

Chapter 3 Section 2

Andrew Taylor

April 30 2022

Problem 1. Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0 \right\}$ a subspace of \mathbb{R}^2 ?

Solution. W contains the zero vector and is closed under addition. But W is not closed under scalar multiplication. Therefore W is not a subspace of \mathbb{R}^2 .

Problem 2. Show that the only subspaces of \mathbb{R}^2 are \mathbb{R}^2 itself, the set $\{\vec{0}\}$, and any of the lines through the origin.

Solution. Let W be a subspace of \mathbb{R}^2 that is neither a line through the origin nor the set $\{\vec{0}\}$. Then we can choose two nonzero nonparallel vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ from our subspace W . Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a vector in \mathbb{R}^2 . We will show that we can write \vec{u} as a linear combination of \vec{v} and \vec{w} .

If \vec{u} can be written as a linear combination of \vec{v} and \vec{w} , then there are solutions to the equation

$$x_1 \vec{v} + x_2 \vec{w} = \vec{u}$$

where x_1 and x_2 are real numbers. We can write this equation in matrix form

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This equation has solutions when $A = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ is invertible. We know that A is invertible when $\det A$ is nonzero.

The components v_1, v_2, w_1, w_2 can either be zero or nonzero. There is a small number of possible cases, since both vectors are not the zero vector, and since the two vectors are not parallel.

Case 1: $v_1 = 0, v_2 \neq 0, w_1 \neq 0, w_2 = 0$

Case 2: $v_1 \neq 0, v_2 = 0, w_1 = 0, w_2 \neq 0$

In both of these cases, the determinant of A is nonzero, and the matrix A is invertible.

Case 3: *At least one of the vectors (\vec{v} and \vec{w}) has two nonzero components.*

Let \vec{v} be the vector with two nonzero components.

There exist real numbers c_1 and c_2 such that $c_1 v_1 = w_1$ and $c_2 v_2 = w_2$. We know that $c_1 \neq c_2$ since the two vectors are not scalar multiples of each other. We can substitute these expressions when we calculate the determinant of A .

$$\begin{aligned}\det A &= v_1 w_2 - v_2 w_1 \\ &= v_1 (c_2 v_2) - v_2 (c_1 v_1) \\ &= c_2 v_1 v_2 - c_1 v_1 v_2 \\ &= v_1 v_2 (c_2 - c_1)\end{aligned}$$

Since $v_1 \neq 0$, $v_2 \neq 0$ and $c_2 \neq c_1$, the determinant of A is nonzero. Thus the matrix A is invertible, and the equation

$$x_1 \vec{v} + x_2 \vec{w} = \vec{u}$$

has solutions for x_1 and x_2 .

Since W is closed under linear combinations, the vector \vec{u} is in the subspace W . This means that W contains every real number, so $W = \mathbb{R}^2$.

We can also express this using a linear transformation. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation

$$\begin{aligned}T(\vec{x}) &= \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\end{aligned}$$

We have shown that the matrix $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ is invertible.

This means that there is a unique solution \vec{x} for every vector \vec{u} in \mathbb{R}^2 .

This is equivalent to saying any vector \vec{u} in \mathbb{R}^2 can be written as a linear combination of \vec{v} and \vec{w} .

Problem 3. Consider the plane V in \mathbb{R}^3 given by the equation

$$x_1 + 2x_2 + 3x_3 = 0$$

- Find a matrix A such that $V = \ker A$
- Find a matrix B such that $V = \operatorname{im} B$

Solution. Let $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. We can write

$$\begin{aligned} V &= \ker A \\ &= \ker \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \end{aligned}$$

We can find two nonparallel vectors \vec{v} and \vec{w} in V .

$$\text{Let } \vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

We can construct our matrix B from the column vectors \vec{v} and \vec{w} .

$$B = [\vec{v} \quad \vec{w}] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 1 \end{bmatrix}$$

Now we can say

$$\begin{aligned} V &= \operatorname{im} B \\ &= \operatorname{im} \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

Problem 4. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Find vectors in \mathbb{R}^3 that span the image of A . What is the smallest number of vectors needed to span the image of A ?

