

Problem 13: Find the slope of the tangent line to $r = 3 \cos 2\theta$ at $\theta = \frac{\pi}{3}$.

(Source: AoPS Calculus)

We will use the formula $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

From our knowledge of polar coordinates, we know that $y = r \sin \theta$ and $x = r \cos \theta$. Since r is a function of θ (and not a constant) we will have to use the product rule when differentiating these equations.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} \\ &= \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}\end{aligned}$$

Now we have $r' = -6 \sin 2\theta$ by the chain rule. Substituting for r and r' , we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} \\ &= \frac{(-6 \sin 2\theta) (\sin \theta) + (3 \cos 2\theta) (\cos \theta)}{(-6 \sin 2\theta) (\cos \theta) - (3 \cos 2\theta) (\sin \theta)}\end{aligned}$$

We arrive at an equation for the slope of the tangent line in terms of the variable θ . We can plug in $\theta = \frac{\pi}{3}$ to get the slope of the tangent line at $\frac{\pi}{3}$ radians.

$$\begin{aligned}
\frac{dy}{dx} \left(\frac{\pi}{3} \right) &= \frac{\left(-6 \sin \frac{2\pi}{3} \right) \left(\sin \frac{\pi}{3} \right) + \left(3 \cos \frac{2\pi}{3} \right) \left(\cos \frac{\pi}{3} \right)}{\left(-6 \sin \frac{2\pi}{3} \right) \left(\cos \frac{\pi}{3} \right) - \left(3 \cos \frac{2\pi}{3} \right) \left(\sin \frac{\pi}{3} \right)} \\
&= \frac{(-3\sqrt{3}) \left(\frac{\sqrt{3}}{2} \right) + \left(-\frac{3}{2} \right) \left(\frac{1}{2} \right)}{(-3\sqrt{3}) \left(\frac{1}{2} \right) - \left(-\frac{3}{2} \right) \left(\frac{\sqrt{3}}{2} \right)} \\
&= \frac{\left(-\frac{18}{4} \right) + \left(-\frac{3}{4} \right)}{\left(-\frac{6\sqrt{3}}{4} \right) + \left(\frac{3\sqrt{3}}{4} \right)} \\
&= \frac{-18 + -3}{-6\sqrt{3} + 3\sqrt{3}} \\
&= \frac{-21}{-3\sqrt{3}} \\
&= \frac{7}{\sqrt{3}} \\
&= \boxed{\frac{7\sqrt{3}}{3}}
\end{aligned}$$

The slope of the tangent line at $\theta = \frac{\pi}{3}$ is $\boxed{\frac{7\sqrt{3}}{3}}$.