Integrating a polynomial

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Lemma 1. *Let n and m be natural numbers. Then*

$$\sum_{k=1}^{n} k^m = \frac{n^{m+1} + \sum_{k=1}^{n} \left(\binom{m+1}{2} k^{m-1}(-1)^2 + \binom{m+1}{3} k^{m-2}(-1)^3 + ... + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)}{m+1}$$

Proof. Consider the sum

$$\sum_{k=1}^{n} \left(k^{m+1} - (k-1)^{m+1}\right)$$

We know that this sum telescopes, which is to say, all but the first and last terms cancel.

$$\sum_{k=1}^{n} \left(k^{m+1} - (k-1)^{m+1}\right) = n^{m+1} - 0^{m+1} = n^{m+1}$$

We can expand the binomial $(k-1)^{m+1}$ and get

$$\begin{aligned}
n^{m+1} &= \sum_{k=1}^{n} \left(k^{m+1} - (k-1)^{m+1}\right) \\
&= \sum_{k=1}^{n} \left(k^{m+1} - \left(\binom{m+1}{0} k^{m+1}(-1)^0 + \binom{m+1}{1} k^m (-1)^1 + ... + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right) \right) \\
&= \sum_{k=1}^{n} -1 \left(\binom{m+1}{1} k^m (-1)^1 + \binom{m+1}{2} k^{m-1}(-1)^2 + ... + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)
\end{aligned}$$

We can solve the above equation for the term $\sum_{k=1}^{n} k^m$

$$\sum_{k=1}^{n} \binom{m+1}{1} k^m = n^{m+1} + \sum_{k=1}^{n} \left(\binom{m+1}{2} k^{m-1}(-1)^2 + ... + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)$$

We can divide both sides by $m+1$.

$$\sum_{k=1}^{n} k^m = \frac{n^{m+1} + \sum_{k=1}^{n} \left(\binom{m+1}{2} k^{m-1}(-1)^2 + \binom{m+1}{3} k^{m-2}(-1)^3 + ... + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)}{m+1}$$

$\square$

Lemma 2. *Let b_n be the sequence*

$$b_n = \frac{1}{m+1} \sum_{k=1}^{n} \left(\binom{m+1}{2} k^{m-1}(-1)^2 + \binom{m+1}{3} k^{m-2}(-1)^3 + ... + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)$$

for some $m \in \mathbb{N}$. Then there exists a $C \in \mathbb{R}$ such that

$$-C\, n^m <= b_n <= C\, n^m$$

Proof. Define the polynomial $P(k)$ as

$$P(k) = \binom{m+1}{2} k^{m-1}(-1)^2 + \binom{m+1}{3} k^{m-2}(-1)^3 + ... + \binom{m+1}{m+1} k^0 (-1)^{m+1}$$

Now we can write

$$b_n = \frac{1}{m+1} \sum_{k=1}^{n} P(k)$$

There exists a polynomial

$$Q(k) = C\, k^{m-1}$$

such that

$$|P(k)| \leq |Q(k)|$$

for some $C \in \mathbb{R}$.

Let c_n be the sequence defined by

$$c_n = \frac{1}{m+1} \sum_{k=1}^{n} Q(k)$$

We know that $|b_n| \leq |c_n|$ since $|P(k)| \leq |Q(k)|$

Furthermore we know that

$$\left|\sum_{k=1}^{n} Q(k)\right| \leq |Cn^m|$$

Thus

$$|c_n| \leq \left|\frac{Cn^m}{m+1}\right|$$

By the transitive property we get

$$\frac{-Cn^m}{m+1} \leq b_n \leq \frac{Cn^m}{m+1}$$

$\square$

Lemma 3. *Let n and m be natural numbers. Let b be a real number.*

$$\lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^{n} k^m = \frac{b^{m+1}}{m+1}$$

Proof. Define A and B as

$$A = \frac{n^{m+1}}{m+1}$$
$$B = \frac{1}{m+1}\sum_{k=1}^{n}\left(\binom{m+1}{2}k^{m-1}(-1)^2 + \binom{m+1}{3}k^{m-2}(-1)^3 + ... + \binom{m+1}{m+1}k^0(-1)^{m+1}\right)$$

Then

$$\begin{aligned}\lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^{n} k^m &= \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}}(A+B)\\ &= \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}}A + \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}}B\end{aligned}$$

Now we can solve each limit separately.

$$\begin{aligned}\lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}}A &= \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}}\frac{n^{m+1}}{m+1}\\ &= \frac{b^{m+1}}{m+1}\end{aligned}$$

We know from lemma 2 that there exists a $C \in \mathbb{R}$ such that

$$|B| \leq |C\,n^m|$$

It follows that

$$\left|\frac{b^{m+1}}{n^{m+1}}B\right| \leq \left|\frac{b^{m+1}}{n^{m+1}}Cn^m\right|$$

We also know that

$$\lim_{n\to\infty}\frac{b^{m+1}}{n^{m+1}}Cn^m = \lim_{n\to\infty}\frac{Cb^{m+1}}{n} = 0$$

Thus, by the pinching theorem:

$$\lim_{n\to\infty}\frac{b^{m+1}}{n^{m+1}}B = 0$$

This gives us

$$\begin{aligned}
\lim_{n\to\infty}\frac{b^{m+1}}{n^{m+1}}\sum_{k=1}^{n}k^m &= \lim_{n\to\infty}\frac{b^{m+1}}{n^{m+1}}(A+B) \\
&= \lim_{n\to\infty}\frac{b^{m+1}}{n^{m+1}}A + \lim_{n\to\infty}\frac{b^{m+1}}{n^{m+1}}B \\
&= \frac{b^{m+1}}{m+1} + 0 \\
&= \frac{b^{m+1}}{m+1}
\end{aligned}$$

□

Theorem 1. *Let $m \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^m$. Then*

$$\int_0^b f(x)\,dx = \frac{b^{m+1}}{m+1}$$

Proof. We can form an equation using the Riemann integral.

$$\begin{aligned}
\int_0^b f(x)\,dx &= \lim_{n\to\infty}\frac{b-0}{n}\sum_{k=1}^{n}f(0+\frac{k(b-0)}{n}) \\
&= \lim_{n\to\infty}\frac{b}{n}\sum_{k=1}^{n}\left(\frac{kb}{n}\right)^m \\
&= \lim_{n\to\infty}\frac{b^{m+1}}{n^{m+1}}\sum_{k=1}^{n}k^m
\end{aligned}$$

We can substitute the result from lemma 3, which gives us

$$\int_0^b f(x)\,dx = \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^{n} k^m$$
$$= \frac{b^{m+1}}{m+1}$$

$\square$