Problem 17: Prove that the polar equation $r = \cos t + \sin t$ is the graph of a circle. (Source: AoPS Calculus)

Converting from polar coordinates to rectangular coordinates, we have

$$\begin{aligned} x(t) &= r\cos t \\ &= (\cos t + \sin t)\cos t \\ &= \cos^2 t + \cos t \sin t \end{aligned}$$

$$\begin{aligned} y(t) &= r\sin t \\ &= (\cos t + \sin t)\sin t \\ &= \cos t \sin t + \sin^2 t \end{aligned}$$

We hypothesize that $r = \cos t + \sin t$ is a circle centered at $(1/2, 1/2)$. Our hypothesis is true if and only if

$$(x(t) - 1/2)^2 + (y(t) - 1/2)^2 = R^2$$

for some constant $R \in \mathbb{R}$. We will solve for R and show that R is a constant.

$$\begin{aligned} (x - 1/2)^2 + (y - 1/2)^2 &= R^2 \\ x^2 - x + 1/4 + y^2 - y + 1/4 &= R^2 \\ x^2 - x + y^2 - y + 1/2 &= R^2 \end{aligned}$$

Substituting $\cos t \sin t + \cos^2 t$ for x and $\cos t \sin t + \sin^2 t$ for y, we get

$$
\begin{aligned}
x^2 - x + y^2 - y + 1/2 &= R^2 \\
(\cos t \sin t + \cos^2 t)^2 - \cos t \sin t - \cos^2 t + (\cos t \sin t + \sin^2 t)^2 - \cos t \sin t - \sin^2 t + 1/2 &= R^2 \\
(\cos t \sin t + \cos^2 t)^2 + (\cos t \sin t + \sin^2 t)^2 - 2\cos t \sin t - 1 + 1/2 &= R^2 \\
(\cos t \sin t + \cos^2 t)^2 + (\cos t \sin t + \sin^2 t)^2 - 2\cos t \sin t - 1/2 &= R^2 \\
\cos^2 t \sin^2 t + 2\cos^3 t \sin t + \cos^4 t + \cos^2 t \sin^2 t + \cos t \sin^3 t + \sin^4 t - 2\cos t \sin t - 1/2 &= R^2 \\
\cos^2 t \sin^2 t + \cos^4 t + \cos^2 t \sin^2 t + \sin^4 t + 2\cos t \sin^3 t + 2\cos^3 t \sin t - 2\cos t \sin t - 1/2 &= R^2 \\
\cos^2 t(\sin^2 t + \cos^2 t) + \sin^2 t(\cos^2 t + \sin^2 t) + 2\cos t \sin^3 t + 2\cos^3 t \sin t - 2\cos t \sin t - 1/2 &= R^2 \\
\cos^2 t + \sin^2 t + 2\cos t \sin^3 t + 2\cos^3 t \sin t - 2\cos t \sin t - 1/2 &= R^2 \\
1 + 2\cos t \sin^3 t + 2\cos^3 t \sin t - 2\cos t \sin t - 1/2 &= R^2 \\
2\cos t \sin^3 t + 2\cos^3 t \sin t - 2\cos t \sin t + 1/2 &= R^2 \\
\cos t \sin t(2\sin^2 t + 2\cos^2 t - 2) + 1/2 &= R^2 \\
\cos t \sin t(2(\sin^2 t + \cos^2 t) - 2) + 1/2 &= R^2 \\
\cos t \sin t(2 - 2) + 1/2 &= R^2 \\
1/2 &= R^2 \\
R &= \sqrt{\frac{1}{2}} \\
R &= \frac{1}{\sqrt{2}} \\
R &= \frac{\sqrt{2}}{2}
\end{aligned}
$$

We have shown that R is a constant, and that $R = \frac{\sqrt{2}}{2}$. Thus the graph of $r = \cos t + \sin t$ is a circle centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$ with a radius of $\frac{\sqrt{2}}{2}$. We can represent this circle in rectangular coordinates with the equation

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

The rectangular equation $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$ and the polar equation $r = \cos\theta + \sin\theta$ both describe the same circle, a circle centered at $(1/2, 1/2)$ with a radius of $\frac{\sqrt{2}}{2}$.