

## Chapter 3 Section 2

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**Problem 1.** Is  $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0 \right\}$  a subspace of  $\mathbb{R}^2$ ?

**Solution.**  $W$  contains the zero vector and is closed under addition. But  $W$  is not closed under scalar multiplication. Therefore  $W$  is not a subspace of  $\mathbb{R}^2$ .

**Problem 2.** Show that the only subspaces of  $\mathbb{R}^2$  are  $\mathbb{R}^2$  itself, the set  $\{\vec{0}\}$ , and any of the lines through the origin.

**Solution.** Let  $W$  be a subspace of  $\mathbb{R}^2$  that is neither a line through the origin nor the set  $\{\vec{0}\}$ . Then we can choose two nonzero nonparallel vectors  $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$  from our subspace  $W$ . Let  $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  be a vector in  $\mathbb{R}^2$ . We will show that we can write  $\vec{u}$  as a linear combination of  $\vec{v}$  and  $\vec{w}$ .

If  $\vec{u}$  can be written as a linear combination of  $\vec{v}$  and  $\vec{w}$ , then there are solutions to the equation

$$x_1 \vec{v} + x_2 \vec{w} = \vec{u}$$

where  $x_1$  and  $x_2$  are real numbers. We can write this equation in matrix form

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This equation has solutions when  $A = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$  is invertible. We know that  $A$  is invertible when  $\det A$  is nonzero.

The components  $v_1, v_2, w_1, w_2$  can either be zero or nonzero. There is a small number of possible cases, since both vectors are not the zero vector, and since the two vectors are not parallel.

Case 1:  $v_1 = 0, v_2 \neq 0, w_1 \neq 0, w_2 = 0$

Case 2:  $v_1 \neq 0, v_2 = 0, w_1 = 0, w_2 \neq 0$

*In both of these cases, the determinant of  $A$  is nonzero, and the matrix  $A$  is invertible.*

Case 3: *At least one of the vectors ( $\vec{v}$  and  $\vec{w}$ ) has two nonzero components.*

*Let  $\vec{v}$  be the vector with two nonzero components.*

*There exist real numbers  $c_1$  and  $c_2$  such that  $c_1 v_1 = w_1$  and  $c_2 v_2 = w_2$ . We know that  $c_1 \neq c_2$  since the two vectors are not scalar multiples of each other. We can substitute these expressions when we calculate the determinant of  $A$ .*

$$\begin{aligned}\det A &= v_1 w_2 - v_2 w_1 \\ &= v_1 (c_2 v_2) - v_2 (c_1 v_1) \\ &= c_2 v_1 v_2 - c_1 v_1 v_2 \\ &= v_1 v_2 (c_2 - c_1)\end{aligned}$$

*Since  $v_1 \neq 0$ ,  $v_2 \neq 0$  and  $c_2 \neq c_1$ , the determinant of  $A$  is nonzero. Thus the matrix  $A$  is invertible, and the equation*

$$x_1 \vec{v} + x_2 \vec{w} = \vec{u}$$

*has solutions for  $x_1$  and  $x_2$ .*

*Since  $W$  is closed under linear combinations, the vector  $\vec{u}$  is in the subspace  $W$ . This means that  $W$  contains every real number, so  $W = \mathbb{R}^2$ .*

*We can also express this using a linear transformation. Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation*

$$\begin{aligned}T(\vec{x}) &= \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\end{aligned}$$

*We have shown that the matrix  $\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$  is invertible.*

*This means that there is a unique solution  $\vec{x}$  for every vector  $\vec{u}$  in  $\mathbb{R}^2$ .*

*This is equivalent to saying any vector  $\vec{u}$  in  $\mathbb{R}^2$  can be written as a linear combination of  $\vec{v}$  and  $\vec{w}$ .*

**Problem 3.** Consider the plane  $V$  in  $\mathbb{R}^3$  given by the equation

$$x_1 + 2x_2 + 3x_3 = 0$$

- Find a matrix  $A$  such that  $V = \ker A$
- Find a matrix  $B$  such that  $V = \operatorname{im} B$

**Solution.** Let  $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ . We can write

$$\begin{aligned} V &= \ker A \\ &= \ker \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \end{aligned}$$

We can find two nonparallel vectors  $\vec{v}$  and  $\vec{w}$  in  $V$ .

$$\text{Let } \vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

We can construct our matrix  $B$  from the column vectors  $\vec{v}$  and  $\vec{w}$ .

$$B = [\vec{v} \quad \vec{w}] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 1 \end{bmatrix}$$

Now we can say

$$\begin{aligned} V &= \operatorname{im} B \\ &= \operatorname{im} \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

**Problem 4.** Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Find vectors in  $\mathbb{R}^3$  that span the image of  $A$ . What is the smallest number of vectors needed to span the image of  $A$ ?

**Solution.** The vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

span the image of  $A$ .

The vector  $v_2$  is redundant because  $v_2 = 2v_1$ . The vector  $v_4$  is redundant because  $v_1 + v_3 = v_4$ . Thus only two vectors are needed to span the image of  $A$ , the vectors  $v_1$  and  $v_3$ .

$$\text{im } A = \text{span}(v_1, v_3)$$

We can verify this algebraically. We know that the image of  $A$  is spanned by the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ .

Let  $\vec{u}$  be a vector in  $\mathbb{R}^3$ . Then

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$$

for some real coefficients  $c_1, c_2, c_3, c_4$ . Substituting we get

$$\begin{aligned}\vec{u} &= c_1\vec{v}_1 + c_2(2\vec{v}_1) + c_3\vec{v}_3 + c_4(v_1 + v_3) \\ &= (c_1 + c_4 + 2c_2)\vec{v}_1 + (c_3 + c_4)\vec{v}_3\end{aligned}$$

Thus  $\vec{u}$  is a linear combination of vectors  $\vec{v}_1$  and  $\vec{v}_3$ . The vectors  $\vec{v}_1$  and  $\vec{v}_3$  form a basis for the image of  $A$ .

**Problem 5.** Are the following vectors in  $\mathbb{R}^7$  linearly independent?

$$\vec{v}_1 = \begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \\ 1 \\ 9 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 6 \\ 0 \\ 7 \\ 1 \\ 4 \\ 8 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 6 \\ 2 \\ 3 \\ 1 \\ 7 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

**Solution.**  $\vec{v}_2$  cannot be a linear combination of  $\vec{v}_1$ , since  $\vec{v}_1$  has a 0 in the fourth component and  $\vec{v}_2$  has a 1 in the fourth component.  $\vec{v}_3$  cannot be a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ , since  $\vec{v}_1$  and  $\vec{v}_2$  have zeros in the last component, and  $\vec{v}_3$  has a seven in the last component.  $\vec{v}_4$  cannot be a linear combination of  $\vec{v}_1, \vec{v}_2$ , and  $\vec{v}_3$ , since  $\vec{v}_4$  has a 5 in the second component and the other three vectors all have zero in the second component.

Thus the four vectors are linearly independent.

**Problem 6.** Are the following vectors in  $\mathbb{R}^7$  linearly independent?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

**Solution.** *The vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  are not linearly independent since  $2\vec{v}_2 - \vec{v}_1 = \vec{v}_3$ .*