Chapter 2 Section 2

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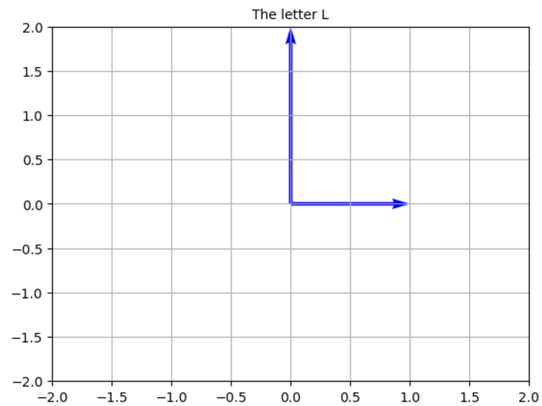
April 6 2022

The letter L can be represented by the vectors $(0, 2)$ and $(1, 0)$.



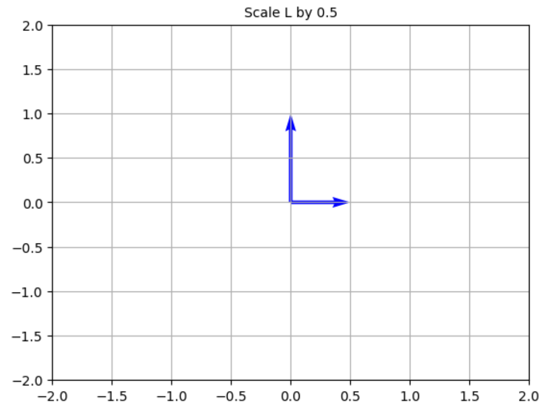
The following problems ask for a linear transformation of the letter L. In the following problems, give the matrix of the transformation and plot the result.

Problem 1. *Scale L by a factor of* $\frac{1}{2}$

Solution. *The matrix of the transformation is*

$$\begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}$$

After the scaling, the L looks like this



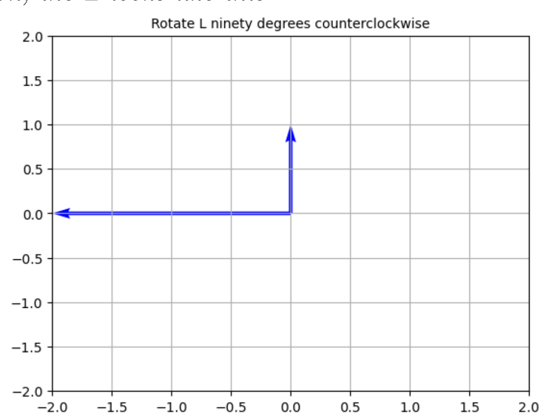
Note that in creating this shape, we scaled both vectors that make up the L.

Problem 2. *Rotate L ninety degrees counterclockwise*

Solution. *The matrix of the transformation is*

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

After the rotation, the L looks like this



Problem 3. *Reflect L about the Y axis*

Solution. *The matrix of the transformation is*

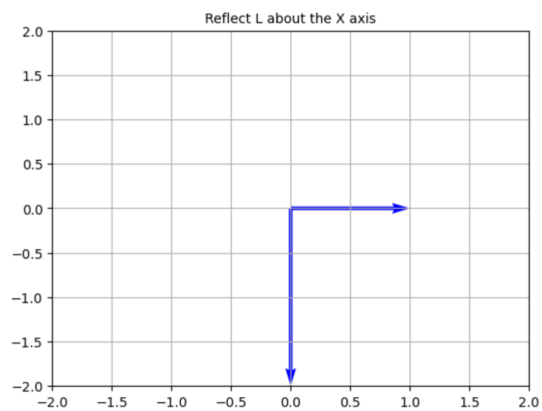
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The plot looks like this

