

Problem 4: Find the roots of $g(z) = z^6 + z^4 + z^2 + 1$. (Source: AoPS Precalculus)

Let $u = z^2$. Then $g(z) = u^3 + u^2 + u + 1$ and $ug(z) = u^4 + u^3 + u^2 + u$.

Subtracting the first equation from the second, we get

$$\begin{aligned} ug(z) - g(z) &= u^4 - 1 \\ g(z)(u - 1) &= u^4 - 1 \\ g(z) &= \frac{u^4 - 1}{u - 1} && \text{provided } z \neq 1 \text{ and } z \neq -1 \\ g(z) &= \frac{(z^2)^4 - 1}{z^2 - 1} \\ g(z) &= \frac{z^8 - 1}{z^2 - 1} \end{aligned}$$

The roots of $g(z)$ are all of the 8th roots of unity except for ± 1 .

$$z = \{e^{2\pi i/8}, e^{4\pi i/8}, e^{6\pi i/8}, e^{10\pi i/8}, e^{12\pi i/8}, e^{14\pi i/8}\}$$