

Chapter 2 Section 4

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Theorem 1. *An $n \times n$ matrix A is invertible if and only if*

$$\text{rref}(A) = I_n$$

or, equivalently, if

$$\text{rank}(A) = n$$

Theorem 2. *To find the inverse of an $n \times n$ matrix A , form the $n \times (2n)$ matrix $[A \mid I_n]$ and compute $\text{rref}[A \mid I_n]$.*

- *If $\text{rref}[A \mid I_n]$ is of the form $[I_n \mid B]$ then A is invertible and $A^{-1} = B$.*
- *If $\text{rref}[A \mid I_n]$ is of another form (i.e., its left half fails to be I_n) then A is not invertible.*

Theorem 3. *For an invertible $n \times n$ matrix A ,*

$$A^{-1}A = I_n \quad \text{and} \quad AA^{-1} = I_n$$

Theorem 4. *If A and B are invertible $n \times n$ matrices, then BA is invertible as well, and*

$$(BA)^{-1} = A^{-1}B^{-1}$$

Theorem 5. *Let A and B be two $n \times n$ matrices such that $BA = I_n$. Then*

- *A and B are both invertible*
- *$A^{-1} = B$ and $B^{-1} = A$*
- *$AB = I_n$*

Problem 1. *Is the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

invertible? If so, find the inverse of A .

Solution.

$$\begin{aligned} & \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} \\ & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

We see that

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus A is invertible.

Note that $\text{rref}(A)$ is an acronym that refers to the reduced row echelon form of matrix A . The computation $\text{rref}(A)$ tells us whether A is invertible.

To invert the matrix, let's calculate $\text{rref}[A \mid I_n]$.

$$\begin{aligned}
& \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array} \right] \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array} \right] \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array} \right] \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right] \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 8 & -5 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right] \\
& \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right]
\end{aligned}$$

Thus

$$rref[A \mid I_n] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right]$$

and

$$A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$$

Problem 2. Suppose A , B , and C are three $n \times n$ matrices such that $ABC = I_n$. Show that B is invertible, and express B^{-1} in terms of A and C .

Solution. By the associative property of matrices

$$\begin{aligned}
ABC &= I_n \\
(AB)C &= I_n \\
A(BC) &= I_n
\end{aligned}$$

Thus matrices A and C are invertible.

$$\begin{aligned} ABC &= I_n \\ A^{-1}ABC &= A^{-1}I_n \\ BC &= A^{-1} \\ BCA &= A^{-1}A \\ B(CA) &= I_n \end{aligned}$$

Thus matrix B is invertible and $B^{-1} = CA$.

Problem 3. For an arbitrary 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ compute the product $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. When is A invertible? If so, what is A^{-1} ?

Solution.

$$\begin{aligned} &\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \end{aligned}$$

When $ad - bc$ is nonzero, we can form the product

$$\begin{aligned} &\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Thus A is invertible when the determinant $ad - bc \neq 0$, and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Problem 4. Is the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ invertible? If so, find the inverse. Interpret $\det A$ geometrically.

Solution.

$$\det A = 1 * 1 - 2 * 3 = -5$$

Since $\det A = -5$ is nonzero, the matrix is invertible.

$$A^{-1} = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

The quantity $|\det A|$ is the area of the shaded parallelogram constructed from the vectors $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. The determinant is negative since the angle from \vec{v} to \vec{w} is negative.

Problem 5. For which values of the constant k is the matrix $A = \begin{bmatrix} 1-k & 2 \\ 4 & 3-k \end{bmatrix}$ invertible?

Solution. The matrix A is invertible when $\det A$ is nonzero.

$$\begin{aligned} \det A &= (1-k)(3-k) - 2 \cdot 4 \\ &= 3 - 4k + k^2 - 8 \\ &= k^2 - 4k - 5 \\ &= (k-5)(k+1) \end{aligned}$$

The matrix A is invertible when $k \neq 5$ and when $k \neq -1$.

In other words, A is invertible for all values of k except $k = 5$ and $k = -1$.

Problem 6. Consider a matrix A that represents the reflection about a line L in the plane. Use the determinant to verify that A is invertible. Find A^{-1} . Explain your answer conceptually, and interpret the determinant geometrically.

Solution. Since A is a reflection matrix, we know that A is of the form

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

We can calculate the determinant of A .

$$\begin{aligned} \det A &= -a^2 - b^2 \\ &= -(a^2 + b^2) \end{aligned}$$

Since the determinant is only zero when $a = b = 0$, we can say that A is invertible for all values except $a = b = 0$.

$$\begin{aligned} A^{-1} &= -\frac{1}{a^2 + b^2} \begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} \\ &= \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \end{aligned}$$

Strictly speaking, a reflection matrix is of the form

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

where $a^2 + b^2 = 1$. So we can substitute $a^2 + b^2 = 1$ into our inverse. We can also assume that the determinant is nonzero since $a^2 + b^2 = 1$. Thus every reflection matrix is invertible.

$$\begin{aligned} A^{-1} &= \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \\ &= \frac{1}{1} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \\ &= A \end{aligned}$$

We find that every reflection matrix A is invertible, and that $A^{-1} = A$.

The determinant of A is actually the area of a unit square. Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} b \\ -a \end{bmatrix}$. The vectors \vec{v} and \vec{w} form a parallelogram, and this parallelogram is a unit square.

