

Problem 3: Find the roots of the polynomial  $p(x) = x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ .

(Source: AoPS Precalculus)

Multiplying  $p(x)$  by  $x$  we get

$$xp(x) = x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x$$

Subtracting  $p(x)$  from  $xp(x)$ , we get

$$\begin{aligned} xp(x) - p(x) &= x^9 - 1 \\ p(x)(x - 1) &= x^9 - 1 \\ p(x) &= \frac{x^9 - 1}{x - 1} \end{aligned} \quad \text{provided } x \neq 1$$

Now we know that  $x = 1$  is not a root of  $p(x)$  since  $p(1) = 9$ . But from our equations above, we see that all of the ninth roots of unity except for  $x = 1$  are roots of  $p(x)$ . Thus the roots of  $p(x)$  are

$$\boxed{e^{2\pi i/9}, e^{4\pi i/9}, e^{6\pi i/9}, e^{8\pi i/9}, e^{10\pi i/9}, e^{12\pi i/9}, e^{14\pi i/9}, e^{16\pi i/9}}$$