Problem 16: Let f and g be real-valued functions such that

$$\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M$$

where a, L, and M are real numbers. Prove that

$$\lim_{x \to a} (fg)(x) = LM$$

Proof:

Let $\epsilon_1 > 0$. Let $\epsilon_2 > 0$ be a number that we will choose later.

For all $\epsilon_2 > 0$ there exists a $\delta_f > 0$ and a $\delta_g > 0$ such that

$$\begin{aligned} 0 < |x - a| < \delta_f &\implies |f(x) - L| < \epsilon_2 \\ 0 < |x - a| < \delta_g &\implies |g(x) - M| < \epsilon_2 \end{aligned}$$

Let $\delta = \min(\delta_f, \delta_g)$. Then

$$\begin{aligned} 0 < |x - a| < \delta &\implies |f(x) - L| < \epsilon_2 \\ 0 < |x - a| < \delta &\implies |g(x) - M| < \epsilon_2 \end{aligned}$$

Case 1: $L > 0$ and $M > 0$

We can choose an $\epsilon_2 > 0$ small enough so that $L - \epsilon_2 > 0$ and $M - \epsilon_2 > 0$

For all x in $(a - \delta, a + \delta)$ we have

$$\begin{aligned} 0 < L - \epsilon_2 < f(x) < L + \epsilon_2 \\ 0 < M - \epsilon_2 < g(x) < M + \epsilon_2 \end{aligned}$$

It follows that

$$\begin{aligned} 0 < (L - \epsilon_2)(M - \epsilon_2) < f(x)g(x) < (L + \epsilon_2)(M + \epsilon_2) \\ 0 < LM - (L + M)\epsilon_2 + (\epsilon_2)^2 < f(x)g(x) < LM + (L + M)\epsilon_2 + (\epsilon_2)^2 \end{aligned}$$

Observe that $|(L + M)\epsilon_2 + (\epsilon_2)^2| > | - (L + M)\epsilon_2 + (\epsilon_2)^2|$. So

$$|f(x)g(x) - LM| < (L + M)\epsilon_2 + (\epsilon_2)^2$$

Now comes the part where we choose an ϵ_2.

We want the expression $(L + M)\epsilon_2 + (\epsilon_2)^2$ to be equal to ϵ_1.

$$(\epsilon_2)^2 + (L+M)\epsilon_2 = \epsilon_1$$
$$(\epsilon_2)^2 + (L+M)\epsilon_2 - \epsilon_1 = 0$$

We can get the solutions to this equation by using the quadratic formula.

$$\epsilon_2 = \frac{-(L+M) \pm \sqrt{(L+M)^2 + 4\epsilon_1}}{2}$$

The discriminant $(L+M)^2 + 4\epsilon_1$ is positive, so the equation has two real solutions.

Let ϵ_2 be one of the solutions to this equation. Then

$$0 < |x-a| < d \implies |f(x)g(x) - LM| < (L+M)\epsilon_2 + (\epsilon_2)^2 = \epsilon_1$$

Thus $\lim_{x\to a} f(x)g(x) = LM$

Case 2: $L < 0$ and $M < 0$

We can choose an $\epsilon_2 > 0$ small enough so that $L + \epsilon_2 < 0$ and $M + \epsilon_2 < 0$

For all x in $(a-\delta, a+\delta)$ we have

$$L - \epsilon_2 < f(x) < L + \epsilon_2 < 0$$
$$M - \epsilon_2 < g(x) < M + \epsilon_2 < 0$$

It follows that

$$0 < (L+\epsilon_2)(M+\epsilon_2) < f(x)g(x) < (L-\epsilon_2)(M-\epsilon_2)$$
$$0 < LM + \epsilon_2(L+M) + (\epsilon_2)^2 < f(x)g(x) < LM - \epsilon_2(L+M) + (\epsilon_2)^2$$

Observe that

$$|-\epsilon_2(L+M) + (\epsilon_2)^2| > |\epsilon_2(L+M) + (\epsilon_2)^2|$$

So

$$|f(x)g(x) - LM| < (\epsilon_2)^2 - \epsilon_2(L+M)$$

We want the expression $(\epsilon_2)^2 - \epsilon_2(L+M)$ to equal ϵ_1

$$(\epsilon_2)^2 - \epsilon_2(L+M) = \epsilon_1$$
$$(\epsilon_2)^2 - \epsilon_2(L+M) - \epsilon_1 = 0$$

The quadratic equation above has a discriminant of

$$(-(L+M))^2 + 4\epsilon_1$$

The discriminant is positive, so the equation has two real solutions.

