The sum of the first n squares

Andrew Taylor

August 25 2022

In this paper we will find a formula for the sum of the first n squares:

$$\sum_{k=0}^{n} k^2$$

The trick to solving this problem is writing n^3 as a series.

We can do this using a well-known identity.

$$a_n - a_0 = \sum_{k=1}^{n} (a_k - a_{k-1})$$

In the equation above, a_n is a real-valued sequence.

The expression on the right is called a telescoping series, because it collapses like a folding telescope.

(When a folding telescope collapses, you are left with the front part and the back part joined together. When a telescoping series collapses, you are left with the first and last terms of the series.)

Now we can write n^3 as a telescoping series, using the sequence $a_n = n^3$.

$$\begin{aligned} n^3 &= n^3 - 0^3 \\ &= \sum_{k=0}^{n} \left(k^3 - (k-1)^3\right) \end{aligned}$$

Expanding the right side of the equation (we can calculate $(k-1)^3$ quickly using Pascal's triangle) we get:

$$
\begin{aligned}
n^3 &= \sum_{k=0}^{n} \left(k^3 - (k-1)^3\right) \\
&= \sum_{k=0}^{n} \left(k^3 - (k^3 - 3k^2 + 3k - 1)\right) \\
&= \sum_{k=0}^{n} \left(k^3 - k^3 + 3k^2 - 3k + 1\right) \\
&= \sum_{k=0}^{n} \left(3k^2 - 3k + 1\right)
\end{aligned}
$$

Using the laws for series, we can write each term on the right hand side as its own series, and factor out the constants.

$$
\begin{aligned}
n^3 &= \sum_{k=0}^{n} \left(3k^2 - 3k + 1)\right) \\
&= \sum_{k=0}^{n} 3k^2 - \sum_{k=0}^{n} 3k + \sum_{k=0}^{n} 1 \\
&= 3\sum_{k=0}^{n} k^2 - 3\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 \\
&= 3\sum_{k=0}^{n} k^2 - \frac{3n(n+1)}{2} + n
\end{aligned}
$$

Now we can solve for the series

$$
\sum_{k=0}^{n} k^2
$$

.

$$
\begin{aligned}
n^3 &= 3\sum_{k=0}^{n} k^2 - \frac{3n(n+1)}{2} + n \\
n^3 + \frac{3n(n+1)}{2} - n &= 3\sum_{k=0}^{n} k^2 \\
\frac{2n^3 + 3n(n+1) - 2n}{2} &= 3\sum_{k=0}^{n} k^2 \\
\frac{2n^3 + 3n(n+1) - 2n}{6} &= \sum_{k=0}^{n} k^2 \\
\frac{n(2n^2 + 3(n+1) - 2)}{6} &= \sum_{k=0}^{n} k^2 \\
\frac{n(2n^2 + 3n + 1)}{6} &= \sum_{k=0}^{n} k^2 \\
\frac{n(n+1)(2n+1)}{6} &= \sum_{k=0}^{n} k^2
\end{aligned}
$$

After factoring the lefthand side, we arrive at the identity

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

In summary, we solved this problem by using a trick: we wrote n^3 as a telescoping series.

$$
\begin{aligned}
n^3 &= n^3 - 0^3 \\
&= \sum_{k=0}^{n} \left(k^3 - (k-1)^3\right)
\end{aligned}
$$

Expanding the right hand side of the equation, we get the term we are looking for.

$$n^3 = \sum_{k=0}^{n} \left(3k^2 - 3k + 1\right)$$

The term we are looking for is

$$\sum_{k=0}^{n} k^2$$

Now we just have to solve for this term.

Doing so gives us the identity

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$