Problem 23: Let $x_n + iy_n = (1 + i\sqrt{3})^n$, where x_n and y_n are real and n is a positive integer. If $x_{19}y_{91} + x_{91}y_{19} = 2^k\sqrt{3}$, compute k. (Source: ARML)

We have

$$x_n + iy_n = \left(2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right)^n = 2^n\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^n = 2^n e^{n\pi i/3}$$

Thus $x_{19} + iy_{19} = 2^{19}e^{19\pi i/3} = 2^{19}e^{\pi i/3}$ and $x_{91} + iy_{91} = 2^{91}e^{91\pi i/3} = 2^{91}e^{\pi i/3}$.

So

$$\begin{aligned} x_{19} &= 2^{19}\cos\frac{\pi}{3} = 2^{18} \\ y_{19} &= 2^{19}\sin\frac{\pi}{3} = 2^{18}\sqrt{3} \\ x_{91} &= 2^{91}\cos\frac{\pi}{3} = 2^{90} \\ y_{91} &= 2^{19}\sin\frac{\pi}{3} = 2^{90}\sqrt{3} \end{aligned}$$

So

$$\begin{aligned} x_{19}y_{91} + x_{91}y_{19} &= 2^{18} \cdot 2^{90}\sqrt{3} + 2^{90} \cdot 2^{18}\sqrt{3} \\ &= 2^{108}\sqrt{3} + 2^{108}\sqrt{3} \\ &= 2^{109}\sqrt{3} \end{aligned}$$

Thus $\boxed{k = 109}$.