Integrating a polynomial

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Lemma 1. *Let n and m be natural numbers. Then*

$$\sum_{k=1}^{n} k^m = \frac{n^{m+1} + \sum_{k=1}^{n}\left(\binom{m+1}{2}k^{m-1}(-1)^2 + \binom{m+1}{3}k^{m-2}(-1)^3 + ... + \binom{m+1}{m+1}k^0(-1)^{m+1}\right)}{m+1}$$

Proof. Consider the sum

$$\sum_{k=1}^{n}\left(k^{m+1} - (k-1)^{m+1}\right)$$

We know that this sum telescopes, which is to say, all but the first and last terms cancel.

$$\sum_{k=1}^{n}\left(k^{m+1} - (k-1)^{m+1}\right) = n^{m+1} - 0^{m+1} = n^{m+1}$$

We can expand the binomial $(k-1)^{m+1}$ and get

$$\begin{aligned}
n^{m+1} &= \sum_{k=1}^{n}\left(k^{m+1} - (k-1)^{m+1}\right) \\
&= \sum_{k=1}^{n}\left(k^{m+1} - \left(\binom{m+1}{0}k^{m+1}(-1)^0 + \binom{m+1}{1}k^{m}(-1)^1 + ... + \binom{m+1}{m+1}k^0(-1)^{m+1}\right)\right) \\
&= \sum_{k=1}^{n} -1\left(\binom{m+1}{1}k^{m}(-1)^1 + \binom{m+1}{2}k^{m-1}(-1)^2 + ... + \binom{m+1}{m+1}k^0(-1)^{m+1}\right)
\end{aligned}$$

We can solve the above equation for the term $\sum_{k=1}^{n} k^m$

$$\sum_{k=1}^{n}\binom{m+1}{1}k^m = n^{m+1} + \sum_{k=1}^{n}\left(\binom{m+1}{2}k^{m-1}(-1)^2 + ... + \binom{m+1}{m+1}k^0(-1)^{m+1}\right)$$

We can divide both sides by $m+1$.

$$\sum_{k=1}^{n} k^m = \frac{n^{m+1} + \sum_{k=1}^{n}\left(\binom{m+1}{2}k^{m-1}(-1)^2 + \binom{m+1}{3}k^{m-2}(-1)^3 + ... + \binom{m+1}{m+1}k^0(-1)^{m+1}\right)}{m+1}$$

$\square$

Lemma 2. *Let n and m be natural numbers. Let b be a real number.*

$$\lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^{n} k^m = \frac{b^{m+1}}{m+1}$$

Proof. Define A and B as

$$A = \frac{n^{m+1}}{m+1}$$
$$B = \frac{1}{m+1}\sum_{k=1}^{n}\left(\binom{m+1}{2}k^{m-1}(-1)^2 + \binom{m+1}{3}k^{m-2}(-1)^3 + ... + \binom{m+1}{m+1}k^0(-1)^{m+1}\right)$$

Then

$$\begin{aligned}\lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^{n} k^m &= \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}}(A+B) \\ &= \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}}A + \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}}B\end{aligned}$$

Now we can solve each limit separately.

$$\begin{aligned}\lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}}A &= \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}}\frac{n^{m+1}}{m+1} \\ &= \frac{b^{m+1}}{m+1}\end{aligned}$$

Let C be the highest magnitude coefficient of the polynomial in B. We know that B is bounded above and below by $\pm(Cn^m)$.

It follows that $\frac{b^{m+1}}{n^{m+1}}B$ is bounded above and below by $\pm\frac{b^{m+1}}{n^{m+1}}Cn^m$

We also know that

$$\lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} Cn^m = \lim_{n\to\infty} \frac{Cb^{m+1}}{n} = 0$$

Thus, by the pinching theorem:

$$\lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} B = 0$$

This gives us

$$\begin{aligned}
\lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^{n} k^m &= \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}}(A+B) \\
&= \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} A + \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} B \\
&= \frac{b^{m+1}}{m+1} + 0 \\
&= \frac{b^{m+1}}{m+1}
\end{aligned}$$

□

Theorem 1. *Let $m \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^m$. Then*

$$\int_0^b f(x)\,dx = \frac{b^{m+1}}{m+1}$$

Proof. We can form an equation using the Riemann integral.

$$\begin{aligned}
\int_0^b f(x)\,dx &= \lim_{n\to\infty} \frac{b-0}{n} \sum_{k=1}^{n} f(0 + \frac{k(b-0)}{n}) \\
&= \lim_{n\to\infty} \frac{b}{n} \sum_{k=1}^{n} \left(\frac{kb}{n}\right)^m \\
&= \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^{n} k^m
\end{aligned}$$

We can substitute the result from lemma 2, which gives us

$$\begin{aligned}
\int_0^b f(x)\,dx &= \lim_{n\to\infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^{n} k^m \\
&= \frac{b^{m+1}}{m+1}
\end{aligned}$$

□