Problem 13: Find all values of x such that $x^4 + 5x^2 + 4x + 5 = 0$. (Source: AoPS Precalculus)

Let $f(x) = x^4 + 5x^2 + 4x + 5$.

Suppose $\frac{p}{q}$ is a rational root of $f(x)$, where p and q are relatively prime. Since $f(x)$ is monic, $\frac{p}{q}$ has to be an integer. Thus $q = 1$ and $\frac{p}{q} = p$. Now 5 has to be a multiple of p. So $p = \pm 1$ or $p = \pm 5$. But if we try plugging $1, -1, 5, -5$ into $f(x)$, the output is not zero. Since these are the only possible rational roots, the quartic polynomial $f(x)$ has no rational roots.

Now let's try factoring $f(x)$ as the product of two quadratics. If we can do this, then we can use the quadratic formula to get the roots. Suppose $f(x) = (x^2 + ax + b)(x^2 + cx + d)$ for real numbers a, b, c, d. Expanding this gives us

$$\begin{aligned} f(x) &= (x^2 + ax + b)(x^2 + cx + d) \\ &= x^4 + cx^3 + dx^2 + ax^3 + acx^2 + adx + bx^2 + bcx + bd \\ &= x^4 + (a+c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd \end{aligned}$$

So $a = -c$ and $bd = 5$. Suppose $b = 1$ and $d = 5$.

$$\begin{aligned} ac + b + d &= 5 \\ ac + 1 + 5 &= 5 \\ ac + 6 &= 5 \\ ac &= -1 \\ -a^2 &= -1 \\ a^2 &= 1 \\ a &= \pm 1 \end{aligned}$$

Let's try the values $a = 1, b = 1, c = -1, d = 5$.

$$\begin{aligned} f(x) &= (x^2 + x + 1)(x^2 - x + 5) \\ &= x^4 - x^3 + 5x^2 + x^3 - x^2 + 5x + x^2 - x + 5 \\ &= x^4 + 5x^2 + 4x + 5 \end{aligned}$$

The values $a = 1, b = 1, c = -1, d = 5$ give us our original polynomial. So we have factored $f(x)$ successfully. Now we can use the quadratic formula to find the roots. We'll start with the first quadratic.

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1 - 4}}{2} \\ &= \frac{-1 \pm i\sqrt{3}}{2} \end{aligned}$$

Now onto the second quadratic.

$$
\begin{aligned}
x &= \frac{1 \pm \sqrt{1 - 4(5)}}{2} \\
&= \frac{1 \pm i\sqrt{19}}{2}
\end{aligned}
$$

Thus

$$
\boxed{x = \frac{-1 + i\sqrt{3}}{2}, \quad \frac{-1 - i\sqrt{3}}{2}, \quad \frac{1 + i\sqrt{19}}{2}, \quad \text{and} \quad \frac{1 - i\sqrt{19}}{2}}
$$