Problem 29: Prove de Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all positive integers n.

We wish to prove by induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all positive integers n.

Let θ be a real number. Suppose that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for some positive integer n. Then

$$\begin{aligned}
(\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta) \\
&\quad \text{by the properties of exponents} \\
&= (\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta) \\
&\quad \text{this follows from our inductive hypothesis} \\
&= \cos\theta\cos n\theta + i\cos\theta\sin n\theta + i\sin\theta\cos n\theta - \sin\theta\sin n\theta \\
&\quad \text{expanding} \\
&= \cos\theta\cos n\theta - \sin\theta\sin n\theta + i(\sin\theta\cos n\theta + \cos\theta\sin n\theta) \\
&\quad \text{rearranging terms} \\
&= \cos(\theta + n\theta) + i\sin(\theta + n\theta) \\
&\quad \text{by the angle sum and angle difference identities} \\
&= \cos((n+1)\theta) + i\sin((n+1)\theta) \\
&\quad \text{simplifying}
\end{aligned}$$

This proves our inductive step. If $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for some positive integer n, then $(\cos\theta + i\sin\theta)^{n+1} = \cos((n+1)\theta) + i\sin((n+1)\theta)$.

Now this proposition is true for the base case $n = 1$, since $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta = \cos(1\cdot\theta) + i\sin(1\cdot\theta)$.

Since we have proven the base case ($n = 1$) and the inductive step, we have completed our proof by induction.

Thus $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all positive integers n.