Rational numbers

Andrew Taylor

March 12 2022

Definition 1. *Let A and B be sets. The Cartesian product of A and B, denoted $A \times B$, is the set*

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Definition 2. *Let X be a set. A relation R on X is a subset of $X \times X$.*

If $(a, b) \in R$, we say "a is related to b", and write $a \sim b$.

Definition 3. *Let X be a set. An equivalence relation R on X is a relation such that*

1. *For all $a \in X$, $(a, a) \in R$*
2. *For all $a, b \in X$, if $(a, b) \in R$, then $(b, a) \in R$*
3. *For all $a, b, c \in X$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$*

If $(a, b) \in R$, we say "a is equivalent to b", and write $a \sim b$.

Definition 4. *Let X be a set and let R be an equivalence relation on X. Let $a \in X$. The equivalence class of a, denoted $C(a)$, is defined by*

$$C(a) = \{b \in X \mid b \sim a\}$$

Definition 5. *Let $F = \{(a, b) \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$. Let $(a, b), (c, d) \in X$.*

We define the relation $\sim$ on F as $(a, b) \sim (c, d)$ if $ad = bc$.

Theorem 1. *The relation $\sim$ is an equivalence relation on F.*

Proof. Let $(a, b) \in F$. We know that $(a, b) \sim (a, b)$ because $a * b = b * a$.

Let $(a, b), (c, d) \in F$ such that $(a, b) \sim (c, d)$. By definition, $ad = bc$. Remember that $=$ is an equivalence relation on the integers. By the symmetric property, $bc = ad$. Since multiplication is commutative, we can write, $cb = da$. This is the equation we want. The equation $cb = da$ tells us that $(c, d) \sim (a, b)$.

Let $(a, b), (c, d), (e, f) \in F$ such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.

Then $ad = bc$ (1) and $cf = de$ (2).

We can multiply both sides of equation (1) by f.

$$adf = bcf$$

Now we can substitute de for cf in the above equation.

$$adf = bde$$

We know that d is nonzero by definition, so we can divide both sides by d.

$$af = be$$

Thus $(a, b) \sim (e, f)$.

The relation $\sim$ satisfies the reflexive property, the symmetric property, and the transitive property. Therefore $\sim$ is an equivalence relation on F. $\square$

Definition 6. *We define the rational numbers $\mathbb{Q}$ as the set of equivalence classes in F determined by the equivalence relation $\sim$.*

$$\mathbb{Q} = \{C((a, b)) \mid (a, b) \in F\}$$