Finding a formula for the sum of squares

Andrew Taylor

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1 Introduction

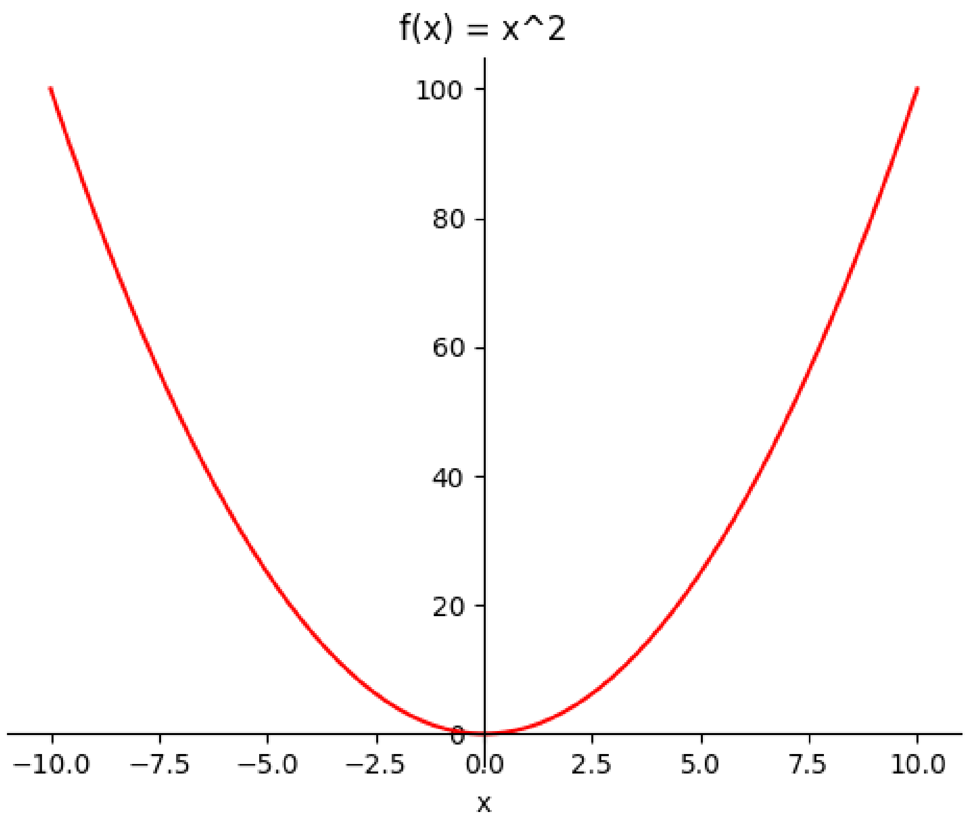
The goal of this paper is to derive a formula for the sum of squares. This means that we will discover a formula for the sum of squares, and prove that the formula is correct.

This paper will use many mathematical words, symbols, and ideas. We will often take time to explain and define these ideas.

2 Formulas

A formula is a function. A function is a correspondence between two sets X and Y such that each element in X is associated with exactly one element in Y.

On the next page there is a graph of a function.


f(x) = x^2
100
80
60
40
20
0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x

The graph in the previous page illustrates the function $f(x) = x^2$.

One thing to notice in the graph is that the x and y axes both visualize the real numbers. The x axis visualizes the interval $[-10, 10]$ and the y axis visualizes the interval $[0, 100]$.

The plot of the graph visualizes the function $f(x)$.

The function $f(x)$ is a correspondence between the real numbers, so we can write, symbolically, that $f : \mathbb{R} \to \mathbb{R}$ defined by the formula $f(x) = x^2$.

The symbol $\mathbb{R}$ is a symbol that represents the real numbers.

If we travel along the x axis left or right, and stop at any point, we can see that there is one function value corresponding to that x coordinate.

This is visual evidence that $f(x)$ is a function.

3 Sequences

A sequence is a list of numbers. A sequence can be finite or infinite.

One example of a sequence is the first ten positive squares.

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100$$

We can write the sequence of squares symbolically as:

$$a_n = n^2$$

The subscript n can be any natural number.

Thus $a_{10} = 10^2 = 100$ is the tenth term in the sequence.

A sequence is a kind of function. For every input we get exactly one output. For every index in the sequence, we get a value.

4 Summation

Let's write out the sum of the first ten squares.

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$$

This takes a lot of time and effort. What if we wanted to sum the first one hundred squares? Is there a concise way of writing this?

We can write this concisely using summation.

$$\sum_{k=1}^{100} k^2$$

The summation operator $\sum$ gets its symbol from the Greek letter sigma. The summation operator accepts a sequence, a lower bound, and an upper bound, and sums every term in the sequence from the lower bound to the upper bound, up to and including the upper bound.

$$\sum_{k=1}^{10} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$$

In the above equation, we see a concise expression on the left hand side, and an expanded expression on the right hand side.

