Chapter 1 Section 2

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Problem 1. *Find all solutions for the equations*

$$\begin{vmatrix} x + y - 2z = 5 \\ 2x + 3y + 4z = 2 \end{vmatrix}$$

Solution. *We can use Gauss-Jordan elimination.*

$$\begin{vmatrix} 1 & 1 & -2 & 5 \\ 2 & 3 & 4 & 2 \end{vmatrix}$$
$$\begin{vmatrix} 1 & 1 & -2 & 5 \\ 0 & 1 & 8 & -8 \end{vmatrix}$$
$$\begin{vmatrix} 1 & 0 & -10 & 13 \\ 0 & 1 & 8 & -8 \end{vmatrix}$$

This gives us the equations $x - 10z = 13$ *and* $y + 8z = -8$.

Thus there are infinitely many solutions. The solutions are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 + 10t \\ -8 - 8t \\ t \end{pmatrix}$$

for an arbitrary real number t.

Problem 2. *Find all solutions for the equations*

$$\begin{vmatrix} 3x + 4y - z = 8 \\ 6x + 8y - 2z = 3 \end{vmatrix}$$

Solution. *We can use Gauss-Jordan elimination once again (and after this problem we may use Gauss-Jordan elimination without saying it explicitly).*

$$\begin{vmatrix} 3 & 4 & -1 & 8 \\ 6 & 8 & -2 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 4 & -1 & 8 \\ 0 & 0 & 0 & -13 \end{vmatrix}$$

This gives us the equations $3x + 4y - z = 8$ *and* $0 = -13$. *The second equation is a contradiction.*

Why is the second equation a contradiction? It's because each equation represents a line, and the two lines do not intersect.

When we do Gauss-Jordan elimination, we assume there is a solution. When this assumption is wrong, we can get a contradiction like $0 = 13$.

Our method shows us that the system is inconsistent, so there are no solutions.

Problem 3. *Find all solutions for the equation*

$$x + 2y + 3z = 4$$

Solution. *Let* $s = y$ *and* $t = z$ *be arbitrary real numbers. Then the solutions to the equation are:*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - 2s - 3t \\ s \\ t \end{pmatrix}$$

Problem 4. *Find all solutions to the equations*

$$\begin{vmatrix} x + y = 1 \\ 2x - y = 5 \\ 3x + 4y = 2 \end{vmatrix}$$

Solution.

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 3 & 4 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & 3 \\ 3 & 4 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 4 & -4 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix}$$

Thus the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Problem 5. *Find the solution to the equations*

$$\begin{vmatrix} x_3 + x_4 = 0 \\ x_2 + x_3 = 0 \\ x_1 + x_2 = 0 \end{vmatrix}$$

Solution. *Remember that in Gauss-Jordan elimination we can divide a row by a scalar, subtract a multiple of a row from another row, and swap rows. These are three elementary row operations we can use on linear systems to get a matrix in reduced row echelon form. In other words, we are computing rref(A) for the augmented matrix A.*

$$\begin{vmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{vmatrix}$$

Now let $r = x_4$ be any arbitrary real number. Then the solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -r \\ r \\ -r \\ r \end{pmatrix}$$

It may not have been necessary to get the augmented matrix in reduced row echelon form (rref) but it's good practice. We could have solved the problem without row operations by making a clever substitution ($r = x_4$) at the very beginning.

Problem 6. *Find solutions for the equations*

$$\begin{vmatrix} x_1 - 7x_2 + x_5 = 3 \\ x_3 - 2x_5 = 2 \\ x_4 + x_5 = 1 \end{vmatrix}$$

Solution.

$$\begin{vmatrix} 1 & -7 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{vmatrix}$$

As you can see the matrix is already in reduced row echelon form. Let $r = x_2$ and $s = x_5$ be arbitrary real numbers. Then the solutions are:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3+7r-s \\ r \\ 2+2s \\ 1-s \\ s \end{pmatrix}$$

Problem 7.

$$\left| \begin{matrix} x_1 + 2x_2 + 2x_4 + 3x_5 = 0 \\ x_3 + 3x_4 + 2x_5 = 0 \\ x_3 + 4x_4 - x5 = 0 \\ x_5 = 0 \end{matrix} \right|$$

Solution.

$$\left| \begin{matrix} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix} \right|$$

$$\left| \begin{matrix} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix} \right|$$

$$\left| \begin{matrix} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 & 11 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix} \right|$$

$$\left| \begin{matrix} 1 & 2 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 11 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix} \right|$$

$$\left| \begin{matrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix} \right|$$

Let $r = x_2$ be any arbitrary real number. Then we have the solutions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2r \\ r \\ 0 \\ 0 \\ 0 \end{pmatrix}$$