Problem 6: Find all solutions to $z^6 + z^4 + z^3 + z^2 + 1 = 0$. (Source: AIME)

Let $p(z) = z^6 + z^4 + z^3 + z^2 + 1$. The sixth roots of unity are

$$1, e^{2\pi i/6}, e^{4\pi i/6}, e^{6\pi i/6}, e^{8\pi i/6}, e^{10\pi i/6}$$

In rectangular form, they are

$$1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Let's guess that $e^{2\pi i/6}$ is a root of $p(z)$. Plugging this in, we get

$$\begin{aligned}
(e^{2\pi i/6})^6 + (e^{2\pi i/6})^4 + (e^{2\pi i/6})^3 + (e^{2\pi i/6})^2 + 1 &= e^{2\pi i} + e^{8\pi i/6} + e^{6\pi i/6} + e^{4\pi i/6} + 1 \\
&= 1 + -\frac{1}{2} - \frac{\sqrt{3}}{2} + -1 - \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 \\
&= 0
\end{aligned}$$

Our guess was right. So $e^{2\pi i/6} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and its conjugate $e^{10\pi i/6} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ are both roots of $p(z)$.

Now $(z - (\frac{1}{2} + \frac{\sqrt{3}}{2}i))(z - (\frac{1}{2} - \frac{\sqrt{3}}{2}i)) = z^2 - z + 1$. Dividing $p(z)$ by $z^2 - z + 1$ gives us

$$p(z) = (z^2 - z + 1)(z^4 + z^3 + z^2 + z + 1)$$

When $z \neq 1$, we have $z^4 + z^3 + z^2 + z + 1 = \dfrac{z^5 - 1}{z - 1}$. So the roots of $z^4 + z^3 + z^2 + z + 1$ are all the fifth roots of unity except for 1. The fifth roots of unity (excepting 1) are

$$e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$$

Combining these roots with the two roots we found earlier, we now have all six roots of the polynomial $p(z)$.

$$\boxed{e^{2\pi i/6}, 2^{10\pi i/6}, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}}$$