

## Chapter 3 Section 1

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**Problem 1.** Find the kernel of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$$

from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .

**Solution.** Let's solve the linear system  $T(\vec{x}) = 0$  to get the kernel of  $T$ .

$$\begin{array}{l} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array} \right) \\ \left( \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \\ \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \end{array}$$

This tells us that  $x_1 = x_3$  and  $x_2 = -2x_3$ .

Let  $t = x_3$  be an arbitrary real number. Then the solutions to the linear system are

$$\ker(T) = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The kernel of  $T$  is the line spanned by the vector  $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .

**Problem 2.** Find the kernel of the linear transformation

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from  $\mathbb{R}^5$  to  $\mathbb{R}^4$ .

**Solution.** Let's solve the linear system  $T(\vec{x}) = Ax = 0$ .

We can solve this linear system by creating the augmented matrix  $[A \mid \vec{0}]$  and calculating  $\text{rref } [A \mid \vec{0}]$ .

$$\begin{pmatrix} 1 & 2 & 2 & -5 & 6 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 4 & 8 & 5 & -8 & 9 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & -4 & 5 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 4 & 8 & 5 & -8 & 9 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Adding row2 to row1}$$

$$\begin{pmatrix} 0 & 0 & 1 & -4 & 5 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Adding row2 to row1}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 3 & 6 & 1 & 5 & -7 & | & 0 \end{pmatrix} \text{ Subtracting row3 from row1}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & -2 & 8 & -10 & | & 0 \end{pmatrix} \text{ Adding row2 to row4}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & -1 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Adding row2 to row4}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & | & 0 \\ -1 & -2 & 0 & -3 & 4 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Adding row3 to row2}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & -4 & | & 0 \\ 0 & 0 & 1 & -4 & 5 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ Swapping rows and multiplying row1 by -1}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_4 - 4x_5 &= 0 \\ x_3 - 4x_4 + 5x_5 &= 0 \end{aligned}$$

$$\begin{aligned}x_1 &= -2x_2 - 3x_4 + 4x_5 \\x_3 &= 4x_4 - 5x_5\end{aligned}$$

Let  $r = x_2$ , let  $s = x_4$ , let  $t = x_5$ .

Then the solutions to the linear system (the kernel) are of the form:

$$\begin{aligned}\ker(T) &= \begin{bmatrix} -2r - 3s + 4t \\ r \\ 4s - 5t \\ s \\ t \end{bmatrix} \\&= \begin{bmatrix} -2r \\ r \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3s \\ 0 \\ 4s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 4t \\ 0 \\ -5t \\ 0 \\ t \end{bmatrix} \\&= r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \\&= \text{span} \left( \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right)\end{aligned}$$

**Problem 3.** For an invertible  $n \times n$  matrix find  $\ker A$ .

**Solution.** Let  $T(\vec{x}) = A\vec{x}$  be a linear transformation with an invertible  $n \times n$  matrix  $A$ . We know that  $T$  is invertible because  $A$  is invertible. Thus there can only be one unique solution to the equation  $A\vec{x} = \vec{0}$ . Since  $\vec{x} = \vec{0}$  is the unique solution, we know that  $\ker A = \{\vec{0}\}$ .

**Problem 4.** For which  $n \times m$  matrices is  $\ker A = \{\vec{0}\}$ . Give your answer in terms of the rank of  $A$ .

**Solution.** Let  $A$  be a  $n \times m$  matrix. When  $\text{rank } A = m$ , we get the unique solution  $\vec{x} = \vec{0}$ . Thus  $\ker A = \{\vec{0}\}$  when  $\text{rank } A = m$ .

In the following problems, find vectors that span the kernel of  $A$  and the image of  $A$ .

**Problem 5.**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

**Solution.** Let  $T(\vec{x}) = A\vec{x}$ . Let's solve the equation  $A\vec{x} = \vec{0}$ .

$$\begin{array}{ll} \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array} \right] & \text{Augmented matrix } \left[ A \mid \vec{0} \right] \\ \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \end{array} \right] & \text{Subtract 3 times row1 from row2} \\ \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -2 & 0 \end{array} \right] & \text{Add row2 to row1} \\ \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] & \text{Divide row2 by -2} \end{array}$$

The solution set is  $\vec{x} = \vec{0}$ , thus  $\ker A = \{0\}$ .

The kernel of  $A$  is spanned by the zero vector. The image of  $A$  is the span of the column vectors  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ .

In other words,  $T(\vec{x}) = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ .

**Problem 6.**

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

**Solution.** Let's solve the equation

$$x_1 + 2x_2 + 3x_3 = 0$$

From inspection, we can find two nonparallel vectors in the solution set, the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

There is no constant  $c$  such that  $\vec{v}_2 = c\vec{v}_1$ , thus the two vectors are linearly independent and form a basis for the kernel.

Thus the kernel of  $A$  is the plane through the origin spanned by the vectors  $\vec{v}_1$  and  $\vec{v}_2$ .

$$\ker A = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right)$$

The image of  $A$  is  $\mathbb{R}$ , since for any  $y \in \mathbb{R}$ , we have the solution  $\vec{x} = \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix}$ .

**Problem 7.**

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**Solution.** The kernel of  $A$  is  $\mathbb{R}^2$  and the image of  $A$  is  $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . We can write the kernel of  $A$  as the span of the unit vectors  $e_1$  and  $e_2$ .

$$\ker A = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$