

## Chapter 2 Section 4

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**Theorem 1.** *An  $n \times n$  matrix  $A$  is invertible if and only if*

$$\text{rref}(A) = I_n$$

*or, equivalently, if*

$$\text{rank}(A) = n$$

**Theorem 2.** *To find the inverse of an  $n \times n$  matrix  $A$ , form the  $n \times (2n)$  matrix  $[A \mid I_n]$  and compute  $\text{rref}[A \mid I_n]$ .*

- *If  $\text{rref}[A \mid I_n]$  is of the form  $[I_n \mid B]$  then  $A$  is invertible and  $A^{-1} = B$ .*
- *If  $\text{rref}[A \mid I_n]$  is of another form (i.e., its left half fails to be  $I_n$ ) then  $A$  is not invertible.*

**Theorem 3.** *For an invertible  $n \times n$  matrix  $A$ ,*

$$A^{-1}A = I_n \quad \text{and} \quad AA^{-1} = I_n$$

**Theorem 4.** *If  $A$  and  $B$  are invertible  $n \times n$  matrices, then  $BA$  is invertible as well, and*

$$(BA)^{-1} = A^{-1}B^{-1}$$

**Theorem 5.** *Let  $A$  and  $B$  be two  $n \times n$  matrices such that  $BA = I_n$ . Then*

- *$A$  and  $B$  are both invertible*
- *$A^{-1} = B$  and  $B^{-1} = A$*
- *$AB = I_n$*

**Problem 1.** *Is the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

*invertible? If so, find the inverse of  $A$ .*

**Solution.**

$$\begin{aligned} & \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} \\ & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

*We see that*

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*Thus  $A$  is invertible.*

*Note that  $\text{rref}(A)$  is an acronym that refers to the reduced row echelon form of matrix  $A$ . The computation  $\text{rref}(A)$  tells us whether  $A$  is invertible.*

*To invert the matrix, let's calculate  $\text{rref}[A \mid I_n]$ .*

$$\begin{aligned}
& \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array} \right] \\
& \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array} \right] \\
& \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array} \right] \\
& \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right] \\
& \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 8 & -5 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right] \\
& \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right]
\end{aligned}$$

Thus

$$rref [A \mid I_n] = \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right]$$

and

$$A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$$

**Problem 2.** Suppose  $A$ ,  $B$ , and  $C$  are three  $n \times n$  matrices such that  $ABC = I_n$ . Show that  $B$  is invertible, and express  $B^{-1}$  in terms of  $A$  and  $C$ .

**Solution.** By the associative property of matrices

$$\begin{aligned}
ABC &= I_n \\
(AB)C &= I_n \\
A(BC) &= I_n
\end{aligned}$$

Thus matrices  $A$  and  $C$  are invertible.

$$\begin{aligned} ABC &= I_n \\ A^{-1}ABC &= A^{-1}I_n \\ BC &= A^{-1} \\ BCA &= A^{-1}A \\ B(CA) &= I_n \end{aligned}$$

Thus matrix  $B$  is invertible and  $B^{-1} = CA$ .

**Problem 3.** For an arbitrary  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  compute the product  $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . When is  $A$  invertible? If so, what is  $A^{-1}$ ?

**Solution.**

$$\begin{aligned} &\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \end{aligned}$$

When  $ad - bc$  is nonzero, we can form the product

$$\begin{aligned} &\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Thus  $A$  is invertible when the determinant  $ad - bc \neq 0$ , and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Problem 4.** Is the matrix  $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$  invertible? If so, find the inverse. Interpret  $\det A$  geometrically.

**Solution.**

$$\det A = 1 * 1 - 2 * 3 = -5$$

Since  $\det A = -5$  is nonzero, the matrix is invertible.

$$A^{-1} = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

The quantity  $|\det A|$  is the area of the shaded parallelogram constructed from the vectors  $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . The determinant is negative since the angle from  $\vec{v}$  to  $\vec{w}$  is negative.

