

The quadratic formula

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We can derive the quadratic formula by solving the quadratic equation $ax^2 + bx + c = 0$ for x .

We solve this equation by completing the square.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} &= \left(\frac{b}{2a}\right)^2 \\ \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} &= \left(\frac{b}{2a}\right)^2 \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

This technique (completing the square) was discovered by the Arabic mathematician al-Khwarizmi in the early 9th century CE.

You can see how we use the rules of algebra to get a square on the lefthand side of the equation. When we take the square root of both sides, we are left with only one term that has the variable x .

Our equations reveal the journey of isolating the variable x , so that only one term contains the variable x .

The expression $b^2 - 4ac$ is called the discriminant. When $b^2 - 4ac$ is negative, the solutions are imaginary numbers. When $b^2 - 4ac$ is nonnegative, the

solutions are real numbers.

The only constraint on this formula is that $a \neq 0$. When $a = 0$ the equation is linear, not quadratic.

Thus we have discovered the quadratic formula by using algebraic operations on the quadratic equation $ax^2 + bx + c = 0$ to isolate the variable x . In other words, we have discovered the quadratic formula by completing the square.

The quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$