

Chapter 3 Section 1

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Problem 1. Find the kernel of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$$

from \mathbb{R}^3 to \mathbb{R}^2 .

Solution. Let's solve the linear system $T(\vec{x}) = 0$ to get the kernel of T .

$$\begin{array}{l} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array} \right) \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \\ \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \end{array}$$

This tells us that $x_1 = x_3$ and $x_2 = -2x_3$.

Let $t = x_3$ be an arbitrary real number. Then the solutions to the linear system are

$$\ker(T) = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel of T is the line spanned by the vector $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Problem 2. Find the kernel of the linear transformation

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from \mathbb{R}^5 to \mathbb{R}^4 .

Solution. Let's solve the linear system $T(\vec{x}) = Ax = 0$.

We can solve this linear system by creating the augmented matrix $[A \mid \vec{0}]$ and calculating $\text{rref } [A \mid \vec{0}]$.

$$\begin{pmatrix} 1 & 2 & 2 & -5 & 6 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 4 & 8 & 5 & -8 & 9 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & -4 & 5 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 4 & 8 & 5 & -8 & 9 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Adding row2 to row1}$$

$$\begin{pmatrix} 0 & 0 & 1 & -4 & 5 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Adding row2 to row1}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Subtracting row3 from row1}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & -2 & 8 & -10 & | & 0 \end{pmatrix} \text{ Adding row2 to row4}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Adding row2 to row4}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & 0 & -3 & 4 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Adding row3 to row2}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & -4 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Swapping rows and multiplying row1 by -1}$$

$$\begin{aligned}
x_1 + 2x_2 + 3x_4 - 4x_5 &= 0 \\
x_3 - 4x_4 + 5x_5 &= 0
\end{aligned}$$

$$x_1 = -2x_2 - 3x_4 + 4x_5$$

$$x_3 = 4x_4 - 5x_5$$

Let $r = x_2$, let $s = x_4$, let $t = x_5$.

Then the solutions to the linear system (the kernel) are of the form:

$$\begin{aligned} \ker(T) &= \begin{bmatrix} -2r - 3s + 4t \\ r \\ 4s - 5t \\ s \\ t \end{bmatrix} \\ &= \begin{bmatrix} -2r \\ r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3s \\ 0 \\ 4s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 4t \\ 0 \\ -5t \\ 0 \\ t \end{bmatrix} \\ &= r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \\ &= \text{span} \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Problem 3. For an invertible $n \times n$ matrix find $\ker A$.

Solution. Let $T(\vec{x}) = A\vec{x}$ be a linear transformation with an invertible $n \times n$ matrix A . We know that T is invertible because A is invertible. Thus there can only be one unique solution to the equation $A\vec{x} = \vec{0}$. Since $\vec{x} = \vec{0}$ is the unique solution, we know that $\ker A = \{\vec{0}\}$.

Problem 4. For which $n \times m$ matrices is $\ker A = \{\vec{0}\}$. Give your answer in terms of the rank of A .

Solution. Let A be a $n \times m$ matrix. When $\text{rank } A = m$, we get the unique solution $\vec{x} = \vec{0}$. Thus $\ker A = \{\vec{0}\}$ when $\text{rank } A = m$.

In the following problems, find vectors that span the kernel of A and the image of A .

Problem 5.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution. Let $T(\vec{x}) = A\vec{x}$. Let's solve the equation $A\vec{x} = \vec{0}$.

$$\begin{array}{ll} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array} \right] & \text{Augmented matrix } \left[A \mid \vec{0} \right] \\ \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \end{array} \right] & \text{Subtract 3 times row1 from row2} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -2 & 0 \end{array} \right] & \text{Add row2 to row1} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] & \text{Divide row2 by -2} \end{array}$$

The solution set is $\vec{x} = \vec{0}$, thus $\ker A = \{0\}$.

The kernel of A is spanned by the zero vector. The image of A is the span of the column vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

In other words, $T(\vec{x}) = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Problem 6.

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Solution. Let's solve the equation

$$x_1 + 2x_2 + 3x_3 = 0$$

From inspection, we can find two nonparallel vectors in the solution set, the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

There is no constant c such that $\vec{v}_2 = c\vec{v}_1$, thus the two vectors are linearly independent and form a basis for the kernel.

Thus the kernel of A is the plane through the origin spanned by the vectors \vec{v}_1 and \vec{v}_2 .

$$\ker A = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right)$$

The image of A is \mathbb{R} , since for any $y \in \mathbb{R}$, we have the solution $\vec{x} = \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix}$.

Problem 7.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution. The kernel of A is \mathbb{R}^2 and the image of A is $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We can write the kernel of A as the span of the unit vectors e_1 and e_2 .

$$\ker A = \text{span} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Problem 8.

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$$

Solution. The image is the span of the column vectors. Since the second column vector is redundant, the image of A is the line spanned by the vector $\begin{bmatrix} 2 & 3 \end{bmatrix}$.