Let ϵ_2 be one of these solutions. Then

$$|f(x)g(x) - LM| < (\epsilon_2)^2 - \epsilon_2(L+M) = \epsilon_1$$

Thus $\lim_{x\to a} f(x)g(x) = LM$

Case 3: $L < 0$ and $M > 0$ =======

We can choose an $\epsilon_2 > 0$ small enough so that $L + \epsilon_2 < 0 and M - \epsilon 2 > 0$

For all x in $(a-\delta, a+\delta)$ we have

$$\begin{aligned}
L - \epsilon_2 < f(x) < L + \epsilon_2 < 0 \\
0 < M - \epsilon_2 < f(x) < M + \epsilon_2
\end{aligned}$$

It follows that

$$\begin{aligned}
&(L-\epsilon_2)(M+\epsilon_2) < f(x)g(x) < (L+\epsilon_2)(M-\epsilon_2) \\
&LM + \epsilon_2 L - \epsilon_2 M - (\epsilon_2)^2 < f(x)g(x) < LM - \epsilon 2L + \epsilon_2 M - (\epsilon_2)^2 \\
&LM + \epsilon_2(L-M) - (\epsilon_2)^2 < f(x)g(x) < LM + \epsilon_2(M-L) - (\epsilon_2)^2
\end{aligned}$$

Observe that

$$|\epsilon_2(L-M) - (\epsilon_2)^2| > |\epsilon_2(M-L) - (\epsilon_2)^2|$$

So

$$\begin{aligned}
|f(x)g(x) - LM| < -(\epsilon_2(L-M) - (\epsilon_2)^2) \\
|f(x)g(x) - LM| < \epsilon_2(M-L) + (\epsilon_2)^2
\end{aligned}$$

We want the expression $\epsilon_2(M-L) + (\epsilon_2)^2$ to be equal to ϵ_1

$$\begin{aligned}
(\epsilon_2)^2 + (M-L)\epsilon_2 = e1 \\
(\epsilon_2)^2 + (M-L)\epsilon_2 - e1 = 0
\end{aligned}$$

The discriminant is $(M-L)^2 + 4\epsilon_1$.

Since the discriminant is positive, the quadratic equation has two real solutions.

Let ϵ_2 be one of these solutions. Then

$$|f(x)g(x) - LM| < \epsilon_2(M - L) + (\epsilon_2)^2 = \epsilon 1$$

Thus $\lim_{x\to a} f(x)g(x) = LM$

Case 4: $L = M = 0$ =======

For all x in $(a - \delta, a + \delta)$, we have

$$\begin{aligned} &-\epsilon_2 < f(x) < \epsilon_2 \\ &-\epsilon_2 < g(x) < \epsilon_2 \end{aligned}$$

It follows that

$$-\epsilon^2 < f(x)g(x) < \epsilon^2$$

Thus

$$|f(x)g(x) - 0| < (\epsilon_2)^2$$

Now it's time to choose a number for ϵ_2.

Let $\epsilon = sqrt(\epsilon_1)$. Conveniently we have

$$|f(x)g(x) - 0| < \epsilon_1$$

Thus $lim_{x\to a} f(x)g(x) = LM = 0$

Case 5: $L = 0$ and $M > 0$ =======

We can choose an $\epsilon_2 > 0$ that is small enough so that $M - \epsilon_2 > 0$

For all x in $(a - \delta, a + \delta)$, we have

$$\begin{aligned} &-\epsilon_2 < f(x) < \epsilon_2 \\ &0 < M - \epsilon_2 < g(x) < M + \epsilon_2 \end{aligned}$$

It follows that

$$-\epsilon_2(M + \epsilon_2) < f(x)g(x) < \epsilon_2(M + \epsilon_2)$$

Thus

$$|f(x)g(x) - 0| < \epsilon_2(M + \epsilon_2)$$

Now it's time to choose a number for ϵ_2.

We want the expression $\epsilon_2(M + \epsilon 2)$ to be equal to ϵ_1

$$\epsilon_2(M + \epsilon_2) = \epsilon_1$$
$$(\epsilon_2)^2 + M\epsilon_2 - \epsilon_1 = 0$$

The discriminant of this quadratic equation is

$$M^2 + 4\epsilon_1$$

Since the discriminant is positive, the equation has two real solutions.

Let ϵ_2 be one of these solutions. Then

$$|f(x)g(x) - 0| < \epsilon 2(M + \epsilon_2) = \epsilon_1$$

Thus $\lim_{x\to a} f(x)g(x) = LM = 0$

Case 6: $L = 0$ and $M < 0$ =======

We can choose an $\epsilon_2 > 0$ that is small enough so that $M + \epsilon_2 < 0$

For all x in $(a - \delta, a + \delta)$ we have

$$-\epsilon 2 < f(x) < \epsilon_2$$
$$M - \epsilon_2 < g(x) < M + \epsilon_2 < 0$$

It follows that

$$\epsilon_2(M - \epsilon_2) < f(x)g(x) < -\epsilon_2(M - \epsilon_2)$$

Thus

$$|f(x)g(x) - 0| < -\epsilon_2(M - \epsilon_2)$$

We want the expression $-\epsilon_2(M - \epsilon_2)$ to equal ϵ_1

$$(\epsilon_2)^2 - M\epsilon_2 = \epsilon_1$$
$$(\epsilon_2)^2 - M\epsilon_2 - \epsilon_1 = 0$$

The discriminant of the quadratic equation is

$$M^2 + 4\epsilon_1$$

Since the discriminant is positive, the equation has two real solutions

Let ϵ_2 be one of these solutions. Then

$$|f(x)g(x) - 0| < -\epsilon_2(M - \epsilon_2) = \epsilon_1$$

Thus

$$\lim_{x \to a} f(x)g(x) = LM = 0$$

In all cases, we find that $\lim_{x \to a} f(x)g(x) = LM$

We have proven what we set out to prove