Problem 9: The perimeter of a rectangle is 36 mm. What is the greatest possible area of the rectangle?

(Source: Mathcounts 2023 State Sprint Exam)

The perimeter of a rectangle is $2(a+b)$, where a and b are width and height of the rectangle. Thus

$$2(a+b) = 36$$

We can divide both sides by 4 to get the arithmetic mean of a and b.

$$\frac{a+b}{2} = 9$$

Now we can use the AM-GM theorem, which says that the arithmetic mean is greater than or equal to the geometric mean, with equality if and only if $a = b$.

$$\frac{a+b}{2} \geq \sqrt{ab} \quad \Longrightarrow \quad 9 \geq \sqrt{ab} \quad \Longrightarrow \quad 81 \geq ab$$

The product ab is the area of the rectangle, and $ab = 81$ if and only if $a = b$.

Thus the greatest possible area of the rectangle is $\boxed{81\,\text{mm}^2}$. The rectangle is a square with side length 9.