Problem 1: Compute the following definite integral:

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$$

(Source: Putnam)

Problem 2: Prove that the polynomial

$$(a-x)^6 - 3a(a-x)^5 + \frac{5}{2}a^2(a-x)^4 - \frac{1}{2}a^4(a-x)^2$$

takes only negative values for $0 < x < a$.

(Source: Putnam 1941)

Problem 3: How many positive integers N satisfy all of the following three conditions?

(i) N is divisible by 2020.
(ii) N has at most 2020 decimal digits.
(iii) The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

(Source: Putnam 2020)

Problem 4: Let $f(x)$ be a cubic polynomial and let $a > 0$ be a positive real number. Show that the average value of $f(x)$ on the interval $[-a, a]$ can be computed by taking the average of $f(\frac{a}{\sqrt{3}})$ and $f(-\frac{a}{\sqrt{3}})$.

(Source: Putnam)

Problem 5: Find all continuous positive functions $f(x)$, for $0 \leq x \leq 1$, such that

$$\int_0^1 f(x)\,dx = 1, \quad \int_0^1 f(x)\,x\,dx = \alpha, \quad \int_0^1 f(x)\,x^2\,dx = \alpha^2$$

where α is a given real number.

(Source: Putnam)

Problem 6: Compute

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

(Source: Art of Problem Solving Calculus Textbook)