Problem 20: Express $\sin 5\theta$ in terms of $\sin\theta$. (Source: AoPS Precalculus)

By de Moivre's theorem, we have

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

Now we can apply the Binomial Theorem.

$$\begin{aligned}(\cos\theta + i\sin\theta)^5 &= (\cos\theta)^5 + 5(\cos\theta)^4(i\sin\theta) + 10(\cos\theta)^3(i\sin\theta)^2 + 10(\cos\theta)^2(i\sin\theta)^3 + 5(\cos\theta)(i\sin\theta)^4 + (i\sin\theta)^5 \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta\end{aligned}$$

We can equate $\sin 5\theta$ with the imaginary part of the equation.

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

Now we can substitute $1 - \cos^2\theta$ for $\sin^2\theta$.

$$\begin{aligned}\sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\ &= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta \\ &= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10(\sin^3\theta - \sin^5\theta) + \sin^5\theta \\ &= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta\end{aligned}$$

We arrive at the equation

$$\boxed{\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta}$$