Problem 4. *Reflect L about the X axis*

Solution. *The matrix of the transformation is*

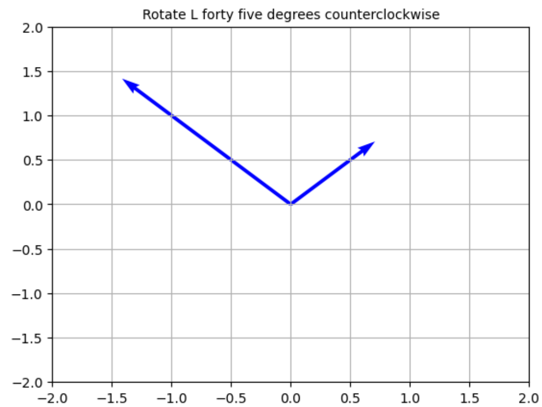
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



Problem 5. *Rotate L forty five degrees counterclockwise*

Solution. *The matrix of the transformation is*

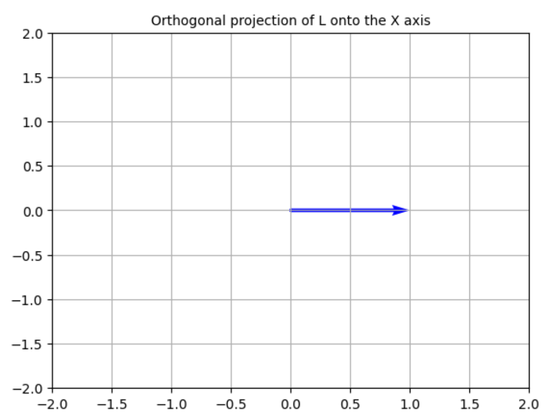
$$\begin{bmatrix} \cos(\frac{\pi}{4}) & -1 * \sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix}$$



Problem 6. *Find the orthogonal projection of L onto the x-axis*

Solution. *The matrix of the transformation is*

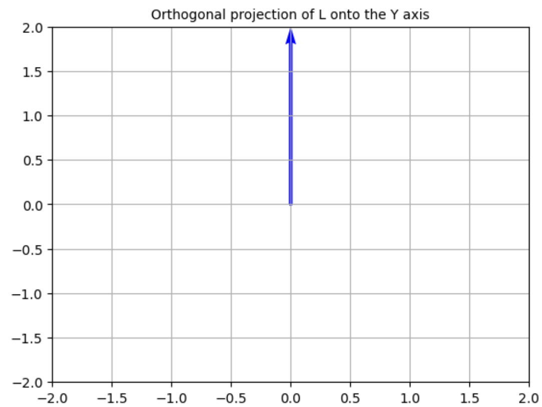
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



Problem 7. *Find the orthogonal projection of L onto the y-axis*

Solution. *The matrix of the transformation is*

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



Problem 8. *Find the matrix P of the orthogonal projection onto the line L spanned by $\vec{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$*

Solution.

$$\begin{aligned} P &= \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} \\ &= \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \\ &= \begin{bmatrix} 0.36 & 0.48 \\ 0.48 & 0.64 \end{bmatrix} \end{aligned}$$

Problem 9. *Let V be the plane defined by $2x_1 + x_2 - 2x_3 = 0$ and let $\vec{x} = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$. Find $\mathrm{ref}_V\, \vec{x}$.*

Solution. *The vector $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is perpendicular to the plane V.*

We can get the unit vector $\vec{u}$ perpendicular to the plane by

$$\begin{aligned} \vec{u} &= \frac{1}{\|\vec{v}\|} \vec{v} \\ &= \frac{1}{\sqrt{2^2 + 1^2 + (-2)^2}} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \end{aligned}$$

We can now use the formula

$$\begin{aligned}
\operatorname{ref}_V \vec{x} &= \operatorname{proj}_V \vec{x} - \operatorname{proj}_L \vec{x} \\
&= \vec{x} - 2 \operatorname{proj}_L \vec{x} \\
&= \vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u} \\
&= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - 2 \left(\begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right) \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \\ -8 \end{bmatrix} \\
&= \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}
\end{aligned}$$

The reflection of the vector $\vec{x}$ over the plane V is the vector

$$\operatorname{ref}_V \vec{x} = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$$

