Problem 18: Compute $\sum_{n=1}^{\infty} \frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)}$.

Solution:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)} &= \lim_{k\to\infty} \sum_{n=1}^{k} \frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)} \\ &= \lim_{k\to\infty} \sum_{n=1}^{k} \left(\frac{A}{3^{n+1}-2^{n+1}} + \frac{B}{3^n-2^n} \right)\end{aligned}$$

Now let's find solutions for A and B.

$$\begin{aligned}\frac{A}{3^{n+1}-2^{n+1}} + \frac{B}{3^n-2^n} &= \frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)} \\ \frac{A(3^n-2^n)}{(3^{n+1}-2^{n+1})(3^n-2^n)} + \frac{B(3^{n+1}-2^{n+1})}{(3^{n+1}-2^{n+1})(3^n-2^n)} &= \frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)} \\ A(3^n-2^n) + B(3^{n+1}-2^{n+1}) = 6^n \\ A3^n - A2^n + B3^{n+1} - B2^{n+1} = 6^n \\ 3^n(A+3B) - 2^n(A+2B) = 6^n\end{aligned}$$

We can set $A+3B = 2^n$ and $A+2B = 0$.

$$\begin{aligned}A+3B &= 2^n \\ A+2B &= 0 \\ B &= 2^n \\ A &= -2^{n+1}\end{aligned}$$

Now let's plug in A and B.

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)} &= \lim_{k\to\infty} \sum_{n=1}^{k} \left(\frac{A}{3^{n+1}-2^{n+1}} + \frac{B}{3^n-2^n} \right) \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} \left(\frac{-2^{n+1}}{3^{n+1}-2^{n+1}} + \frac{2^n}{3^n-2^n} \right) \\
&= \lim_{k\to\infty} \sum_{n=1}^{k} \left(\frac{2^n}{3^n-2^n} - \frac{2^{n+1}}{3^{n+1}-2^{n+1}} \right) \\
&= \lim_{k\to\infty} \left(\frac{2}{3-2} - \frac{2^{k+1}}{3^{k+1}-2^{k+1}} \right) \\
&= \lim_{k\to\infty} \frac{2}{3-2} - \lim_{k\to\infty} \frac{2^{k+1}}{3^{k+1}-2^{k+1}} \\
&= 2 - \lim_{k\to\infty} \frac{2^{k+1}}{3^{k+1}-2^{k+1}} \\
&= 2 - \lim_{k\to\infty} \frac{1}{(\frac{3}{2})^{k+1}-1} \\
&= 2 - 0 \\
&= \boxed{2}
\end{aligned}
$$