

Problem 14: Find the equation of the tangent line to $r = 3 \cos 2\theta$ at $\theta = \frac{\pi}{3}$.

(Source: AoPS Calculus)

In Problem 13, we computed the slope of the tangent line to be $\frac{7\sqrt{3}}{3}$. Now all that remains is to find a point on the tangent line, and give the equation of the tangent line in point-slope form.

We know that the tangent line intersects the graph of $r = 3 \cos 2\theta$ at $\theta = \frac{\pi}{3}$, so we can compute the x and y coordinates of $r = 3 \cos 2\theta$ at $\theta = \frac{\pi}{3}$, in order to find a point on the tangent line.

$$\begin{aligned}x &= r \cos \theta \\&= 3 \cos 2\theta \cos \theta \\x\left(\frac{\pi}{3}\right) &= 3 \cos \frac{2\pi}{3} \cos \frac{\pi}{3} \\&= -\frac{3}{4}\end{aligned}$$

$$\begin{aligned}y &= r \sin \theta \\&= 3 \cos 2\theta \sin \theta \\y\left(\frac{\pi}{3}\right) &= 3 \cos \frac{2\pi}{3} \sin \frac{\pi}{3} \\&= -\frac{3\sqrt{3}}{4}\end{aligned}$$

The tangent line intersects the graph of $r = 3 \cos 2\theta$ at the point $(x, y) = \left(-\frac{3}{4}, -\frac{3\sqrt{3}}{4}\right)$.

Knowing the slope of the tangent line and a point on the tangent line, we can give the equation of the tangent line in point-slope form.

$$\begin{aligned}y - \frac{3\sqrt{3}}{4} &= \frac{7\sqrt{3}}{3} \left(x + \frac{3}{4}\right) \\y &= \frac{7\sqrt{3}}{3} \left(x + \frac{3}{4}\right) + \frac{3\sqrt{3}}{4}\end{aligned}$$

Thus the tangent line has the equation

$$\boxed{y = \frac{7\sqrt{3}}{3} \left(x + \frac{3}{4}\right) + \frac{3\sqrt{3}}{4}}$$