Problem 13: Find the slope of the tangent line to $r = 3\cos 2\theta$ at $\theta = \dfrac{\pi}{3}$.

We will use the formula $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$.

From our knowledge of polar coordinates, we know that $y = r\sin\theta$ and $x = r\cos\theta$. Since r is a function of θ (and not a constant) we will have to use the product rule when differentiating these equations.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\
&= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} \\
&= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}
\end{aligned}$$

Now we have $r' = -6\sin 2\theta$ by the chain rule. Plugging that in we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} \\
&= \frac{(-6\sin 2\theta)(\sin\theta) + (3\cos 2\theta)(\cos\theta)}{(-6\sin 2\theta)(\cos\theta) - (3\cos 2\theta)(\sin\theta)}
\end{aligned}$$

We arrive at an equation for the slope of the tanget line in terms of the variable θ. We can plug in $\theta = \dfrac{\pi}{3}$ to get the slope of the tangent line at $\dfrac{\pi}{3}$ radians.

$$\frac{dy}{dx}\left(\frac{\pi}{3}\right) = \frac{\left(-6\sin\frac{2\pi}{3}\right)\left(\sin\frac{\pi}{3}\right) + \left(3\cos\frac{2\pi}{3}\right)\left(\cos\frac{\pi}{3}\right)}{\left(-6\sin\frac{2\pi}{3}\right)\left(\cos\frac{\pi}{3}\right) - \left(3\cos\frac{2\pi}{3}\right)\left(\sin\frac{\pi}{3}\right)}$$

$$= \frac{(-3\sqrt{3})\left(\frac{\sqrt{3}}{2}\right) + \left(-\frac{3}{2}\right)\left(\frac{1}{2}\right)}{(-3\sqrt{3})\left(\frac{1}{2}\right) - \left(-\frac{3}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{\left(-\frac{18}{4}\right) + \left(-\frac{3}{4}\right)}{\left(-\frac{6\sqrt{3}}{4}\right) + \left(\frac{3\sqrt{3}}{4}\right)}$$

$$= \frac{-18 + -3}{-6\sqrt{3} + 3\sqrt{3}}$$

$$= \frac{-21}{-3\sqrt{3}}$$

$$= \frac{7}{\sqrt{3}}$$

$$= \boxed{\frac{7\sqrt{3}}{3}}$$

The slope of the tangent line at $\theta = \frac{\pi}{3}$ is $\boxed{\frac{7\sqrt{3}}{3}}$.