1 The derivative

Definition 1. *Let f be a real-valued function and let $x_0 \in \mathbb{R}$. The function f is differentiable at x_0 if and only if the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. When the limit exists, we denote this limit by $f'(x_0)$.

Definition 2. *Let f be a real-valued function. The derivative of f is the function f' whose domain $D_{f'}$ is the set of all real numbers on which f is differentiable and which assigns to each $x \in D_{f'}$ the number*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let $y = f(x)$. Another notation for the derivative is

$$f'(x) = \frac{dy}{dx} = \frac{df}{dx}$$

Theorem 1. *Let $f(x) = x^n$. Then the derivative of f is*

$$f'(x) = nx^{n-1}$$

Proof.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{\binom{n}{0} x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + ... + \binom{n}{n} h^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{\binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + ... + \binom{n}{n} h^n}{h} \\
&= \lim_{h \to 0} \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + ... + \binom{n}{n} h^{n-1} \\
&= \binom{n}{1} x^{n-1} \\
&= nx^{n-1}
\end{aligned}$$

In the above proof we see that $\binom{n}{0} x^n = x^n$ cancels out with $-x^n$

In addition, after dividing by h, every term in the polynomial that has a coefficient of h goes to zero as h approaches zero.

We are left with the term $\binom{n}{0} x^{n-1} = nx^{n-1}$

Thus

$$f'(x) = \frac{df}{dx} = nx^{n-1}$$

$\square$