A = \operatorname{span}(\vec{w_1}, \vec{w_2}, \vec{w_3})$$

$$= \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

Problem 2. *For which values of the constant k do the following vectors form a basis of $\mathbb{R}^3$?*

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

Solution. *We need to determine when the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & k \\ 1 & 1 & k^2 \end{bmatrix}$$

is invertible. This matrix reduces to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & (1-k)/2 \\ 0 & 0 & k^2 - 1 \end{bmatrix}$$

When $k = 1$ or $k = -1$, the rank of the matrix is 2. For all other values of k, we are able to reduce the matrix to I_3, simply by dividing the third row by the scalar $k^2 - 1$ and getting a leading 1 in the third column. Thus the matrix A is invertible when $k^2 - 1 \neq 0$, that is, when $k \neq 1$ and $k \neq -1$.

In the following problems, find the redundant column vector of the given matrix A by inspection. Then find a basis of the image of A and a basis of the kernel of A.

Problem 3.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Solution. *The second column vector is redundant. The first column vector is a basis of the image of A. From the redundant column vector, we get the relation*

$$3v_1 = v_2$$
$$3v_1 - v_2 = 0$$
$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 0$$

Thus the vector $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ *is in the kernel of A.*

By the rank-nullity theorem, we know that $\dim \operatorname{im} A + \dim \ker A = m$ *where* $m = 2$ *(because the matrix A has two columns).*

Thus the kernel of A has dimension 1.

This means that the vector $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ *forms a basis for the kernel of A.*

In conclusion, the vector

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is a basis for the image of A, and the vector

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

is a basis for the kernel of A.

Problem 4.

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

Solution. *The first column vector is redundant because it's the zero vector. The second column vector* $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *forms a basis for the image of A.*

We get the relation

$$\vec{v_1} = 0\vec{v_2}$$
$$\vec{v_1} + 0\vec{v_2} = 0$$
$$\begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

Thus the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *is a basis for the kernel of A.*

Problem 5.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution. *The vectors* $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *and* $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ *form a basis for the image of A, because the two vectors are linearly independent.*

The rank-nullity theorem tells us that $\dim \operatorname{im} A + \dim \ker A = m$. *We know that* $m = 2$ *since A is a* 2×2 *matrix. Since* $\dim \operatorname{im} A = 2$, *we know that* $\dim \ker A = 0$. *Thus* $\ker A = \{\vec{0}\}$ *and the vector* $\vec{0}$ *forms a basis for the kernel of A.*

Problem 6.

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$

Solution. *The second column vector is redundant. Thus the first column vector* $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *forms a basis for the image of A.*

The redundant column vector gives us the relation

$$4\vec{v_1} = \vec{v_2}$$
$$4\vec{v_1} - \vec{v_2} = 0$$
$$\begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = 0$$

Thus the vector $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ *is in the kernel of A, and since the kernel of A has dimension 1, the vector* $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ *forms a basis for the kernel of A.*

Problem 7.

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

Solution. *The column vector* $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ *is redundant. The first two column vectors* $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$ *form a basis for the image of A.*

We get the relation $3\vec{v_1} = \vec{v_3}$. *Thus*

$$3\vec{v_1} - \vec{v_3} = 0$$

$$\begin{bmatrix} \vec{v_1} & \vec{v_2} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 0$$

The kernel has dimension 1. Therefore the vector $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ *forms a basis for the kernel of A.*

Problem 8.

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$$

Solution. *The third column vector is redundant, since*

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

The first two column vectors are linearly independent. Thus $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *form a basis for the image of A.*

The redundant column vector gives us the relation

$$\vec{v_1} + 2\vec{v_2} = \vec{v_3}$$
$$\vec{v_1} + 2\vec{v_2} - \vec{v_3} = 0$$
$$\begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Thus the vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ *is in the kernel of A. The kernel of A has dimension 1, so the vector* $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ *forms a basis for the kernel of A.*