Problem 17: Evaluate

$$\int_1^\infty \frac{dx}{e^{x+1} + e^{3-x}}$$

(Source: Putnam)

$$\begin{aligned}
\int_1^\infty \frac{dx}{e^{x+1} + e^{3-x}} &= \int_1^\infty \frac{dx}{e^{x+1} + e^{-(x-3)}} \\
&= \int_1^\infty \frac{dx}{e^{x+1} + \frac{1}{e^{x-3}}} \\
&= \int_1^\infty \frac{dx}{\frac{e^{x+1}e^{x-3}}{e^{x-3}} + \frac{1}{e^{x-3}}} \\
&= \int_1^\infty \frac{dx}{\frac{e^{x+1}e^{x-3}+1}{e^{x-3}}} \\
&= \int_1^\infty \frac{e^{x-3}}{e^{x+1}e^{x-3}+1}\, dx \\
&= \int_1^\infty \frac{e^{x-3}}{e^{2(x-1)}+1}\, dx \\
&= \frac{1}{e^2} \int_1^\infty \frac{u}{u^2+1} \cdot \frac{du}{u} && \text{substitute } u = e^{x-1} \\
&= \frac{1}{e^2} \int_1^\infty \frac{1}{u^2+1}\, du \\
&= \frac{1}{e^2} \lim_{b\to\infty} \int_1^b \frac{1}{u^2+1}\, du \\
&= \frac{1}{e^2} \lim_{b\to\infty} \left(\arctan u \Big|_{x=1}^{x=b} \right) \\
&= \frac{1}{e^2} \lim_{b\to\infty} \left(\arctan(e^{x-1}) \Big|_{x=1}^{x=b} \right) \\
&= \frac{1}{e^2} \lim_{b\to\infty} \big(\arctan(e^{b-1}) - \arctan(e^{1-1}) \big) \\
&= \frac{1}{e^2} \left(\frac{\pi}{2} - \arctan(1) \right) \\
&= \frac{1}{e^2} \left(\frac{\pi}{2} - \frac{\pi}{4} \right) \\
&= \boxed{\frac{\pi}{4e^2}}
\end{aligned}$$