

Problem 16: Let  $f$  and  $g$  be real-valued functions such that

$$\lim_{x \rightarrow a} f(x) = L \text{ and } \lim_{x \rightarrow a} g(x) = M$$

where  $a$ ,  $L$ , and  $M$  are real numbers. Prove that

$$\lim_{x \rightarrow a} (fg)(x) = LM$$

Proof:

Let  $\epsilon_1 > 0$ . Let  $\epsilon_2 > 0$  be a number that we will choose later.

For all  $\epsilon_2 > 0$  there exists a  $\delta_f > 0$  and a  $\delta_g > 0$  such that

$$\begin{aligned} 0 < |x - a| < \delta_f &\implies |f(x) - L| < \epsilon_2 \\ 0 < |x - a| < \delta_g &\implies |g(x) - M| < \epsilon_2 \end{aligned}$$

Let  $\delta = \min(\delta_f, \delta_g)$ . Then

$$\begin{aligned} 0 < |x - a| < \delta &\implies |f(x) - L| < \epsilon_2 \\ 0 < |x - a| < \delta &\implies |g(x) - M| < \epsilon_2 \end{aligned}$$

Case 1:  $L > 0$  and  $M > 0$

We can choose an  $\epsilon_2 > 0$  small enough so that  $L - \epsilon_2 > 0$  and  $M - \epsilon_2 > 0$

For all  $x$  in  $(a - \delta, a + \delta)$  we have

$$\begin{aligned} 0 < L - \epsilon_2 < f(x) < L + \epsilon_2 \\ 0 < M - \epsilon_2 < g(x) < M + \epsilon_2 \end{aligned}$$

It follows that

$$\begin{aligned} 0 < (L - \epsilon_2)(M - \epsilon_2) < f(x)g(x) < (L + \epsilon_2)(M + \epsilon_2) \\ 0 < LM - (L + M)\epsilon_2 + (\epsilon_2)^2 < f(x)g(x) < LM + (L + M)\epsilon_2 + (\epsilon_2)^2 \end{aligned}$$

Observe that  $|(L + M)\epsilon_2 + (\epsilon_2)^2| > |-(L + M)\epsilon_2 + (\epsilon_2)^2|$ . So

$$|f(x)g(x) - LM| < (L + M)\epsilon_2 + (\epsilon_2)^2$$

Now comes the part where we choose an  $\epsilon_2$ .

We want the expression  $(L + M)\epsilon_2 + (\epsilon_2)^2$  to be equal to  $\epsilon_1$ .

$$\begin{aligned}(\epsilon_2)^2 + (L + M)\epsilon_2 &= \epsilon_1 \\(\epsilon_2)^2 + (L + M)\epsilon_2 - \epsilon_1 &= 0\end{aligned}$$

We can get the solutions to this equation by using the quadratic formula.

$$\epsilon_2 = \frac{-(L + M) \pm \sqrt{(L + M)^2 + 4\epsilon_1}}{2}$$

The discriminant  $(L + M)^2 + 4\epsilon_1$  is positive, so the equation has two real solutions.

Let  $\epsilon_2$  be one of the solutions to this equation. Then

$$0 < |x - a| < d \implies |f(x)g(x) - LM| < (L + M)\epsilon_2 + (\epsilon_2)^2 = \epsilon_1$$

Thus  $\lim_{x \rightarrow a} f(x)g(x) = LM$

Case 2:  $L < 0$  and  $M < 0$

We can choose an  $\epsilon_2 > 0$  small enough so that  $L + \epsilon_2 < 0$  and  $M + \epsilon_2 < 0$

For all  $x$  in  $(a - \delta, a + \delta)$  we have

$$\begin{aligned}L - \epsilon_2 &< f(x) < L + \epsilon_2 < 0 \\M - \epsilon_2 &< g(x) < M + \epsilon_2 < 0\end{aligned}$$

It follows that

$$\begin{aligned}0 &< (L + \epsilon_2)(M + \epsilon_2) < f(x)g(x) < (L - \epsilon_2)(M - \epsilon_2) \\0 &< LM + \epsilon_2(L + M) + (\epsilon_2)^2 < f(x)g(x) < LM - \epsilon_2(L + M) + (\epsilon_2)^2\end{aligned}$$

Observe that

$$| -\epsilon_2(L + M) + (\epsilon_2)^2 | > | \epsilon_2(L + M) + (\epsilon_2)^2 |$$

So

$$|f(x)g(x) - LM| < (\epsilon_2)^2 - \epsilon_2(L + M)$$

We want the expression  $(\epsilon_2)^2 - \epsilon_2(L + M)$  to equal  $\epsilon_1$

$$\begin{aligned}(\epsilon_2)^2 - \epsilon_2(L + M) &= \epsilon_1 \\(\epsilon_2)^2 - \epsilon_2(L + M) - \epsilon_1 &= 0\end{aligned}$$

The quadratic equation above has a discriminant of

$$(-(L + M))^2 + 4\epsilon_1$$

The discriminant is positive, so the equation has two real solutions.

