Finding the derivative of a polynomial function

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February 26 2022

1 The derivative

Definition 1. *Let* f *be a real-valued function and let* $x_0 \in \mathbb{R}$*. The function* f *is differentiable at* x_0 *if and only if the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. When the limit exists, we denote this limit by $f'(x_0)$*.*

Definition 2. *Let* f *be a real-valued function. The derivative of* f *is the function* f' *whose domain* $D_{f'}$ *is the set of all real numbers on which* f *is differentiable and which assigns to each* $x \in D_{f'}$ *the number*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let $y = f(x)$*. Another notation for the derivative is*

$$f'(x) = \frac{dy}{dx} = \frac{df}{dx}$$

Theorem 1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = x^n$$

for some natural number n*. Then*

$$f'(x) = nx^{n-1}$$

Proof.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + ... + \binom{n}{n}h^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + ... + \binom{n}{n}h^n}{h} \\
&= \lim_{h\to 0} \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + ... + \binom{n}{n}h^{n-1} \\
&= \binom{n}{1}x^{n-1} \\
&= nx^{n-1}
\end{aligned}
$$

In the above proof we see that $\binom{n}{0}x^n = x^n$ cancels out with $-x^n$

In addition, after dividing by h, every term in the polynomial that has a coefficient of h goes to zero as h approaches zero.

We are left with the term $\binom{n}{0}x^{n-1} = nx^{n-1}$

Thus

$$f'(x) = \frac{df}{dx} = nx^{n-1}$$

The function f' has $\mathbb{R}$ as its domain and $\mathbb{R}$ as its range.

$\square$