

Problem 18: Suppose we define a sequence of complex numbers by  $z_1 = 0$  and  $z_{n+1} = z_n^2 + i$  for  $n \geq 1$ . How far away from the origin is  $z_{111}$ ? (Source: AHSME)

Let's figure out the first five terms, and see if there is a pattern.

$$z_1 = 0$$

$$z_2 = (z_1)^2 + i = 0^2 + i = i$$

$$z_3 = (z_2)^2 + i = i^2 + i = i - 1$$

$$z_4 = (z_3)^2 + i = (i - 1)^2 + i = (i^2 - 2i + 1) + i = -i$$

$$z_5 = (z_4)^2 + i = (-i)^2 + i = i - 1$$

There is a pattern. The sequence contains the cycle  $i - 1, -i, i - 1$ .

So we can write the following:

$$z_n = \begin{cases} 0 & \text{if } n = 1 \\ i & \text{if } n = 2 \\ i - 1 & \text{if } n \text{ is odd and } n > 2 \\ -i & \text{if } n \text{ is even and } n > 2 \end{cases}$$

According to our formula,  $z_{111} = i - 1$ .

So  $|z_{111}| = |i - 1| = \sqrt{2}$ .

The complex number  $z_{111}$  is a distance of  $\boxed{\sqrt{2}}$  from the origin.