Problem 7 lemma: Let $p(t) = \prod_{k=1}^{n}(t+k)$ where $n \in \mathbb{N}$ and $n \geq 2$. Prove that $p'(t) = \left(\prod_{k=1}^{n}(t+k)\right)\left(\sum_{k=1}^{n}\frac{1}{t+k}\right)$.

Proof. Let $p_m(t) = \prod_{k=1}^{m}(t+k)$ be a sequence of polynomials starting at $m = 2$.

Suppose that $(p_n)'(t) = \left(\prod_{k=1}^{n}(t+k)\right)\left(\sum_{k=1}^{n}\frac{1}{t+k}\right)$ for some natural number $n \geq 2$. Then

$$\begin{aligned}
(p_{(n+1)})'(t) &= (p_n(t) \cdot (t+n+1))' \\
&= (p_n)'(t) \cdot (t+n+1) + p_n(t) \\
&= \left(\prod_{k=1}^{n}(t+k)\right)\left(\sum_{k=1}^{n}\frac{1}{t+k}\right) \cdot (t+n+1) + \left(\prod_{k=1}^{n}(t+k)\right) \\
&= \left(\prod_{k=1}^{n}(t+k)\right)\left((t+n+1)\left(\sum_{k=1}^{n}\frac{1}{t+k}\right) + 1\right) \\
&= \left(\prod_{k=1}^{n}(t+k)\right)\left((t+n+1)\left(\sum_{k=1}^{n}\frac{1}{t+k}\right) + \frac{t+n+1}{t+n+1}\right) \\
&= \left(\prod_{k=1}^{n}(t+k)\right)(t+n+1)\left(\sum_{k=1}^{n+1}\frac{1}{t+k}\right) \\
&= \left(\prod_{k=1}^{n+1}(t+k)\right)\left(\sum_{k=1}^{n+1}\frac{1}{t+k}\right)
\end{aligned}$$

This proves the inductive step. If the lemma is true for some natural number $n \geq 2$, then it is true for $n+1$.

Now let's prove the base case for $n = 2$.

$$\begin{aligned}
p_2(t) &= (t+1)(t+2) \\
&= t^2 + 3t + 2
\end{aligned}$$

$$(p_2)'(t) = 2t + 3$$

$$\begin{aligned}
\left(\prod_{k=1}^{2}(t+k)\right)\left(\sum_{k=1}^{2}\frac{1}{t+k}\right) &= (t+1)(t+2)\left(\frac{1}{t+1} + \frac{1}{t+2}\right) \\
&= (t+2) + (t+1) \\
&= (2t+3)
\end{aligned}$$

This shows that $(p_2)'(t) = \left(\prod_{k=1}^{2}(t+k)\right)\left(\sum_{k=1}^{2}\frac{1}{t+k}\right) = 2t + 3$.

Now that we have proven the inductive step, and the base case where $n = 2$, we have completed our proof by induction. We have shown by induction that $p'(t) = \left(\prod_{k=1}^{n}(t+k)\right)\left(\sum_{k=1}^{n}\frac{1}{t+k}\right)$.

$\square$