

## Chapter 2 Section 3

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**Problem 1.** *Calculate the matrix product*

$$\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

**Solution.**

$$\begin{aligned} \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} &= \begin{bmatrix} 6*1 + 7*3 & 6*2 + 7*5 \\ 8*1 + 9*3 & 8*2 + 9*5 \end{bmatrix} \\ &= \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix} \end{aligned}$$

**Problem 2.** *Compute the products  $BA$  and  $AB$  for*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

*Interpret your answers geometrically, as composites of linear transformation.*

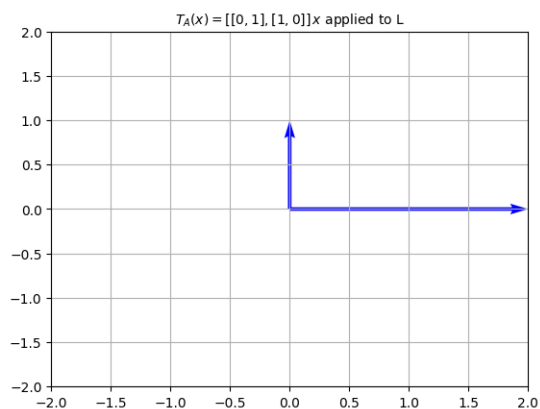
**Solution.** Let  $T_A(\vec{x}) = A\vec{x}$  and  $T_B(\vec{y}) = B\vec{y}$ . We can write

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [T_A(e_1) \quad T_A(e_2)] \\ B &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = [T_B(e_1) \quad T_B(e_2)] \end{aligned}$$

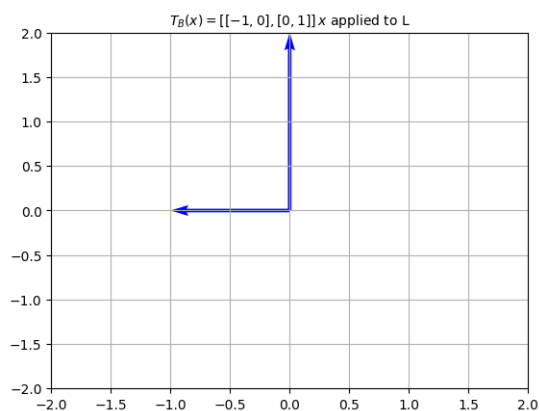
*Thus*

$$\begin{aligned} T_A(\vec{e}_1) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, T_A(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T_B(\vec{e}_1) &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, T_B(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

We see that  $T_A$  is a reflection about the line  $y = x$ . In other words,  $T_A$  is a reflection about the line spanned by the vector  $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .



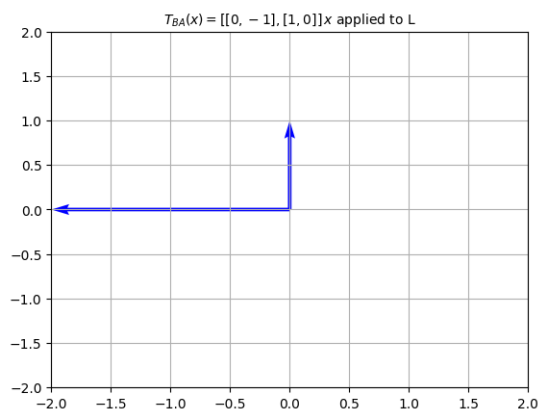
We see that  $T_B$  is a reflection about the line  $x = 0$ . In other words,  $T_B$  is a reflection about the line spanned by the vector  $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .



Now let's compute the products  $BA$  and  $AB$ .

