Problem 15: The polynomial

$$P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$$

has 34 roots of the form

$$z_k = r_k[\cos(2\pi\alpha_k) + i\sin(2\pi\alpha_k)],$$

$k = 1, 2, \ldots, 34$, with

$$0 < \alpha_1 \le \alpha_2 \le \alpha_3 \le \cdots \le \alpha_{34} < 1$$

and $r_k > 0$. Find $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$. (Source: AIME)

First notice that the primitive 17th roots of unity are all roots of $P(x)$. This accounts for 16 of the 34 roots of $P(x)$.

We can quickly demonstrate this.

Let ω be a primitive 17th root of unity. Then $1 + \omega + \omega^2 + \cdots + \omega^{17} = 1$, and $(1 + \omega + \omega^2 + \cdots + \omega^{17})^2 - \omega^{17} = 0$.

Now observe that

$$\begin{aligned} P(x) &= (1 + x + x^2 + \cdots + x^{17})^2 - x^{17} \\ &= \left(\frac{x^{18} - 1}{x - 1}\right)^2 - x^{17} \\ &= \frac{x^{36} - 2x^{18} + 1}{(x-1)^2} - x^{17} \\ &= \frac{x^{36} - 2x^{18} + 1}{x^2 - 2x + 1} - \frac{x^{19} - 2x^{18} + x^{17}}{x^2 - 2x + 1} \\ &= \frac{x^{36} - x^{19} - x^{17} + 1}{(x-1)^2} \end{aligned}$$

The primitive 19th roots of unity are also roots of $P(x)$. Let's demonstrate this.

Let ω be one of the primitive 19th roots of unity. Then

$$\begin{aligned} P(\omega) &= \frac{\omega^{36} - \omega^{19} - \omega^{17} + 1}{(\omega - 1)^2} \\ &= \frac{\omega^{17} - 1 - \omega^{17} + 1}{(\omega - 1)^2} \\ &= 0 \end{aligned}$$

There are 18 primitive 19th roots of unity. So we have found $16 + 18 = 34$ roots of $P(x)$. Note that there is no overlap between the primitive 19th roots of unity and the primitive 17th roots of unity, so we can be sure that we have found all 34 roots of $P(x)$.

Listing the roots of $P(x)$ in order, we get

$$e^{2\pi i/19}, e^{2\pi i/17}, e^{4\pi i/19}, e^{4\pi i/17}, e^{6\pi i/19}$$

Thus

$$\begin{aligned}
\alpha_1 &= \frac{1}{19} \\
\alpha_2 &= \frac{1}{17} \\
\alpha_3 &= \frac{2}{19} \\
\alpha_4 &= \frac{2}{17} \\
\alpha_5 &= \frac{3}{19}
\end{aligned}$$

We can give these fractions a common denominator, and add them.

$$\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 &= \frac{1}{19} + \frac{1}{17} + \frac{2}{19} + \frac{2}{17} + \frac{3}{19} \\
&= \frac{17}{323} + \frac{19}{323} + \frac{34}{323} + \frac{38}{323} + \frac{51}{323} \\
&= \boxed{\frac{159}{323}}
\end{aligned}$$

Finally, we get

$$\boxed{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = \frac{159}{323}}$$