Problem 12: Let z be such that $z^7 = 1$ and $z \neq 1$. Compute the numerical value of

$$z^{10} + \frac{1}{z^{10}} + z^{30} + \frac{1}{z^{30}} + z^{50} + \frac{1}{z^{50}}.$$

(Source: AoPS Precalculus)

This is a problem where we are adding fractions.

We can give each fraction a common denominator, and then add them.

$$\begin{aligned}
z^{10} + \frac{1}{z^{10}} + z^{30} + \frac{1}{z^{30}} + z^{50} + \frac{1}{z^{50}} &= \frac{z^{60}}{z^{50}} + \frac{z^{40}}{z^{50}} + \frac{z^{80}}{z^{50}} + \frac{z^{20}}{z^{50}} + \frac{z^{100}}{z^{50}} + \frac{1}{z^{50}} \\
&= \frac{1 + z^{20} + z^{40} + z^{60} + z^{80} + z^{100}}{z^{50}} \\
&= \frac{\frac{z^{120}-1}{z^{20}-1}}{z^{50}} \\
&= \frac{\frac{z-1}{z^{6}-1}}{z} \\
&= \frac{\frac{z-1}{(z-1)(z^5+z^4+z^3+z^2+z+1)}}{z} \\
&= \frac{1}{z^6 + z^5 + z^4 + z^3 + z^2 + z} \\
&= \frac{1}{-1} \\
&= \boxed{-1}
\end{aligned}$$

Since the nonreal seventh roots of unity are all primitive roots of unity, the series $z^6 + z^5 + z^4 + z^3 + z^2 + z$ represents the sum of the nonreal seventh roots of unity. Thus $z^6 + z^5 + z^4 + z^3 + z^2 + z = -1$.

After a series of algebraic manipulations, we discover that

$$\boxed{z^6 + z^5 + z^4 + z^3 + z^2 + z = -1}$$