Problem 21. *Show that if $w = r \operatorname{cis} \alpha$ and $z = s \operatorname{cis} \beta$ (and $z \neq 0$), then*

$$\frac{w}{z} = \frac{r}{s} \operatorname{cis}(\alpha - \beta)$$

(Source: AoPS Precalculus)

Proof. First we will compute the reciprocal of z.

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{s \operatorname{cis} \beta} \\
&= \frac{1}{s(\cos\theta + i \sin\theta)} \\
&= \frac{1}{s(\cos\theta + i \sin\theta)} \cdot \frac{s(\cos\theta - i\sin\theta)}{s(\cos\theta - i\sin\theta)} \\
&= \frac{s(\cos\theta - i\sin\theta)}{s^2(\cos^2\theta + \sin^2\theta)} \\
&= \frac{s(\cos\theta - i\sin\theta)}{s^2} \\
&= \frac{\cos\theta - i\sin\theta}{s}
\end{aligned}$$

This yields the equation

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} \qquad \text{for all complex numbers } z \text{ where } z \neq 0$$

Dividing w by z, we get

$$\begin{aligned}
\frac{w}{z} &= r(\cos\alpha + i\sin\alpha) \cdot \left(\frac{\cos\beta - i\sin\beta}{s}\right) \\
&= \frac{r}{s}(\cos\alpha\cos\beta - i\cos\alpha\sin\beta + i\cos\beta\sin\alpha + \sin\alpha\sin\beta) \\
&= \frac{r}{s}(\cos\alpha\cos\beta + \sin\alpha\sin\beta + i(\sin\alpha\cos\beta - \cos\alpha\sin\beta)) \\
&= \frac{r}{s}(\cos(\alpha)\cos(-\beta) - \sin(\alpha)\sin(-\beta) + i(\sin(\alpha)\cos(-\beta) + \cos(\alpha)\sin(-\beta))) \\
&= \frac{r}{s}(\cos(\alpha - \beta) + i\sin(\alpha - \beta)) \\
&= \boxed{\frac{r}{s}\operatorname{cis}(\alpha - \beta)}
\end{aligned}$$

We can also do it this way.

$$\operatorname{cis}(\beta)\operatorname{cis}(\alpha - \beta) = \operatorname{cis}(\beta + \alpha - \beta) = \operatorname{cis}(\alpha)$$

Thus

$$\frac{\operatorname{cis}\alpha}{\operatorname{cis}\beta} = \operatorname{cis}(\alpha - \beta)$$

Now we can compute $\dfrac{w}{z}$.

$$\begin{aligned}
\frac{w}{z} &= \frac{r\operatorname{cis}\alpha}{s\operatorname{cis}\beta} \\
&= \frac{r}{s}\cdot\frac{\operatorname{cis}\alpha}{\operatorname{cis}\beta} \\
&= \boxed{\frac{r}{s}\operatorname{cis}(\alpha - \beta)}
\end{aligned}$$

$\square$