Chapter 2 Section 4

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Theorem 1. *An $n \times n$ matrix A is invertible if and only if*

$$rref(A) = I_n$$

or, equivalently, if

$$rank(A) = n$$

Theorem 2. *To find the inverse of an $n \times n$ matrix A, form the $n \times (2n)$ matrix $\begin{bmatrix} A \mid I_n \end{bmatrix}$ and compute $rref[A \mid I_n]$.*

- *If $rref[A \mid I_n]$ is of the form $\begin{bmatrix} I_n \mid B \end{bmatrix}$ then A is invertible and $A^{-1} = B$.*
- *If $rref[A \mid I_n]$ is of another form (i.e., its left half fails to be I_n) then A is not invertible.*

Theorem 3. *For an invertible $n \times n$ matrix A,*

$$A^{-1}A = I_n \quad and \quad AA^{-1} = I_n$$

Theorem 4. *If A and B are invertible $n \times n$ matrices, then BA is invertible as well, and*

$$(BA)^{-1} = A^{-1}B^{-1}$$

Theorem 5. *Let A and B be two $n \times n$ matrices such that $BA = I_n$. Then*

- *A and B are both invertible*
- *$A^{-1} = B$ and $B^{-1} = A$*
- *$AB = I_n$*

Problem 1. *Is the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

invertible? If so, find the inverse of A.

Solution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We see that

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus A is invertible.

Note that $rref(A)$ is an acronym that refers to the reduced row echelon form of matrix A. The computation $rref(A)$ tells us whether A is invertible.

To invert the matrix, let's calculate $rref\left[A \mid I_n\right]$.

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 2 & 3 & 2 & | & 0 & 1 & 0 \\ 3 & 8 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 5 & -1 & | & -3 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & 7 & -5 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 0 & | & 8 & -5 & 1 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & | & 10 & -6 & 1 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{bmatrix}$$

Thus

$$rref \begin{bmatrix} A \mid I_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 10 & -6 & 1 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{bmatrix}$$

and

$$A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$$

Problem 2. *Suppose A, B, and C are three $n \times n$ matrices such that $ABC = I_n$. Show that B is invertible, and express B^{-1} in terms of A and C.*

Solution. *By the associative property of matrices*

$$\begin{aligned} ABC &= I_n \\ (AB)C &= I_n \\ A(BC) &= I_n \end{aligned}$$

Thus matrices A and C are invertible.

$$\begin{aligned} ABC &= I_n \\ A^{-1}ABC &= A^{-1}I_n \\ BC &= A^{-1} \\ BCA &= A^{-1}A \\ B(CA) &= I_n \end{aligned}$$

Thus matrix B is invertible and $B^{-1} = CA$.

Problem 3. *For an arbitrary 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ compute the product $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. When is A invertible? If so, what is A^{-1}?*

Solution.

$$\begin{aligned} &\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \end{aligned}$$

When $ad - bc$ is nonzero, we can form the product

$$\begin{aligned} &\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Thus A is invertible when the determinant $ad - bc \neq 0$, and

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Problem 4. *Is the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ invertible? If so, find the inverse. Interpret $\det A$ geometrically.*

Solution.

$$\det A = 1 * 1 - 2 * 3 = -5$$

Since $\det A = -5$ is nonzero, the matrix is invertible.

$$A^{-1} = \frac{1}{-5}\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

The quantity $|\det A|$ *is the area of the shaded parallelogram constructed from the vectors* $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. *The determinant is negative since the angle from* $\vec{v}$ *to* $\vec{w}$ *is negative.*

Problem 5. *For which values of the constant* k *is the matrix* $A = \begin{bmatrix} 1-k & 2 \\ 4 & 3-k \end{bmatrix}$ *invertible?*

Solution. *The matrix* A *is invertible when* $\det A$ *is nonzero.*

$$\begin{aligned} \det A &= (1-k)(3-k) - 2 * 4 \\ &= 3 - 4k + k^2 - 8 \\ &= k^2 - 4k - 5 \\ &= (k-5)(k+1) \end{aligned}$$

The matrix A *is invertible when* $k \neq 5$ *and when* $k \neq -1$.

In other words, A *is invertible for all values of* k *except* $k = 5$ *and* $k = -1$.

