Problem 5: Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that $\sum\limits_{n=1}^{\infty} a_n$ diverges. (Source: Putnam)

Let $b_n = \sum\limits_{k=f(n)}^{g(n)} a_k$, where $f(n) = 2^{n-1}$ and $g(n) = 2^n - 1$.

We can write out the first four terms of b_n to make the sequence easier to understand.

$$\begin{aligned}
b_1 &= a_1 \\
b_2 &= a_2 + a_3 \\
b_3 &= a_4 + a_5 + a_6 + a_7 \\
b_4 &= a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15}
\end{aligned}$$

For all positive integers n we have $b_n \geq a_1$. (This is because of the condition $0 < a_n \leq a_{2n} + a_{2n+1}$.)

Since $\sum\limits_{n=1}^{\infty} a_1$ diverges, we know by the Series Comparison Test that $\sum\limits_{n=1}^{\infty} b_n$ also diverges.

But $\sum\limits_{n=1}^{\infty} b_n$ is another way of writing $\sum\limits_{n=1}^{\infty} a_n$. That is, $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} a_n$. So $\sum\limits_{n=1}^{\infty} a_n$ diverges.