

Problem 9: Evaluate the sum

$$\cos \frac{2\pi}{2n+1} + \cos \frac{2 \cdot 2\pi}{2n+1} + \cos \frac{3 \cdot 2\pi}{2n+1} + \cdots + \cos \frac{n \cdot 2\pi}{2n+1},$$

where n is a positive integer. (Source: AoPS Precalculus)

$$\text{Let } S = \cos \frac{2\pi}{2n+1} + \cos \frac{2 \cdot 2\pi}{2n+1} + \cos \frac{3 \cdot 2\pi}{2n+1} + \cdots + \cos \frac{n \cdot 2\pi}{2n+1}.$$

Since $\cos(2\pi - \theta) = \cos(\theta)$, we have $\cos \frac{(n)2\pi}{2n+1} = \frac{(n+1)2\pi}{2n+1}$ and $\cos \frac{(n-1)2\pi}{2n+1} = \cos \frac{(n+2)2\pi}{2n+1}$ and so on.

Thus

$$\begin{aligned} & \cos \frac{(n+1)2\pi}{2n+1} + \cos \frac{(n+2)2\pi}{2n+1} + \cdots + \cos \frac{(2n)2\pi}{2n+1} \\ &= \cos \frac{(n)2\pi}{2n+1} + \cos \frac{(n-1)2\pi}{2n+1} + \cdots + \cos \frac{2\pi}{2n+1} \end{aligned}$$

where each term is equal to the term above it.

$$\text{Adding these two equations, we get } \cos \frac{2\pi}{2n+1} + \cos \frac{2 \cdot 2\pi}{2n+1} + \cos \frac{3 \cdot 2\pi}{2n+1} + \cdots + \cos \frac{(2n) \cdot 2\pi}{2n+1} = 2S.$$

The series above is the sum of all the $(n+1)^{\text{th}}$ roots of unity except for 1. Adding 1 gives us the sum of the $(n+1)^{\text{th}}$ roots of unity, which is 0.

$$1 + \cos \frac{2\pi}{2n+1} + \cos \frac{2 \cdot 2\pi}{2n+1} + \cos \frac{3 \cdot 2\pi}{2n+1} + \cdots + \cos \frac{(2n) \cdot 2\pi}{2n+1} = 1 + 2S = 0$$

Now we can solve for S .

$$0 = 1 + 2S \implies S = -\frac{1}{2}$$

Thus

$$\boxed{\cos \frac{2\pi}{2n+1} + \cos \frac{2 \cdot 2\pi}{2n+1} + \cos \frac{3 \cdot 2\pi}{2n+1} + \cdots + \cos \frac{n \cdot 2\pi}{2n+1} = -\frac{1}{2}}$$