Problem 16: Describe the graph of the equation $r = \cos\theta + \sin\theta$.

(Source: AoPS Calculus)

First let's find the slope of the line tangent to $r = \cos\theta + \sin\theta$ at any given point θ.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

To compute $dy/d\theta$ and $dx/d\theta$, we need equations for y and x.

$$\begin{aligned} y &= r\sin\theta \\ &= (\cos\theta + \sin\theta)\sin\theta \\ &= \cos\theta\sin\theta + \sin^2\theta \end{aligned}$$

$$\begin{aligned} x &= r\cos\theta \\ &= (\cos\theta + \sin\theta)\cos\theta \\ &= \cos^2\theta + \cos\theta\sin\theta \end{aligned}$$

Now we are ready to compute $dy/d\theta$ and $dx/d\theta$.

$$\begin{aligned} \frac{dy}{d\theta} &= \frac{d}{d\theta}\left(\cos\theta\sin\theta + \sin^2\theta\right) \\ &= -\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta && \text{by the product rule and the chain rule} \end{aligned}$$

$$\begin{aligned} \frac{dx}{d\theta} &= \frac{d}{d\theta}\left(\cos^2\theta + \cos\theta\sin\theta\right) \\ &= -2\sin\theta\cos\theta + -\sin^2\theta + \cos^2\theta && \text{by the product rule and the chain rule} \end{aligned}$$

Thus

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{-\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta}{-2\sin\theta\cos\theta + -\sin^2\theta + \cos^2\theta} \end{aligned}$$

Setting $dy/d\theta = 0$, we get

$$\begin{aligned}
-\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta &= 0 \\
\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta &= 2\sin^2\theta \\
(\cos\theta + \sin\theta)^2 &= 2\sin^2\theta \\
\cos\theta + \sin\theta &= \pm\sqrt{2}\sin\theta \\
\cos\theta &= \pm\sqrt{2}\sin\theta - \sin\theta \\
\cos\theta &= \sin\theta\left(\pm\sqrt{2} - 1\right) \\
1 &= \tan\theta\left(\pm\sqrt{2} - 1\right) \\
\tan\theta &= \frac{1}{\pm\sqrt{2} - 1} \\
\tan\theta &= \left\{\sqrt{2} + 1, 1 - \sqrt{2}\right\} \\
\theta &= \left\{1.17809725, -0.392699082\right\}
\end{aligned}$$

The tangent line to the graph of $r = \cos\theta + \sin\theta$ has a slope of 0 at $\theta_1 = 1.17809725$ and $\theta_2 = -0.392699082$.

The rectangular coordinates corresponding to these polar coordinates are

$$\begin{aligned}
(x_1, y_1) &\approx (0.5, 1.2071067811865475) \\
(x_2, y_2) &\approx (0.5, -0.2071067811865475)
\end{aligned}$$

The graph of $r = \cos\theta + \sin\theta$ appears to be a circle centered at $(0.5, 0.5)$ with a radius of 0.7071067811865475.

I have not proven that the graph of $r = \cos\theta + \sin\theta$ is a circle, but it appears to be a circle.

What I have done is find the points on the graph of $r = \cos\theta + \sin\theta$ where the tangent line has a slope of 0. These points represent the coordinates on the graph where the y-values are smallest and largest. This allows us to find the radius of the circle (assuming that the graph is in fact a circle). It also allows us to find the center of the circle (since the diameter passes through the center).

Assuming that $r = \cos\theta + \sin\theta$ is the graph of a circle, we are able to conclude that the circle has a center of $(0.5, 0.5)$ in rectangular coordinates and a radius of approximately 0.7071067811865475.