1 Find a formula for $\sum_{k=1}^{n} k$

First let's look at the binomial expansion of $(k-1)^2$:

$$(k-1)^2 = k^2 - 2k + 1 \tag{1}$$

Let $a_k = k^2$. Let's create the telescoping sequence $t_k = a_k - a_{k-1}$.

$$\begin{aligned} t_k &= a_k - a_{k-1} \\ &= k^2 - (k-1)^2 \\ &= k^2 - (k^2 - 2k + 1) \\ &= 2k - 1 \end{aligned}$$

Thus we have the equation:

$$t_k = 2k - 1 \tag{2}$$

We can sum both sides of equation (2) to get a telescoping series.

$$\sum_{k=1}^{n} t_k = \sum_{k=1}^{n} (2k-1) \tag{3}$$

All terms in the series $\sum_{k=1}^{n} t_k$ cancel out, except for the first and the last:

$$\begin{aligned} \sum_{k=1}^{n} t_k &= \sum_{k=1}^{n} (a_k - a_{k-1}) \\ &= a_n - a_0 \\ &= n^2 - 0^2 \\ &= n^2 \end{aligned}$$

Now we have the equation:

$$\sum_{k=1}^{n} t_k = n^2 \tag{4}$$

We can perform substitution in equation (3)

$$\begin{aligned} n^2 &= \sum_{k=1}^{n} (2k-1) \\ &= \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 \\ &= 2 * \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \end{aligned}$$

Thus we have the equation:

$$n^2 = 2 * \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \tag{5}$$

We can solve equation (5) for $\sum_{k=1}^{n} k$.

$$\begin{aligned} 2 * \sum_{k=1}^{n} k &= n^2 + \sum_{k=1}^{n} 1 \\ &= n^2 + n \\ &= n(n+1) \end{aligned}$$

And we get the formula:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

2 Find a formula for $\sum_{k=1}^{n} k^2$

We can apply the same technique as before.

Let's look at the binomial expansion of $(k-1)^3$.

$$\begin{aligned} (k-1)^3 &= (k-1)(k^2 - 2k + 1) \\ &= k(k^2 - 2k + 1) - (k^2 - 2k + 1) \\ &= k^3 - 2k^2 + k - k^2 + 2k - 1 \\ &= k^3 - 3k^2 + 3k - 1 \end{aligned}$$

The coefficients are what we expect from Pascal's triangle.

This gives us the equation:

$$(k-1)^3 = k^3 - 3k^2 + 3k - 1 \tag{7}$$

Now let's look at the general term of a telescoping series.

Let $t_k = k^3 - (k-1)^3$

$$\begin{aligned} t_k &= k^3 - (k-1)^3 \\ &= k^3 - (k^3 - 3k^2 + 3k - 1) \\ &= 3k^2 - 3k + 1 \end{aligned}$$

Thus we have the equation:

$$t_k = 3k^2 - 3k + 1 \tag{8}$$

The left hand side of the equation is the term of a sequence. The right hand side is a polynomial. We can take the summation of both sides of the equation.

$$\sum_{k=1}^{n} t_k = \sum_{k=1}^{n} (3k^2 - 3k + 1) \tag{9}$$

The left hand side of the equation telescopes.

$$\begin{aligned} n^3 &= \sum_{k=1}^{n} (3k^2 - 3k + 1) \\ &= \sum_{k=1}^{n} (3k^2) - \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 1 \\ &= 3 * \sum_{k=1}^{n} k^2 - 3 * \sum_{k=1}^{n} k + n \end{aligned}$$

We can simplify this equation more using the result from section 1.

$$n^3 = 3 * \sum_{k=1}^{n} k^2 - \frac{3n(n+1)}{2} + n$$

Rearranging we get:

$$3 * \sum_{k=1}^{n} k^2 = n^3 - n + \frac{3n(n+1)}{2}$$

Let's give the right hand side a common denominator, and simplify.

$$\begin{aligned} 3 * \sum_{k=1}^{n} k^2 &= \frac{2n^3 - 2n + 3n(n+1)}{2} \\ &= \frac{2n^3 - 2n + 3n^2 + 3n}{2} \\ &= \frac{2n^3 + 3n^2 + n}{2} \\ &= \frac{n(2n^2 + 3n + 1)}{2} \\ &= \frac{n(n+1)(2n+1)}{2} \end{aligned}$$

Now we can divide both sides of the equation by 3.

This gives us the formula for the sum of squares:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{10}$$

The same technique we used in the first section (telescoping) helps us find a formula for the sum of squares. We can use this technique to find a formula for the sum of cubes, fourths, and so on.