Problem 21: Show that $\sin 4\theta$ cannot be expressed as a function of $\sin\theta$. (Source: AoPS Precalculus)

Suppose there is a function f such that $f(\sin\theta) = \sin 4\theta$.

Let $\theta = \dfrac{\pi}{3}$. Then $f\left(\sin\dfrac{\pi}{3}\right) = f\left(\dfrac{\sqrt{3}}{2}\right) = \sin\dfrac{4\pi}{3} = -\dfrac{\sqrt{3}}{2}$.

Let $\theta = \dfrac{2\pi}{3}$. Then $f\left(\sin\dfrac{2\pi}{3}\right) = f\left(\dfrac{\sqrt{3}}{2}\right) = \sin\dfrac{8\pi}{3} = \dfrac{\sqrt{3}}{2}$.

But $f\left(\dfrac{\sqrt{3}}{2}\right)$ cannot equal both $\dfrac{\sqrt{3}}{2}$ and $-\dfrac{\sqrt{3}}{2}$, since a function must have exactly one output for every input.

Thus there is no function f such that $f(\sin\theta) = \sin 4\theta$. This is equivalent to saying $\sin 4\theta$ cannot be expressed in terms of $\sin\theta$.