Chapter 2 Section 3

Andrew Taylor

April 11 2022

Problem 1. *Calculate the matrix product*

$$\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Solution.

$$\begin{aligned} \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} &= \begin{bmatrix} 6*1+7*3 & 6*2+7*5 \\ 8*1+9*3 & 8*2+9*5 \end{bmatrix} \\ &= \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix} \end{aligned}$$

Problem 2. *Compute the products BA and AB for*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Interpret your answers geometrically, as composites of linear transformation.

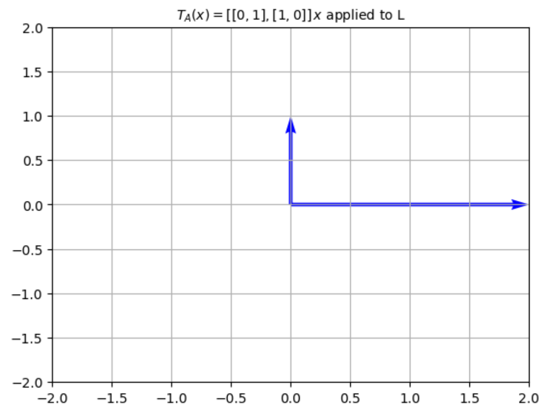
Solution. *Let $T_A(\vec{x}) = A\vec{x}$ and $T_B(\vec{y}) = B\vec{y}$. We can write*

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} T_A(e_1) & T_A(e_2) \end{bmatrix} \\ B &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} T_B(e_1) & T_B(e_2) \end{bmatrix} \end{aligned}$$

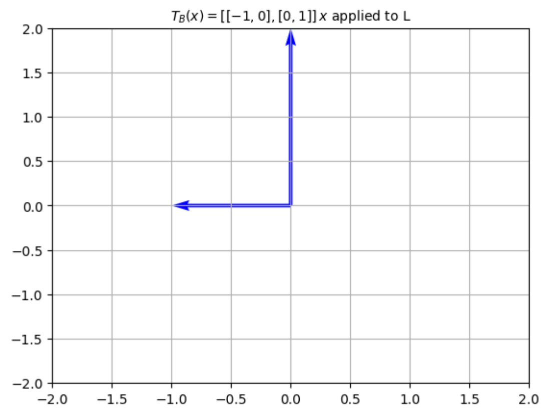
Thus

$$\begin{aligned} T_A(\vec{e_1}) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, T_A(\vec{e_2}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T_B(\vec{e_1}) &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, T_B(\vec{e_2}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

We see that T_A is a reflection about the line $y = x$. In other words, T_A is a reflection about the line spanned by the vector $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



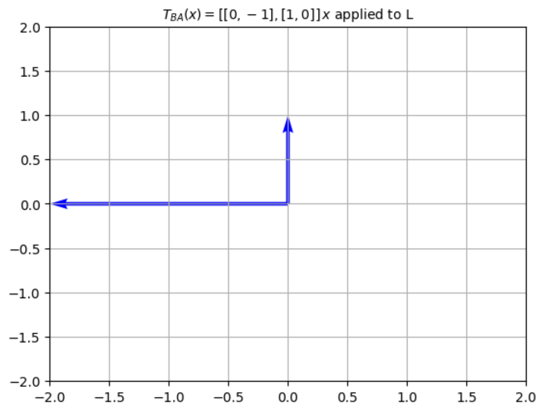
We see that T_B is a reflection about the line $x = 0$. In other words, T_B is a reflection about the line spanned by the vector $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



Now let's compute the products BA and AB.

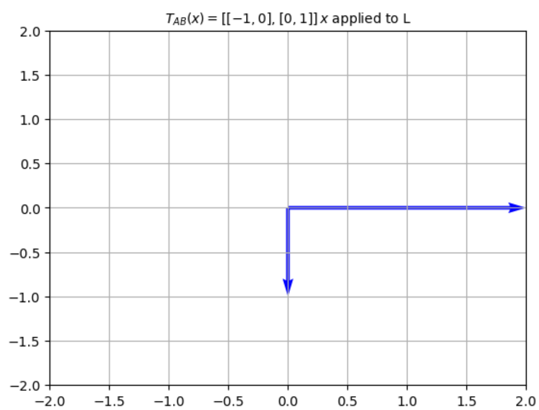
$$\begin{aligned} BA &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

The product BA is a rotation matrix that rotates a vector ninety degrees counterclockwise.



$$
\begin{aligned}
AB &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\end{aligned}
$$

The product AB is a rotation matrix that rotates a vector ninety degrees clockwise.



Problem 3. *Multiply the block matrices*

$$
\left(\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right)
$$

Afterwards, compute the product without a partition and see if you get the same result.

Solution.

$$\left(\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right)$$
$$= \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} 7 & 8 \end{bmatrix} \;\middle|\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} 9 \end{bmatrix}\right)$$
$$= \left(\begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -7 & -8 \\ 7 & 8 \end{bmatrix} \;\middle|\; \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -9 \\ 9 \end{bmatrix}\right)$$
$$= \left(\begin{bmatrix} -3 & -3 \\ 8 & 10 \end{bmatrix} \;\middle|\; \begin{bmatrix} -3 \\ 12 \end{bmatrix}\right)$$
$$= \left(\begin{array}{cc|c} -3 & -3 & -3 \\ 8 & 10 & 12 \end{array}\right)$$

Now let's compute the product without a partition.

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
$$= \begin{bmatrix} 4-7 & 5-8 & 6-9 \\ 1+7 & 2+8 & 3+9 \end{bmatrix}$$
$$= \begin{bmatrix} -3 & -3 & -3 \\ 8 & 10 & 12 \end{bmatrix}$$

We get the same result.

In the following problems, multiply the matrices and describe the linear transformations geometrically.

Problem 4.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution. *The first matrix is a horizontal shear with $k = 1$.*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$$

Problem 5.

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

Solution. *Let A be a n by p matrix and let B be a q by m matrix. The product of a matrix AB is defined if and only if $p = q$.*

In the matrices below, $p = q$, so the product of the matrices is defined.

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3-1 & 2-0 \\ 2 & 0 \\ 6+1 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}$$

Problem 6.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution. *The matrix product is not defined, since the number of columns in the first matrix (3) does not equal the number of rows in the second matrix (2)*

Problem 7.

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 7-3 & 5-1 \\ -14+6 & -10+2 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 4 \\ -8 & -8 \end{bmatrix}$$

Problem 8.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix}$$

Problem 9.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Solution.

$$\begin{aligned} &\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$