Problem 28: A sequence of complex numbers z_0, z_1, z_2, ... satisfies the rule

$$z_{n+1} = \frac{iz_n}{\overline{z}_n}.$$

Suppose that $|z_0| = 1$ and $z_{2005} = 1$. How many possible values are there for z_0? (Source: AMC 12)

We can assume that $z_0 = e^{i\theta}$ for some real number θ since $|z_0| = 1$.

Now let's write out some terms of the sequence.

$$\begin{aligned}
z_1 &= \frac{iz_0}{\overline{z_0}} \\
&= \frac{ie^{i\theta}}{\overline{e^{i\theta}}} \\
&= \frac{ie^{i\theta}}{e^{-i\theta}} \\
&= ie^{2i\theta} \\
z_2 &= \frac{iz_1}{\overline{z_1}} \\
&= \frac{i(ie^{2i\theta})}{\overline{(ie^{2i\theta})}} \\
&= \frac{i(ie^{2i\theta})}{-ie^{-2i\theta}} \\
&= -ie^{4i\theta} \\
z_3 &= \frac{iz_2}{\overline{z_2}} \\
&= \frac{i(-ie^{4i\theta})}{\overline{(-ie^{4i\theta})}} \\
&= \frac{i(-ie^{4i\theta})}{ie^{-4i\theta}} \\
&= -ie^{8i\theta}
\end{aligned}$$

The first four terms of the sequence are $e^{i\theta}, ie^{2i\theta}, -ie^{4i\theta}, -ie^{8i\theta}$.

We can deduce that $z_n = -i(e^{i\theta})^{2^n} = -i(z_0)^{2^n}$ for $n \geq 2$.

Since $z_{2005} = 1$, we get the equation

$$z_{2005} = -i(z_0)^{2^{2005}} = 1$$

Multiplying both sides by i, we get

$$(z_0)^{2^{2005}} = i$$

There are 2^{2005} solutions to this equation, so there are $\boxed{2^{2005}}$ possible values of z_0.