Area of a regular polygon

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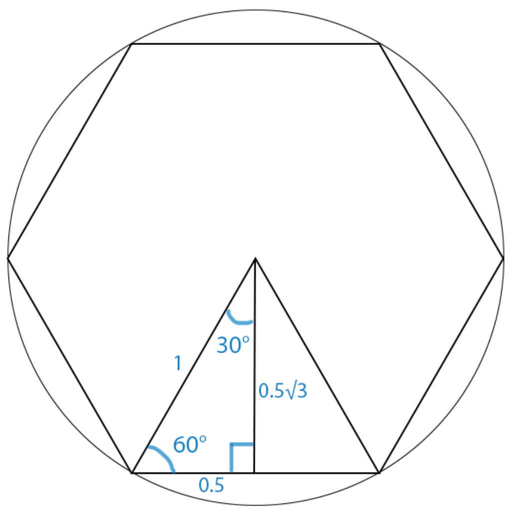

In the figure above, the area of the equilateral triangle is given by:

$$\begin{aligned} A_T &= \frac{1}{2}bh \\ &= \frac{1}{2}\left(2\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{6}\right) \\ &= \sin\frac{\pi}{6}\cos\frac{\pi}{6} \end{aligned}$$

The area of the regular hexagon is six times the area of the triangle.

$$\begin{aligned} A_H &= 6A_T \\ &= 6\sin\frac{\pi}{6}\cos\frac{\pi}{6} \\ &= \frac{3\sqrt{3}}{2} \end{aligned}$$

This formula generalizes for a regular n-sided polygon. The area of a regular n-sided polygon inscribed in a unit circle is given by:

$$A_P = n \sin \frac{\pi}{n} \cos \frac{\pi}{n} \tag{1}$$

As n approaches infinity, the area of the polygon approaches π.

$$\lim_{n \to \infty} n \sin \frac{\pi}{n} \cos \frac{\pi}{n} = \pi \tag{2}$$

When the circle has radius r, we get the equations:

$$A_P = \left(n \sin \frac{\pi}{n} \cos \frac{\pi}{n} \right) r^2 \tag{3}$$

$$\lim_{n \to \infty} \left(n \sin \frac{\pi}{n} \cos \frac{\pi}{n} \right) r^2 = \pi r^2 \tag{4}$$

These equations give us an approximation of π. For example:

$$\begin{aligned}
A_{10} &= 10 \sin \frac{\pi}{10} \cos \frac{\pi}{10} &&= 2.93892626146236546347 \\
A_{100} &= 10^2 \sin \frac{\pi}{10^2} \cos \frac{\pi}{10^2} &&= 3.13952597646566866629 \\
A_{1000} &= 10^3 \sin \frac{\pi}{10^3} \cos \frac{\pi}{10^3} &&= 3.14157198277947591336 \\
A_{10,000} &= 10^4 \sin \frac{\pi}{10^4} \cos \frac{\pi}{10^4} &&= 3.14159244688128591605 \\
A_{100,000} &= 10^5 \sin \frac{\pi}{10^5} \cos \frac{\pi}{10^5} &&= 3.14159265152270750221 \\
A_{1,000,000} &= 10^6 \sin \frac{\pi}{10^6} \cos \frac{\pi}{10^6} &&= 3.14159265356912253964
\end{aligned}$$

In conclusion, we have discovered equations one through four by looking at the special case of a regular hexagon inscribed in a unit circle.

These equations give us the area of a regular n-sided polygon inscribed in a circle, as well as an approximation of π.