Problem 22: Let $z = e^{i\theta}$, where $z \neq 1$. Show that $\frac{1+z}{1-z} = i \cot \frac{\theta}{2}$. (Source: AoPS Precalculus)

Let $z = a + bi$. We have

$$
\begin{aligned}
\frac{1+z}{1-z} &= \frac{1 + (a+bi)}{1 - (a+bi)} \\
&= \frac{(1+a) + bi}{(1-a) - bi} \\
&= \frac{((1+a) + bi)((1-a) + bi)}{((1-a) - bi)((1-a) + bi)} \\
&= \frac{((1+a) + bi)((1-a) + bi)}{(1-a)^2 + b^2} \\
&= \frac{1 - a^2 + (1+a)bi + (1-a)bi - b^2}{(1-a)^2 + b^2} \\
&= \frac{1 - a^2 + 2bi - b^2}{(1-a)^2 + b^2} \\
&= \frac{1 - (a^2 + b^2) + 2bi}{1 - 2a + a^2 + b^2} \\
&= \frac{1 - 1 + 2bi}{1 - 2a + 1} \\
&= \frac{2bi}{2 - 2a} \\
&= \frac{b}{1-a} i
\end{aligned}
$$

Now $a = \cos\theta$ and $b = \sin\theta$. Making these substitutions we get

$$
\frac{1+z}{1-z} = \frac{b}{1-a} i = \frac{\sin\theta}{1 - \cos\theta} i
$$

.

Applying the half-angle identity for cot, we have

$$
\frac{1+z}{1-z} = \frac{\sin\theta}{1 - \cos\theta} i = \boxed{i \cot \frac{\theta}{2}}
$$

.