Problem 32: Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Find the probability that $\sqrt{2+\sqrt{3}} \leq |v+w|$. (Source: AIME)

Let $v = \cos x + i \sin x$ and $w = \cos y + i \sin y$ be two distinct roots of the equation $z^{1997} - 1 = 0$.

We will start by assuming $\sqrt{2+\sqrt{3}} \leq |v+w|$.

$$\begin{aligned}
&\sqrt{2+\sqrt{3}} \leq |v+w| \\
&\sqrt{2+\sqrt{3}} \leq |(\cos x + \cos y) + i(\sin x + \sin y)| \\
&\sqrt{2+\sqrt{3}} \leq \sqrt{(\cos x + \cos y)^2 + (\sin x + \sin y)^2} \\
&2+\sqrt{3} \leq (\cos x + \cos y)^2 + (\sin x + \sin y)^2 \\
&2+\sqrt{3} \leq \cos^2 x + \cos^2 y + 2\cos x \cos y + \sin^2 x + \sin^2 y + 2 \sin x \sin y \\
&2+\sqrt{3} \leq \cos^2 x + \sin^2 x + \cos^2 y + \sin^2 y + 2\cos x \cos y + 2 \sin x \sin y \\
&2+\sqrt{3} \leq 2 + 2\cos x \cos y + 2 \sin x \sin y \\
&2+\sqrt{3} \leq 2 + 2(\cos x \cos y + \sin x \sin y) \\
&2+\sqrt{3} \leq 2 + 2\cos(x-y) \\
&\sqrt{3} \leq 2\cos(x-y) \\
&\frac{\sqrt{3}}{2} \leq \cos(x-y)
\end{aligned}$$

We have arrived at the result $\frac{\sqrt{3}}{2} \leq \cos(x-y)$. For this to be true, it is necessary that $-\frac{\pi}{6} \leq x - y \leq \frac{\pi}{6}$.

Now x is the argument of v, and y is the argument of w.

Since v and w are 1997th roots of unity, we can write $x = 2k\pi/1997$ and $y = 2l\pi/1997$ for integers $0 \leq k, l \leq 1996$.

Since $v \neq w$, we know that $k \neq l$.

Now we can solve for k and l.

$$\begin{aligned}
&-\frac{\pi}{6} \leq x - y \leq \frac{\pi}{6} \\
&-\frac{\pi}{6} \leq \frac{2k\pi}{1997} - \frac{2l\pi}{1997} \leq \frac{\pi}{6} \\
&-\frac{\pi}{6} \leq \frac{2(k-l)\pi}{1997} \leq \frac{\pi}{6} \\
&-\frac{1997}{12} \leq k - l \leq \frac{1997}{12} \\
&-166 \leq k - l \leq 166 && \text{we are able to round since k and l are integers}
\end{aligned}$$

We know that the difference $k - l$ is in the range of integers $[-166, 0) \cup (0, +166]$.

After randomly choosing a value for k, there are 332 values of l that meet this requirement, out of 1996 possible values of l.

Thus the probability that $-166 \leq k - l \leq 166$ (for two distinct integers $0 \leq k, l \leq 1996$) is $\frac{332}{1996}$.

This condition is strong enough that it implies both $-\frac{\pi}{6} \leq x - y \leq \frac{\pi}{6}$ and $\sqrt{2+\sqrt{3}} \leq |v+w|$.

Thus the probability that $\sqrt{2+\sqrt{3}} \leq |v+w|$ is $\boxed{\frac{332}{1996} = \frac{83}{499}}$