Problem 1: Find the area of one leaf of the rose given by $r = 3 \sin 2\theta$. (Source: AoPS Calculus)

We can divide the leaf into a number of small sectors, where each sector has an area of $\dfrac{d\theta}{2\pi}(\pi r^2)$. We can integrate this expression from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$ to get the area of the leaf.

$$\int_0^{\frac{\pi}{2}} (\pi r^2) \frac{d\theta}{2\pi} = \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (3 \sin 2\theta)^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} 9 \sin^2 2\theta \, d\theta$$

$$= \frac{9}{2} \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$$

Now we are going to prove a trig identity that lets us integrate $\sin^2 2\theta$.

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin^2 x = \cos^2 x - \cos 2x$$

$$\sin^2 x = (1 - \sin^2 x) - \cos 2x$$

$$2 \sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Let $u = 2\theta$. Then $\dfrac{du}{d\theta} = 2$.

$$\int_0^{\frac{\pi}{2}} (\pi r^2) \frac{d\theta}{2\pi} = \frac{9}{2} \int_0^{\frac{\pi}{2}} \sin^2 2\theta \, d\theta$$

$$= \frac{9}{2} \int_0^{\pi} \sin^2 u \, \frac{du}{2}$$

$$= \frac{9}{4} \int_0^{\pi} \sin^2 u \, du$$

$$= \frac{9}{4} \int_0^{\pi} \frac{1}{2}(1 - \cos 2u) \, du$$

$$= \frac{9}{8} \int_0^{\pi} (1 - \cos 2u) \, du$$

$$= \frac{9}{8} \int_0^{\pi} (1 - \cos 2u) \, du$$

$$= \frac{9}{8}\left(u - \frac{1}{2} \sin 2u\right)\Big|_0^{\pi}$$

$$= \boxed{\frac{9\pi}{8}}$$