Problem 5: Find all continuous positive functions $f(x)$, for $0 \leq x \leq 1$, such that

$$\int_0^1 f(x)\,dx = 1, \quad \int_0^1 f(x)\,x\,dx = \alpha, \quad \int_0^1 f(x)\,x^2\,dx = \alpha^2$$

where α is a given real number.

(Source: Putnam)

Solution:

Suppose $f(x)$ is a continuous positive function that satisfies these conditions.

We have

$$\int_0^1 f(x)(\alpha^2 - 2x\alpha + x^2)\,dx = \alpha^2 - 2\alpha^2 + \alpha^2 = 0$$

But the function $f(x)\,(\alpha - x)^2$ is positive for all $x \neq \alpha$, so

$$\int_0^1 f(x)\,(\alpha - x)^2\,dx > 0$$

.

This contradicts our previous equation.

Thus there are no continuous positive functions $f(x)$ that satisfy the three conditions from the problem statement.