

Problem 16: Describe the graph of the polar equation $r = \cos \theta + \sin \theta$.

(Source: AoPS Calculus)

First let's find the slope of the line tangent to $r = \cos \theta + \sin \theta$ at any given point θ .

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

To compute $dy/d\theta$ and $dx/d\theta$, we need equations for y and x .

$$\begin{aligned} y &= r \sin \theta \\ &= (\cos \theta + \sin \theta) \sin \theta \\ &= \cos \theta \sin \theta + \sin^2 \theta \end{aligned}$$

$$\begin{aligned} x &= r \cos \theta \\ &= (\cos \theta + \sin \theta) \cos \theta \\ &= \cos^2 \theta + \cos \theta \sin \theta \end{aligned}$$

Now we are ready to compute $dy/d\theta$ and $dx/d\theta$.

$$\begin{aligned} \frac{dy}{d\theta} &= \frac{d}{d\theta} (\cos \theta \sin \theta + \sin^2 \theta) \\ &= -\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta \end{aligned} \quad \text{by the product rule and the chain rule}$$

$$\begin{aligned} \frac{dx}{d\theta} &= \frac{d}{d\theta} (\cos^2 \theta + \cos \theta \sin \theta) \\ &= -2 \sin \theta \cos \theta + -\sin^2 \theta + \cos^2 \theta \end{aligned} \quad \text{by the product rule and the chain rule}$$

Thus

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{-\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta}{-2 \sin \theta \cos \theta + -\sin^2 \theta + \cos^2 \theta} \end{aligned}$$

Setting $dy/d\theta = 0$, we get

$$\begin{aligned}
-\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta &= 0 \\
\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta &= 2 \sin^2 \theta \\
(\cos \theta + \sin \theta)^2 &= 2 \sin^2 \theta \\
\cos \theta + \sin \theta &= \pm \sqrt{2} \sin \theta \\
\cos \theta &= \pm \sqrt{2} \sin \theta - \sin \theta \\
\cos \theta &= \sin \theta (\pm \sqrt{2} - 1) \\
1 &= \tan \theta (\pm \sqrt{2} - 1) \\
\tan \theta &= \frac{1}{\pm \sqrt{2} - 1} \\
\tan \theta &= \left\{ \sqrt{2} + 1, 1 - \sqrt{2} \right\} \\
\theta &= \left\{ 1.17809725, -0.392699082 \right\}
\end{aligned}$$

The tangent line to the graph of $r = \cos \theta + \sin \theta$ has a slope of 0 at $\theta_1 = 1.17809725$ and $\theta_2 = -0.392699082$.

The rectangular coordinates corresponding to these polar coordinates are

$$\begin{aligned}
(x_1, y_1) &\approx (0.5, 1.2071067811865475) \\
(x_2, y_2) &\approx (0.5, -0.2071067811865475)
\end{aligned}$$

The graph of $r = \cos \theta + \sin \theta$ appears to be a circle centered at $(0.5, 0.5)$ with a radius of 0.7071067811865475.

I have not proven that the graph of $r = \cos \theta + \sin \theta$ is a circle, but it appears to be a circle.

What I have done is find the points on the graph of $r = \cos \theta + \sin \theta$ where the tangent line has a slope of 0. These points represent the coordinates on the graph where the y -values are smallest and largest. This allows us to find the radius of the circle (assuming that the graph is in fact a circle). It also allows us to find the center of the circle (since the diameter passes through the center).

Assuming that $r = \cos \theta + \sin \theta$ is the graph of a circle, we are able to conclude that the circle has a center of $(0.5, 0.5)$ in rectangular coordinates and a radius of approximately 0.7071067811865475.