Problem 2: We say that a polynomial is irreducible over the rational numbers if it cannot be factored as the product of polynomials with smaller positive degree and rational coefficients. Find a degree 4 polynomial with leading coefficient 1 and rational coefficients such that the polynomial is irreducible over the rational numbers and all the polynomial's roots are twelfth roots of unity. (Source: HMMT)

The twelfth roots of unity are

$$1, e^{2\pi i/12}, e^{4\pi i/12}, e^{6\pi i/12}, e^{8\pi i/12}, e^{10\pi i/12}, e^{12\pi i/12}, e^{14\pi i/12}, e^{16\pi i/12}, e^{18\pi i/12}, e^{20\pi i/12}, e^{22\pi i/12}$$

Let's label the twelfth roots of unity $\omega_1, \omega_2, ..., \omega_{12}$. We can write them in rectangular form.

$$\begin{aligned}
\omega_1 &= 1 \\
\omega_2 &= \frac{\sqrt{3}}{2} + \frac{1}{2}i \\
\omega_3 &= \frac{1}{2} + \frac{\sqrt{3}}{2}i \\
\omega_4 &= i \\
\omega_5 &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\
\omega_6 &= -\frac{\sqrt{3}}{2} + \frac{1}{2}i \\
\omega_7 &= -1 \\
\omega_8 &= -\frac{\sqrt{3}}{2} - \frac{1}{2}i \\
\omega_9 &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\
\omega_{10} &= -i \\
\omega_{11} &= \frac{1}{2} - \frac{\sqrt{3}}{2}i \\
\omega_{12} &= \frac{\sqrt{3}}{2} - \frac{1}{2}i
\end{aligned}$$

Let $p(x)$ be a fourth degree monic polynomial with rational coefficients such that all of the polynomial's roots are twelfth roots of unity. We can write

$$\begin{aligned}
p(x) &= (x - r_1)(x - r_2)(x - r_3)(x - r_4) \\
&= x^4 - (r_1 + r_2 + r_3 + r_4)x^3 + (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)x^2 - (r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)x + r_1r_2r_3r_4
\end{aligned}$$

Let

$$
\begin{aligned}
r_1 &= \frac{\sqrt{3}}{2} + \frac{1}{2}i \\
r_2 &= \frac{\sqrt{3}}{2} - \frac{1}{2}i \\
r_3 &= -\frac{\sqrt{3}}{2} + \frac{1}{2}i \\
r_4 &= -\frac{\sqrt{3}}{2} - \frac{1}{2}i
\end{aligned}
$$

Then

$$
\begin{aligned}
p(x) &= x^4 - (r_1 + r_2 + r_3 + r_4)x^3 + (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)x^2 - (r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)x + r_1r_2r_3r_4 \\
&= x^4 - 0x^3 + (1 + -1 - \frac{1 + i\sqrt{3}}{2} - \frac{1 - i\sqrt{3}}{2} - 1 + 1)x^2 - (-\sqrt{3} + \sqrt{3})x + 1 \\
&= x^4 - x^2 + 1
\end{aligned}
$$

We have $p(x) = x^4 - x^2 + 1$ as our result. Let's plug in r_1, r_2, r_3, r_4 to confirm that the polynomial is correct.

$$
\begin{aligned}
(e^{2\pi i/12})^4 - (e^{2\pi i/12})^2 + 1 = e^{8\pi i/12} - e^{4\pi i/12} + 1 = 0 \\
(e^{22\pi i/12})^4 - (e^{22\pi i/12})^2 + 1 = e^{88\pi i/12} - e^{44\pi i/12} + 1 = 0 \\
(e^{10\pi i/12})^4 - (e^{10\pi i/12})^2 + 1 = e^{40\pi i/12} - e^{20\pi i/12} + 1 = 0 \\
(e^{14\pi i/12})^4 - (e^{14\pi i/12})^2 + 1 = e^{56\pi i/12} - e^{28\pi i/12} + 1 = 0
\end{aligned}
$$

Doing some arithmetic in my head, I feel more confident that $p(x) = x^4 - x^2 + 1$. Now I want to confirm that $p(x)$ is irreducible over the rationals. Recall that

$$
p(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)
$$

If we combine any two or three linear factors to form a quadratic or a cubic, we would end up getting some irrational coefficients.

So $\boxed{p(x) = x^4 - x^2 + 1}$ is a monic polynomial with rational coefficients that is irreducible over the rational numbers, and all of the polynomial's roots are twelfth roots of unity.