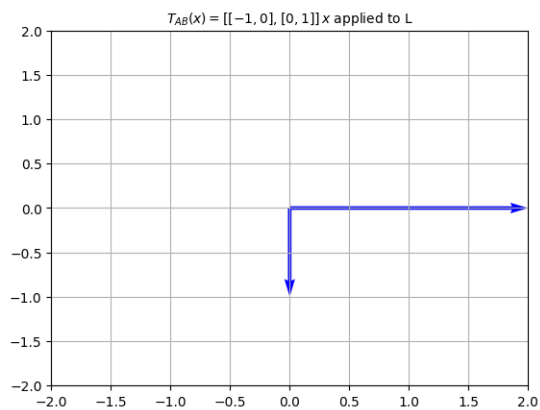
$$\begin{aligned} BA &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

The product  $BA$  is a rotation matrix that rotates a vector ninety degrees counterclockwise.



$$\begin{aligned}
 AB &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
 \end{aligned}$$

The product  $AB$  is a rotation matrix that rotates a vector ninety degrees clockwise.



**Problem 3.** Multiply the block matrices

$$\left( \begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \end{array} \right) \left( \begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array} \right)$$

Afterwards, compute the product without a partition and see if you get the same result.

**Solution.**

$$\begin{aligned}
& \left( \begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \end{array} \right) \left( \begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array} \right) \\
&= \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 7 & 8 \end{bmatrix} \mid \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 9 \end{bmatrix} \right) \\
&= \left( \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -7 & -8 \\ 7 & 8 \end{bmatrix} \mid \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -9 \\ 9 \end{bmatrix} \right) \\
&= \left( \begin{bmatrix} -3 & -3 \\ 8 & 10 \end{bmatrix} \mid \begin{bmatrix} -3 \\ 12 \end{bmatrix} \right) \\
&= \left( \begin{array}{cc|c} -3 & -3 & -3 \\ 8 & 10 & 12 \end{array} \right)
\end{aligned}$$

Now let's compute the product without a partition.

$$\begin{aligned}
& \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\
&= \begin{bmatrix} 4-7 & 5-8 & 6-9 \\ 1+7 & 2+8 & 3+9 \end{bmatrix} \\
&= \begin{bmatrix} -3 & -3 & -3 \\ 8 & 10 & 12 \end{bmatrix}
\end{aligned}$$

We get the same result.

In the following problems, multiply the matrices and describe the linear transformations geometrically.

**Problem 4.**

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

**Solution.** The first matrix is a horizontal shear with  $k = 1$ .

$$\begin{aligned}
& \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}
\end{aligned}$$

**Problem 5.**

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

**Solution.** Let  $A$  be a  $n$  by  $p$  matrix and let  $B$  be a  $q$  by  $m$  matrix. The product of a matrix  $AB$  is defined if and only if  $p = q$ .

In the matrices below,  $p = q$ , so the product of the matrices is defined.

$$\begin{aligned} & \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3-1 & 2-0 \\ 2 & 0 \\ 6+1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix} \end{aligned}$$

**Problem 6.**

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

**Solution.** The matrix product is not defined, since the number of columns in the first matrix (3) does not equal the number of rows in the second matrix (2)

**Problem 7.**

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix}$$

**Solution.**

$$\begin{aligned} & \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7-3 & 5-1 \\ -14+6 & -10+2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 4 \\ -8 & -8 \end{bmatrix} \end{aligned}$$

**Problem 8.**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

**Solution.**

$$\begin{aligned} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix} \end{aligned}$$

**Problem 9.**

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

**Solution.**

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**Problem 10.**

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

**Solution.**

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \\ = \begin{bmatrix} 1-2 & 2-1 & 3-3 \\ 3+2 & 2+1 & 1+3 \\ 1-3-4 & 2-2-2 & 3-1-6 \end{bmatrix} \\ = \begin{bmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{bmatrix}$$

**Problem 11.**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Solution.**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

*The matrix product represents a scaling by a factor of  $ad - bc$ .*

**Problem 12.**

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -6 & 8 \\ 3 & -4 \end{bmatrix}$$

**Solution.**

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -6 & 8 \\ 3 & -4 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**Problem 13.**

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \quad 2 \quad 3]$$

**Solution.** *Multiplying the column vector by the row vector, we get*

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \quad 2 \quad 3] \\ = 1 + 4 + 9 \\ = 14$$

**Problem 14.**

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [3 \quad 2 \quad 1]$$

**Solution.**

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [3 \quad 2 \quad 1] \\ = 3 + 4 + 3 \\ = 10$$