Problem 14: Find the equation of the tangent line to $r = 3\cos 2\theta$ at $\theta = \dfrac{\pi}{3}$.

(Source: AoPS Calclulus)

In Problem 13, we computed the slope of the tangent line to be $\dfrac{7\sqrt{3}}{3}$. Now all that remains is to find a point on the tangent line, and give the equation of the tangent line in point-slope form.

We know that the tangent line intersects the graph of $r = 3\cos 2\theta$ at $\theta = \dfrac{\pi}{3}$, so we can compute the x and y coordinates of $r = 3\cos 2\theta$ at $\theta = \dfrac{\pi}{3}$, in order to find a point on the tangent line.

$$\begin{aligned} x &= r\cos\theta \\ &= 3\cos 2\theta\cos\theta \\ x\left(\frac{\pi}{3}\right) &= 3\cos\frac{2\pi}{3}\cos\frac{\pi}{3} \\ &= -\frac{3}{4} \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= 3\cos 2\theta\sin\theta \\ y\left(\frac{\pi}{3}\right) &= 3\cos\frac{2\pi}{3}\sin\frac{\pi}{3} \\ &= -\frac{3\sqrt{3}}{4} \end{aligned}$$

The tangent line intersects the graph of $r = 3\cos 2\theta$ at the point $(x, y) = \left(-\dfrac{3}{4}, -\dfrac{3\sqrt{3}}{4}\right)$.

Knowing the slope of the tangent line and a point on the tangent line, we can give the equation of the tangent line in point-slope form.

$$\begin{aligned} y + \frac{3\sqrt{3}}{4} &= \frac{7\sqrt{3}}{3}\left(x + \frac{3}{4}\right) \\ y &= \frac{7\sqrt{3}}{3}\left(x + \frac{3}{4}\right) - \frac{3\sqrt{3}}{4} \end{aligned}$$

Thus the tangent line has the equation

$$\boxed{y = \frac{7\sqrt{3}}{3}\left(x + \frac{3}{4}\right) - \frac{3\sqrt{3}}{4}}$$