Problem 7: Find the range of

$$f(A) = \frac{\sin A(3\cos^2 A + \cos^4 A + 3\sin^2 A + \sin^2 A \cos^2 A)}{\tan A(\sec A - \sin A \tan A)},$$

where A is not an integer multiple of $\pi/2$. (Source: AoPS Precalculus Ex 3.1.6 and HMMT)

Solution:

$$\begin{aligned}
f(A) &= \frac{\sin A(3\cos^2 A + \cos^4 A + 3\sin^2 A + \sin^2 A \cos^2 A)}{\tan A(\sec A - \sin A \tan A)} \\
&= \frac{\cos A(3\cos^2 A + \cos^4 A + 3\sin^2 A + \sin^2 A \cos^2 A)}{\sec A - \sin A \tan A} \\
&= \frac{\cos A(3\cos^2 A + \cos^4 A + 3\sin^2 A + \sin^2 A \cos^2 A)}{\frac{1-\sin^2 A}{\cos A}} \\
&= \frac{\cos A(3\cos^2 A + \cos^4 A + 3\sin^2 A + \sin^2 A \cos^2 A)}{\frac{\cos^2 A}{\cos A}} \\
&= \frac{\cos A(3\cos^2 A + \cos^4 A + 3\sin^2 A + \sin^2 A \cos^2 A)}{\cos A} \\
&= 3\cos^2 A + \cos^4 A + 3\sin^2 A + \sin^2 A \cos^2 A \\
&= 3(\cos^2 A + \sin^2 A) + \cos^2 A(\cos^2 A + \sin^2 A) \\
&= 3 + \cos^2 A
\end{aligned}$$

Since A is not an integer multiple of $\dfrac{\pi}{2}$, we know that $\cos^2 A \in (0, 1)$. Thus $\boxed{f(A) \in (3, 4)}$.