

Problem 14: Factor $x^7 + x^4 + x^3 + x + 1$ into the product of two polynomials with integer coefficients and positive degrees. (Source: AoPS Precalculus)

We can manipulate the expression $x^7 + x^4 + x^3 + x + 1$ until we get an expression that factors nicely.

$$\begin{aligned}x^7 + x^4 + x^3 + x + 1 &= x^4 + x^3 + x^2 + x + 1 + (x^7 - x^2) \\&= x^4 + x^3 + x^2 + x + 1 + x^2(x^5 - x^1) \\&= x^4 + x^3 + x^2 + x + 1 + x^2(x - 1)(x^4 + x^3 + x^2 + x + 1) \\&= (x^4 + x^3 + x^2 + x + 1)(1 + x^2(x - 1)) \\&= (x^4 + x^3 + x^2 + x + 1)(1 + x^3 - x^2) \\&= \boxed{(x^3 - x^2 + 1)(x^4 + x^3 + x^2 + x + 1)}\end{aligned}$$