Problem 4: Find the shaded area, shown at right, inside the limacon given by the graph of $r = 1 + 2\sin\theta$. (Source: AoPS Calculus)

First we will compute the area of the non-shaded region. We can solve the inequality $r \le 0$ for θ to get the bounds we need for integration.

$$\begin{aligned}
1 + 2\sin\theta &\le 0 \\
2\sin\theta &\le -1 \\
\sin\theta &\le -\frac{1}{2} \\
\frac{7\pi}{6} \le \theta &\le \frac{11\pi}{6}
\end{aligned}$$

Now we can integrate from $\theta = \frac{7\pi}{6}$ to $\frac{11\pi}{6}$ to get the area of the non-shaded region.

$$\begin{aligned}
A_{ns} &= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} r^2\, d\theta \\
&= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 2\sin\theta)^2\, d\theta \\
&= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 4\sin\theta + 4\sin^2\theta)\, d\theta \\
&= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 4\sin\theta + 4(\frac{1}{2}(1 - \cos 2\theta))\, d\theta \\
&= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 4\sin\theta + 2(1 - \cos 2\theta))\, d\theta \\
&= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 4\sin\theta + 2 - 2\cos 2\theta)\, d\theta \\
&= \frac{1}{2}\left(\theta - 4\cos\theta + 2\theta - 2(\frac{1}{2}\sin 2\theta)\right)\Bigg|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \\
&= \frac{1}{2}(\theta - 4\cos\theta + 2\theta - \sin 2\theta)\Bigg|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \\
&= \frac{1}{2}\left(\frac{11\pi}{6} - 4(\frac{\sqrt{3}}{2}) + 2(\frac{11\pi}{6}) - \sin\frac{22\pi}{6}\right) - \frac{1}{2}\left(\frac{7\pi}{6} - 4(-\frac{\sqrt{3}}{2}) + 2(\frac{7\pi}{6}) - \sin\frac{14\pi}{6}\right) \\
&= \frac{1}{2}\left(\frac{11\pi}{6} - 2\sqrt{3} + \frac{22\pi}{6} + \frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\frac{7\pi}{6} + 2\sqrt{3} + \frac{14\pi}{6} - \frac{\sqrt{3}}{2}\right) \\
&= \frac{1}{2}\left(\frac{4\pi}{6} - 4\sqrt{3} + \frac{8\pi}{6} + \sqrt{3}\right) \\
&= \frac{1}{2}\left(\frac{12\pi}{6} - 3\sqrt{3}\right) \\
&= \pi - \frac{3\sqrt{3}}{2}
\end{aligned}$$

To get the area inside the limacon, we can just integrate from $\theta = -\frac{\pi}{6}$ to $\frac{7\pi}{6}$.

$$
\begin{aligned}
A_l &= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} r^2\, d\theta \\
&= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (1+2\sin\theta)^2\, d\theta \\
&= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (1+4\sin\theta+4\sin^2\theta)\, d\theta \\
&= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (1+4\sin\theta+4(\frac{1}{2}(1-\cos 2\theta))\, d\theta \\
&= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (1+4\sin\theta+2(1-\cos 2\theta))\, d\theta \\
&= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (1+4\sin\theta+2-2\cos 2\theta)\, d\theta \\
&= \frac{1}{2}\left(\theta-4\cos\theta+2\theta-2(\frac{1}{2}\sin 2\theta)\right)\Bigg|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\
&= \frac{1}{2}\left(\theta-4\cos\theta+2\theta-\sin 2\theta\right)\Bigg|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\
&= \frac{1}{2}\left(\frac{7\pi}{6}+2\sqrt{3}+\frac{14\pi}{6}-\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\left(-\frac{\pi}{6}-2\sqrt{3}-\frac{2\pi}{6}+\frac{\sqrt{3}}{2}\right) \\
&= \frac{1}{2}\left(\frac{21\pi}{6}+\frac{3\sqrt{3}}{2}\right)-\frac{1}{2}\left(-\frac{3\pi}{6}-\frac{3\sqrt{3}}{2}\right) \\
&= \frac{1}{2}\left(\frac{21\pi}{6}+\frac{3\sqrt{3}}{2}+\frac{3\pi}{6}+\frac{3\sqrt{3}}{2}\right) \\
&= \frac{1}{2}\left(\frac{24\pi}{6}+3\sqrt{3}\right) \\
&= \frac{1}{2}\left(4\pi+3\sqrt{3}\right) \\
&= 2\pi+\frac{3\sqrt{3}}{2}
\end{aligned}
$$

To get the shaded area, we subtract the area of the non-shaded region from the area inside the limacon.

$$
\begin{aligned}
A_s &= A_l - A_{ns} \\
&= \left(2\pi+\frac{3\sqrt{3}}{2}\right)-\left(\pi-\frac{3\sqrt{3}}{2}\right) \\
&= \boxed{\pi+3\sqrt{3}}
\end{aligned}
$$