Problem 18: The roots of $x^3 + ax^2 + bx + c = 0$ are $\alpha, \beta,$ and γ. Find the cubic polynomial whose roots are $\alpha^3, \beta^3,$ and γ^3. (Source: Putnam 1939 A3)

Using Vieta's formulas, we can write the cubic polynomial in terms of its roots. Letting $q(x)$ be the cubic polynomial, we have

$$q(x) = x^3 - (\alpha^3 + \beta^3 + \gamma^3)x^2 + (\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3)x - \alpha^3\beta^3\gamma^3$$

Now we want to write the coefficients in terms of $a, b,$ and c. First we will find an expression for $\alpha^3 + \beta^3 + \gamma^3$. We can expand $(\alpha + \beta + \gamma)^3$ using the multinomial theorem.

$$(\alpha + \beta + \gamma)^3 = \alpha^3 + \beta^3 + \gamma^3 + 3(\alpha\beta^2 + \alpha\gamma^2 + \beta\alpha^2 + \beta\gamma^2 + \gamma\alpha^2 + \gamma\beta^2) + 6\alpha\beta\gamma$$

Now let $p(x) = x^3 + ax^2 + bx + c$ be the original polynomial. By Vieta's formulas, we have $a = -(\alpha + \beta + \gamma)$, and $b = \alpha\beta + \alpha\gamma + \beta\gamma$, and $c = -\alpha\beta\gamma$. This allows us to make many substitutions in our equation involving $(\alpha + \beta + \gamma)^3$.

$$(-a)^3 = \alpha^3 + \beta^3 + \gamma^3 + 3(\alpha\beta^2 + \alpha\gamma^2 + \beta\alpha^2 + \beta\gamma^2 + \gamma\alpha^2 + \gamma\beta^2) - 6c \qquad \text{(Equation 1)}$$

We want to isolate the expression $\alpha^3 + \beta^3 + \gamma^3$, but first we have to find a way of writing $\alpha\beta^2 + \alpha\gamma^2 + \beta\alpha^2 + \beta\gamma^2 + \gamma\alpha^2 + \gamma\beta^2$ in terms of $a, b,$ and c. We can accomplish this with the following trick.

$$(\alpha + \beta + \gamma)(\alpha\beta + \alpha\gamma + \beta\gamma) = \alpha\beta^2 + \alpha\gamma^2 + \beta\alpha^2 + \beta\gamma^2 + \gamma\alpha^2 + \gamma\beta^2 + 3\alpha\beta\gamma$$

Now we can substitute $a, b,$ and c into the above equation.

$$\begin{aligned} -ab &= \alpha\beta^2 + \alpha\gamma^2 + \beta\alpha^2 + \beta\gamma^2 + \gamma\alpha^2 + \gamma\beta^2 - 3c \\ -ab + 3c &= \alpha\beta^2 + \alpha\gamma^2 + \beta\alpha^2 + \beta\gamma^2 + \gamma\alpha^2 + \gamma\beta^2 \qquad \text{(Result 1)} \end{aligned}$$

Now we can simplify Equation 1 by substituting the expression from our first result.

$$\begin{aligned} (-a)^3 &= \alpha^3 + \beta^3 + \gamma^3 + 3(-ab + 3c) - 6c \\ -a^3 &= \alpha^3 + \beta^3 + \gamma^3 - 3ab + 9c - 6c \\ -a^3 &= \alpha^3 + \beta^3 + \gamma^3 - 3ab + 3c \\ -a^3 + 3ab - 3c &= \alpha^3 + \beta^3 + \gamma^3 \qquad \text{(Result 2)} \end{aligned}$$

This brings us to our second result: we have written the first symmetric sum of the roots of $q(x)$ in terms of $a, b,$ and c. Now we want to write the second symmetric sum of the roots of $q(x)$ in terms of $a, b,$ and c. We can do this by expanding the expression $(\alpha\beta + \alpha\gamma + \beta\gamma)^3$ and manipulating the equation that results.

$$(\alpha\beta + \alpha\gamma + \beta\gamma)^3 = \alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3 + 3\alpha\beta\gamma(\alpha\beta^2 + \alpha\gamma^2 + \beta\alpha^2 + \beta\gamma^2 + \gamma\alpha^2 + \gamma\beta^2) + 6\alpha^2\beta^2\gamma^2$$

Now $b = \alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3$, and $c^2 = \alpha^2\beta^2\gamma^2$. Also, $-c = \alpha\beta\gamma$. Furthermore, our first result gives us $-ab + 3c = \alpha\beta^2 + \alpha\gamma^2 + \beta\alpha^2 + \beta\gamma^2 + \gamma\alpha^2 + \gamma\beta^2$. So we can make these substitutions in the equation involving $(\alpha\beta + \alpha\gamma + \beta\gamma)^3$. After making these substitutions, we can isolate the expression $\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3$.

$$
\begin{aligned}
(\alpha\beta+\alpha\gamma+\beta\gamma)^3 &= \alpha^3\beta^3+\alpha^3\gamma^3+\beta^3\gamma^3+3\alpha\beta\gamma(\alpha\beta^2+\alpha\gamma^2+\beta\alpha^2+\beta\gamma^2+\gamma\alpha^2+\gamma\beta^2)+6\alpha^2\beta^2\gamma^2 \\
b^3 &= \alpha^3\beta^3+\alpha^3\gamma^3+\beta^3\gamma^3-3c(-ab+3c)+6c^2 \\
b^3+3c(-ab+3c)-6c^2 &= \alpha^3\beta^3+\alpha^3\gamma^3+\beta^3\gamma^3 \\
b^3-3abc+9c^2-6c^2 &= \alpha^3\beta^3+\alpha^3\gamma^3+\beta^3\gamma^3 \\
b^3-3abc+3c^2 &= \alpha^3\beta^3+\alpha^3\gamma^3+\beta^3\gamma^3 \qquad \text{(Result 3)}
\end{aligned}
$$

In our third result, we have written the second symmetric sum $\alpha^3\beta^3+\alpha^3\gamma^3+\beta^3\gamma^3$ in terms of $a, b,$ and c. Now we are ready to rewrite the polynomial $q(x)$ in terms of $a, b,$ and c.

$$
\begin{aligned}
q(x) &= x^3-(\alpha^3+\beta^3+\gamma^3)x^2+(\alpha^3\beta^3+\alpha^3\gamma^3+\beta^3\gamma^3)x-\alpha^3\beta^3\gamma^3 \\
&= x^3-(-a^3+3ab-3c)x^2+(b^3-3abc+3c^2)x+c^3 \\
&= x^3+(a^3-3ab+3c)x^2+(b^3-3abc+3c^2)x+c^3
\end{aligned}
$$

Finally we arrive at the cubic polynomial

$$
\boxed{q(x)=x^3+(a^3-3ab+3c)x^2+(b^3-3abc+3c^2)x+c^3}
$$

whose roots are $\alpha^3, \beta^3,$ and γ^3.