Problem 9: Evaluate the sum

$$\cos\frac{2\pi}{2n+1}+\cos\frac{2\cdot 2\pi}{2n+1}+\cos\frac{3\cdot 2\pi}{2n+1}+\cdots+\cos\frac{n\cdot 2\pi}{2n+1},$$

where n is a positive integer. (Source: AoPS Precalculus)

Let $S=\cos\frac{2\pi}{2n+1}+\cos\frac{2\cdot 2\pi}{2n+1}+\cos\frac{3\cdot 2\pi}{2n+1}+\cdots+\cos\frac{n\cdot 2\pi}{2n+1}$.

Since $\cos(2\pi-\theta)=\cos(\theta)$, we have $\cos\frac{(n)2\pi}{2n+1}=\frac{(n+1)2\pi}{2n+1}$ and $\cos\frac{(n-1)2\pi}{2n+1}=\cos\frac{(n+2)2\pi}{2n+1}$ and so on.

Thus

$$\begin{aligned}&\cos\frac{(n+1)2\pi}{2n+1}+\cos\frac{(n+2)2\pi}{2n+1}+\cdots+\cos\frac{(2n)2\pi}{2n+1}\\=&\cos\frac{(n)2\pi}{2n+1}+\cos\frac{(n-1)2\pi}{2n+1}+\cdots+\cos\frac{2\pi}{2n+1}\end{aligned}$$

where each term is equal to the term above it.

Adding these two equations, we get $\cos\frac{2\pi}{2n+1}+\cos\frac{2\cdot 2\pi}{2n+1}+\cos\frac{3\cdot 2\pi}{2n+1}+\cdots+\cos\frac{(2n)\cdot 2\pi}{2n+1}=2S$.

The series above is the sum of all the $(2n+1)^{\text{th}}$ roots of unity except for 1. Adding 1 gives us the sum of the $(2n+1)^{\text{th}}$ roots of unity, which is 0.

$$1+\cos\frac{2\pi}{2n+1}+\cos\frac{2\cdot 2\pi}{2n+1}+\cos\frac{3\cdot 2\pi}{2n+1}+\cdots+\cos\frac{(2n)\cdot 2\pi}{2n+1}=1+2S=0$$

Now we can solve for S.

$$1+2S=0 \implies S=-\frac{1}{2}$$

Thus

$$\boxed{\cos\frac{2\pi}{2n+1}+\cos\frac{2\cdot 2\pi}{2n+1}+\cos\frac{3\cdot 2\pi}{2n+1}+\cdots+\cos\frac{n\cdot 2\pi}{2n+1}=-\frac{1}{2}}$$