**Problem 5.** For which values of the constant  $k$  is the matrix  $A = \begin{bmatrix} 1-k & 2 \\ 4 & 3-k \end{bmatrix}$  invertible?

**Solution.** The matrix  $A$  is invertible when  $\det A$  is nonzero.

$$\begin{aligned} \det A &= (1-k)(3-k) - 2 \cdot 4 \\ &= 3 - 4k + k^2 - 8 \\ &= k^2 - 4k - 5 \\ &= (k-5)(k+1) \end{aligned}$$

The matrix  $A$  is invertible when  $k \neq 5$  and when  $k \neq -1$ .

In other words,  $A$  is invertible for all values of  $k$  except  $k = 5$  and  $k = -1$ .

**Problem 6.** Consider a matrix  $A$  that represents the reflection about a line  $L$  in the plane. Use the determinant to verify that  $A$  is invertible. Find  $A^{-1}$ . Explain your answer conceptually, and interpret the determinant geometrically.

**Solution.** Since  $A$  is a reflection matrix, we know that  $A$  is of the form

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

We can calculate the determinant of  $A$ .

$$\begin{aligned} \det A &= -a^2 - b^2 \\ &= -(a^2 + b^2) \end{aligned}$$

Since the determinant is only zero when  $a = b = 0$ , we can say that  $A$  is invertible for all values except  $a = b = 0$ .

$$\begin{aligned} A^{-1} &= -\frac{1}{a^2 + b^2} \begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} \\ &= \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \end{aligned}$$

Strictly speaking, a reflection matrix is of the form

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

where  $a^2 + b^2 = 1$ . So we can substitute  $a^2 + b^2 = 1$  into our inverse. We can also assume that the determinant is nonzero since  $a^2 + b^2 = 1$ . Thus every reflection matrix is invertible.

$$\begin{aligned} A^{-1} &= \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \\ &= \frac{1}{1} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \\ &= A \end{aligned}$$

We find that every reflection matrix  $A$  is invertible, and that  $A^{-1} = A$ .

The determinant of  $A$  is actually the area of a unit square. Let  $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} b \\ -a \end{bmatrix}$ . The vectors  $\vec{v}$  and  $\vec{w}$  form a parallelogram, and this parallelogram is a unit square.

We can write the determinant as

$$\begin{aligned} \det A &= \|\vec{v}\| \sin \theta \|\vec{w}\| \\ &= \|\sqrt{a^2 + b^2}\| \sin \left(-\frac{\pi}{2}\right) \|\sqrt{b^2 + (-a)^2}\| \\ &= -1 \end{aligned}$$

where  $\theta$  is the angle between vectors  $\vec{v}$  and  $\vec{w}$ .

In summary, every reflection matrix  $A$  is invertible, since  $\det A = -1$  for all reflection matrices. The inverse of a reflection matrix  $A$  is  $A^{-1} = A$ . The determinant of a reflection matrix is the area of a unit square formed by the two column vectors of the reflection matrix. This area is negative because the angle between the vectors  $\left(-\frac{\pi}{2}\right)$  is negative.

**Problem 7.** Let  $A$  be a block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where  $A_{11}$  is an  $n \times n$  matrix,  $A_{22}$  is an  $m \times m$  matrix, and  $A_{12}$  is an  $n \times m$  matrix.

- For which choices of  $A_{11}$ ,  $A_{12}$ , and  $A_{22}$  is  $A$  invertible?
- If  $A$  is invertible, what is  $A^{-1}$  (in terms of  $A_{11}$ ,  $A_{12}$ ,  $A_{22}$ )?

**Solution.** We are looking for a matrix  $B$  such that

$$BA = I_{n+m} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}$$

Let us partition  $B$  in the same way as  $A$ .

