

Repeating decimals

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March 13 2022

Definition 1. *A terminating decimal is a rational number that has a finite number of nonzero digits.*

Example 1. *The number 1.125 is a terminating decimal.*

$$1.125 = \frac{10}{8} = \frac{5}{4}$$

Definition 2. *A repeating decimal is a rational number whose decimal representation repeats in regular cycles.*

Example 2. *The fraction $\frac{1}{3}$ is a repeating decimal.*

$$\frac{1}{3} = 0.33333... = 0.\overline{3}$$

Example 3. *The number $\frac{1}{7}$ is a repeating decimal.*

$$\frac{1}{7} = 0.142857142857... = 0.\overline{142857}$$

Theorem 1. *Every rational number can be written as either a terminating decimal or a repeating decimal.*

Proof. Let $\frac{a}{b}$ be a rational number. The division algorithm lets us write the equation

$$a = bq + r$$

for unique integers q and r .

We can apply the division algorithm repeatedly, and each time we do, we will get a remainder r from the set $\{0, 1, 2, \dots, b-1\}$. Since this set of possible remainders has b elements, we are guaranteed after b applications of the division algorithm to get a cycle or the remainder 0. If we get a cycle that does not end with zero, we have a repeating decimal. If we get zero, we have a terminating decimal. Therefore a rational number is either a terminating decimal or a repeating decimal. \square

Problem 1. *Show that $0.136136136\dots$ is a rational number.*

Proof. Let $x = 0.136136136\dots$

Then $1000x = 136.136136\dots$

Subtracting, we get:

$$1000x - x = 136$$

$$999x = 136$$

$$x = \frac{136}{999}$$

Therefore $x = \frac{136}{999}$ and x is a rational number.

□