Problem 18: Prove that the polar equation $r = a\cos t + b\sin t$, where a, b, and t are real numbers, and either a or b is nonzero, is the graph of a circle. (Source: AoPS Calculus)

We can show that the polar equation $r = a\cos t + b\sin t$ is the graph of a circle by finding an equation that proves it's a circle.

Let $x(t) = r\cos t$ and $y(t) = r\sin t$. These parametric equations describe the same figure as the polar equation.

We hypothesize that the figure is a circle centered at $(a/2, b/2)$. If $(x - a/2)^2 + (y - b/2)^2$ is equal to a constant, then the figure is indeed a circle.

$$\begin{aligned}
\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 &= x^2 - ax + \frac{a^2}{4} + y^2 - by + \frac{b^2}{4} \\
&= (r\cos t)^2 - a(r\cos t) + \frac{a^2}{4} + (r\sin t)^2 - b(r\sin t) + \frac{b^2}{4} \\
&= r^2(\cos^2 t + \sin^2 t) - r(a\cos t + b\sin t) + \frac{a^2 + b^2}{4} \\
&= r^2 - r(a\cos t + b\sin t) + \frac{a^2 + b^2}{4} \\
&= (a\cos t + b\sin t)^2 - (a\cos t + b\sin t)^2 + \frac{a^2 + b^2}{4} \\
&= \frac{a^2 + b^2}{4}
\end{aligned}$$

We have shown that $\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}$. This equation proves that the polar equation $r = a\cos t + b\sin t$ is the graph of a circlc centered at $\left(\frac{a}{2}, \frac{b}{2}\right)$ with a radius of $\frac{\sqrt{a^2 + b^2}}{2}$.