Chapter 3 Section 2

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Problem 1. *Is* $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2\text{: } x \geq 0 \text{ and } y \geq 0 \right\}$ *a subspace of* $\mathbb{R}^2$*?*

Solution. *W contains the zero vector and is closed under addition. But W is not closed under scalar multiplication. Therefore W is not a subspace of $\mathbb{R}^2$.*

Problem 2. *Show that the only subspaces of $\mathbb{R}^2$ are $\mathbb{R}^2$ itself, the set $\{\vec{0}\}$, and any of the lines through the origin.*

Solution. *Let W be a subspace of $\mathbb{R}^2$ that is neither a line through the origin nor the set $\{\vec{0}\}$. Then we can choose two nonzero nonparallel vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ from our subspace W. Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. We will show that we can write $\vec{u}$ as a linear combination of $\vec{v}$ and $\vec{w}$.*

If $\vec{u}$ can be written as a linear combination of $\vec{v}$ and $\vec{w}$, then there are solutions to the equation

$$x_1\vec{v} + x_2\vec{w} = \vec{u}$$

where x_1 and x_2 are real numbers. We can write this equation in matrix form

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This equation has solutions when $A = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ is invertible. We know that A is invertible when $\det A$ is nonzero.

The components v_1, v_2, w_1, w_2 can either be zero or nonzero. There is a small number of possible cases, since both vectors are not the zero vector, and since the two vectors are not parallel.

Case 1: $v_1 = 0, v_2 \neq 0, w_1 \neq 0, w_2 = 0$
Case 2: $v_1 \neq 0, v_2 = 0, w_1 = 0, w_2 \neq 0$

In both of these cases, the determinant of A is nonzero, and the matrix A is invertible.

Case 3: *At least one of the vectors ($\vec{v}$ and $\vec{w}$) has two nonzero components.*

Let $\vec{v}$ be the vector with two nonzero components.

There exist real numbers c_1 and c_2 such that $c_1 v_1 = w_1$ and $c_2 v_2 = w_2$. We know that $c_1 \neq c_2$ since the two vectors are not scalar multiples of each other. We can substitute these expressions when we calculate the determinant of A.

$$\begin{aligned} \det A &= v_1 w_2 - v_2 w_1 \\ &= v_1(c_2 v_2) - v_2(c_1 v_1) \\ &= c_2 v_1 v_2 - c_1 v_1 v_2 \\ &= v_1 v_2 (c_2 - c_1) \end{aligned}$$

Since $v_1 \neq 0$, $v_2 \neq 0$ and $c_2 \neq c_1$, the determinant of A is nonzero. Thus the matrix A is invertible, and the equation

$$x_1 \vec{v} + x_2 \vec{w} = \vec{u}$$

has solutions for x_1 and x_2.

Since W is closed under linear combinations, the vector $\vec{u}$ is in the subspace W. This means that W contains every real number, so $W = \mathbb{R}^2$.

We can also express this using a linear transformation. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

We have shown that the matrix $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ is invertible.

This means that there is a unique solution $\vec{x}$ for every vector $\vec{u}$ in $\mathbb{R}^2$.

This is equivalent to saying any vector $\vec{u}$ in $\mathbb{R}^2$ can be written as a linear combination of $\vec{v}$ and $\vec{w}$.

Problem 3. *Consider the plane V in $\mathbb{R}^3$ given by the equation*

$$x_1 + 2x_2 + 3x_3 = 0$$

- *Find a matrix A such that $V = \ker A$*
- *Find a matrix B such that $V = \operatorname{im} B$*

Solution. *Let $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. We can write*

$$\begin{aligned} V &= \ker A \\ &= \ker \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \end{aligned}$$

We can find two nonparallel vectors $\vec{v}$ and $\vec{w}$ in V.

Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ *and* $\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

We can construct our matrix B from the column vectors $\vec{v}$ and $\vec{w}$.

$$B = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 1 \end{bmatrix}$$

Now we can say

$$\begin{aligned} V &= \operatorname{im} B \\ &= \operatorname{im} \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

Problem 4. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Find vectors in $\mathbb{R}^3$ that span the image of A. What is the smallest number of vectors needed to span the image of A?

