Problem 10: Find the circle that runs through the points $(5,5)$, $(4,6)$, and $(6,2)$. Write your equation in the form $(x-a)^2+(y-b)^2=c^2$. Find the center and radius of this circle.

(Source: Linear Algebra with Applications, Exercise 1.1.39)

The equation of a circle with center (a,b) and radius c is

$$(x-a)^2+(y-b)^2=c^2$$

.

After expanding, and rearranging terms, we get

$$\begin{aligned}(x-a)^2+(y-b)^2&=c^2\\ x^2-2ax+a^2+y^2-2by+b^2&=c^2\\ x^2-2ax+a^2+y^2-2by+b^2-c^2&=0\end{aligned}$$

Now we can plug in the coordinates of our three points to get a system of three equations.

$$\begin{aligned}5^2-2a(5)+a^2+5^2-2b(5)+b^2-c^2&=0\\ 4^2-2a(4)+a^2+6^2-2b(6)+b^2-c^2&=0\\ 6^2-2a(6)+a^2+2^2-2b(2)+b^2-c^2&=0\end{aligned}$$

Simplifying, we get

$$\begin{aligned}-10a+a^2-10b+b^2-c^2&=-50 && \text{Equation 1}\\ -8a+a^2-12b+b^2-c^2&=-52 && \text{Equation 2}\\ -12a+a^2-4b+b^2-c^2&=-40 && \text{Equation 3}\end{aligned}$$

We can subtract Equation 1 from Equations 2 and 3 to eliminate some variables.

$$\begin{aligned}2a-2b&=-2 && \text{Equation 2}\\ -2a+6b&=10 && \text{Equation 3}\end{aligned}$$

Now we can add equations 2 and 3 to solve for b.

$$\begin{aligned}4b&=8\\ b&=2\\ 2a-2(2)&=-2\\ a-2&=-1\\ a&=1\end{aligned}$$

We arrive at the result $a=1$ and $b=2$. Let's plug this into equation 1 to solve for c.

$$\begin{aligned}
-10a + a^2 - 10b + b^2 - c^2 &= -50 && \text{Equation 1} \\
-10(1) + 1^2 - 10(2) + 2^2 - c^2 &= -50 \\
-10 + 1 - 20 + 4 - c^2 &= -50 \\
-25 - c^2 &= -50 \\
-c^2 &= -25 \\
c^2 &= 25 \\
c &= 5
\end{aligned}$$

Finally, we get $c = 5$.

Thus the circle we seek has center $(1, 2)$ and radius 5, and is given by the equation

$$(x - 1)^2 + (y - 2)^2 = 25$$