Chapter 2 Section 2

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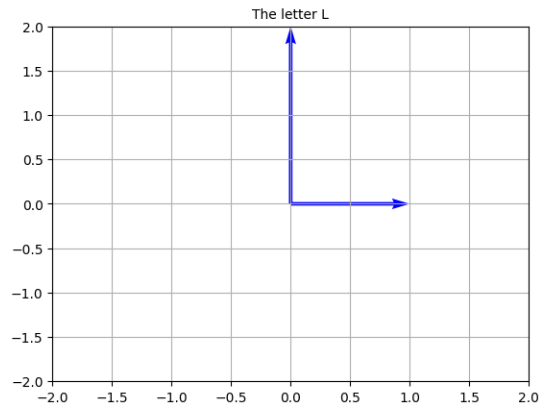
The letter L can be represented by the vectors $(0, 2)$ and $(1, 0)$.



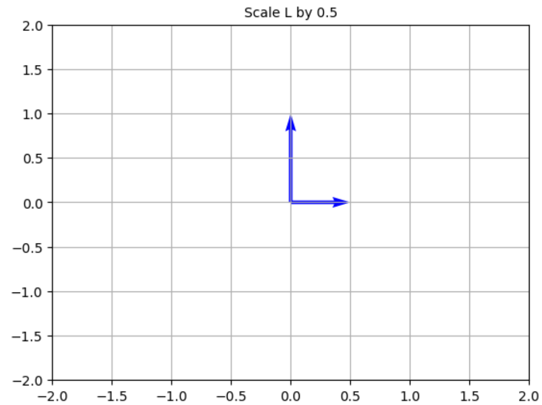
The following problems ask for a linear transformation of the letter L. In the following problems, give the matrix of the transformation and plot the result.

Problem 1. *Scale L by a factor of* $\frac{1}{2}$

Solution. *The matrix of the transformation is*

$$\begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}$$

After the scaling, the L looks like this



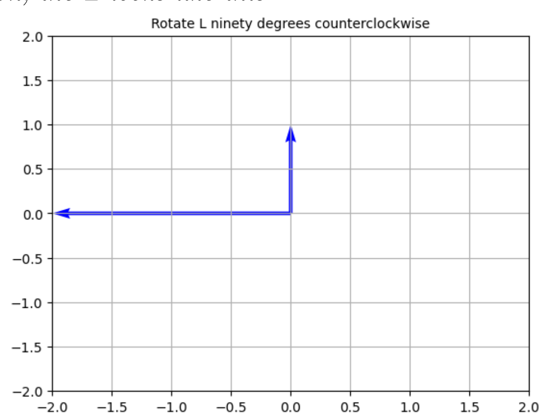
Note that in creating this shape, we scaled both vectors that make up the L.

Problem 2. *Rotate L ninety degrees counterclockwise*

Solution. *The matrix of the transformation is*

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

After the rotation, the L looks like this



Problem 3. *Reflect L about the Y axis*

Solution. *The matrix of the transformation is*

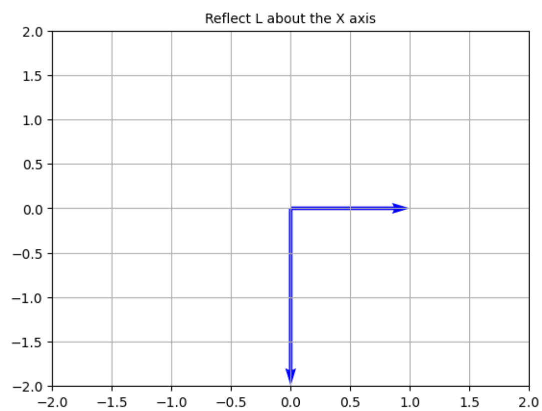
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The plot looks like this

Problem 4. *Reflect L about the X axis*

Solution. *The matrix of the transformation is*

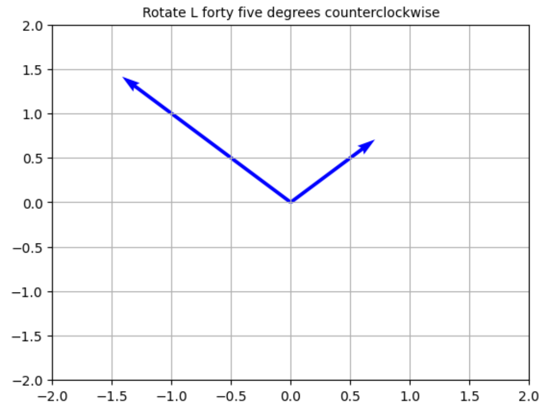
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