Problem 10. *Give the matrix of a counterclockwise rotation through $\frac{\pi}{6}$.*

Solution. *The matrix of a counterclockwise rotation through $\frac{\pi}{6}$ is*

$$\begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Problem 11. *Let $T(\vec{x})$ be the linear transformation*

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{x} \\ &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

How does the linear transformation $T(\vec{x})$ affect the letter L? Remember that our letter L is composed of vectors $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution. *The vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be written in polar coordinates as*

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$$

We can substitute these polar coordinates into the linear transformation, and write it as

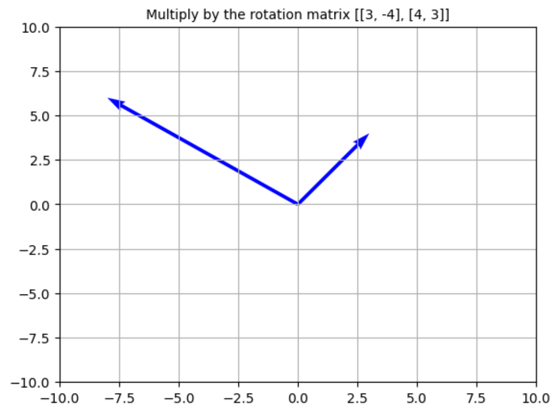
$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

The linear transformation rotates the vector counterclockwise through an angle of θ and scales the vector by a factor of r.

For example, let $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Then we have

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &\approx 5 \begin{bmatrix} \cos(53.1^\circ) & -\sin(53.1^\circ) \\ \sin(53.1^\circ) & \cos(53.1^\circ) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

This rotates L approximately 53.1° and scales L by a factor of 5.



Thus we see that the linear transformation

$$
\begin{aligned}
T(\vec{x}) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

rotates L counterclockwise through θ degrees and scales L by a factor of r.

Problem 12. *Sketch the image of the standard L under the linear transformation*

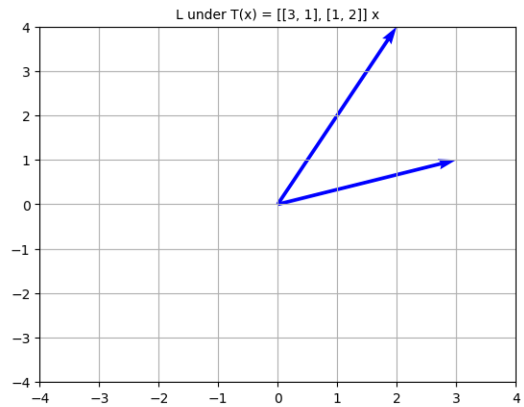
$$T(\vec{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}$$

Solution. *We can apply the linear transformation to both vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ that make up our L.*

$$
\begin{aligned}
T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 4 \end{bmatrix}
\end{aligned}
$$

We can now draw these vectors on the Cartesian plane.



Problem 13. *Find the matrix of a rotation through an angle of* 60° *in the counterclockwise direction.*

Solution. *The matrix of the rotation is*

$$\begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Problem 14. *Consider a linear transformation from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$*. Use* $T(\vec{e_1})$ *and* $T(\vec{e_2})$ *to describe the image of the unit square geometrically.*

Solution. *Let* $\vec{x}$ *be a vector in the unit square. We can write the vector* $\vec{x}$ *as*

$$\vec{x} = x_1\vec{e_1} + x_2\vec{e_2}$$

where $0 \leq x_1, x_2 \leq 1$ *and* $\vec{e_1}$*,* $\vec{e_2}$ *are the unit vectors. Thus*

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e_1} + x_2\vec{e_2}) \\ &= T(x_1\vec{e_1}) + T(x_2\vec{e_2}) \\ &= x_1T(\vec{e_1}) + x_2T(\vec{e_2}) \end{aligned}$$

The vectors $T(\vec{e_1})$ *and* $T(\vec{e_2})$ *describe a parallelogram in* $\mathbb{R}^3$*.*

Thus $T(\vec{x}) = x_1T(\vec{e_1}) + x_2T(\vec{e_2})$ *is a vector in that parallelogram in* $\mathbb{R}^3$*.*

Note that we got our equation for $T(\vec{x})$ using the properties of linear transformations.

