Chapter 3 Section 2

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April 30 2022

Problem 1. *Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 \colon x \geq 0 \text{ and } y \geq 0 \right\}$ a subspace of $\mathbb{R}^2$?*

Solution. *W contains the zero vector and is closed under addition. But W is not closed under scalar multiplication. Therefore W is not a subspace of $\mathbb{R}^2$.*

Problem 2. *Show that the only subspaces of $\mathbb{R}^2$ are $\mathbb{R}^2$ itself, the set $\{\vec{0}\}$, and any of the lines through the origin.*

Solution. *Let W be a subspace of $\mathbb{R}^2$ that is neither a line through the origin nor the set $\{\vec{0}\}$. Then we can choose two nonzero nonparallel vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ from our subspace W. Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. We will show that we can write $\vec{u}$ as a linear combination of $\vec{v}$ and $\vec{w}$.*

If $\vec{u}$ can be written as a linear combination of $\vec{v}$ and $\vec{w}$, then there are solutions to the equation

$$x_1 \vec{v} + x_2 \vec{w} = \vec{u}$$

where x_1 and x_2 are real numbers. We can write this equation in matrix form

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This equation has solutions when $A = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ is invertible. We know that A is invertible when $\det A$ is nonzero.

The components v_1, v_2, w_1, w_2 can either be zero or nonzero. There is a small number of possible cases, since both vectors are not the zero vector, and since the two vectors are not parallel.

Case 1: $v_1 = 0, v_2 \neq 0, w_1 \neq 0, w_2 = 0$
Case 2: $v_1 \neq 0, v_2 = 0, w_1 = 0, w_2 \neq 0$

In both of these cases, the determinant of A is nonzero, and the matrix A is invertible.

Case 3:   At least one of the vectors ($\vec{v}$ and $\vec{w}$) has two nonzero components.

Let $\vec{v}$ be the vector with two nonzero components.

There exist real numbers c_1 and c_2 such that $c_1 v_1 = w_1$ and $c_2 v_2 = w_2$. We know that $c_1 \neq c_2$ since the two vectors are not scalar multiples of each other. We can substitute these expressions when we calculate the determinant of A.

$$
\begin{aligned}
\det A &= v_1 w_2 - v_2 w_1 \\
&= v_1(c_2 v_2) - v_2(c_1 v_1) \\
&= c_2 v_1 v_2 - c_1 v_1 v_2 \\
&= v_1 v_2 (c_2 - c_1)
\end{aligned}
$$

Since $v_1 \neq 0$, $v_2 \neq 0$ and $c_2 \neq c_1$, the determinant of A is nonzero. Thus the matrix A is invertible, and the equation

$$x_1 \vec{v} + x_2 \vec{w} = \vec{u}$$

has solutions for x_1 and x_2.

Since W is closed under linear combinations, the vector $\vec{u}$ is in the subspace W. This means that W contains every real number, so $W = \mathbb{R}^2$.

We can also express this using a linear transformation. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$
\begin{aligned}
T(\vec{x}) &= \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

We have shown that the matrix $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ is invertible.

This means that there is a unique solution $\vec{x}$ for every vector $\vec{u}$ in $\mathbb{R}^2$.

This is equivalent to saying any vector $\vec{u}$ in $\mathbb{R}^2$ can be written as a linear combination of $\vec{v}$ and $\vec{w}$.

Problem 3. *Consider the plane V in $\mathbb{R}^3$ given by the equation*

$$x_1 + 2x_2 + 3x_3 = 0$$

- *Find a matrix A such that $V = \ker A$*

- *Find a matrix B such that $V = \operatorname{im} B$*

Solution. *Let $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. We can write*

$$
\begin{aligned}
V &= \ker A \\
&= \ker \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}
\end{aligned}
$$

We can find two nonparallel vectors $\vec{v}$ and $\vec{w}$ in V.

Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

We can construct our matrix B from the column vectors $\vec{v}$ and $\vec{w}$.

$$B = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 1 \end{bmatrix}$$

Now we can say

$$
\begin{aligned}
V &= \operatorname{im} B \\
&= \operatorname{im} \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 1 \end{bmatrix}
\end{aligned}
$$

Problem 4. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Find vectors in $\mathbb{R}^3$ that span the image of A. What is the smallest number of vectors needed to span the image of A?

Solution. *The vectors*

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v_4} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

span the image of A.

The vector v_2 is redundant because $v_2 = 2v_1$. The vector v_4 is redundant because $v_1 + v_3 = v_4$. Thus only two vectors are needed to span the image of A, the vectors v_1 and v_3.

$$\operatorname{im} A = \operatorname{span}(v_1, v_3)$$

We can verify this algebraically. We know that the image of A is spanned by the vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$.

Let $\vec{u}$ be a vector in $\mathbb{R}^3$. Then

$$\vec{u} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4}$$

for some real coefficients c_1, c_2, c_3, c_4. Substituting we get

$$\vec{u} = c_1\vec{v_1} + c_2(2\vec{v_1}) + c_3\vec{v_3} + c_4(v_1 + v_3)$$
$$= (c_1 + c_4 + 2c_1)\vec{v_1} + (c_3 + c_4)\vec{v_3}$$

Thus $\vec{u}$ is a linear combination of vectors $\vec{v_1}$ and $\vec{v_3}$. The vectors $\vec{v_1}$ and $\vec{v_3}$ form a basis for the image of A.

Problem 5. *Are the following vectors in $\mathbb{R}^7$ linearly independent?*

$$\vec{v_1} = \begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \\ 1 \\ 9 \\ 0 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 6 \\ 0 \\ 7 \\ 1 \\ 4 \\ 8 \\ 0 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 5 \\ 0 \\ 6 \\ 2 \\ 3 \\ 1 \\ 7 \end{bmatrix} \quad \vec{v_4} = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

Solution. *$\vec{v_2}$ cannot be a linear combination of $\vec{v_1}$, since $\vec{v_1}$ has a 0 in the fourth component and $\vec{v_2}$ has a 1 in the fourth component. $\vec{v_3}$ cannot be a linear combination of $\vec{v_1}$ and $\vec{v_2}$, since $\vec{v_1}$ and $\vec{v_2}$ have zeros in the last component, and $\vec{v_3}$ has a seven in the last component. $\vec{v_4}$ cannot be a linear combination of $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$, since $\vec{v_4}$ has a 5 in the second component and the other three vectors all have zero in the second component.*

Thus the four vectors are linearly independent.

Problem 6. *Are the following vectors in $\mathbb{R}^7$ linearly independent?*

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Solution. *The vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are not linearly independent since $2\vec{v_2} - \vec{v_1} = \vec{v_3}$.*

Problem 7. *Suppose the column vectors of an $n \times m$ matrix A are linearly independent. Find the kernel of matrix A.*

Solution. *Let*

$$A = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

We can find the kernel of A by solving the equation

$$x_1 \vec{v_1} + x_2 \vec{v_2} + \cdots + x_m \vec{v_m} = 0$$

Since the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ are linearly independent, there is only the trivial relation, with $x_1 = x_2 = \cdots = x_m = 0$, thus the kernel of A is $\{\vec{0}\}$.

Problem 8. *If $v_1, \ldots, v_m$ is a basis of a subspace V of $\mathbb{R}^n$, and if $\vec{v}$ is a vector in V, how many solutions $c_1, \ldots, c_m$ does the equation*

$$\vec{v} = c_1 \vec{v_1} + \cdots + c_m \vec{v_m}$$

have?

Solution. *Let $c_1, \ldots, c_m$ and $d_1, \ldots, d_m$ be solutions to the equation. Then*

$$\vec{v} = c_1 \vec{v_1} + \cdots + c_m \vec{v_m}$$
$$\vec{v} = d_1 \vec{v_1} + \cdots + d_m \vec{v_m}$$

Subtracting we get

$$0 = (c_1 - d_1)\vec{v_1} + \cdots + (c_m - d_m)\vec{v_m}$$

Since the vectors $v_1, \ldots, v_m$ are linearly independent, we can conclude that $c_1 = d_1$. Thus there is only one solution to the equation.

