Problem 7: Let $n \geq 2$ be an integer, and for any real number $0 \leq \alpha \leq 1$, let $C(\alpha)$ be the coefficient of x^n in the power series expansion of $(1+x)^\alpha$. Prove that

$$\int_0^1 \left(C(-t-1) \left(\sum_{k=1}^{n} \frac{1}{t+k} \right) \right) dt = (-1)^n n.$$

(Source: Putnam)

Proof. First we'll find the coefficient of x^n in the power series expansion of $(1+x)^\alpha$. We'll do this by giving the Taylor series for $(1+x)^\alpha$ at $x=0$. Let $f(x) = (1+x)^\alpha$. Then

$$(1+x)^\alpha = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n$$

Thus $C(\alpha) = \dfrac{f^n(0)}{n!}$.

Now let's compute the nth derivative of $f(x) = (1+x)^\alpha$.

$$\begin{aligned}
f(x) &= (1+x)^\alpha \\
f'(x) &= \alpha(1+x)^{\alpha-1} \\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \\
f^n(x) &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))(1+x)^{\alpha-n}
\end{aligned}$$

So the coefficient of x^n is

$$C(\alpha) = \frac{f^n(0)}{n!} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{n!}$$

Plugging in $-t-1$ for α, we get

$$C(-t-1) = \frac{(-t-1)(-t-2)\cdots(-t-n)}{n!} = \frac{(-1)^n(t+1)(t+2)\cdots(t+n)}{n!}$$

Now we can evaluate the integral.

$$\begin{aligned}
\int_0^1 \left(C(-t-1) \sum_{k=1}^{n} \frac{1}{t+k} \right) dt &= \int_0^1 \left(\frac{(-1)^n(t+1)(t+2)\cdots(t+n)}{n!} \sum_{k=1}^{n} \frac{1}{t+k} \right) dt \\
&= \frac{(-1)^n}{n!} \int_0^1 \left((t+1)(t+2)\cdots(t+n) \sum_{k=1}^{n} \frac{1}{t+k} \right) dt
\end{aligned}$$

Let $g(t) = (t+1)(t+2)\cdots(t+n)$. Then $\dfrac{dg}{dt} = (t+1)(t+2)\cdots(t+n)\displaystyle\sum_{k=1}^{n} \frac{1}{t+k}$. (We can prove this lemma by using the product rule for derivatives and inducting on n.) Thus

$$
\begin{aligned}
\int_0^1 \left(C(-t-1) \sum_{k=1}^{n} \frac{1}{t+k} \right) dt &= \frac{(-1)^n}{n!} \int_0^1 \left((t+1)(t+2)\cdots(t+n) \sum_{k=1}^{n} \frac{1}{t+k} \right) dt \\
&= \frac{(-1)^n}{n!} \int_0^1 \left(\frac{dg}{dt} \right) dt \\
&= \frac{(-1)^n}{n!} g(t) \bigg|_0^1 \\
&= \frac{(-1)^n}{n!} (g(1) - g(0)) \\
&= \frac{(-1)^n}{n!} ((n+1)! - n!) \\
&= \frac{(-1)^n}{n!} (n!((n+1) - 1)) \\
&= \frac{(-1)^n}{n!} (n! \cdot n) \\
&= (-1)^n \cdot n
\end{aligned}
$$

Quod erat demonstrandum. $\square$