Problem 11: Find the ellipse centered at the origin that runs through the points $(1,2)$ and $(3,1)$.

The equation of an ellipse centered at the origin with a width of $2a$ and a height of $2b$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Multiplying both sides by a^2b^2 we get

$$b^2x^2+a^2y^2=a^2b^2$$

Now we can plug in the coordinates of the points we are given to get a system of equations.

$$\begin{aligned} b^2+4a^2&=a^2b^2 && \text{Equation 1}\\ 9b^2+a^2&=a^2b^2 && \text{Equation 2}\end{aligned}$$

Now we can multiply both sides of Equation 2 by 4 in order to eliminate a variable.

$$\begin{aligned} b^2+4a^2&=a^2b^2 && \text{Equation 1}\\ 36b^2+4a^2&=4a^2b^2 && \text{Equation 2}\end{aligned}$$

Subtracting Equation 1 from Equation 2, we get

$$35b^2=3a^2b^2$$

Now we can divide both sides by b^2.

$$35=3a^2$$

This gives us

$$a^2=\frac{35}{3}\implies a=\sqrt{\frac{35}{3}}$$

Plugging our value for a into Equation 1 gives us

$$
\begin{aligned}
b^2 + 4\left(\frac{35}{3}\right) &= \frac{35b^2}{3} \\
4\left(\frac{35}{3}\right) &= \frac{32b^2}{3} \\
4 \times 35 &= 32b^2 \\
35 &= 8b^2 \\
b^2 &= \frac{35}{8} \\
b &= \sqrt{\frac{35}{8}}
\end{aligned}
$$

We have arrived at the result $a = \sqrt{\frac{35}{3}}$ and $b = \sqrt{\frac{35}{8}}$. Thus our equation is

$$\boxed{\frac{3x^2}{35} + \frac{8y^2}{35} = 1}$$