Problem 5. *Rotate L forty five degrees counterclockwise*

Solution. *The matrix of the transformation is*

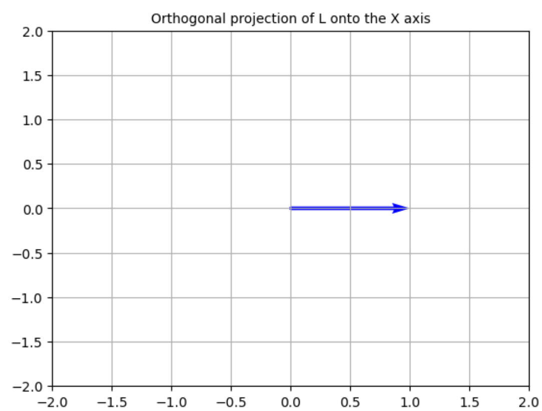
$$\begin{bmatrix} \cos(\frac{\pi}{4}) & -1 * \sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix}$$



Problem 6. *Find the orthogonal projection of L onto the x-axis*

Solution. *The matrix of the transformation is*

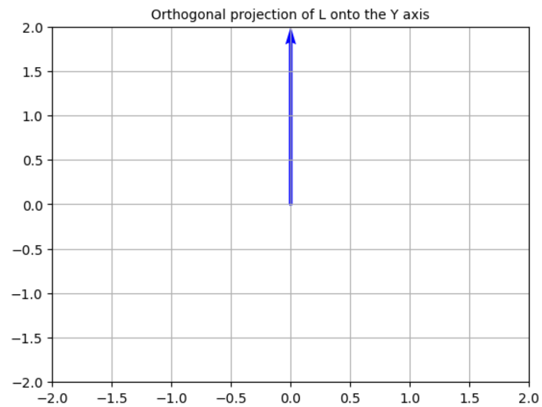
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Problem 7. *Find the orthogonal projection of L onto the y-axis*

Solution. *The matrix of the transformation is*

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



Problem 8. *Find the matrix P of the orthogonal projection onto the line L spanned by $\vec{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$*

Solution.

$$\begin{aligned} P &= \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} \\ &= \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \\ &= \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix} \end{aligned}$$

Problem 9. *Let V be the plane defined by $2x_1 + x_2 - 2x_3 = 0$ and let $\vec{x} = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$. Find $\mathrm{ref}_V\, \vec{x}$.*

Solution. *The vector $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is perpendicular to the plane V.*

We can get the unit vector $\vec{u}$ perpendicular to the plane by

$$\begin{aligned} \vec{u} &= \frac{1}{\|\vec{v}\|} \vec{v} \\ &= \frac{1}{\sqrt{2^2 + 1^2 + (-2)^2}} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \end{aligned}$$

We can now use the formula

$$\begin{aligned}
\operatorname{ref}_V \vec{x} &= \operatorname{proj}_V \vec{x} - \operatorname{proj}_L \vec{x} \\
&= \vec{x} - 2\operatorname{proj}_L \vec{x} \\
&= \vec{x} - 2(\vec{x}\cdot\vec{u})\vec{u} \\
&= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - 2\left(\begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}\right)\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \\ -8 \end{bmatrix} \\
&= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}
\end{aligned}$$

The reflection of the vector $\vec{x}$ over the plane V is the vector

$$\operatorname{ref}_V \vec{x} = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$$

Problem 10. *Give the matrix of a counterclockwise rotation through $\dfrac{\pi}{6}$.*

Solution. *The matrix of a counterclockwise rotation through $\dfrac{\pi}{6}$ is*

$$\begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Problem 11. *Let $T(\vec{x})$ be the linear transformation*

$$
\begin{aligned}
T(\vec{x}) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{x} \\
&= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

How does the linear transformation $T(\vec{x})$ affect the letter L? Remember that our letter L is composed of vectors $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution. *The vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be written in polar coordinates as*

$$
\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}
$$

We can substitute these polar coordinates into the linear transformation, and write it as

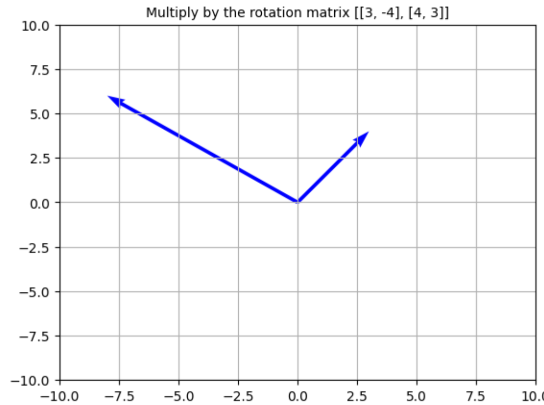
$$
\begin{aligned}
T(\vec{x}) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

The linear transformation rotates the vector counterclockwise through an angle of θ and scales the vector by a factor of r.

For example, let $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Then we have

$$
\begin{aligned}
T(\vec{x}) &= \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&\approx 5 \begin{bmatrix} \cos(53.1^\circ) & -\sin(53.1^\circ) \\ \sin(53.1^\circ) & \cos(53.1^\circ) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

This rotates L approximately 53.1° and scales L by a factor of 5.



Thus we see that the linear transformation

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

rotates L *counterclockwise through* θ *degrees and scales* L *by a factor of* r.