Repeating decimals

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Definition 1. *A terminating decimal is a rational number that has a finite number of nonzero digits.*

Example 1. *The number 1.125 is a terminating decimal.*

$$1.125 = \frac{10}{8} = \frac{5}{4}$$

Definition 2. *A repeating decimal is a rational number whose decimal representation repeats in regular cycles.*

Example 2. *The fraction* $\frac{1}{3}$ *is a repeating decimal.*

$$\frac{1}{3} = 0.33333... = 0.\overline{3}$$

Example 3. *The number* $\frac{1}{7}$ *is a repeating decimal.*

$$\frac{1}{7} = 0.142857142857... = 0.\overline{142857}$$

Theorem 1. *Every rational number can be written as either a terminating decimal or a repeating decimal.*

Proof. Let $\frac{a}{b}$ be a rational number. The division algorithm lets us write the equation

$$a = bq + r$$

for a unique pair of integers q and r with $0 \leq r < b$.

We can apply the division algorithm repeatedly, and each time we do, we will get a remainder r from the set {0, 1, 2, ... b-1}. Since this set of possible remainders has b elements, we are guaranteed after b applications of the division algorithm to get a cycle or the remainder 0. If we get a cycle that does not end with zero, we have a repeating decimal. If we get zero, we have a terminating decimal. Therefore a rational number is either a terminating decimal or a repeating decimal. □

Problem 1. *Show that 0.136136136... is a rational number.*

Proof. Let $x = 0.136136136...$

Then $1000x = 136.136136....$

Subtracting, we get:

$$\begin{aligned} 1000x - x &= 136 \\ 999x &= 136 \\ x &= \frac{136}{999} \end{aligned}$$

Therefore $x = \dfrac{136}{999}$ and x is a rational number. $\square$