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Problem 3: How many positive integers N satisfy all of the following three conditions?

(i) N is divisible by 2020.
(ii) N has at most 2020 decimal digits.
(iii) The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

(Source: Putnam 2020)

Solution:

Let $a_0 = 2020 * 55 = 111100$.

Let $a_n = a_{n-1} * 10^4 + a_0$.

We can write out some terms of this sequence.

$$(a_n) = 111100, 1111111100, 11111111111100, ...$$

We want our sequence (a_n) to be finite, since N has at most 2020 decimal digits.

We can verify that a_k has $4(k+1)+2$ digits.

Let's find the maximum value of k.

$$\begin{aligned}
4(k+1)+2 &= 2020 \\
4k+6 &= 2020 \\
4k &= 2014 \\
k &= 503.5
\end{aligned}$$

Our equations show that a_{503} has 2018 digits.

Thus the sequence (a_n) stops at a_{503} and has a total of 504 terms.

Every term in (a_n) satisfies the three conditions set out in the problem statement. Moreover, we can pad each term in (a_n) with zeroes to create new numbers that satisfy the three conditions.

A term a_n has $4n+6$ digits.

A term a_n can be used to generate $2020-(4n+6)+1$ numbers that satisfy the three conditions set out in the problem statement.

Let $f(n) = 2020-(4n+6)+1 = 2015-4n$.

The function $f(n)$ gives us the number of positive integers generated by a_n.

The number of positive integers N that satisfy all three conditions in the problem statement is given by the sum

$$\begin{aligned}\sum_{n=0}^{503} f(n) &= \sum_{n=0}^{503} 2015 - 4n \\ &= 504(2015+3)/2 \\ &= 504(2018)/2 \\ &= 504(1009) \\ &= 504,000 + 9 * 504 \\ &= 508,536\end{aligned}$$

There are $\boxed{508,536}$ positive integers that satisfy all three conditions.