

Integrating a polynomial

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Lemma 1. *Let n and m be natural numbers. Then*

$$\sum_{k=1}^n k^m = \frac{n^{m+1} + \sum_{k=1}^n \left(\binom{m+1}{2} k^{m-1} (-1)^2 + \binom{m+1}{3} k^{m-2} (-1)^3 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)}{m+1}$$

Proof. Consider the sum

$$\sum_{k=1}^n (k^{m+1} - (k-1)^{m+1})$$

We know that this sum telescopes, which is to say, all but the first and last terms cancel.

$$\sum_{k=1}^n (k^{m+1} - (k-1)^{m+1}) = n^{m+1} - 0^{m+1} = n^{m+1}$$

We can expand the binomial $(k-1)^{m+1}$ and get

$$\begin{aligned} n^{m+1} &= \sum_{k=1}^n (k^{m+1} - (k-1)^{m+1}) \\ &= \sum_{k=1}^n \left(k^{m+1} - \left(\binom{m+1}{0} k^{m+1} (-1)^0 + \binom{m+1}{1} k^m (-1)^1 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right) \right) \\ &= \sum_{k=1}^n -1 \left(\binom{m+1}{1} k^m (-1)^1 + \binom{m+1}{2} k^{m-1} (-1)^2 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right) \end{aligned}$$

We can solve the above equation for the term $\sum_{k=1}^n k^m$

$$\sum_{k=1}^n \binom{m+1}{1} k^m = n^{m+1} + \sum_{k=1}^n \left(\binom{m+1}{2} k^{m-1} (-1)^2 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)$$

We can divide both sides by $m + 1$.

$$\sum_{k=1}^n k^m = \frac{n^{m+1} + \sum_{k=1}^n \left(\binom{m+1}{2} k^{m-1} (-1)^2 + \binom{m+1}{3} k^{m-2} (-1)^3 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)}{m+1}$$

□

Lemma 2. *Let n and m be natural numbers. Let b be a real number.*

$$\lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^n k^m = \frac{b^{m+1}}{m+1}$$

Proof. Define A and B as

$$A = \frac{n^{m+1}}{m+1}$$

$$B = \frac{1}{m+1} \sum_{k=1}^n \left(\binom{m+1}{2} k^{m-1} (-1)^2 + \binom{m+1}{3} k^{m-2} (-1)^3 + \dots + \binom{m+1}{m+1} k^0 (-1)^{m+1} \right)$$

Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^n k^m &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} (A + B) \\ &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} A + \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} B \end{aligned}$$

Now we can solve each limit separately.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} A &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \frac{n^{m+1}}{m+1} \\ &= \frac{b^{m+1}}{m+1} \end{aligned}$$

We know that B is bounded above by $|c * n^m|$ for some $c \in \mathbb{R}$. (Let c be the highest magnitude coefficient of the polynomial in B.)

$$\text{It follows that } \frac{b^{m+1}}{n^{m+1}} B \text{ is bounded above by } \left| \frac{c * n^m * b^{m+1}}{n^{m+1}} \right|.$$

We also know that

$$\lim_{n \rightarrow \infty} \left| \frac{c * n^m * b^{m+1}}{n^{m+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{c * b^{m+1}}{n} \right| = 0$$

Thus, by the pinching theorem:

$$\lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} B = 0$$

This gives us

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^n k^m &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} (A + B) \\ &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} A + \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} B \\ &= \frac{b^{m+1}}{m+1} + 0 \\ &= \frac{b^{m+1}}{m+1} \end{aligned}$$

□

Theorem 1. Let $m \in \mathbb{N}$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^m$. Then

$$\int_0^b f(x) dx = \frac{b^{m+1}}{m+1}$$

Proof. We can form an equation using the Riemann integral.

$$\begin{aligned} \int_0^b f(x) dx &= \lim_{n \rightarrow \infty} \frac{b-0}{n} \sum_{k=1}^n f\left(0 + \frac{k(b-0)}{n}\right) \\ &= \lim_{n \rightarrow \infty} \frac{b}{n} \sum_{k=1}^n \left(\frac{kb}{n}\right)^m \\ &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^n k^m \end{aligned}$$

We can substitute the result from lemma 2, which gives us

$$\begin{aligned} \int_0^b f(x) dx &= \lim_{n \rightarrow \infty} \frac{b^{m+1}}{n^{m+1}} \sum_{k=1}^n k^m \\ &= \frac{b^{m+1}}{m+1} \end{aligned}$$

□