Problem 1: Find the area of one leaf of the rose given by $r = 3\sin 2\theta$. (Source: AoPS Calculus)

We can divide the leaf into a number of small sectors, where each sector has an area of $\frac{d\theta}{2\pi}(\pi r^2)$. We can integrate this expression from $\theta = 0$ to $\theta = \frac{\pi}{2}$ to get the area of the leaf.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} (\pi r^2)\,\frac{d\theta}{2\pi} &= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} (3\sin 2\theta)^2\,d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} 9\sin^2 2\theta\,d\theta \\
&= \frac{9}{2}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\,d\theta
\end{aligned}$$

Now we are going to prove a trig identity that lets us integrate $\sin^2 2\theta$.

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
\sin^2 x &= \cos^2 x - \cos 2x \\
\sin^2 x &= (1 - \sin^2 x) - \cos 2x \\
2\sin^2 x &= 1 - \cos 2x \\
\sin^2 x &= \frac{1}{2}(1 - \cos 2x)
\end{aligned}$$

Let $u = 2\theta$. Then $\frac{du}{d\theta} = 2$.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} (\pi r^2)\,\frac{d\theta}{2\pi} &= \frac{9}{2}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\,d\theta \\
&= \frac{9}{2}\int_0^{\pi} \sin^2 u\,\frac{du}{2} \\
&= \frac{9}{4}\int_0^{\pi} \sin^2 u\,du \\
&= \frac{9}{4}\int_0^{\pi} \frac{1}{2}(1 - \cos 2u)\,du \\
&= \frac{9}{8}\int_0^{\pi} (1 - \cos 2u)\,du \\
&= \frac{9}{8}(u - \frac{1}{2}\sin 2u)\Big|_0^{\pi} \\
&= \boxed{\frac{9\pi}{8}}
\end{aligned}$$