Problem 12: Find the ellipse centered at the origin that runs through the points $(1, 2)$ and $(5, 1)$.

The equation of an ellipse centered at the origin with a width of $2a$ and a height of $2b$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Multiplying both sides by a^2b^2 we get

$$b^2x^2 + a^2y^2 = a^2b^2$$

Now we can plug in the coordinates of the points we are given to get a system of equations.

$$\begin{aligned} b^2 + 4a^2 &= a^2b^2 && \text{Equation 1} \\ 25b^2 + a^2 &= a^2b^2 && \text{Equation 2} \end{aligned}$$

Now we can multiply both sides of Equation 2 by 4 in order to eliminate a variable.

$$\begin{aligned} b^2 + 4a^2 &= a^2b^2 && \text{Equation 1} \\ 100b^2 + 4a^2 &= 4a^2b^2 && \text{Equation 2} \end{aligned}$$

Subtracting Equation 1 from Equation 2, we get

$$99b^2 = 3a^2b^2$$

Now we can divide both sides by $3b^2$.

$$33 = a^2$$

This gives us

$$a = \sqrt{33}$$

Plugging our value for a into Equation 1 gives us

$$\begin{aligned} b^2 + 4(33) &= 33b^2 \\ 4(33) &= 32b^2 \\ 33 &= 8b^2 \\ \frac{33}{8} &= b^2 \\ b &= \sqrt{\frac{33}{8}} \end{aligned}$$

We have arrived at the result $a = \sqrt{33}$ and $b = \sqrt{\frac{33}{8}}$. Thus our equation is

$$\boxed{\frac{x^2}{33} + \frac{8y^2}{33} = 1}$$