

Problem 4: Find the shaded area, shown at right, inside the limaçon given by the graph of $r = 1 + 2 \sin \theta$. (Source: AoPS Calculus)

First we will compute the area of the non-shaded region. We can solve the inequality $r \leq 0$ for θ to get the bounds we need for integration.

$$\begin{aligned} 1 + 2 \sin \theta &\leq 0 \\ 2 \sin \theta &\leq -1 \\ \sin \theta &\leq -\frac{1}{2} \\ \frac{7\pi}{6} &\leq \theta \leq \frac{11\pi}{6} \end{aligned}$$

Now we can integrate from $\theta = \frac{7\pi}{6}$ to $\frac{11\pi}{6}$ to get the area of the non-shaded region.

$$\begin{aligned} A_{ns} &= \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} r^2 d\theta \\ &= \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 2 \sin \theta)^2 d\theta \\ &= \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 4 \sin \theta + 4 \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 4 \sin \theta + 4(\frac{1}{2}(1 - \cos 2\theta))) d\theta \\ &= \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 4 \sin \theta + 2(1 - \cos 2\theta)) d\theta \\ &= \frac{1}{2} \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1 + 4 \sin \theta + 2 - 2 \cos 2\theta) d\theta \\ &= \frac{1}{2} \left(\theta - 4 \cos \theta + 2\theta - 2(\frac{1}{2} \sin 2\theta) \right) \Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \\ &= \frac{1}{2} (\theta - 4 \cos \theta + 2\theta - \sin 2\theta) \Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \\ &= \frac{1}{2} \left(\frac{11\pi}{6} - 4(\frac{\sqrt{3}}{2}) + 2(\frac{11\pi}{6}) - \sin \frac{22\pi}{6} \right) - \frac{1}{2} \left(\frac{7\pi}{6} - 4(-\frac{\sqrt{3}}{2}) + 2(\frac{7\pi}{6}) - \sin \frac{14\pi}{6} \right) \\ &= \frac{1}{2} \left(\frac{11\pi}{6} - 2\sqrt{3} + \frac{22\pi}{6} + \frac{\sqrt{3}}{2} \right) - \frac{1}{2} \left(\frac{7\pi}{6} + 2\sqrt{3} + \frac{14\pi}{6} - \frac{\sqrt{3}}{2} \right) \\ &= \frac{1}{2} \left(\frac{4\pi}{6} - 4\sqrt{3} + \frac{8\pi}{6} + \sqrt{3} \right) \\ &= \frac{1}{2} \left(\frac{12\pi}{6} - 3\sqrt{3} \right) \\ &= \pi - \frac{3\sqrt{3}}{2} \end{aligned}$$

To get the area inside the limaçon, we can just integrate from $\theta = -\frac{\pi}{6}$ to $\frac{7\pi}{6}$.

$$\begin{aligned}
A_l &= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} r^2 d\theta \\
&= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (1 + 2 \sin \theta)^2 d\theta \\
&= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (1 + 4 \sin \theta + 4 \sin^2 \theta) d\theta \\
&= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (1 + 4 \sin \theta + 4(\frac{1}{2}(1 - \cos 2\theta))) d\theta \\
&= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (1 + 4 \sin \theta + 2(1 - \cos 2\theta)) d\theta \\
&= \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} (1 + 4 \sin \theta + 2 - 2 \cos 2\theta) d\theta \\
&= \frac{1}{2} \left(\theta - 4 \cos \theta + 2\theta - 2(\frac{1}{2} \sin 2\theta) \right) \Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\
&= \frac{1}{2} (\theta - 4 \cos \theta + 2\theta - \sin 2\theta) \Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\
&= \frac{1}{2} \left(\frac{7\pi}{6} + 2\sqrt{3} + \frac{14\pi}{6} - \frac{\sqrt{3}}{2} \right) - \frac{1}{2} \left(-\frac{\pi}{6} - 2\sqrt{3} - \frac{2\pi}{6} + \frac{\sqrt{3}}{2} \right) \\
&= \frac{1}{2} \left(\frac{21\pi}{6} + \frac{3\sqrt{3}}{2} \right) - \frac{1}{2} \left(-\frac{3\pi}{6} - \frac{3\sqrt{3}}{2} \right) \\
&= \frac{1}{2} \left(\frac{21\pi}{6} + \frac{3\sqrt{3}}{2} + \frac{3\pi}{6} + \frac{3\sqrt{3}}{2} \right) \\
&= \frac{1}{2} \left(\frac{24\pi}{6} + 3\sqrt{3} \right) \\
&= \frac{1}{2} (4\pi + 3\sqrt{3}) \\
&= 2\pi + \frac{3\sqrt{3}}{2}
\end{aligned}$$

To get the area of the shaded region, we subtract the area of the non-shaded region from the area inside the limaçon.

$$\begin{aligned}
A_s &= A_l - A_{ns} \\
&= \left(2\pi + \frac{3\sqrt{3}}{2} \right) - \left(\pi - \frac{3\sqrt{3}}{2} \right) \\
&= \boxed{\pi + 3\sqrt{3}}
\end{aligned}$$