Problem 26: Let x and y be two k^{th} roots of unity. Prove that $(x+y)^k$ is real. (Source: HMMT)

We can apply the Binomial Theorem.

$$(x+y)^k = \binom{k}{0}x^k + \binom{k}{1}x^{k-1}y + \cdots + \binom{k}{k}y^k \tag{1}$$

Notice that $\binom{k}{n}x^n y^{k-n} = \overline{\binom{k}{k-n}x^{k-n}y^n}$ for all integers $0 \le n \le k$.

Thus $\binom{k}{n}x^n y^{k-n} + \binom{k}{k-n}x^{k-n}y^n$ has to be a real number.

If k is odd, we can pair the $\binom{k}{n}$ and $\binom{k}{k-n}$ terms in Equation 1 so that each term is paired with its conjugate. The resulting sum has to be real.

If k is even, then the middle term of Equation 1 will be

$$\binom{k}{k/2}x^{k/2}y^{k/2} = \binom{k}{k/2}(-1)(-1) = \binom{k}{k/2}$$

which is real. We can pair each of the remaining terms with its conjugate. The resulting sum has to be real.

In both cases we are able to turn Equation 1 into a sum of real numbers. Thus $(x+y)^k$ is real.