In the following three problems, which of the sets W are subspaces of $\mathbb{R}^3$?

Problem 9.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$$

Solution.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$$

is not a subspace because it does not contain the zero vector.

Problem 10.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$$

Solution.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$$

is not a subspace since it is not closed under scalar multiplication.

Problem 11.

$$W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ are arbitrary constants} \right\}$$

Solution. *The first two column vectors are linearly independent, and the third is redundant. Thus W is the span of the first two column vectors, a plane passing through the origin. A plane that passes through the origin contains the zero vector, is closed under addition, and is closed under scalar multiplication. Thus W is a subspace of $\mathbb{R}^3$.*

Problem 12. *Consider the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ in $\mathbb{R}^n$. Is span $(\vec{v_1}, \ldots, \vec{v_m})$ necessarily a subspace of $\mathbb{R}^n$? Justify your answer.*

Solution. *The span of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ is necessarily a subspace of $\mathbb{R}^n$. The span of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ is the set of all linear combinations of these vectors, which means it has to be closed under addition and multiplication. Furthermore, the zero vector is included in the span of these vectors, since we can multiply any vector by the scalar zero to get the zero vector. Thus the span of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ is a subspace of $\mathbb{R}^n$, as it meets every requirement of a subspace.*

Problem 13. *Give a geometrical description of all subspaces of $\mathbb{R}^3$. Justify your answer.*

Solution. *The subspaces of $\mathbb{R}^3$ can be $\mathbb{R}^3$ itself, a plane passing through the origin, a line passing through the origin, or the set $\{\vec{0}\}$.*

Problem 14. *Consider two subspaces V and W of $\mathbb{R}^n$. Is the intersection $V \cap W$ necessarily a subspace of $\mathbb{R}^n$?*

Solution. *Let $\vec{v}, \vec{w}$ be vectors in $V \cap W$. By the definition of intersection, the vectors $\vec{v}, \vec{w}$ are in V and W. Since V and W are closed under addition, the vector $\vec{v} + \vec{w}$ is in both V and W. Thus $\vec{v} + \vec{w}$ is in $V \cap W$. This means that $V \cap W$ is closed under addition.*

Let $k \in \mathbb{R}$ and let $\vec{v} \in V \cap W$. We know from the definition of intersection that $\vec{v} \in V$ and $\vec{v} \in W$. Since V and W are closed under scalar multiplication, the vector $k\vec{v}$ is in both V and W. Thus the vector $k\vec{v}$ is in $V \cap W$. This means that $V \cap W$ is closed under scalar multiplication.

Since both V and W are subspaces, both V and W must contain the zero vector. Thus the zero vector is also in the intersection $V \cap W$.

The intersection $V \cap W$ is closed under addition, is closed under scalar multiplication, and contains the zero vector. Thus $V \cap W$ is a subspace of $\mathbb{R}^n$.

Problem 15. *Consider two subspaces V and W of $\mathbb{R}^n$. Is the union $V \cup W$ necessarily a subspace of $\mathbb{R}^n$?*

Solution. *The union of two subspaces is not necessarily a subspace.*

Let $V = \operatorname{span}[1, 0]$ and $W = \operatorname{span}[0, 1]$. The vectors $[1, 0]$ and $[0, 1]$ are in the union of V and W, but their sum $[1, 1]$ is not in the union of V and W. The set $V \cup W$ is not closed under addition, so $V \cup W$ is not a subspace.

