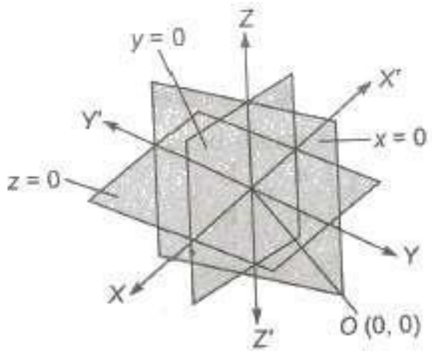
Three Dimensional Geometry

Coordinate System

The three mutually perpendicular lines in a space which divides the space into eight parts and if these perpendicular lines are the coordinate axes, then it is said to be a coordinate system.



Sign Convention

Octant Coordinate	x	y	z
OXYZ	+	+	+
OX'YZ	–	+	+
OXY'Z	+	–	+
OXYZ'	+	+	–
OX'Y'Z	–	–	+
OX'YZ'	–	+	–
OXY'Z'	+	–	–
OX'Y'Z'	–	–	–

Distance between Two Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. The distance between these points is given by

PQ $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

The distance of a point $P(x, y, z)$ from origin O is

$OP = \sqrt{x^2 + y^2 + z^2}$

Section Formulae

(i) The coordinates of any point, which divides the join of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio m : n internally are

$(mx_2 + nx_1 / m + n, my_2 + ny_1 / m + n, mz_2 + nz_1 / m + n)$

(ii) The coordinates of any point, which divides the join of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio m : n externally are

$(mx_2 – nx_1 / m – n, my_2 – ny_1 / m – n, mz_2 – nz_1 / m – n)$

(iii) The coordinates of mid-point of P and Q are

$(x_1 + x_2 / 2 , y_1 + y_2 / 2, z_1 + z_2 / 2)$

(iv) Coordinates of the centroid of a triangle formed with vertices $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ are

$(x_1 + x_2 + x_3 / 3 , y_1 + y_2 + y_3 / 3, z_1 + z_2 + z_3 / 3)$

(v) Centroid of a Tetrahedron

If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are the vertices of a tetrahedron, then its centroid G is given by

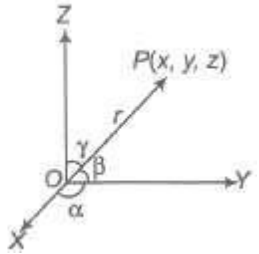
$(x_1 + x_2 + x_3 + x_4 / 4 , y_1 + y_2 + y_3 + y_4 / 4, z_1 + z_2 + z_3 + z_4 / 4)$

Direction Cosines

If a directed line segment OP makes angle α, β and γ with OX , OY and OZ respectively, then Cos α, cos β and cos γ are called direction cosines of up and it is represented by l, m, n.

i.e.,

$l = \cos \alpha$
$m = \cos \beta$
and $n = \cos \gamma$



If OP = r, then coordinates of OP are (lr, mr , nr)

(i) If 1, m, n are direction cosines of a vector r, then

(a) $r = |r| (li + mj + nk) \Rightarrow r = li + mj + nk$

(b) $l^2 + m^2 + n^2 = 1$

(c) Projections of r on the coordinate axes are

(d) $|r| = l|r|, m|r|, n|r| / \sqrt{\text{sum of the squares of projections of r on the coordinate axes}}$

(ii) If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, such that the direction cosines of PQ are l, m, n. Then,

$x_2 - x_1 = l|PQ|$, $y_2 - y_1 = m|PQ|$, $z_2 - z_1 = n|PQ|$

These are projections of PQ on X , Y and Z axes, respectively.

(iii) If 1, m, n are direction cosines of a vector r and a b, c are three numbers, such that l / a = m / b = n / c.

Then, we say that the direction ratio of r are proportional to a, b, c.