The summation operator is called a trinary operator because it accepts three operands: a sequence, a lower bound, and an upper bound.

$$\sum_{k=L}^{M} a_k$$

In the expression above, a_k is the sequence, L is the lower bound of summation, and M is the upper bound of summation. The variable k is the index of summation.

We can also write the operation using a second notation, shown below.

$$\sum(a_k, L, M)$$

Summation is an operation that adds the terms of a sequence or a subsequence. It can add all of the terms of a sequence, or some of the terms of a sequence. It can add a finite number of terms, or an infinite number of terms. When we use summation on an infinite number of terms, we call it a series.

We are looking for a formula that sums the first n squares. We can write the sum of the first n squares concisely in this way:

$$\sum_{k=1}^{n} k^2$$

5 The sum of the first n natural numbers

Lemma 1.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Proof. Let's write out the sum of the first n natural numbers.

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + n$$

By rearranging terms, we can write the equation in two ways.

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + n$$
$$\sum_{k=1}^{n} k = n + (n-1) + (n-2) + ... + 1$$

Now let's add the two equations, so that 1 is paired with n, 2 is paired with (n-1), 3 is paired with (n-2), and so on.

$$2 * \sum_{k=1}^{n} k = (1+n) + (2+(n-1)) + (3+(n-2)) + ... + (n+1)$$

This gives us n pairs of numbers that sum to $n+1$.

$$2 * \sum_{k=1}^{n} k = n(n+1)$$

Now let's divide both sides of the equation by 2.

$$\sum_{k=1}^{n} k = \frac{n(1+n)}{2}$$

□

6 Telescoping sums

Lemma 2. *Let a_k be a sequence of real numbers. Then*

$$\sum_{k=1}^{n}(a_k - a_{k-1}) = a_n - a_0$$

Proof. We can expand the sum, and rearrange the terms, so that every term but the first and the last cancel out.

$$\begin{aligned}\sum_{k=1}^{n}(a_k - a_{k-1}) &= (a_n - a_{n-1}) + (a_{n-1} + a_{n-2}) + ... + (a_1 - a_0) \\ &= a_n + (-a_{n-1} + a_{n-1}) + (-a_{n-2} + a_{n-2}) + ... + (-a_1 + a_1) - a_0 \\ &= a_n - a_0\end{aligned}$$

□

7 The sum of the first n squares

Theorem 1. *Let $a_k = k^2$ be a sequence of real numbers. Then*

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof. Let $b_k = k^3$ be a sequence of real numbers.

From lemma 2 we know that

$$\begin{aligned}\sum_{k=1}^{n}(b_k - b_{k-1}) &= b_n - b_0 \\ &= n^3 - 0^3 \\ &= n^3\end{aligned}$$

Thus

$$\sum_{k=1}^{n}(b_k - b_{k-1}) = n^3 \tag{1}$$

We also know that

$$
\begin{aligned}
\sum_{k=1}^{n}(b_k - b_{k-1}) &= \sum_{k=1}^{n}[k^3 - (k-1)^3] \\
&= \sum_{k=1}^{n}[k^3 - (k^3 - 3k^2 + 3k - 1)] \\
&= \sum_{k=1}^{n}(3k^2 - 3k + 1) \\
&= 3 * \sum_{k=1}^{n} k^2 - 3 * \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1
\end{aligned}
$$

Substituting our result from lemma 1 into the above equation, we get

$$
\begin{aligned}
\sum_{k=1}^{n}(b_k - b_{k-1}) &= 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= 3\sum_{k=1}^{n} k^2 - \frac{3n(n+1)}{2} + n \\
&= 3\sum_{k=1}^{n} k^2 - \frac{3n^2 + 3n}{2} + \frac{2n}{2} \\
&= 3\sum_{k=1}^{n} k^2 - \frac{3n^2 + n}{2}
\end{aligned}
$$

Thus

$$
\sum_{k=1}^{n}(b_k - b_{k-1}) = 3\sum_{k=1}^{n} k^2 - \frac{3n^2 + n}{2} \tag{2}
$$

We can now set equations 1 and 2 equal to each other.

$$
3\sum_{k=1}^{n} k^2 - \frac{3n^2 + n}{2} = n^3
$$

Adding to both sides of the equation, and simplifying, we get

$$
\begin{aligned}
3\sum_{k=1}^{n} k^2 &= n^3 + \frac{3n^2 + n}{2} \\
&= \frac{2n^3 + 3n^2 + n}{2}
\end{aligned}
$$

Dividing both sides of the equation by 3, we get

$$
\begin{aligned}
\sum_{k=1}^{n} k^2 &= \frac{2n^3 + 3n^2 + n}{6} \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$

$\square$