Problem 14: Prove that

$$\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

Proof. Let $\epsilon > 0$. Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

Then there exist δ_f and δ_g such that

$$0 < |x-a| < \delta_f \quad \implies \quad |f(x) - L| < \frac{\epsilon}{2} \tag{1}$$
$$0 < |x-a| < \delta_g \quad \implies \quad |g(x) - M| < \frac{\epsilon}{2} \tag{2}$$

Let $\delta = \min(\delta_f, \delta_g)$. Then

$$0 < |x-a| < \delta \quad \implies \quad |f(x) - L| < \frac{\epsilon}{2} \tag{3}$$
$$0 < |x-a| < \delta \quad \implies \quad |g(x) - M| < \frac{\epsilon}{2} \tag{4}$$

We can add the inequalities on the right.

$$0 < |x-a| < \delta \quad \implies \quad |f(x) - L| + |g(x) - M| < \epsilon$$

Now we can use the triangle inequality. By the triangle inequality,

$$0 < |x-a| < \delta \quad \implies \quad |f(x) + g(x) - (L+M)| \leq |f(x) - L| + |g(x) - M| < \epsilon$$

Thus

$$\lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

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