Limits

Andrew Taylor

February 27 2022

1 Limits

Definition 1. *Let a_n be a sequence. The limit of a_n is A if and only if for all $\epsilon > 0$ there exists a natural number N such that*

$$|a_n - A| < \epsilon$$

for all $n > N$.

We write this as

$$\lim_{n \to \infty} a_n = A$$

and say the sequence a_n converges to A.

Definition 2. *Let f be a real-valued function and let a be a real number. The limit of f as x approaches a is L if and only if for all $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x) - L| < \epsilon$$

for all

$$0 < |x - a| < \delta$$

We write this as

$$\lim_{x \to a} f(x) = L$$

Problem 1. *Let $f : \mathbb{R} \setminus 0 \to \mathbb{R}$ be defined by*

$$f(x) = \frac{x}{x}$$

Find the limit of f as x approaches zero, if it exists.

Proof. Let $\epsilon > 0$ and $\delta = 10$. Let x be a real number such that $0 < |x| < \delta$.

$$
\begin{aligned}
|f(x) - 1| &= |\frac{x}{x} - 1| \\
&= |1 - 1| \\
&= 0
\end{aligned}
$$

As zero is always less than ϵ, the limit of f as x approaches zero is one.

$$\lim_{x \to 0} f(x) = 1$$

In this example we chose $\delta = 10$, but we can choose any $\delta > 0$ (no matter how small or big) because of the betweenness property of real numbers.

$\square$