Chapter 2 Section 4

Andrew Taylor

April 16 2022

Theorem 1. *An $n \times n$ matrix A is invertible if and only if*

$$rref(A) = I_n$$

or, equivalently, if

$$rank(A) = n$$

Theorem 2. *To find the inverse of an $n \times n$ matrix A, form the $n \times (2n)$ matrix $\begin{bmatrix} A \mid I_n \end{bmatrix}$ and compute $rref[A \mid I_n]$.*

- *If $rref[A \mid I_n]$ is of the form $\begin{bmatrix} I_n \mid B \end{bmatrix}$ then A is invertible and $A^{-1} = B$.*
- *If $rref[A \mid I_n]$ is of another form (i.e., its left half fails to be I_n) then A is not invertible.*

Theorem 3. *For an invertible $n \times n$ matrix A,*

$$A^{-1}A = I_n \quad \textit{and} \quad AA^{-1} = I_n$$

Theorem 4. *If A and B are invertible $n \times n$ matrices, then BA is invertible as well, and*

$$(BA)^{-1} = A^{-1}B^{-1}$$

Theorem 5. *Let A and B be two $n \times n$ matrices such that $BA = I_n$. Then*

- *A and B are both invertible*
- *$A^{-1} = B$ and $B^{-1} = A$*
- *$AB = I_n$*

Problem 1. *Is the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

invertible? If so, find the inverse of A.

Solution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We see that

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus A is invertible.

Note that $rref(A)$ is an acronym that refers to the reduced row echelon form of matrix A. The computation $rref(A)$ tells us whether A is invertible.

To invert the matrix, let's calculate $rref\left[A \mid I_n\right]$.

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 2 & 3 & 2 & | & 0 & 1 & 0 \\ 3 & 8 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 5 & -1 & | & -3 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & -1 & | & 7 & -5 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 0 & | & 8 & -5 & 1 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & | & 10 & -6 & 1 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{bmatrix}$$

Thus

$$rref\left[A \mid I_n\right] = \begin{bmatrix} 1 & 0 & 0 & | & 10 & -6 & 1 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -7 & 5 & -1 \end{bmatrix}$$

and

$$A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$$

Problem 2. *Suppose A, B, and C are three $n \times n$ matrices such that $ABC = I_n$. Show that B is invertible, and express B^{-1} in terms of A and C.*

Solution. *By the associative property of matrices*

$$\begin{aligned} ABC &= I_n \\ (AB)C &= I_n \\ A(BC) &= I_n \end{aligned}$$

Thus matrices A and C are invertible.

$$
\begin{aligned}
ABC &= I_n \\
A^{-1}ABC &= A^{-1}I_n \\
BC &= A^{-1} \\
BCA &= A^{-1}A \\
B(CA) &= I_n
\end{aligned}
$$

Thus matrix B is invertible and $B^{-1} = CA$.

Problem 3. *For an arbitrary 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ compute the product $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. When is A invertible? If so, what is A^{-1}?*

Solution.

$$
\begin{aligned}
&\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}
\end{aligned}
$$

When $ad - bc$ is nonzero, we can form the product

$$
\begin{aligned}
&\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Thus A is invertible when the determinant $ad - bc \neq 0$, and

$$
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
$$

Problem 4. *Is the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ invertible? If so, find the inverse. Interpret $\det A$ geometrically.*

Solution.

$$
\det A = 1 * 1 - 2 * 3 = -5
$$

Since $\det A = -5$ is nonzero, the matrix is invertible.

$$A^{-1} = \frac{1}{-5}\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \\ \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

The quantity $|\det A|$ *is the area of the shaded parallelogram constructed from the vectors* $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. *The determinant is negative since the angle from* $\vec{v}$ *to* $\vec{w}$ *is negative.*