Problem 5: Let $\omega = e^{4\pi i/7}$. Evaluate

$$(2+\omega)(2+\omega^2)(2+\omega^3)(2+\omega^4)(2+\omega^5)(2+\omega^6).$$

(Source: AoPS Precalculus)

We can write ω as $\omega = e^{4\pi i/7} = e^{2\pi ik/7}$ where $k = 2$. Since $\omega^7 = 1$ and $\gcd(k, 7) = 1$, we know that ω is a primitive seventh root of unity.

By the fundamental theorem of algebra, we have

$$z^7 - 1 = (z-1)(z-\omega)(z-\omega^2)(z-\omega^3)(z-\omega^4)(z-\omega^5)(z-\omega^6)$$

Substituting $z = -2$ gives us an expression like the one in the problem statement.

$$\begin{aligned}
(-2)^7 - 1 &= (-2-1)(-2-\omega)(-2-\omega^2)(-2-\omega^3)(-2-\omega^4)(-2-\omega^5)(-2-\omega^6) \\
(-2)^7 - 1 &= (-1)^7(2+1)(2+\omega)(2+\omega^2)(2+\omega^3)(2+\omega^4)(2+\omega^5)(2+\omega^6) \\
(-2)^7 - 1 &= -3(2+\omega)(2+\omega^2)(2+\omega^3)(2+\omega^4)(2+\omega^5)(2+\omega^6) \\
-128 - 1 &= -3(2+\omega)(2+\omega^2)(2+\omega^3)(2+\omega^4)(2+\omega^5)(2+\omega^6) \\
-129 &= -3(2+\omega)(2+\omega^2)(2+\omega^3)(2+\omega^4)(2+\omega^5)(2+\omega^6) \\
43 &= (2+\omega)(2+\omega^2)(2+\omega^3)(2+\omega^4)(2+\omega^5)(2+\omega^6)
\end{aligned}$$

Thus $\boxed{(2+\omega)(2+\omega^2)(2+\omega^3)(2+\omega^4)(2+\omega^5)(2+\omega^6) = 43}$.