Problem 16. *Consider a nonempty subset W of $\mathbb{R}^n$ that is closed under addition and under scalar multiplication. Is W necessarily a subspace of $\mathbb{R}^n$? Explain*

Solution. *W is necessarily a subspace of $\mathbb{R}^n$. Let $\vec{v}$ be a vector in W. Since W is closed under scalar multiplication, the additive inverse $-\vec{v}$ is in W. Since W is closed under addition, $\vec{v} + -\vec{v} = \vec{0}$ is in W. Since W contains the zero vector and is closed under addition and scalar multiplication, W is a subspace of $\mathbb{R}^n$.*

Problem 17. *Find a nontrivial relation among the following vectors:*

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Solution.

$$2 * \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Problem 18. *Consider the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ in $\mathbb{R}^n$, with $\vec{v_m} = \vec{0}$. Are these vectors linearly independent?*

Solution. *The vectors are not linearly independent, since we can form a nontrivial relation by setting $c_1 = c_2 = \cdots = c_{m-1} = 0$ and $c_m = 1$.*

$$c_1 \vec{v_1} + c_2 \vec{v_2} + \cdots + c_m \vec{v_m} = \vec{0}$$

is a nontrivial relation, where $c_1 = c_2 = \cdots = c_{m-1} = 0$ and $c_m = 1$

In the following exercises, identify the redundant vectors and determine whether the given vectors are linearly independent.

Problem 19.

$$\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution. *The two vectors are not linearly independent, as the zero vector is redundant. We can write the zero vector as a linear combination of $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$, thus the zero vector is redundant.*

Problem 20.

$$\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 11 \\ 7 \end{bmatrix}$$

Solution. *The two vectors are linearly independent, as we cannot write the second vector as a linear combination of the first.*

Problem 21.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

Solution. *The two vectors are not linearly independent. The second vector is redundant: the second vector is a scalar multiple of the first.*

Problem 22.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

Solution. *The two vectors are not linearly independent. The second vector is redundant.*

Problem 23.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

Solution. *The three vectors are not linearly independent, as a nontrivial relation exists among them.*

$$\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{0}$$

The third vector is redundant.

Problem 24.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Solution. *The three vectors are linearly dependent. The third vector is redundant, as shown.*

$$2 * \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{0}$$

Problem 25.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution. *The three vectors are linearly independent. The zero components of each vector show that no vector is redundant.*

Problem 26.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

Solution. *Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$. Let's find the kernel of A.*

$$
\left(\begin{array}{ccc|c}
1 & 1 & 1 & 0 \\
1 & 2 & 3 & 0 \\
1 & 3 & 6 & 0
\end{array}\right)
$$

$$
\left(\begin{array}{ccc|c}
1 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 2 & 5 & 0
\end{array}\right)
$$

$$
\left(\begin{array}{ccc|c}
1 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

$$
\left(\begin{array}{ccc|c}
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

$$
\left(\begin{array}{ccc|c}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

The kernel of A is $\{\vec{0}\}$. Thus the three vectors are linearly independent, and none of the vectors are redundant.

Problem 27.

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

Solution. *The first vector (the zero vector) is redundant. The third vector (3, 0, 0) is redundant. The fifth and sixth vectors (4, 5, 0) and (6, 7, 0) are redundant. The seven vectors are linearly dependent.*

The vectors (1, 0, 0) (0, 1, 0) and (0, 0, 1) are linearly independent.

Problem 28.

$$
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}
$$

Solution. *The second and fifth vectors are redundant. The five vectors are linearly dependent.*

Problem 29.

$$
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}
$$

Solution. *Let* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \\ 1 & 4 & 10 \end{bmatrix}$

Let's find the kernel of A.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 3 & 7 & 0 \\ 1 & 4 & 10 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 3 & 9 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

This gives us the equations $x_1 = 2x_3$ *and* $x_2 = -3x_3$.

$$\begin{aligned} \ker A &= \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix} \\ &= t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \\ &= \mathrm{span} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \end{aligned}$$

The kernel of A *is the line spanned by the vector* $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$.

This means that the three column vectors are linearly dependent, since there are many nontrivial relations among the three vectors. The third column vector is redundant.