Also, we have

$l = a / \sqrt{a^2 + b^2 + c^2}$, $m = b / \sqrt{a^2 + b^2 + c^2}$, $n = c / \sqrt{a^2 + b^2 + c^2}$

(iv) If θ is the angle between two lines having direction cosines l_1, m_1, n_1 and $1_2, m_2, n_2$, then

$\cos\theta = l_1 1_2 + m_1 m_2 + n_1 n_2$

(a) Lines are parallel, if $l_1 / 1_2 = m_1 / m_2 = n_1 / n_2$

(b) Lines are perpendicular, if $l_1 1_2 + m_1 m_2 + n_1 n_2$

(v) If θ is the angle between two lines whose direction ratios are proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then the angle θ between them is given by

$\cos\theta = a_1a_2 + b_1b_2 + c_1c_2 / \sqrt{a^2_1 + b^2_1 + c^2_1} \sqrt{a^2_2 + b^2_2 + c^2_2}$

Lines are parallel, if $a_1 / a_2 = b_1 / b_2 = c_1 / c_2$

Lines are perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

(vi) The projection of the line segment joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ to the line having direction cosines 1, m, n is

$|(x_2 – x_1)l + (y_2 – y_1)m + (z_2 – z_1)n|$.

(vii) The direction ratio of the line passing through points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are proportional to $x_2 – x_1, y_2 – y_1 – z_2 – z_1$ Then, direction cosines of PQ are

$x_2 – x_1 / |PQ|, y_2 – y_1 / |PQ|, z_2 – z_1 / |PQ|$

Area of Triangle

If the vertices of a triangle be $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, then

$$\text{Area of } \Delta ABC = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

$$\text{where, } \Delta x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \Delta y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} \text{ and } \Delta z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Angle Between Two Intersecting Lines

If $l(x_1, m_1, n_1)$ and $l(x_2, m_2, n_2)$ be the direction cosines of two given lines, then the angle θ between them is given by

$\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

(i) The angle between any two diagonals of a cube is $\cos^{-1} (1 / 3)$.

(ii) The angle between a diagonal of a cube and the diagonal of a face (of the cube is $\cos^{-1} (\sqrt{2} / 3)$

Straight Line in Space

The two equations of the line $ax + by + cz + d = 0$ and $a' x + b' y + c' z + d' = 0$ together represents a straight line.

1. Equation of a straight line passing through a fixed point $A(x_1, y_1, z_1)$ and having direction ratios a, b, c is given by

$x – x_1 / a = y – y_1 / b = z – z_1 / c$, it is also called the symmetrically form of a line.

Any point P on this line may be taken as $(x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$, where $\lambda \in R$ is parameter. If a, b, c are replaced by direction cosines 1, m, n, then λ, represents distance of the point P from the fixed point A.

2. Equation of a straight line joining two fixed points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$x - x_1 / x_2 - x_1 = y - y_1 / y_2 - y_1 = z - z_1 / z_2 - z_1$

3. Vector equation of a line passing through a point with position vector a and parallel to vector b is r = a + λ b, where A, is a parameter.

4. Vector equation of a line passing through two given points having position vectors a and b is r = a + λ (b – a) , where λ is a parameter.

5. (a) The length of the perpendicular from a point $P(\vec{\alpha})$ on the line r – a + λ b is given by

$$\sqrt{|\vec{\alpha} - \mathbf{a}|^2 - \left\{\frac{(\vec{\alpha} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|}\right\}^2}$$

(b) The length of the perpendicular from a point $P(x_1, y_1, z_1)$ on the line

$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$ is given by

$$\sqrt{\{(a-x_1)^2 + (b-y_1)^2 + (c-z_1)^2\} - \{(a-x_1)\,l + (b-y_1)\,m + (c-z_1)\,n\}^2}$$

where, 1, m, n are direction cosines of the line.

6. **Skew Lines** Two straight lines in space are said to be skew lines, if they are neither parallel nor intersecting.

7. **Shortest Distance** If l_1 and l_2 are two skew lines, then a line perpendicular to each of lines 4 and 12 is known as the line of shortest distance.