Solution. *The vectors*

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \; \vec{v_2} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \; \vec{v_3} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \; \vec{v_4} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

span the image of A.

The vector v_2 is redundant because $v_2 = 2v_1$. The vector v_4 is redundant because $v_1 + v_3 = v_4$. Thus only two vectors are needed to span the image of A, the vectors v_1 and v_3.

$$\operatorname{im} A = \operatorname{span}(v_1, v_3)$$

We can verify this algebraically. We know that the image of A is spanned by the vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$, $\vec{v_4}$.

Let $\vec{u}$ be a vector in $\mathbb{R}^3$. Then

$$\vec{u} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4}$$

for some real coefficients c_1, c_2, c_3, c_4. Substituting we get

$$\begin{aligned} \vec{u} &= c_1\vec{v_1} + c_2(2\vec{v_1}) + c_3\vec{v_3} + c_4(v_1 + v_3) \\ &= (c_1 + c_4 + 2c_1)\vec{v_1} + (c_3 + c_4)\vec{v_3} \end{aligned}$$

Thus $\vec{u}$ is a linear combination of vectors $\vec{v_1}$ and $\vec{v_3}$. The vectors $\vec{v_1}$ and $\vec{v_3}$ form a basis for the image of A.

Problem 5. *Are the following vectors in $\mathbb{R}^7$ linearly independent?*

$$\vec{v_1} = \begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \\ 1 \\ 9 \\ 0 \end{bmatrix} \vec{v_2} = \begin{bmatrix} 6 \\ 0 \\ 7 \\ 1 \\ 4 \\ 8 \\ 0 \end{bmatrix} \vec{v_3} = \begin{bmatrix} 5 \\ 0 \\ 6 \\ 2 \\ 3 \\ 1 \\ 7 \end{bmatrix} \vec{v_4} = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

Solution. *$\vec{v_2}$ cannot be a linear combination of $\vec{v_1}$, since $\vec{v_1}$ has a 0 in the fourth component and $\vec{v_2}$ has a 1 in the fourth component. $\vec{v_3}$ cannot be a linear combination of $\vec{v_1}$ and $\vec{v_2}$, since $\vec{v_1}$ and $\vec{v_2}$ have zeros in the last component, and $\vec{v_3}$ has a seven in the last component. $\vec{v_4}$ cannot be a linear combination of $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$, since $\vec{v_4}$ has a 5 in the second component and the other three vectors all have zero in the second component.*

Thus the four vectors are linearly independent.

Problem 6. *Are the following vectors in $\mathbb{R}^7$ linearly independent?*

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \vec{v_2} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \vec{v_3} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Solution. *The vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are not linearly independent since $2\vec{v_2} - \vec{v_1} = \vec{v_3}$.*

Problem 7. *Suppose the column vectors of an $n \times m$ matrix A are linearly independent. Find the kernel of matrix A.*

Solution. *Let*

$$A = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

We can find the kernel of A by solving the equation

$$x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_m\vec{v_m} = 0$$

Since the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ are linearly independent, there is only the trivial relation, with $x_1 = x_2 = \cdots = x_m = 0$, thus the kernel of A is $\{\vec{0}\}$.

Problem 8. *If $v_1, \ldots, v_m$ is a basis of a subspace V of $\mathbb{R}^n$, and if $\vec{v}$ is a vector in V, how many solutions $c_1, \ldots, c_m$ does the equation*

$$\vec{v} = c_1\vec{v_1} + \cdots + c_m\vec{v_m}$$

have?

Solution. *Let $c_1, \ldots, c_m$ and $d_1, \ldots, d_m$ be solutions to the equation. Then*

$$\vec{v} = c_1\vec{v_1} + \cdots + c_m\vec{v_m}$$
$$\vec{v} = d_1\vec{v_1} + \cdots + d_m\vec{v_m}$$

Subtracting we get

$$0 = (c_1 - d_1)\vec{v_1} + \cdots + (c_m - d_m)\vec{v_m}$$

Since the vectors $v_1, \ldots, v_m$ are linearly independent, we can conclude that $c_1 = d_1$. Thus there is only one solution to the equation.