Problem 22: For how many positive integers n less than or equal to 1000 is

$$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$

true for all real t? (Source: AIME)

We have $\sin t = \cos(t - \frac{\pi}{2})$ and $\cos t = -\sin(t - \frac{\pi}{2})$. Thus

$$\begin{aligned}(\sin t + i\cos t)^n &= (\cos(t - \frac{\pi}{2}) - i\sin(t - \frac{\pi}{2}))^n \\ &= \cos(n(t - \frac{\pi}{2})) - i\sin(n(t - \frac{\pi}{2})) \\ &= \cos(nt - \frac{n\pi}{2}) - i\sin(nt - \frac{n\pi}{2})\end{aligned}$$

Now we have $\sin nt = \cos(nt - \frac{\pi}{2})$ and $\cos nt = -\sin(nt - \frac{\pi}{2})$.

So

$$\sin nt + i\cos nt = \cos(nt - \frac{\pi}{2}) - i\sin(nt - \frac{\pi}{2})$$

We want to find the positive integers $n \le 1000$ for which

$$\cos(nt - \frac{n\pi}{2}) - i\sin(nt - \frac{n\pi}{2}) = \cos(nt - \frac{\pi}{2}) - i\sin(nt - \frac{\pi}{2})$$

First we observe that the equation is true for $n = 1, 5, 9, ..., 997$.

It is not true for the even positive integers, that is, $n = 2, 4, 6....$

It is also not true for the sequence $n = 3, 7, 11, ...$

So the solutions are $n = 1, 5, 9, ..., 997$.

There are 250 numbers in the sequence $n = 1, 5, 9, ..., 997$.

So the equation

$$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$

is true for $\boxed{250}$ positive integers less than or equal to 1000.