

Problem 13: The nonzero complex numbers a , b , and c satisfy $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$. What are the possible values of $\frac{a+b-c}{a-b+c}$? (Source: AoPS Precalculus)

Let $k = \frac{a}{b} = \frac{b}{c} = \frac{c}{a}$. Then $a = bk$, $b = ck$, and $c = ak$. Substituting these values in the denominator we get

$$\begin{aligned}\frac{a+b-c}{a-b+c} &= \frac{a+b-c}{bk - ck + ak} \\ &= \frac{a+b-c}{k(b-c+a)} \\ &= \frac{a+b-c}{k(a+b-c)} \\ &= \frac{1}{k}\end{aligned}$$

Now $k^3 = \frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a} = 1$, so k is a primitive cube root of unity.

$$\begin{aligned}\frac{a+b-c}{a-b+c} &= \frac{1}{k} \\ &= \frac{k^2}{k^3} \\ &= k^2\end{aligned}$$

The possible values of $\frac{a+b-c}{a-b+c}$ are the possible values of k^2 , which are

$$\boxed{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \text{ and } -\frac{1}{2} - \frac{\sqrt{3}}{2}i}$$