Problem 6: The function f defined by

$$f(x) = \frac{ax+b}{cx+d},$$

where a, b, c, and d are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$, and $f(f(x)) = x$ for all values of x except $-d/c$. Find the unique number that is not in the range of f. (Source: AIME)

Let's invert f by swapping x and y and solving for y.

$$\begin{aligned} x &= \frac{ay+b}{cy+d} \\ x(cy+d) &= ay+b \\ cxy+dx &= ay+b \\ ay-cxy &= dx-b \\ y(a-cx) &= dx-b \\ y &= \frac{dx-b}{a-cx} \end{aligned}$$

Thus $f^{-1}(x) = \frac{dx-b}{a-cx}$.

This equation tells us there is a discontinuity in f^{-1} at $x = \frac{a}{c}$.

Since $f(f(x)) = x$, we know that $f(x) = f^{-1}(x)$. So f and f^{-1} must have the same discontinuity.

It follows that $\frac{a}{c} = -\frac{d}{c}$. This brings us to our first result: $a = -d$.

Now we can create a system of two equations and three variables, using the information we are given.

$$f(x) = \frac{ax+b}{cx+d} = \frac{b-dx}{cx+d}$$

$$\begin{aligned} 19 &= \frac{b-19d}{19c+d} \\ 19(19c+d) &= b-19d \\ b - 19^2c - 38d = 0& \end{aligned}$$

$$\begin{aligned} 97 &= \frac{b-97d}{97c+d} \\ 97(97c+d) &= b-97d \\ b - 97^2c - 194d = 0& \end{aligned}$$

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\begin{aligned}
b - 19^2c - 38d &= 0 \\
b - 97^2c - 194d &= 0 \\
(97^2 - 19^2)c + 156d &= 0 \\
(97 - 19)(97 + 19)c + 156d &= 0 \\
(78)(116)c + 156d &= 0 \\
(78)(29)c + 39d &= 0 \\
(6)(29)c + 3d &= 0 \\
(2)(29)c + d &= 0 \\
d &= -58c \\
\frac{d}{c} &= -58 \\
-\frac{d}{c} &= 58
\end{aligned}
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Since f^{-1} is not continuous at $x = \frac{a}{c} = -\frac{d}{c} = 58$, we know that 58 is not in the range of f.

And f^{-1} has only one discontinuity.

So $\boxed{58}$ is the unique value that is not in the range of f.