

Integrating a polynomial

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Lemma 1. *Let n and m be natural numbers. Then*

$$\sum_{k=1}^n k^m = \frac{n^{m+1} + \sum_{k=1}^n \left(\binom{m+1}{2} k^{m-1} (-1)^2 + \binom{m+1}{3} k^{m-2} (-1)^3 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)}{m+1}$$

Proof. Consider the sum

$$\sum_{k=1}^n (k^{m+1} - (k-1)^{m+1})$$

We know that this sum telescopes, which is to say, all but the first and last terms cancel.

$$\sum_{k=1}^n (k^{m+1} - (k-1)^{m+1}) = n^{m+1} - 0^{m+1} = n^{m+1}$$

We can expand the binomial $(k-1)^{m+1}$ and get

$$\begin{aligned} n^{m+1} &= \sum_{k=1}^n (k^{m+1} - (k-1)^{m+1}) \\ &= \sum_{k=1}^n \left(k^{m+1} - \left(\binom{m+1}{0} k^{m+1} (-1)^0 + \binom{m+1}{1} k^m (-1)^1 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right) \right) \\ &= \sum_{k=1}^n -1 \left(\binom{m+1}{1} k^m (-1)^1 + \binom{m+1}{2} k^{m-1} (-1)^2 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right) \end{aligned}$$

We can solve the above equation for the term $\sum_{k=1}^n k^m$

$$\sum_{k=1}^n \binom{m+1}{1} k^m = n^{m+1} + \sum_{k=1}^n \left(\binom{m+1}{2} k^{m-1} (-1)^2 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)$$

We can divide both sides by $m + 1$.

$$\sum_{k=1}^n k^m = \frac{n^{m+1} + \sum_{k=1}^n \left(\binom{m+1}{2} k^{m-1} (-1)^2 + \binom{m+1}{3} k^{m-2} (-1)^3 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)}{m+1}$$

□

Lemma 2. *Let b_n be the sequence*

$$b_n = \frac{1}{m+1} \sum_{k=1}^n \left(\binom{m+1}{2} k^{m-1} (-1)^2 + \binom{m+1}{3} k^{m-2} (-1)^3 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)$$

for some $m \in \mathbb{N}$. Then there exists a $C \in \mathbb{R}$ such that

$$-C n^m \leq b_n \leq C n^m$$

Proof. Define the polynomial $P(k)$ as

$$P(k) = \binom{m+1}{2} k^{m-1} (-1)^2 + \binom{m+1}{3} k^{m-2} (-1)^3 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1}$$

Now we can write

$$b_n = \frac{1}{m+1} \sum_{k=1}^n P(k)$$

There exists a polynomial

$$Q(k) = C k^{m-1}$$

such that

$$|P(k)| \leq |Q(k)|$$

for some $C \in \mathbb{R}$.

Let c_n be the sequence defined by

$$c_n = \frac{1}{m+1} \sum_{k=1}^n Q(k)$$

We know that $|b_n| \leq |c_n|$ since $|P(k)| \leq |Q(k)|$

Furthermore we know that

$$\left| \sum_{k=1}^n Q(k) \right| \leq |Cn^m|$$

Thus

$$|c_n| \leq \left| \frac{Cn^m}{m+1} \right|$$

By the transitive property we get

$$\frac{-Cn^m}{m+1} \leq b_n \leq \frac{Cn^m}{m+1}$$

□

Lemma 3. *Let n and m be natural numbers. Let b be a real number.*

$$\lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^n k^m = \frac{b^{m+1}}{m+1}$$

Proof. Define A and B as

$$A = \frac{n^{m+1}}{m+1}$$

$$B = \frac{1}{m+1} \sum_{k=1}^n \left(\binom{m+1}{2} k^{m-1} (-1)^2 + \binom{m+1}{3} k^{m-2} (-1)^3 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)$$

Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^n k^m &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} (A + B) \\ &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} A + \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} B \end{aligned}$$

Now we can solve each limit separately.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} A &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \frac{n^{m+1}}{m+1} \\ &= \frac{b^{m+1}}{m+1} \end{aligned}$$

We know from lemma 2 that there exists a $C \in \mathbb{R}$ such that

$$|B| \leq |C n^m|$$

It follows that

$$\left| \frac{b^{m+1}}{n^{m+1}} B \right| \leq \left| \frac{b^{m+1}}{n^{m+1}} C n^m \right|$$

We also know that

$$\lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} C n^m = \lim_{n \rightarrow \infty} \frac{C b^{m+1}}{n} = 0$$

Thus, by the pinching theorem:

$$\lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} B = 0$$

This gives us

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^n k^m &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} (A + B) \\ &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} A + \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} B \\ &= \frac{b^{m+1}}{m+1} + 0 \\ &= \frac{b^{m+1}}{m+1} \end{aligned}$$

□

Theorem 1. Let $m \in \mathbb{N}$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^m$. Then

$$\int_0^b f(x) dx = \frac{b^{m+1}}{m+1}$$

Proof. We can form an equation using the Riemann integral.

$$\begin{aligned} \int_0^b f(x) dx &= \lim_{n \rightarrow \infty} \frac{b-0}{n} \sum_{k=1}^n f\left(0 + \frac{k(b-0)}{n}\right) \\ &= \lim_{n \rightarrow \infty} \frac{b}{n} \sum_{k=1}^n \left(\frac{kb}{n}\right)^m \\ &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^n k^m \end{aligned}$$

We can substitute the result from lemma 3, which gives us

$$\begin{aligned}\int_0^b f(x) dx &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^n k^m \\ &= \frac{b^{m+1}}{m+1}\end{aligned}$$

□