Problem 6. *Consider a matrix* A *that represents the reflection about a line* L *in the plane. Use the determinant to verify that* A *is invertible. Find* A^{-1}. *Explain your answer conceptually, and interpret the determinant geometrically.*

Solution. *Since* A *is a reflection matrix, we know that* A *is of the form*

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

We can calculate the determinant of A.

$$\begin{aligned} \det A &= -a^2 - b^2 \\ &= -(a^2 + b^2) \end{aligned}$$

Since the determinant is only zero when $a = b = 0$, we can say that A is invertible for all values except $a = b = 0$.

$$
\begin{aligned}
A^{-1} &= -\frac{1}{a^2+b^2}\begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} \\
&= \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ b & -a \end{bmatrix}
\end{aligned}
$$

Strictly speaking, a reflection matrix is of the form

$$
\begin{bmatrix} a & b \\ b & -a \end{bmatrix}
$$

where $a^2 + b^2 = 1$. So we can substitute $a^2 + b^2 = 1$ into our inverse. We can also assume that the determinant is nonzero since $a^2 + b^2 = 1$. Thus every reflection matrix is invertible.

$$
\begin{aligned}
A^{-1} &= \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \\
&= \frac{1}{1}\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \\
&= A
\end{aligned}
$$

We find that every reflection matrix A is invertible, and that $A^{-1} = A$.

The determinant of A is actually the area of a unit square. Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} b \\ -a \end{bmatrix}$. The vectors $\vec{v}$ and $\vec{w}$ form a parallelogram, and this parallelogram is a unit square.

We can write the determinant as

$$
\begin{aligned}
\det A &= \|\vec{v}\| \sin\theta \|\vec{w}\| \\
&= \|\sqrt{a^2+b^2}\| \sin\left(-\frac{\pi}{2}\right) \|\sqrt{b^2+(-a)^2}\| \\
&= -1
\end{aligned}
$$

where θ is the angle between vectors $\vec{v}$ and $\vec{w}$.

In summary, every reflection matrix A is invertible, since $\det A = -1$ for all reflection matrices. The inverse of a reflection matrix A is $A^{-1} = A$. The determinant of a reflection matrix is the area of a unit square formed by the two column vectors of the reflection matrix. This area is negative because the angle between the vectors $\left(-\frac{\pi}{2}\right)$ is negative.

Problem 7. *Let A be a block matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is an $n \times n$ matrix, A_{22} is an $m \times m$ matrix, and A_{12} is an $n \times m$ matrix.

- *For which choices of A_{11}, A_{12}, and A_{22} is A invertible?*
- *If A is invertible, what is A^{-1} (in terms of A_{11}, A_{12}, A_{22})?*

Solution. *We are looking for a matrix B such that*

$$BA = I_{n+m} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}$$

Let us partition B in the same way as A.

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11} is $n \times n$, B_{12} is $m \times m$, and so on. We know that

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}$$

Thus

$$\begin{aligned} B_{11}A_{11} &= I_n \\ B_{11}A_{12} + B_{12}A_{22} &= 0 \\ B_{21}A_{11} + B_{22}0 &= 0 \\ B_{21}A_{12} + B_{22}A_{22} &= I_m \end{aligned}$$

This gives us

$$
\begin{aligned}
B_{11} &= A_{11}^{-1} \\
B_{12} &= -A_{11}^{-1} A_{12} A_{22}^{-1} \\
B_{21} &= 0 \\
B_{22} &= A_{22}^{-1}
\end{aligned}
$$

We find that A is invertible when A_{11} and A_{22} are invertible.

When A is invertible, the inverse of A is given by

$$
B = A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ & \\ 0 & A_{22}^{-1} \end{bmatrix}
$$

Problem 8. *Verify this result for the following example:*

$$
\left(\begin{array}{cc|ccc} 1 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 5 & 6 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)^{-1} = \left(\begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 \\ -1 & 1 & -3 & -3 & -3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)
$$

Solution.

$$\left(\begin{array}{cc|ccc} 1 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 5 & 6 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 \\ -1 & 1 & -3 & -3 & -3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{c|c} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -3 & -3 & -3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}\right)$$

$$= \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Since the product of the two matrices is I_5, the identity matrix, we know that

$$\left(\begin{array}{cc|ccc} 1 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 5 & 6 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)^{-1} = \left(\begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 \\ -1 & 1 & -3 & -3 & -3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

In the next fifteen problems, decide whether the matrices are invertible. If they are, find the inverse.

Problem 9.

$$\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$$

Solution. *The determinant is* $2*8-5*3=16-15=1$*. Since the determinant is nonzero, the matrix is invertible.*

The inverse is given by swapping the entries $(1,1)$ *and* $(2,2)$*, negating the entries* $(1,2)$ *and* $(2,1)$*, and dividing by the determinant.*

$$\begin{aligned}
\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}^{-1} &= \frac{1}{1}\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}
\end{aligned}$$

Theorem 6. *If a matrix* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is invertible, then the inverse of* A *is*

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We will use this theorem in the following problems, as we did in the previous problem.

Problem 10.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution. *The matrix is not invertible, because the determinant is zero.*

$$\det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 1*1-1*1 = 0$$

Problem 11.

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

Solution.

$$\det \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = 0*1-1*2 = -2$$

Thus the matrix is invertible.

$$\begin{aligned}
\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^{-1} &= \frac{1}{-2}\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -0.5 & 1 \\ 0.5 & 0 \end{bmatrix}
\end{aligned}$$