Problem 4: Let $c > 0$ be a real number. Prove that $\sum_{n=1}^{\infty} \frac{n!}{(cn)^n}$ converges if $c > \frac{1}{e}$ and diverges if $c < \frac{1}{e}$.

We can use the Ratio Test.

$$\begin{aligned} r &= \lim_{n\to\infty} \frac{\frac{(n+1)!}{(c(n+1))^{n+1}}}{\frac{n!}{(cn)^n}} \\ &= \lim_{n\to\infty} \frac{n^n}{c(n+1)^n} \\ &= \frac{1}{c} \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \end{aligned}$$

Let $y = \left(\frac{n}{n+1}\right)^n$. Then

$$\begin{aligned} \lim_{n\to\infty} \log y &= \lim_{n\to\infty} n \log \frac{n}{n+1} \\ &= \lim_{n\to\infty} \frac{\log \frac{n}{n+1}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{\frac{n+1}{n}\left(\frac{n+1-n}{(n+1)^2}\right)}{-n^{-2}} \\ &= \lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{-\frac{1}{n^2}} \\ &= \lim_{n\to\infty} -\frac{n}{n+1} \\ &= -1 \end{aligned}$$

Now

$$\lim_{n\to\infty} y = \lim_{n\to\infty} e^{\log y} = e^{\lim_{n\to\infty} \log y} = e^{-1} = \frac{1}{e}$$

So

$$\begin{aligned} r &= \frac{1}{c} \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\ &= \frac{1}{ec} \end{aligned}$$

Now let's find the values of c for which $r = \frac{1}{ec} < 1$.

$$\frac{1}{ec} < 1 \iff \frac{1}{c} < e \iff c > \frac{1}{e}$$

Thus the series $\sum_{n=1}^{\infty} \frac{n!}{(cn)^n}$ converges for $c > \frac{1}{e}$ and diverges for $c < \frac{1}{e}$ by the Ratio Test.