Problem 6: Compute

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

Let $x = \tan\theta$. We will use the quotient rule to get the derivative of $\tan\theta$.

$$(fg)' = \frac{f'g - fg'}{g^2} \qquad \text{(Quotient rule)}$$

Thus

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\tan\theta) = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

This means that

$$dx = \sec^2\theta\,d\theta$$

Substituting, we have

$$\int \frac{1}{1+x^2}\,dx = \int \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta$$

Note that

$$1+\tan^2\theta = \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

Using this identity, we have

$$\int \frac{1}{1+x^2}\,dx = \int \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta = \int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \int 1\,d\theta = \theta + C$$

Now

$$x = \tan\theta \quad \Longrightarrow \quad \theta = \arctan x$$

Thus

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

We have found the antiderivative of our function. Now it's time to compute the definite integral.

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left(\arctan x + C\right)\Bigg|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

To find this answer, we had to know the derivative of $\tan x$, the identity $1+\tan^2 x = \sec^2 x$, and the method of integrating by substitution.

Observe that the function

$$f(x) = \frac{1}{1+x^2}$$

is continuous. It is an even function because $f(x) = f(-x)$. The maximum value of this function is 1.