

Chapter 3 Section 4

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Problem 1. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Let $V = \text{span}(\vec{v}_1, \vec{v}_2)$. Is the vector

$\vec{w} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$ on the plane V ?

Solution. If the vector \vec{w} is on the plane V , then there exist some $x_1, x_2 \in \mathbb{R}$ such that $\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2$. This gives us the equations

$$x_1 + x_2 = 5$$

$$x_1 + 2x_2 = 7$$

$$x_1 + 3x_2 = 9$$

We can solve these equations using a matrix.

$$\begin{bmatrix} 1 & 1 & | & 5 \\ 1 & 2 & | & 7 \\ 1 & 3 & | & 9 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & | & 5 \\ 0 & 1 & | & 2 \\ 0 & 2 & | & 4 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & | & 5 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$$

This gives us $x_1 = 3$ and $x_2 = 2$.

Thus \vec{w} is on the plane V because $\vec{w} = 3\vec{v}_1 + 2\vec{v}_2$.

Problem 2. Consider the basis \mathfrak{B} of \mathbb{R}^2 consisting of the vectors $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

- If $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ find $\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}$
- If $\begin{bmatrix} \vec{y} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ find \vec{y}

Solution. We can find the coordinates of \vec{x} with respect to \mathfrak{B} by means of an equation.

$$\begin{aligned}\vec{x} &= c_1 \vec{v}_1 + c_2 \vec{v}_2 \\ \begin{bmatrix} 10 \\ 10 \end{bmatrix} &= c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 10 \\ 10 \end{bmatrix} &= \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\end{aligned}$$

We can solve this equation using elementary row operations.

$$\begin{aligned}&\left(\begin{array}{cc|c} 3 & -1 & 10 \\ 1 & 3 & 10 \end{array} \right) \\&\left(\begin{array}{cc|c} 0 & -10 & -20 \\ 1 & 3 & 10 \end{array} \right) \\&\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 3 & 10 \end{array} \right) \\&\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 4 \end{array} \right) \\&\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array} \right)\end{aligned}$$

By reducing the matrix, we find that $c_1 = 4$ and $c_2 = 2$. Thus $\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

We can also use an equation to solve for \vec{y} .

$$\begin{aligned}
\vec{y} &= 2\vec{v}_1 - \vec{v}_2 \\
&= 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 6 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 7 \\ -1 \end{bmatrix}
\end{aligned}$$

Problem 3. Let \vec{v}_1 and \vec{v}_2 be perpendicular unit vectors in \mathbb{R}^3 . Let \vec{v}_3 be the cross product of \vec{v}_1 and \vec{v}_2 , that is, $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$. We know from the properties of the cross product that \vec{v}_3 is perpendicular to \vec{v}_1 and \vec{v}_2 . Thus the three vectors are linearly independent. The three vectors form a basis for \mathbb{R}^3 .

1. What is $\vec{v}_1 \times \vec{v}_3$?
2. Find the \mathfrak{B} -matrix of the linear transformation $T(x) = \vec{v}_1 \times \vec{x}$.

Solution. $\vec{v}_1 \times \vec{v}_3 = -\vec{v}_2$.

The \mathfrak{B} -matrix is the matrix B such that

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = B \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}$$

We can find the coordinates of \vec{x} with respect to \mathfrak{B} by means of an equation.

$$\begin{aligned}
\vec{x} &= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \\
&= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
T(x) &= \vec{v}_1 \times \vec{x} \\
&= \vec{v}_1 \times (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \\
&= c_1(\vec{v}_1 \times \vec{v}_1) + c_2(\vec{v}_1 \times \vec{v}_2) + c_3(\vec{v}_1 \times \vec{v}_3) \\
&= c_2\vec{v}_3 - c_3\vec{v}_2 \\
&= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix}
\end{aligned}$$

Thus

$$\begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

and

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix}$$

Now let's find the matrix B such that

$$\begin{bmatrix} 0 \\ -c_3 \\ c_2 \end{bmatrix} = B \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

By inspection, we see that

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}$$

and the \mathfrak{B} -matrix of T is

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 4. Let T be the linear transformation from \mathbb{R}^2 to \mathbb{R}^2 that projects any vector onto the line L spanned by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Earlier we found that the \mathfrak{B} -matrix of T with respect to the basis $\mathfrak{B} = \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right)$ is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

What is the relationship between B and the standard matrix A of T (such that $T(x) = Ax$)?

Solution.

$$\begin{aligned}
 \text{proj}_L(\vec{x}) &= \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} \\
 &= \frac{1}{3^2 + 1^2} \begin{bmatrix} 3^2 & 3 * 1 \\ 3 * 1 & 1^2 \end{bmatrix} \\
 &= \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{9}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix} \\
 &= \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}
 \end{aligned}$$

Thus the standard matrix of T is

$$A = \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix}$$

and

$$\begin{aligned}
 T(\vec{x}) &= A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
 &= \begin{bmatrix} 0.9 & 0.3 \\ 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
 \end{aligned}$$

Now we're going to find the relationship between A and B .

$$\text{Let } S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

Let's write $T(\vec{x})$ in terms of A and S .

$$\begin{aligned}
 \vec{x} &= S \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} \\
 T(\vec{x}) &= A\vec{x} \\
 T(\vec{x}) &= AS \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}
 \end{aligned}$$

Now let's write $T(\vec{x})$ in terms of B and S .

$$\begin{aligned}\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} &= B \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}} \\ T(\vec{x}) &= S \begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathfrak{B}} \\ T(\vec{x}) &= SB \begin{bmatrix} \vec{x} \end{bmatrix}_{\mathfrak{B}}\end{aligned}$$

The above equations show that $AS = SB$ and $A = SBS^{-1}$.

The equation $A = SBS^{-1}$ gives us another way of finding A (since we know S , we know B , and we can calculate S^{-1}).

Definition 1. Two $n \times n$ matrices A and B are similar if there exists an invertible matrix S such that

$$AS = SB, \text{ or } B = S^{-1}AS$$

Problem 5. Is matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ similar to $B = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$?

Solution. We're looking for a matrix $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that

$$\begin{bmatrix} x + 2z & y + 2t \\ 4x + 3z & 4y + 3t \end{bmatrix} = \begin{bmatrix} 5x & -y \\ 5z & -t \end{bmatrix}$$

By inspection we see that $z = 2x$ and $t = -y$. Therefore

$$S = \begin{bmatrix} x & y \\ 2x & -y \end{bmatrix}$$

Now let's look at the determinant of S .

$$\det(S) = -3xy$$

The matrix S is invertible when $x \neq 0$ and $y \neq 0$. Thus

$$S = \begin{bmatrix} x & y \\ 2x & -y \end{bmatrix}$$

where $x \neq 0$ and $y \neq 0$.

We have found invertible matrices S such that $AS = SB$, and we know that matrix A is similar to matrix B .