

Chapter 3 Section 2

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Problem 1. Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0 \right\}$ a subspace of \mathbb{R}^2 ?

Solution. W contains the zero vector and is closed under addition. But W is not closed under scalar multiplication. Therefore W is not a subspace of \mathbb{R}^2 .

Problem 2. Show that the only subspaces of \mathbb{R}^2 are \mathbb{R}^2 itself, the set $\{\vec{0}\}$, and any of the lines through the origin.

Solution. Let W be a subspace of \mathbb{R}^2 that is neither a line through the origin nor the set $\{\vec{0}\}$. Then we can choose two nonparallel vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ from our subspace W . Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a vector in \mathbb{R}^2 . We will show that we can write \vec{u} as a linear combination of \vec{v} and \vec{w} .

If \vec{u} can be written as a linear combination of \vec{v} and \vec{w} , then there are solutions to the equation

$$x_1 \vec{v} + x_2 \vec{w} = \vec{u}$$

where x_1 and x_2 are real numbers. We can write this equation in matrix form

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This equation has solutions when $A = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ is invertible. We know that A is invertible when $\det A$ is nonzero.

There exist real numbers c_1 and c_2 such that $c_1 v_1 = w_1$ and $c_2 v_2 = w_2$. We know that $c_1 \neq c_2$ since the two vectors are not parallel (or else one vector would be a scalar multiple of the other). We can substitute these expressions when we calculate the determinant of A .

$$\begin{aligned}
\det A &= v_1 w_2 - v_2 w_1 \\
&= v_1(c_2 v_2) - v_2(c_1 v_1) \\
&= c_2 v_1 v_2 - c_1 v_1 v_2 \\
&= v_1 v_2 (c_2 - c_1) \neq 0
\end{aligned}$$

We have shown that the determinant of A is not zero, since $c_2 \neq c_1$. Therefore A is invertible, and the equation

$$x_1 \vec{v} + x_2 \vec{w} = \vec{u}$$

has solutions for x_1 and x_2 .

Since W is closed under linear combinations, the vector \vec{u} has to be in the subspace W . This means that every real number is contained in W , so $W = \mathbb{R}^2$.