The kernel of A is the solution set of the equation $2x_1 + 3x_2 = 0$.

$$x_1 = -\frac{3}{2}x_2$$

This equation describes a line that passes through the origin.

The kernel of A is the line spanned by the vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Problem 9.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

Solution. Let's solve the linear system $A\vec{x} = 0$ to get the kernel of A .

$$\begin{aligned}
& \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{array} \right] \\
& \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 3 & 5 & 0 \end{array} \right] \\
& \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array} \right] \\
& \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \\
& \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]
\end{aligned}$$

This gives us the equations

$$\begin{aligned}
x_1 &= x_3 \\
x_2 &= -2x_3
\end{aligned}$$

Let t be an arbitrary real number. Then we can write the kernel of A as

$$\begin{aligned}
\ker A &= \left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \right\} \\
&= \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \\
&= \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)
\end{aligned}$$

Thus the kernel of A is the line spanned by the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

In our matrix A , the third column vector is redundant. Thus

$$\text{im } A = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$$

The image of A is a plane spanned by the above vectors that passes through the origin.

Problem 10.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution. Let's solve the linear system $A\vec{x} = 0$.

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array} \right] \\ & \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \\ & \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \end{aligned}$$

This gives us the equations

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \end{aligned}$$

Let t be an arbitrary real number. Then we can write the kernel of A as

$$\begin{aligned} \ker A &= \left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \right\} \\ &= \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \\ &= \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Thus the kernel of A is the line spanned by the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

In our matrix A , the third column vector is redundant. Thus

$$\text{im } A = \text{span} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

The image of A is a plane that passes through the origin.

Problem 11.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution. *Let's solve the linear system $A\vec{x} = \vec{0}$.*

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 0 \\ 3 & 2 & 1 & 0 \end{array} \right] \\ & \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 \end{array} \right] \\ & \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 \end{array} \right] \\ & \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -14 & 0 \end{array} \right] \\ & \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -7 & 0 \end{array} \right] \\ & \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -6 & 0 \end{array} \right] \\ & \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \\ & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \end{aligned}$$

Since $\text{rref } A = I_3$, $\ker A = \{\vec{0}\}$.

Problem 12.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution. *We can solve the linear system $A\vec{x} = \vec{0}$.*

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This gives us the equation $x_1 + x_2 + x_3 = 0$.

This equation describes a plane passing through the origin. Two solutions to the equation are

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel of A has to be a subspace of \mathbb{R}^3 . Possible subspaces are \mathbb{R}^3 , a plane passing through the origin, a line passing through the origin, and $\{\vec{0}\}$. Since the kernel of A is not the domain, and since it contains two nonparallel vectors, it has to be a plane passing through the origin.

The kernel of A is the span of these vectors.

$$\ker A = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$$

Problem 13.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution.

$$\text{rref } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This gives us the equations $x_1 = 0$ and $x_2 = 0$.

Thus $\ker A = \{\vec{0}\}$.

Problem 14.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution. We can solve the linear system $A\vec{x} = \vec{0}$ with elementary row operations on an augmented matrix.

$$\begin{aligned} & \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{array} \right) \\ & \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{array} \right) \\ & \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right) \\ & \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \\ & \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \end{aligned}$$

We get the equations $x_1 = 0$ and $x_2 = 0$. Thus

$$\ker A = \{\vec{0}\}$$

Problem 15.

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix}$$

Solution. We can solve the linear system $A\vec{x} = \vec{0}$.

$$\left(\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & -2 & 2 & 0 \\ 1 & -1 & -2 & 0 & 3 & 0 \\ 2 & -2 & -1 & 3 & 4 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & -1 & -1 & 3 & 0 \\ -1 & 1 & 0 & -2 & 2 & 0 \\ 1 & -1 & -2 & 0 & 3 & 0 \\ 2 & -2 & -1 & 3 & 4 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & -1 & -1 & 3 & 0 \\ -1 & 1 & 0 & -2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 5 & 0 \\ 2 & -2 & -1 & 3 & 4 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & -1 & -1 & 3 & 0 \\ -1 & 1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 2 & -2 & -1 & 3 & 4 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & -1 & -1 & 3 & 0 \\ -1 & 1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 6 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & -1 & -1 & 3 & 0 \\ -1 & 1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & -1 & -1 & 3 & 0 \\ -1 & 1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccccc|c} -1 & 1 & 0 & -2 & 2 & 0 \\ 0 & 0 & -1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This gives us the equations

$$\begin{aligned}x_3 &= -x_4 \\ x_5 &= 0\end{aligned}$$

Let r, s be real numbers. Then the solutions are

$$\ker A = \begin{bmatrix} s - 2t \\ s \\ -t \\ t \\ 0 \end{bmatrix}$$