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where  $B_{11}$  is  $n \times n$ ,  $B_{12}$  is  $m \times n$ , and so on. We know that

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}$$

Thus

$$\begin{aligned} B_{11}A_{11} &= I_n \\ B_{11}A_{12} + B_{12}A_{22} &= 0 \\ B_{21}A_{11} + B_{22}0 &= 0 \\ B_{21}A_{12} + B_{22}A_{22} &= I_m \end{aligned}$$

This gives us

$$\begin{aligned}
B_{11} &= A_{11}^{-1} \\
B_{12} &= -A_{11}^{-1}A_{12}A_{22}^{-1} \\
B_{21} &= 0 \\
B_{22} &= A_{22}^{-1}
\end{aligned}$$

We find that  $A$  is invertible when  $A_{11}$  and  $A_{22}$  are invertible.

When  $A$  is invertible, the inverse of  $A$  is given by

$$B = A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

**Problem 8.** Verify this result for the following example:

$$\left( \begin{array}{cc|ccc} 1 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 5 & 6 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)^{-1} = \left( \begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 \\ -1 & 1 & -3 & -3 & -3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$



**Solution.**

$$\begin{aligned}
& \left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 3 & \\ 1 & 2 & 4 & 5 & 6 & \\ \hline 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \end{array} \right) \left( \begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 & \\ -1 & 1 & -3 & -3 & -3 & \\ \hline 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \end{array} \right) \\
&= \left( \begin{array}{ccc|ccc} \left[ \begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right] \left[ \begin{array}{cc} 2 & -1 \\ -1 & 1 \end{array} \right] + \left[ \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] & & \left[ \begin{array}{ccc} 1 & 1 & 2 \\ -3 & -3 & -3 \end{array} \right] + \left[ \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \\ \hline & \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] & & \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \end{array} \right) \\
&= \left( \begin{array}{ccc|ccc} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] + \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] & & \left[ \begin{array}{ccc} -1 & -2 & -3 \\ -4 & -5 & -6 \end{array} \right] + \left[ \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \\ \hline & \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] & & \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \end{array} \right) \\
&= \left( \begin{array}{ccc|ccc} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] & & \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \\ \hline & \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] & & \end{array} \right) \\
&= \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)
\end{aligned}$$

Since the product of the two matrices is  $I_5$ , the identity matrix, we know that

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 3 & \\ 1 & 2 & 4 & 5 & 6 & \\ \hline 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \end{array} \right)^{-1} = \left( \begin{array}{cc|ccc} 2 & -1 & 2 & 1 & 0 & \\ -1 & 1 & -3 & -3 & -3 & \\ \hline 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \end{array} \right)$$

In the next fifteen problems, decide whether the matrices are invertible. If they are, find the inverse.

**Problem 9.**

$$\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$$

**Solution.** The determinant is  $2*8-5*3 = 16-15 = 1$ . Since the determinant is nonzero, the matrix is invertible.

The inverse is given by swapping the entries  $(1,1)$  and  $(2,2)$ , negating the entries  $(1,2)$  and  $(2,1)$ , and dividing by the determinant.

$$\begin{aligned}\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}^{-1} &= \frac{1}{1} \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}\end{aligned}$$

**Theorem 6.** If a matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is invertible, then the inverse of  $A$  is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We will use this theorem in the following problems, as we did in the previous problem.

**Problem 10.**

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

**Solution.** The matrix is not invertible, because the determinant is zero.

$$\det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 1*1 - 1*1 = 0$$

**Problem 11.**

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

**Solution.**

$$\det \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = 0*1 - 1*2 = -2$$

Thus the matrix is invertible.

$$\begin{aligned}\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^{-1} &= \frac{1}{-2} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -0.5 & 1 \\ 0.5 & 0 \end{bmatrix}\end{aligned}$$

**Problem 12.**

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution.**

$$\begin{aligned} & \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= 1 * \det \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + 3 \det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= 1 - 0 + 0 \\ &= 1 \end{aligned}$$

*The determinant of the matrix is 1, thus the matrix is invertible.*

*Let's find the inverse using elementary row operations.*

$$\begin{aligned} & \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \\ & \left( \begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \\ & \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \\ & \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \end{aligned}$$

*The inverse of the matrix is*

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$