

Problem 1: If the six solutions of $x^6 = -64$ are written in the form $a + bi$, where a and b are real, then find the product of the solutions with $a > 0$. (Source: AHSME)

The sixth roots of unity are

$$1, e^{2\pi i/6}, e^{4\pi i/6}, e^{6\pi i/6}, e^{8\pi i/6}, e^{10\pi i/6}$$

Now let $x^6 = -64$. We have $x = \sqrt[6]{-64} = 2\sqrt[6]{-1} = 2i$.

We can get all six solutions if we multiply $2i$ by each of the sixth roots of unity.

$$x = 2i, 2ie^{2\pi i/6}, 2ie^{4\pi i/6}, 2ie^{6\pi i/6}, 2ie^{8\pi i/6}, 2ie^{10\pi i/6}$$

Converting these solutions to rectangular form, we get

$$\begin{aligned} x_1 &= 2i & x_2 &= 2i\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\sqrt{3} + i & x_3 &= 2i\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\sqrt{3} - i & x_4 &= 2i(-1 + 0) = -2i \\ x_5 &= 2i\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \sqrt{3} - i & x_6 &= 2i\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \sqrt{3} + i \end{aligned}$$

There are only two solutions with $a > 0$, where a is the real part of the complex number. These solutions are

$$x_5 = \sqrt{3} - i \text{ and } x_6 = \sqrt{3} + i. \text{ Their product is } \boxed{(\sqrt{3} - i)(\sqrt{3} + i) = 4}.$$