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Problem 1: Compute the following definite integral:

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$$

(Source: Putnam)

Solution:

We can expand the numerator and then do long division. This gives us

$$\begin{aligned}
\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx &= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right)\,dx \\
&= \left(\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x - 4\tan^{-1} x\right)\Bigg|_0^1 \\
&= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\cdot\frac{\pi}{4} \\
&= \frac{1}{7} - \frac{6}{3} + 1 + 4 - \pi \\
&= \frac{1}{7} + 3 - \pi \\
&= \boxed{\frac{22}{7} - \pi}
\end{aligned}$$