Problem 10: Find $\lim_{x\to\infty} \dfrac{1}{x}$

(Source: AoPS Calculus)

Let $\epsilon > 0$. We want to find an N such that

$$x > N \quad \implies \quad \left|\frac{1}{x}\right| < \epsilon$$

Note that

$$\left|\frac{1}{x}\right| < \epsilon \quad \Leftrightarrow \quad |x| > \frac{1}{\epsilon}$$

where the $\Leftrightarrow$ symbol means "if and only if".

(We reverse the inequality after taking the reciprocal because both sides of the inequality have the same sign.)

Let $N = \dfrac{1}{\epsilon}$. Then

$$x > N = \frac{1}{\epsilon} \quad \implies \quad \left|\frac{1}{x}\right| < \epsilon$$

(Once again we reverse the inequality after taking the reciprocal, since both sides have the same sign.)

This proves that $\lim_{x\to\infty} \dfrac{1}{x} = 0$. $\square$