

Problem 17: Prove that the polar equation  $r = \cos t + \sin t$  is the graph of a circle. (Source: AoPS Calculus)

Converting from polar coordinates to rectangular coordinates, we have

$$\begin{aligned}x(t) &= r \cos t \\&= (\cos t + \sin t) \cos t \\&= \cos^2 t + \cos t \sin t\end{aligned}$$

$$\begin{aligned}y(t) &= r \sin t \\&= (\cos t + \sin t) \sin t \\&= \cos t \sin t + \sin^2 t\end{aligned}$$

We hypothesize that  $r = \cos t + \sin t$  is a circle centered at  $(1/2, 1/2)$ . Our hypothesis is true if and only if

$$(x(t) - 1/2)^2 + (y(t) - 1/2)^2 = R^2$$

for some constant  $R \in \mathbb{R}$ . We will solve for  $R$  and show that  $R$  is a constant.

$$\begin{aligned}(x - 1/2)^2 + (y - 1/2)^2 &= R^2 \\x^2 - x + 1/4 + y^2 - y + 1/4 &= R^2 \\x^2 - x + y^2 - y + 1/2 &= R^2\end{aligned}$$

Substituting  $\cos t \sin t + \cos^2 t$  for  $x$  and  $\cos t \sin t + \sin^2 t$  for  $y$ , we get

$$\begin{aligned}
x^2 - x + y^2 - y + 1/2 &= R^2 \\
(\cos t \sin t + \cos^2 t)^2 - \cos t \sin t - \cos^2 t + (\cos t \sin t + \sin^2 t)^2 - \cos t \sin t - \sin^2 t + 1/2 &= R^2 \\
(\cos t \sin t + \cos^2 t)^2 + (\cos t \sin t + \sin^2 t)^2 - 2 \cos t \sin t - 1 + 1/2 &= R^2 \\
(\cos t \sin t + \cos^2 t)^2 + (\cos t \sin t + \sin^2 t)^2 - 2 \cos t \sin t - 1/2 &= R^2 \\
\cos^2 t \sin^2 t + 2 \cos^3 t \sin t + \cos^4 t + \cos^2 t \sin^2 t + \cos t \sin^3 t + \sin^4 t - 2 \cos t \sin t - 1/2 &= R^2 \\
\cos^2 t \sin^2 t + \cos^4 t + \cos^2 t \sin^2 t + \sin^4 t + 2 \cos t \sin^3 t + 2 \cos^3 t \sin t - 2 \cos t \sin t - 1/2 &= R^2 \\
\cos^2 t (\sin^2 t + \cos^2 t) + \sin^2 t (\cos^2 t + \sin^2 t) + 2 \cos t \sin^3 t + 2 \cos^3 t \sin t - 2 \cos t \sin t - 1/2 &= R^2 \\
\cos^2 t + \sin^2 t + 2 \cos t \sin^3 t + 2 \cos^3 t \sin t - 2 \cos t \sin t - 1/2 &= R^2 \\
1 + 2 \cos t \sin^3 t + 2 \cos^3 t \sin t - 2 \cos t \sin t - 1/2 &= R^2 \\
2 \cos t \sin^3 t + 2 \cos^3 t \sin t - 2 \cos t \sin t + 1/2 &= R^2 \\
\cos t \sin t (2 \sin^2 t + 2 \cos^2 t - 2) + 1/2 &= R^2 \\
\cos t \sin t (2 (\sin^2 t + \cos^2 t) - 2) + 1/2 &= R^2 \\
\cos t \sin t (2 - 2) + 1/2 &= R^2 \\
1/2 &= R^2 \\
R &= \sqrt{1/2} \\
R &= \frac{1}{\sqrt{2}} \\
R &= \frac{\sqrt{2}}{2}
\end{aligned}$$

We have shown that  $R$  is a constant, and that  $R = \frac{\sqrt{2}}{2}$ . Thus the graph of  $r = \cos t + \sin t$  is a circle centered at  $\left(\frac{1}{2}, \frac{1}{2}\right)$  with a radius of  $\frac{\sqrt{2}}{2}$ . We can represent this circle in rectangular coordinates with the equation

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

The rectangular equation  $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$  and the polar equation  $r = \cos \theta + \sin \theta$  both describe the same circle, a circle centered at  $(1/2, 1/2)$  with a radius of  $\frac{\sqrt{2}}{2}$ .