Problem 4: Let $f(x)$ be a cubic polynomial and let $a > 0$ be a positive real number. Show that the average value of $f(x)$ on the interval $[-a, a]$ can be computed by taking the average of $f(\frac{a}{\sqrt{3}})$ and $f(-\frac{a}{\sqrt{3}})$.

(Source: Putnam)

Solution:

We can write $f(x)$ as $f(x) = c_3x^3 + c_2x^2 + c_1x + c_0$.

The average value of $f(x)$ on the interval $[-a, a]$ is given by

$$\begin{aligned}
\frac{1}{2a}\int_{-a}^{a} f(x)\ dx &= \frac{1}{2a}\int_{-a}^{a} \left(c_3x^3 + c_2x^2 + c_1x + c_0\right)\ dx \\
&= \left(\frac{c_3}{8a}x^4 + \frac{c_2}{6a}x^3 + \frac{c_1}{4a}x^2 + \frac{c_0}{2a}x\right)\Bigg|_{-a}^{a} \\
&= \left(\frac{c_3}{8a}a^4 + \frac{c_2}{6a}a^3 + \frac{c_1}{4a}a^2 + \frac{c_0}{2a}a\right) \\
&- \left(\frac{c_3}{8a}a^4 - \frac{c_2}{6a}a^3 + \frac{c_1}{4a}a^2 - \frac{c_0}{2a}a\right) \\
&= \frac{c_2}{3}a^2 + c_0
\end{aligned}$$

Now let's calculate the average of $f(\frac{a}{\sqrt{3}})$ and $f(-\frac{a}{\sqrt{3}})$.

$$\begin{aligned}
\frac{f(\frac{a}{\sqrt{3}}) + f(-\frac{a}{\sqrt{3}})}{2} &= \frac{1}{2}\Bigg(\frac{c_3}{3\sqrt{3}}a^3 + \frac{c_2}{3}a^2 + \frac{c_1}{\sqrt{3}}a + c_0 \\
&\quad - \frac{c_3}{3\sqrt{3}}a^3 + \frac{c_2}{3}a^2 - \frac{c_1}{\sqrt{3}}a + c_0\Bigg) \\
&= \frac{1}{2}\left(\frac{2c_2}{3}a^2 + 2c_0\right) \\
&= \frac{c_2}{3}a^2 + c_0
\end{aligned}$$

Our work shows us that

$$\frac{1}{2a}\int_{-a}^{a} f(x)\ dx = \frac{c_2}{3}a^2 + c_0 = \frac{f(\frac{a}{\sqrt{3}}) + f(-\frac{a}{\sqrt{3}})}{2}$$

In other words, the average value of $f(x)$ on the interval $[-a, a]$ is equal to the average value of $f(\frac{a}{\sqrt{3}})$ and $f(-\frac{a}{\sqrt{3}})$.

QED