Solution. The vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

span the image of A .

The vector v_2 is redundant because $v_2 = 2v_1$. The vector v_4 is redundant because $v_1 + v_3 = v_4$. Thus only two vectors are needed to span the image of A , the vectors v_1 and v_3 .

$$\text{im } A = \text{span}(v_1, v_3)$$

We can verify this algebraically. We know that the image of A is spanned by the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

Let \vec{u} be a vector in \mathbb{R}^3 . Then

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$$

for some real coefficients c_1, c_2, c_3, c_4 . Substituting we get

$$\begin{aligned}\vec{u} &= c_1\vec{v}_1 + c_2(2\vec{v}_1) + c_3\vec{v}_3 + c_4(v_1 + v_3) \\ &= (c_1 + c_4 + 2c_2)\vec{v}_1 + (c_3 + c_4)\vec{v}_3\end{aligned}$$

Thus \vec{u} is a linear combination of vectors \vec{v}_1 and \vec{v}_3 . The vectors \vec{v}_1 and \vec{v}_3 form a basis for the image of A .

Problem 5. Are the following vectors in \mathbb{R}^7 linearly independent?

$$\vec{v}_1 = \begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \\ 1 \\ 9 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 6 \\ 0 \\ 7 \\ 1 \\ 4 \\ 8 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 6 \\ 2 \\ 3 \\ 1 \\ 7 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

Solution. \vec{v}_2 cannot be a linear combination of \vec{v}_1 , since \vec{v}_1 has a 0 in the fourth component and \vec{v}_2 has a 1 in the fourth component. \vec{v}_3 cannot be a linear combination of \vec{v}_1 and \vec{v}_2 , since \vec{v}_1 and \vec{v}_2 have zeros in the last component, and \vec{v}_3 has a seven in the last component. \vec{v}_4 cannot be a linear combination of \vec{v}_1, \vec{v}_2 , and \vec{v}_3 , since \vec{v}_4 has a 5 in the second component and the other three vectors all have zero in the second component.

Thus the four vectors are linearly independent.

Problem 6. Are the following vectors in \mathbb{R}^7 linearly independent?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Solution. The vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are not linearly independent since $2\vec{v}_2 - \vec{v}_1 = \vec{v}_3$.

Problem 7. Suppose the column vectors of an $n \times m$ matrix A are linearly independent. Find the kernel of matrix A .

Solution. Let

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

We can find the kernel of A by solving the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = 0$$

Since the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are linearly independent, there is only the trivial relation, with $x_1 = x_2 = \cdots = x_m = 0$, thus the kernel of A is $\{\vec{0}\}$.

Problem 8. If v_1, \dots, v_m is a basis of a subspace V of \mathbb{R}^n , and if \vec{v} is a vector in V , how many solutions c_1, \dots, c_m does the equation

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$$

have?

Solution. Let c_1, \dots, c_m and d_1, \dots, d_m be solutions to the equation. Then

$$\begin{aligned} \vec{v} &= c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \\ \vec{v} &= d_1\vec{v}_1 + \cdots + d_m\vec{v}_m \end{aligned}$$

Subtracting we get

$$0 = (c_1 - d_1)\vec{v}_1 + \cdots + (c_m - d_m)\vec{v}_m$$

Since the vectors v_1, \dots, v_m are linearly independent, we can conclude that $c_1 = d_1$. Thus there is only one solution to the equation.

In the following three problems, which of the sets W are subspaces of \mathbb{R}^3 ?

Problem 9.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$$

Solution.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$$

is not a subspace because it does not contain the zero vector.

Problem 10.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$$

Solution.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \leq y \leq z \right\}$$

is not a subspace since it is not closed under scalar multiplication.

Problem 11.

$$W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ are arbitrary constants} \right\}$$

Solution. *The first two column vectors are linearly independent, and the third is redundant. Thus W is the span of the first two column vectors, a plane passing through the origin. A plane that passes through the origin contains the zero vector, is closed under addition, and is closed under scalar multiplication. Thus W is a subspace of \mathbb{R}^3 .*