Problem 12: A polynomial of degree four with leading coefficient 1 and integer coefficients has two real zeros, both of which are integers. Which of the following can also be a zero of the polynomial? (Source: AMC 12)

$$\frac{1+i\sqrt{11}}{2}, \frac{1+i}{2}, \frac{1}{2}+i, 1+\frac{i}{2}, \frac{1+i\sqrt{13}}{2}.$$

Let $f(x)$ be such a polynomial. Let $r$ and $s$ be the integer root of $f(x)$. Then

$$f(x) = (x-r)(x-s)(x^2+ax+b)$$

Expanding this gives

$$\begin{aligned} f(x) &= (x-r)(x-s)(x^2+ax+b) \\ &= (x^2-(r+s)x+rs)(x^2+ax+b) \\ &= x^4+ax^3+bx^2 \\ &\quad -(r+s)x^3-a(r+s)x^2-b(r+s)x \\ &\quad +rsx^2+arsx+brs \\ &= x^4+(a-r-s)x^3+(b-ar-as+rs)x^2+(ars-br-bs)x+brs \end{aligned}$$

Now since the coefficients of $f(x)$ are integers, $a-r-s$ has to be an integer, so $a$ has to be an integer. Furthermore, $b-ar-as+rs$ has to be an integer, so $b$ has to be an integer.

Knowing that $b$ has to be an integer, we can rule out many possibilities.

If $f(x)$ has nonreal roots, then the nonreal roots must form a conjugate pair. Their product must be an integer. The only possibility that meets this criteria is $\frac{1+i\sqrt{11}}{2}$, since $\frac{1+i\sqrt{11}}{2} \cdot \frac{1-i\sqrt{11}}{2} = \frac{12}{4} = 3$ is an integer.

Thus $\boxed{\frac{1+i\sqrt{11}}{2}}$ is the only complex number listed that can be a zero of the polynomial.