Problem 15: Let f be a real-valued function with $\lim_{x \to a} f(x) = L$ and let $c \in \mathbb{R}$. Prove that

$$\lim_{x \to a} (cf)(x) = cL$$

*Proof.* Let $\epsilon > 0$. Since $\lim_{x \to a} f(x) = L$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \implies \quad |f(x) - L| < \frac{\epsilon}{|c|}$$

We can multiply both sides of the inequality on the right by $|c|$.

$$0 < |x - a| < \delta \quad \implies \quad |c| \cdot |f(x) - L| < |c| \cdot \frac{\epsilon}{|c|} \quad \implies \quad |c| \cdot |f(x) - L| < \epsilon$$

Now we can merge the two absolute value operations.

$$0 < |x - a| < \delta \quad \implies \quad |c(f(x) - L)| < \epsilon \quad \implies \quad |cf(x) - cL| < \epsilon$$

Thus

$$\lim_{x \to a} (cf)(x) = cL$$

$\square$