We can write the determinant as

$$\begin{aligned} \det A &= \|\vec{v}\| \sin \theta \|\vec{w}\| \\ &= \|\sqrt{a^2 + b^2}\| \sin \left(-\frac{\pi}{2} \right) \|\sqrt{b^2 + (-a)^2}\| \\ &= -1 \end{aligned}$$

where θ is the angle between vectors \vec{v} and \vec{w} .

In summary, every reflection matrix A is invertible, since $\det A = -1$ for all reflection matrices. The inverse of a reflection matrix A is $A^{-1} = A$. The determinant of a reflection matrix is the area of a unit square formed by the two column vectors of the reflection matrix. This area is negative because the angle between the vectors $\left(-\frac{\pi}{2}\right)$ is negative.

Problem 7. Let A be a block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is an $n \times n$ matrix, A_{22} is an $m \times m$ matrix, and A_{12} is an $n \times m$ matrix.

- For which choices of A_{11} , A_{12} , and A_{22} is A invertible?
- If A is invertible, what is A^{-1} (in terms of A_{11} , A_{12} , A_{22})?

Solution. We are looking for a matrix B such that

$$BA = I_{n+m} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}$$

Let us partition B in the same way as A .

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11} is $n \times n$, B_{12} is $m \times n$, and so on. We know that

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}$$

Thus

$$\begin{aligned} B_{11}A_{11} &= I_n \\ B_{11}A_{12} + B_{12}A_{22} &= 0 \\ B_{21}A_{11} + B_{22}0 &= 0 \\ B_{21}A_{12} + B_{22}A_{22} &= I_m \end{aligned}$$

This gives us

$$\begin{aligned}
B_{11} &= A_{11}^{-1} \\
B_{12} &= -A_{11}^{-1}A_{12}A_{22}^{-1} \\
B_{21} &= 0 \\
B_{22} &= A_{22}^{-1}
\end{aligned}$$

We find that A is invertible when A_{11} and A_{22} are invertible.

When A is invertible, the inverse of A is given by

$$B = A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

Problem 8. Verify this result for the following example:

$$\left(\begin{array}{cc|ccc} 1 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 5 & 6 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)^{-1} = \left(\begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 \\ -1 & 1 & -3 & -3 & -3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

Solution.

$$\begin{aligned}
& \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 3 & \\ 1 & 2 & 4 & 5 & 6 & \\ \hline 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \end{array} \right) \left(\begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 & \\ -1 & 1 & -3 & -3 & -3 & \\ \hline 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \end{array} \right) \\
&= \left(\begin{array}{ccc|ccc} \left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right] \left[\begin{array}{cc} 2 & -1 \\ -1 & 1 \end{array} \right] + \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] & & \left[\begin{array}{ccc} 1 & 1 & 2 \\ -3 & -3 & -3 \end{array} \right] + \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \\ \hline & \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] & & \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \end{array} \right) \\
&= \left(\begin{array}{ccc|ccc} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] & & \left[\begin{array}{ccc} -1 & -2 & -3 \\ -4 & -5 & -6 \end{array} \right] + \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \\ \hline & \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] & & \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \end{array} \right) \\
&= \left(\begin{array}{ccc|ccc} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] & & \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \\ \hline & \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] & & \end{array} \right) \\
&= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)
\end{aligned}$$

Since the product of the two matrices is I_5 , the identity matrix, we know that

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 3 & \\ 1 & 2 & 4 & 5 & 6 & \\ \hline 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \end{array} \right)^{-1} = \left(\begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 & \\ -1 & 1 & -3 & -3 & -3 & \\ \hline 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \end{array} \right)$$

In the next fifteen problems, decide whether the matrices are invertible. If they are, find the inverse.

Problem 9.

$$\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$$

Solution. The determinant is $2*8-5*3 = 16-15 = 1$. Since the determinant is nonzero, the matrix is invertible.

The inverse is given by swapping the entries $(1,1)$ and $(2,2)$, negating the entries $(1,2)$ and $(2,1)$, and dividing by the determinant.

$$\begin{aligned}\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}^{-1} &= \frac{1}{1} \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}\end{aligned}$$

Theorem 6. If a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, then the inverse of A is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We will use this theorem in the following problems, as we did in the previous problem.

Problem 10.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution. The matrix is not invertible, because the determinant is zero.

$$\det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 1*1 - 1*1 = 0$$

Problem 11.

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

Solution.

$$\det \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = 0*1 - 1*2 = -2$$

Thus the matrix is invertible.

$$\begin{aligned}\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^{-1} &= \frac{1}{-2} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -0.5 & 1 \\ 0.5 & 0 \end{bmatrix}\end{aligned}$$

Problem 12.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution.

$$\begin{aligned} & \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= 1 * \det \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + 3 \det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= 1 - 0 + 0 \\ &= 1 \end{aligned}$$

The determinant of the matrix is 1, thus the matrix is invertible.

Let's find the inverse using elementary row operations.

$$\begin{aligned} & \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \\ & \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \\ & \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \\ & \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \end{aligned}$$

The inverse of the matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 13.

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Solution.

$$\begin{aligned} & \det \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \\ &= 1 * \det \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - 2 \det \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} + 2 \det \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \\ &= (9 - 1) - 2(3 - 1) + 2(1 - 3) \\ &= 8 - 4 + -4 \\ &= 0 \end{aligned}$$

The determinant of the matrix is zero, so the matrix is not invertible.