Chapter 3 Section 1

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Problem 1. *Find the kernel of the linear transformation*

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$$

from $\mathbb{R}^3$ *to* $\mathbb{R}^2$.

Solution. *Let's solve the linear system* $T(\vec{x}) = 0$ *to get the kernel of* T.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

This tells us that $x_1 = x_3$ *and* $x_2 = -2x_3$.

Let $t = x_3$ *be an arbitrary real number. Then the solutions to the linear system are*

$$\ker(T) = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel of T *is the line spanned by the vector* $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Problem 2. *Find the kernel of the linear transformation*

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from $\mathbb{R}^5$ *to* $\mathbb{R}^4$.

Solution. *Let's solve the linear system* $T(\vec{x}) = Ax = 0$.

We can solve this linear system by creating the augmented matrix $\begin{bmatrix} A & | & \vec{0} \end{bmatrix}$ *and calculating* $\operatorname{rref} \begin{bmatrix} A & | & \vec{0} \end{bmatrix}$.

$$\left(\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 4 & 8 & 5 & -8 & 9 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & -4 & 5 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 4 & 8 & 5 & -8 & 9 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right) \textit{Adding row2 to row1}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & -4 & 5 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right) \textit{Adding row2 to row1}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right) \textit{Subtracting row3 from row1}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & -2 & 8 & -10 & 0 \end{array}\right) \textit{Adding row2 to row4}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \textit{Adding row2 to row4}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & -3 & 4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \textit{Adding row3 to row2}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \textit{Swapping rows and multiplying row1 by -1}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_4 - 4x_5 &= 0 \\ x_3 - 4x_4 + 5x_5 &= 0 \end{aligned}$$

$$x_1 = -2x_2 - 3x_4 + 4x_5$$
$$x_3 = 4x_4 - 5x_5$$

Let $r = x_2$, let $s = x_4$, let $t = x_5$.

Then the solutions to the linear system (the kernel) are of the form:

$$\begin{aligned}
\ker(T) &= \begin{bmatrix} -2r - 3s + 4t \\ r \\ 4s - 5t \\ s \\ t \end{bmatrix} \\
&= \begin{bmatrix} -2r \\ r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3s \\ 0 \\ 4s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 4t \\ 0 \\ -5t \\ 0 \\ t \end{bmatrix} \\
&= r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \\
&= \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right)
\end{aligned}$$

Problem 3. *For an invertible $n \times n$ matrix find* $\ker A$.

Solution. *Let $T(\vec{x}) = A\vec{x}$ be a linear transformation with an invertible $n \times n$ matrix A. We know that T is invertible because A is invertible. Thus there can only be one unique solution to the equation $A\vec{x} = 0$. Since $\vec{x} = \vec{0}$ is the unique solution, we know that* $\ker A = \{\vec{0}\}$.

Problem 4. *For which $n \times m$ matrices is $\ker A = \{\vec{0}\}$. Give your answer in terms of the rank of A.*

Solution. *Let A be a $n \times m$ matrix. When $\operatorname{rank} A = m$, we get the unique solution $\vec{x} = \vec{0}$. Thus $\ker A = \{\vec{0}\}$ when $\operatorname{rank} A = m$.*

In the following problems, find vectors that span the kernel of A and the image of A.

Problem 5.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution. *Let $T(\vec{x}) = A\vec{x}$. Let's solve the equation $A\vec{x} = \vec{0}$.*

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right] \qquad \textit{Augmented matrix } \begin{bmatrix} A & | & \vec{0} \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \end{array}\right] \qquad \textit{Subtract 3 times row1 from row2}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -2 & 0 \end{array}\right] \qquad \textit{Add row2 to row1}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \qquad \textit{Divide row2 by -2}$$

The solution set is $\vec{x} = \vec{0}$, thus $\ker A = \{0\}$.

The kernel of A is spanned by the zero vector. The image of A is the span of the column vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

In other words, $T(\vec{x}) = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Problem 6.

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Solution. *Let's solve the equation*

$$x_1 + 2x_2 + 3x_3 = 0$$

From inspection, we can find two nonparallel vectors in the solution set, the vectors

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \vec{v_2} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

There is no constant c such that $\vec{v_2} = c\vec{v_1}$, thus the two vectors are linearly independent and form a basis for the kernel.

Thus the kernel of A is the plane through the origin spanned by the vectors $\vec{v_1}$ and $\vec{v_2}$.

$$\ker A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right)$$

The image of A is $\mathbb{R}$, since for any $y \in \mathbb{R}$, we have the solution $\vec{x} = \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix}$.

Problem 7.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution. *The kernel of A is $\mathbb{R}^2$ and the image of A is $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We can write the kernel of A as the span of the unit vectors e_1 and e_2.*

$$\ker A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Problem 8.

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$$

Solution. *The image is the span of the column vectors. Since the second column vector is redundant, the image of A is the line spanned by the vector $\begin{bmatrix} 2 & 3 \end{bmatrix}$.*

The kernel of A is the solution set of the equation $2x_1 + 3x_2 = 0$.

$$x_1 = -\frac{3}{2}x_2$$

This equation describes a line that passes through the origin.

The kernel of A is the line spanned by the vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Problem 9.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

Solution. *Let's solve the linear system $A\vec{x} = 0$ to get the kernel of A.*

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 3 & 5 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives us the equations

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \end{aligned}$$

Let t be an arbitrary real number. Then we can write the kernel of A as

$$\begin{aligned} \ker A &= \left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \right\} \\ &= \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \\ &= \operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Thus the kernel of A is the line spanned by the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

In our matrix A, the third column vector is redundant. Thus

$$\operatorname{im} A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$$

The image of A is a plane spanned by the above vectors that passes through the origin.

Problem 10.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution. *Let's solve the linear system $A\vec{x} = 0$.*

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

This gives us the equations

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \end{aligned}$$

Let t be an arbitrary real number. Then we can write the kernel of A as

$$\begin{aligned} \ker A &= \left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \right\} \\ &= \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \\ &= \operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Thus the kernel of A is the line spanned by the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

In our matrix A, the third column vector is redundant. Thus

$$\operatorname{im} A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

The image of A is a plane that passes through the origin.

Problem 11.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution. *Let's solve the linear system $A\vec{x} = \vec{0}$.*

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -14 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -7 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since rref A $= I_3, \ker A = \{\vec{0}\}$.

Problem 12.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution. *We can solve the linear system $A\vec{x} = \vec{0}$.*

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives us the equation $x_1 + x_2 + x_3 = 0$.

This equation describes a plane passing through the origin. Two solutions to the equation are

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$
$$\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel of A *has to be a subspace of* $\mathbb{R}^3$. *Possible subspaces are* $\mathbb{R}^3$, *a plane passing through the origin, a line passing through the origin, and* $\{\vec{0}\}$. *Since the kernel of* A *is not the domain, and since it contains two nonparallel vectors, it has to be a plane passing through the origin.*

The kernel of A *is the span of these vectors.*

$$\ker A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$$