Problem 15: Let f be a real-valued function with $\lim_{x\to a} f(x) = L$ and let $c \in \mathbb{R}$. Prove that

$$\lim_{x\to a}(cf)(x) = cL$$

Proof. Let $\epsilon > 0$. If $c = 0$ then we can choose $\delta = \epsilon$, since

$$0 < |x-a| < \delta \quad \implies \quad |0 \cdot f(x) - 0 \cdot L| = 0 < \epsilon \quad \implies \quad \lim_{x\to a}(cf)(x) = 0$$

Suppose $c \neq 0$. Since $\lim_{x\to a} f(x) = L$, there exists a $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \implies \quad |f(x) - L| < \frac{\epsilon}{|c|}$$

We can multiply both sides of the inequality on the right by $|c|$.

$$0 < |x-a| < \delta \quad \implies \quad |c| \cdot |f(x) - L| < |c| \cdot \frac{\epsilon}{|c|} \quad \implies \quad |c| \cdot |f(x) - L| < \epsilon$$

We know that $|c| \cdot |f(x) - L| = |c(f(x) - L)|$ by the properties of the absolute value, so

$$0 < |x-a| < \delta \quad \implies \quad |c(f(x) - L)| < \epsilon \quad \implies \quad |cf(x) - cL| < \epsilon$$

By the δ-ϵ definition of the limit, we have

$$\lim_{x\to a}(cf)(x) = cL$$

$\square$