

Chapter 2 Section 4

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Theorem 1. *An $n \times n$ matrix A is invertible if and only if*

$$\text{rref}(A) = I_n$$

or, equivalently, if

$$\text{rank}(A) = n$$

Theorem 2. *To find the inverse of an $n \times n$ matrix A , form the $n \times (2n)$ matrix $[A \mid I_n]$ and compute $\text{rref}[A \mid I_n]$.*

- *If $\text{rref}[A \mid I_n]$ is of the form $[I_n \mid B]$ then A is invertible and $A^{-1} = B$.*
- *If $\text{rref}[A \mid I_n]$ is of another form (i.e., its left half fails to be I_n) then A is not invertible.*

Theorem 3. *For an invertible $n \times n$ matrix A ,*

$$A^{-1}A = I_n \quad \text{and} \quad AA^{-1} = I_n$$

Theorem 4. *If A and B are invertible $n \times n$ matrices, then BA is invertible as well, and*

$$(BA)^{-1} = A^{-1}B^{-1}$$

Theorem 5. *Let A and B be two $n \times n$ matrices such that $BA = I_n$. Then*

- *A and B are both invertible*
- *$A^{-1} = B$ and $B^{-1} = A$*
- *$AB = I_n$*

Problem 1. *Is the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

invertible? If so, find the inverse of A .

Solution.

$$\begin{aligned} & \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} \\ & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

We see that

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus A is invertible.

Note that $\text{rref}(A)$ is an acronym that refers to the reduced row echelon form of matrix A . The computation $\text{rref}(A)$ tells us whether A is invertible.

To invert the matrix, let's calculate $\text{rref}[A \mid I_n]$.

$$\begin{aligned}
& \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array} \right] \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array} \right] \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array} \right] \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right] \\
& \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 8 & -5 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right] \\
& \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right]
\end{aligned}$$

Thus

$$rref[A \mid I_n] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array} \right]$$

and

$$A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$$

Problem 2. Suppose A , B , and C are three $n \times n$ matrices such that $ABC = I_n$. Show that B is invertible, and express B^{-1} in terms of A and C .

Solution. By the associative property of matrices

$$\begin{aligned}
ABC &= I_n \\
(AB)C &= I_n \\
A(BC) &= I_n
\end{aligned}$$

Thus matrices A and C are invertible.

$$\begin{aligned} ABC &= I_n \\ A^{-1}ABC &= A^{-1}I_n \\ BC &= A^{-1} \\ BCA &= A^{-1}A \\ B(CA) &= I_n \end{aligned}$$

Thus matrix B is invertible and $B^{-1} = CA$.

Problem 3. For an arbitrary 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ compute the product $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. When is A invertible? If so, what is A^{-1} ?

Solution.

$$\begin{aligned} &\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \end{aligned}$$

When $ad - bc$ is nonzero, we can form the product

$$\begin{aligned} &\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Thus A is invertible when the determinant $ad - bc \neq 0$, and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Problem 4. Is the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ invertible? If so, find the inverse. Interpret $\det A$ geometrically.

Solution.

$$\det A = 1 * 1 - 2 * 3 = -5$$

Since $\det A = -5$ is nonzero, the matrix is invertible.

$$A^{-1} = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

The quantity $|\det A|$ is the area of the shaded parallelogram constructed from the vectors $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. The determinant is negative since the angle from \vec{v} to \vec{w} is negative.

Problem 5. For which values of the constant k is the matrix $A = \begin{bmatrix} 1-k & 2 \\ 4 & 3-k \end{bmatrix}$ invertible?

Solution. The matrix A is invertible when $\det A$ is nonzero.

$$\begin{aligned} \det A &= (1-k)(3-k) - 2 * 4 \\ &= 3 - 4k + k^2 - 8 \\ &= k^2 - 4k - 5 \\ &= (k-5)(k+1) \end{aligned}$$

The matrix A is invertible when $k \neq 5$ and when $k \neq -1$.

In other words, A is invertible for all values of k except $k = 5$ and $k = -1$.