Problem 27: If $\omega^{1997} = 1$ and $\omega \neq 1$, then evaluate

$$\frac{1}{1+\omega} + \frac{1}{1+\omega^2} + \cdots + \frac{1}{1+\omega^{1997}}.$$

(Source: Mandelbrot)

We can pair the $\dfrac{1}{1+w^k}$ term with the $\dfrac{1}{1+w^{1997-k}}$ term, for each integer $1 \leq k \leq 1996$.

This is because, for all $1 \leq k \leq 1996$, we have the relation

$$\frac{1}{1+w^k} + \frac{1}{1+w^{1997-k}} = \frac{1}{1+w^k} + \frac{1}{1+\overline{w^k}}$$

(This relation is due to the fact that $\overline{w^k} = w^{1997-k}$ for all integers k.)

We can now simplify the expression $\dfrac{1}{1+w^k} + \dfrac{1}{1+w^{1997-k}}$.

Let $w^k = a_k + b_k i$ for all integers $1 \leq k \leq 1996$, where a_k and b_k are real numbers.

$$\begin{aligned}
\frac{1}{1+w^k} + \frac{1}{1+w^{1997-k}} &= \frac{1}{1+w^k} + \frac{1}{1+\overline{w^k}} \\
&= \frac{1}{1+a_k+b_k i} + \frac{1}{1+a_k-b_k i} \\
&= \frac{1+a_k-b_k i}{(1+a_k+b_k i)(1+a_k-b_k i)} + \frac{1+a_k+b_k i}{(1+a_k+b_k i)(1+a_k-b_k i)} \\
&= \frac{1+a_k-b_k i}{(1+a_k)^2+b_k^2} + \frac{1+a_k+b_k i}{(1+a_k)^2+b_k^2} \\
&= \frac{2+2a_k}{(1+a_k)^2+b_k^2} \\
&= \frac{2+2a_k}{1+2a_k+a_k^2+b_k^2} \\
&= \frac{2+2a_k}{1+2a_k+1} \\
&= \frac{2+2a_k}{2+2a_k} \\
&= 1
\end{aligned}$$

Note that $a_k \neq -1$ since -1 is not one of the 1997^{th} roots of unity. This justifies our last step of reducing $\dfrac{2+2a_k}{2+2a_k}$ to 1.

Thus every pair of terms $\dfrac{1}{1+w^k} + \dfrac{1}{1+w^{1997-k}}$ adds up to 1.

There are 998 such pairs. That is,

$$
\begin{aligned}
&\frac{1}{1+\omega}+\frac{1}{1+\omega^2}+\cdots+\frac{1}{1+\omega^{1997}}\\
&=\left(\frac{1}{1+\omega}+\frac{1}{1+\omega^{1996}}\right)+\left(\frac{1}{1+\omega^2}+\frac{1}{1+\omega^{1995}}\right)+\cdots+\left(\frac{1}{1+\omega^{998}}+\frac{1}{1+\omega^{999}}\right)+\frac{1}{1+\omega^{1997}}\\
&=(998)(1)+\frac{1}{1+\omega^{1997}}\\
&=998+\frac{1}{1+1}\\
&=998+\frac{1}{2}\\
&=\frac{1997}{2}\\
&=998.5
\end{aligned}
$$

Therefore,

$$
\boxed{\frac{1}{1+\omega}+\frac{1}{1+\omega^2}+\cdots+\frac{1}{1+\omega^{1997}}=\frac{1997}{2}=998.5}
$$