Problem 7 lemma: Let
$$p(t) = \prod_{k=1}^{n} (t+k)$$
 where $n \in \mathbb{N}$ and $n \ge 2$. Prove that $p'(t) = \prod_{k=1}^{n} (t+k) \sum_{k=1}^{n} \frac{1}{t+k}$.

Proof. Let $p_m(t) = \prod_{k=1}^m (t+k)$ be a sequence of polynomials starting at m=2.

Suppose that $(p_n)'(t) = \prod_{k=1}^n (t+k) \sum_{k=1}^n \frac{1}{t+k}$ for some natural number $n \ge 2$. Then

$$(p_{(n+1)})'(t) = (p_n(t) \cdot (t+n+1))'$$

$$= (p_n)'(t) \cdot (t+n+1) + p_n(t)$$

$$= \left(\prod_{k=1}^n (t+k) \sum_{k=1}^n \frac{1}{t+k}\right) \cdot (t+n+1) + \prod_{k=1}^n (t+k)$$

$$= \prod_{k=1}^n (t+k) \left(\sum_{k=1}^n \frac{1}{t+k} \cdot (t+n+1) + 1\right)$$

$$= \prod_{k=1}^n (t+k) \left(\sum_{k=1}^n \frac{1}{t+k} \cdot (t+n+1) + \frac{t+n+1}{t+n+1}\right)$$

$$= \left(\prod_{k=1}^n (t+k)\right) (t+n+1) \sum_{k=1}^{n+1} \frac{1}{t+k}$$

$$= \prod_{k=1}^{n+1} (t+k) \sum_{k=1}^{n+1} \frac{1}{t+k}$$

This proves the inductive step. If the lemma is true for some natural number $n \ge 2$, then it is true for n + 1. Now let's prove the base case for n = 2.

$$p_2(t) = (t+1)(t+2)$$

= $t^2 + 3t + 2$

$$(p_2)'(t) = 2t + 3$$

$$\prod_{k=1}^{2} (t+k) \sum_{k=1}^{2} \frac{1}{t+k} = (t+1)(t+2) \left(\frac{1}{t+1} + \frac{1}{t+2} \right)$$
$$= (t+2) + (t+1)$$
$$= (2t+3)$$

This shows that $(p_2)'(t) = \prod_{k=1}^{2} (t+k) \sum_{k=1}^{2} = 2t+3$.

Now that we have proven the inductive step, and the base case where n=2, we have completed our proof by induction. We have shown by induction that $p'(t) = \prod_{k=1}^{n} (t+k) \sum_{k=1}^{n} \frac{1}{t+k}$.