Problem 9: Two solutions of $x^4 - 3x^3 + 5x^2 - 27x - 36 = 0$ are pure imaginary numbers. Find these two solutions. (Source: ARML)

Let z_1, z_2, z_3, z_4 be the four complex roots of the polynomial. WLOG, z_1 and z_2 are pure imaginary numbers.

$$\begin{aligned}
x^4 - 3x^3 + 5x^2 - 27x - 36 &= 0 \\
(x - z_1)(x - z_2)(x - z_3)(x - z_4) &= 0 \\
x^4 - (z_1 + z_2 + z_3 + z_4)x^3 + (z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4)x^2 - (z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4)x + z_1z_2z_3z_4 &= 0
\end{aligned}$$

The coefficient of x^3 shows that $z_1 = -z_2$, since the coefficient is real. Also, $z_3 + z_4 = 3$.

We have

$$\begin{aligned}
z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4 &= 27 \\
-z_1^2z_3 - z_1^2z_4 + z_1z_3z_4 - z_1z_3z_4 &= 27 \\
-z_1^2z_3 - z_1^2z_4 &= 27 \\
-z_1^2(z_3 + z_4) &= 27 \\
-3z_1^2 &= 27 \\
z_1^2 &= -9 \\
z_1 &= \pm 3i
\end{aligned}$$

Thus $\boxed{z_1 = 3i \text{ and } z_2 = -3i}$ are the two solutions that are purely imaginary.