Let  $\epsilon_2$  be one of these solutions. Then

$$|f(x)g(x) - LM| < (\epsilon_2)^2 - \epsilon_2(L + M) = \epsilon_1$$

Thus  $\lim_{x \rightarrow a} f(x)g(x) = LM$

Case 3:  $L < 0$  and  $M > 0$

We can choose an  $\epsilon_2 > 0$  small enough so that  $L + \epsilon_2 < 0$  and  $M - \epsilon_2 > 0$

For all  $x$  in  $(a - \delta, a + \delta)$  we have

$$\begin{aligned} L - \epsilon_2 &< f(x) < L + \epsilon_2 < 0 \\ 0 &< M - \epsilon_2 < f(x) < M + \epsilon_2 \end{aligned}$$

It follows that

$$\begin{aligned} (L - \epsilon_2)(M + \epsilon_2) &< f(x)g(x) < (L + \epsilon_2)(M - \epsilon_2) \\ LM + \epsilon_2 L - \epsilon_2 M - (\epsilon_2)^2 &< f(x)g(x) < LM - \epsilon_2 L + \epsilon_2 M - (\epsilon_2)^2 \\ LM + \epsilon_2(L - M) - (\epsilon_2)^2 &< f(x)g(x) < LM + \epsilon_2(M - L) - (\epsilon_2)^2 \end{aligned}$$

Observe that

$$|\epsilon_2(L - M) - (\epsilon_2)^2| > |\epsilon_2(M - L) - (\epsilon_2)^2|$$

So

$$\begin{aligned} |f(x)g(x) - LM| &< -(\epsilon_2(L - M) - (\epsilon_2)^2) \\ |f(x)g(x) - LM| &< \epsilon_2(M - L) + (\epsilon_2)^2 \end{aligned}$$

We want the expression  $\epsilon_2(M - L) + (\epsilon_2)^2$  to be equal to  $\epsilon_1$

$$\begin{aligned} (\epsilon_2)^2 + (M - L)\epsilon_2 &= \epsilon_1 \\ (\epsilon_2)^2 + (M - L)\epsilon_2 - \epsilon_1 &= 0 \end{aligned}$$

The discriminant is  $(M - L)^2 + 4\epsilon_1$ .

Since the discriminant is positive, the quadratic equation has two real solutions.

Let  $\epsilon_2$  be one of these solutions. Then

$$|f(x)g(x) - LM| < \epsilon_2(M - L) + (\epsilon_2)^2 = \epsilon_1$$

Thus  $\lim_{x \rightarrow a} f(x)g(x) = LM$

Case 4:  $L = M = 0$

For all  $x$  in  $(a - \delta, a + \delta)$ , we have

$$\begin{aligned} -\epsilon_2 &< f(x) < \epsilon_2 \\ -\epsilon_2 &< g(x) < \epsilon_2 \end{aligned}$$

It follows that

$$-\epsilon^2 < f(x)g(x) < \epsilon^2$$

Thus

$$|f(x)g(x) - 0| < (\epsilon_2)^2$$

Now it's time to choose a number for  $\epsilon_2$ .

Let  $\epsilon = \sqrt{\epsilon_1}$ . Conveniently we have

$$|f(x)g(x) - 0| < \epsilon_1$$

Thus  $\lim_{x \rightarrow a} f(x)g(x) = LM = 0$

Case 5:  $L = 0$  and  $M > 0$

We can choose an  $\epsilon_2 > 0$  that is small enough so that  $M - \epsilon_2 > 0$

For all  $x$  in  $(a - \delta, a + \delta)$ , we have

$$\begin{aligned} -\epsilon_2 &< f(x) < \epsilon_2 \\ 0 &< M - \epsilon_2 < g(x) < M + \epsilon_2 \end{aligned}$$

It follows that

$$-\epsilon_2(M + \epsilon_2) < f(x)g(x) < \epsilon_2(M + \epsilon_2)$$

Thus

$$|f(x)g(x) - 0| < \epsilon_2(M + \epsilon_2)$$

Now it's time to choose a number for  $\epsilon_2$ .

We want the expression  $\epsilon_2(M + \epsilon_2)$  to be equal to  $\epsilon_1$

$$\begin{aligned}\epsilon_2(M + \epsilon_2) &= \epsilon_1 \\ (\epsilon_2)^2 + M\epsilon_2 - \epsilon_1 &= 0\end{aligned}$$

The discriminant of this quadratic equation is

$$M^2 + 4\epsilon_1$$

Since the discriminant is positive, the equation has two real solutions.

Let  $\epsilon_2$  be one of these solutions. Then

$$|f(x)g(x) - 0| < \epsilon_2(M + \epsilon_2) = \epsilon_1$$

Thus  $\lim_{x \rightarrow a} f(x)g(x) = LM = 0$

Case 6:  $L = 0$  and  $M < 0$

We can choose an  $\epsilon_2 > 0$  that is small enough so that  $M + \epsilon_2 < 0$

For all  $x$  in  $(a - \delta, a + \delta)$  we have

$$\begin{aligned}-\epsilon_2 &< f(x) < \epsilon_2 \\ M - \epsilon_2 &< g(x) < M + \epsilon_2 < 0\end{aligned}$$

It follows that

$$\epsilon_2(M - \epsilon_2) < f(x)g(x) < -\epsilon_2(M - \epsilon_2)$$

Thus

$$|f(x)g(x) - 0| < -\epsilon_2(M - \epsilon_2)$$

We want the expression  $-\epsilon_2(M - \epsilon_2)$  to equal  $\epsilon_1$

$$\begin{aligned}(\epsilon_2)^2 - M\epsilon_2 &= \epsilon_1 \\ (\epsilon_2)^2 - M\epsilon_2 - \epsilon_1 &= 0\end{aligned}$$

The discriminant of the quadratic equation is

$$M^2 + 4\epsilon_1$$

Since the discriminant is positive, the equation has two real solutions

Let  $\epsilon_2$  be one of these solutions. Then

$$|f(x)g(x) - 0| < -\epsilon_2(M - \epsilon_2) = \epsilon_1$$

Thus

$$\lim_{x \rightarrow a} f(x)g(x) = LM = 0$$

In all cases, we find that  $\lim_{x \rightarrow a} f(x)g(x) = LM$

We have proven what we set out to prove