$T(a+b) = T(a) + T(b)$ *and* $T(ka) = kT(a)$ *for vectors* a, b *and scalar* k.

Problem 15. *Interpret the following linear transformation geometrically:*

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$

Solution. *The matrix of the transformation is a rotation matrix combined with a scaling. Since a rotation matrix has the form*

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

We see that $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *forms a* $-45°$ *angle with the* X *axis. The vector has magnitude* $\sqrt{2}$.

Thus the linear transformation T *applies a rotation of* $45°$ *in the clockwise direction and a scaling of* $\sqrt{2}$.

Problem 16. *The matrix*

$$\begin{bmatrix} -0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

represents a rotation. Find the angle of rotation (in radians).

Solution. *We have* $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix}$ *and* $a^2 + b^2 = 1$.

Thus $\cos(\theta) = -0.8$ *and* $\sin(\theta) = 0.6$ *where* θ *is the angle of rotation.*

We can solve for θ.

$$\cos(\theta) = -0.8$$
$$\theta = \cos^{-1}(-0.8) = 2.49809...$$

Thus $\theta \approx 2.49809$ *radians.*

Problem 17. *Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector* $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. *Find the orthogonal projection of the vector* $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *onto L.*

Solution. *Let* $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

$$\begin{aligned}
\operatorname{proj}_L \vec{x} &= \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w} \\
&= \left(\frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}} \right) \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\
&= \left(\frac{5}{9} \right) \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix}
\end{aligned}$$

Since the magnitude of $\vec{x}$ is $\sqrt{3}$, it's fine that the first and third components of $\operatorname{proj}_L \vec{x}$ exceed 1.

The vector $\operatorname{proj}_L \vec{x}$ represents the component of $\vec{x}$ that is parallel to $\vec{w}$.

Let V be the plane perpendicular to L.

$$\begin{aligned}
\vec{x} &= \operatorname{proj}_L \vec{x} + \operatorname{proj}_V \vec{x} \\
&= \begin{bmatrix} \frac{10}{9} \\ \frac{5}{9} \\ \frac{10}{9} \end{bmatrix} + \begin{bmatrix} \frac{-1}{9} \\ \frac{4}{9} \\ \frac{-1}{9} \end{bmatrix}
\end{aligned}$$

Problem 18. *Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector* $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. *Find the reflection of the vector* $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *onto L.*

Solution.

$$\begin{aligned}
\operatorname{ref}_L \vec{x} &= 2 \operatorname{proj}_L \vec{x} - \vec{x} \\
&= 2 \begin{bmatrix} \frac{10}{9} \\ \\ \frac{5}{9} \\ \\ \frac{10}{9} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{20}{9} \\ \\ \frac{10}{9} \\ \\ \frac{20}{9} \end{bmatrix} - \begin{bmatrix} \frac{9}{9} \\ \\ \frac{9}{9} \\ \\ \frac{9}{9} \end{bmatrix} \\
&= \begin{bmatrix} \frac{11}{9} \\ \\ \frac{1}{9} \\ \\ \frac{11}{9} \end{bmatrix}
\end{aligned}$$

Problem 19. *Interpret the following linear transformation geometrically:*

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \vec{x}$$

Solution. *The matrix of the transformation represents a reflection. Recall that a reflection matrix is of the form*

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

Here we have $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

In the linear transformation $T(\vec{x})$*, the vector* $\vec{x}$ *is reflected about the line spanned by the vector* $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Problem 20. *Interpret the following linear transformation geometrically:*

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \vec{x}$$

Solution. *The linear transformation is a vertical shear with* $k = 1$*. Recall that a vertical shear has a matrix of the form*

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

We can apply the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ *to a vector* $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$\begin{aligned} T(\vec{x}) &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix} \end{aligned}$$