Problem 22: Let $z = e^{i\theta}$, where $z \neq 1$. Show that $\dfrac{1+z}{1-z} = i\cot\dfrac{\theta}{2}$. (Source: AoPS Precalculus)

Let $z = a + bi$. We have

$$\begin{aligned}
\frac{1+z}{1-z} &= \frac{1+(a+bi)}{1-(a+bi)} \\
&= \frac{(1+a)+bi}{(1-a)-bi} \\
&= \frac{((1+a)+bi)((1-a)+bi)}{((1-a)-bi)((1-a)+bi)} \\
&= \frac{((1+a)+bi)((1-a)+bi)}{(1-a)^2+b^2} \\
&= \frac{1-a^2+(1+a)bi+(1-a)bi-b^2}{(1-a)^2+b^2} \\
&= \frac{1-a^2+2bi-b^2}{(1-a)^2+b^2} \\
&= \frac{1-(a^2+b^2)+2bi}{1-2a+a^2+b^2} \\
&= \frac{1-1+2bi}{1-2a+1} \\
&= \frac{2bi}{2-2a} \\
&= \frac{b}{1-a}i
\end{aligned}$$

Now $a = \cos\theta$ and $b = \sin\theta$. Making these substitutions we get

$$\frac{1+z}{1-z} = \frac{b}{1-a}i = \frac{\sin\theta}{1-\cos\theta}\,i$$

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Applying the half-angle identity for cot, we have

$$\frac{1+z}{1-z} = \frac{\sin\theta}{1-\cos\theta}\,i = \boxed{i\cot\frac{\theta}{2}}$$

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