Problem 31: If $w = \cos 40^\circ + i \sin 40^\circ$, then express $\left|w + 2w^2 + 3w^3 + \cdots + 9w^9\right|^{-1}$ in the form $a \sin b$.

(Source: AHSME)

Let $S = w + 2w^2 + 3w^3 + \cdots + 9w^9$.

Then $wS = w^2 + 2w^3 + 3w^4 + \cdots + 9w^{10}$, and

$$\begin{aligned}
S - wS &= w + w^2 + w^3 + \cdots + w^9 - 9w^{10} \\
S(1-w) &= w + w^2 + w^3 + \cdots + w^9 - 9w^{10} \\
&= -9w \\
S &= -\frac{9w}{1-w} \\
&= \frac{9w}{w-1} \\
|S|^{-1} &= \frac{1}{|S|} \\
&= \frac{|w-1|}{|9w|} \\
&= \frac{|w-1|}{9} \\
&= \frac{|\cos 40^\circ + i \sin 40^\circ - 1|}{9} \\
&= \frac{|(\cos 40^\circ - 1) + i \sin 40^\circ|}{9} \\
&= \frac{\sqrt{(\cos 40^\circ - 1)^2 + \sin^2 40^\circ}}{9} \\
&= \frac{\sqrt{\cos^2 40^\circ - 2\cos 40^\circ + 1 + \sin^2 40^\circ}}{9} \\
&= \frac{\sqrt{2 - 2\cos 40^\circ}}{9}
\end{aligned}$$

Now we can use the half-angle identity for sine.

$$\begin{aligned}
\sin\frac{\theta}{2} &= \sqrt{\frac{1-\cos\theta}{2}} \\
\sin\frac{40^\circ}{2} &= \sqrt{\frac{1-\cos 40^\circ}{2}} \\
\sin 20^\circ &= \frac{1}{\sqrt{2}}\sqrt{1-\cos 40^\circ} \\
2\sin 20^\circ &= \sqrt{2 - 2\cos 40^\circ} \\
\frac{2}{9}\sin 20^\circ &= \frac{\sqrt{2-2\cos 40^\circ}}{9}
\end{aligned}$$

Thus

$$\boxed{|S|^{-1} = |w + 2w^2 + 3w^3 + \cdots + 9w^9|^{-1} = \frac{2}{9}\sin 20^\circ}$$