Problem 24: The equation  $z^6 + z^3 + 1 = 0$  has one complex root with argument (angle)  $\theta$  between 90° and 180° in the complex plane. Determine the degree measure of  $\theta$ . (Source: AIME)

Let  $u = z^3$ . Then  $u^2 + u + 1 = 0$ .

$$u = \frac{-1 \pm \sqrt{1 - 4}}{2}$$
$$= \frac{-1 \pm \sqrt{-3}}{2}$$
$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Putting u in exponential form, we get

$$u = \{e^{2\pi i/3}, e^{4\pi i/3}\}$$

Now we can solve for z.

$$z^{3} = e^{2\pi i/3} \implies z = \{e^{2\pi i/9}, e^{8\pi i/9}, e^{14\pi i/9}\}$$
$$z^{3} = e^{4\pi i/3} \implies z = \{e^{4\pi i/9}, e^{10\pi i/9}, e^{16\pi i/9}\}$$

Combining all six solutions, we have

$$z = \{e^{2\pi i/9}, e^{4\pi i/9}, e^{8\pi i/9}, e^{10\pi i/9}, e^{14\pi i/9}, e^{16\pi i/9}\}$$

Only one of these roots is between 90° and 180°, and that root is  $z = e^{8\pi i/9}$ .

The angle measure of  $\frac{8\pi}{9}$  radians is equivalent to 160°.

Thus  $\theta = 160^{\circ}$ .