Problem 10: The equation

$$x^{10} + (13x - 1)^{10} = 0$$

has the 10 complex roots r_1, $\overline{r_1}$, r_2, $\overline{r_2}$, r_3, $\overline{r_3}$, r_4, $\overline{r_4}$, r_5, $\overline{r_5}$. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}.$$

(Source: AIME)

First let's find one of the roots.

$$\begin{aligned}
x^{10} + (13x-1)^{10} &= 0 \\
x^{10} &= -(13x-1)^{10} \\
x^{10} &= (i^{10})(13x-1)^{10} \\
x^{10} &= (i(13x-1))^{10} \\
x &= i(13x-1) && \text{this step gives us one of the roots} \\
x &= 13ix - i \\
x - 13ix &= -i \\
x(1-13i) &= -i \\
x(13i-1) &= i \\
x &= \frac{i}{13i-1} \\
x &= \frac{i(13i+1)}{-169-1} \\
x &= \frac{-13+i}{-170} \\
x &= \frac{13}{170} - \frac{1}{170}i
\end{aligned}$$

We have found that $r_1 = \frac{13}{170} - \frac{1}{170}i$ is one of the roots of the equation. We can get all the roots of the equation by taking conjugates and by multiplying r_1 by the tenth roots of unity.

$$
\begin{aligned}
r_1 &= \frac{13}{170} - \frac{1}{170}i \\
\overline{r_1} &= \frac{13}{170} + \frac{1}{170}i \\
r_2 &= e^{2\pi i/10}\left(\frac{13}{170} - \frac{1}{170}i\right) \\
\overline{r_2} &= e^{18\pi i/10}\left(\frac{13}{170} + \frac{1}{170}i\right) \\
r_3 &= e^{4\pi i/10}\left(\frac{13}{170} - \frac{1}{170}i\right) \\
\overline{r_3} &= e^{16\pi i/10}\left(\frac{13}{170} + \frac{1}{170}i\right) \\
r_4 &= e^{6\pi i/10}\left(\frac{13}{170} - \frac{1}{170}i\right) \\
\overline{r_4} &= e^{14\pi i/10}\left(\frac{13}{170} + \frac{1}{170}i\right) \\
r_5 &= e^{8\pi i/10}\left(\frac{13}{170} - \frac{1}{170}i\right) \\
\overline{r_5} &= e^{12\pi i/10}\left(\frac{13}{170} + \frac{1}{170}i\right)
\end{aligned}
$$

We have found all ten roots of the equation. Now we can find the value of the given expression.

$$
\begin{aligned}
\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}} &= 170 + e^{20\pi i/10}170 + e^{20\pi i/10}170 + e^{20\pi i/10}170 + e^{20\pi i/10}170 \\
&= 170 \times 5 \\
&= \boxed{850}
\end{aligned}
$$