If the line of shortest distance intersects the lines l_1 and l_2 at P and Q respectively, then the distance PQ between points P and Q is known as the shortest distance between l_1 and l_2.

8. The shortest distance between the lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$$

and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ is given by

$$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1\,m_2 - l_2\,m_1)^2}}$$

9. The shortest distance between lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$ is given by

$$d = \left| \frac{(\mathbf{b}_1 \times \mathbf{b}_2) \cdot (\mathbf{a}_2 - \mathbf{a}_1)}{|\mathbf{b}_1 \times \mathbf{b}_2|} \right|$$

10. The shortest distance parallel lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$ is given by

$$d = \left| \frac{(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}}{|\mathbf{b}|} \right|$$

11. Lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$ are intersecting lines, if $(b_1 * b_2) * (a_2 – a_1) = 0$.

12. The image or reflection (x, y, z) of a point (x_1, y_1, z_1) in a plane ax + by + cz + d = 0 is given by

$x – x_1 / a = y – y_1 / b = z – z_1 / c = – 2 (ax_1 + by_1 + cz_1 + d) / a^2 + b^2 + c^2$

13. The foot (x, y, z) of a point (x_1, y_1, z_1) in a plane ax + by + cz + d = 0 is given by

$x – x_1 / a = y – y_1 / b = z – z_1 / c = – (ax_1 + by_1 + cz_1 + d) / a^2 + b^2 + c^2$

14. Since, x, y and z-axes pass through the origin and have direction cosines (1, 0, 0), (0, 1, 0) and (0, 0, 1), respectively. Therefore, their equations are

x – axis : x – 0 / 1 = y – 0 / 0 = z – 0 / 0

y – axis : x – 0 / 0 = y – 0 / 1 = z – 0 / 0

z – axis : x – 0 / 0 = y – 0 / 0 = z – 0 / 1

Plane

A plane is a surface such that, if two points are taken on it, a straight line joining them lies wholly in the surface.

General Equation of the Plane

The general equation of the first degree in x, y, z always represents a plane. Hence, the general equation of the plane is ax + by + cz + d = 0. The coefficient of x, y and z in the cartesian equation of a plane are the direction ratios of normal to the plane.

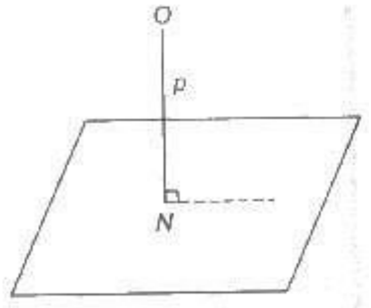
Equation of the Plane Passing Through a Fixed Point

The equation of a plane passing through a given point (x_1, y_1, z_1) is given by $a(x – x_1) + b (y — y_1) + c (z — z_1) = 0$.

Normal Form of the Equation of Plane

(i) The equation of a plane, which is at a distance p from origin and the direction cosines of the normal from the origin to the plane are l, m, n is given by lx + my + nz = p.

(ii) The coordinates of foot of perpendicular N from the origin on the plane are (1p, mp, np).



Intercept Form

The intercept form of equation of plane represented in the form of

x / a + y / b + z / c = 1

where, a, b and c are intercepts on X, Y and Z-axes, respectively.

For x intercept Put y = 0, z = 0 in the equation of the plane and obtain the value of x. Similarly, we can determine for other intercepts.

Equation of Planes with Given Conditions

(i) Equation of a plane passing through the point A(x_1, y_1, z_1) and parallel to two given lines with direction ratios

a_1, b_1, c_1 and a_2, b_2, c_2 is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

(ii) Equation of a plane through two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) and parallel to a line with direction ratios a, b, c is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \end{vmatrix} = 0.$$

(iii) The Equation of a plane passing through three points A(x_1, y_1, z_1), B(x_2, y_2, z_2) and C(x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

(iv) Four points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0.$$

(v) Equation of the plane containing two coplanar lines

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and $$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2} \text{ is}$$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

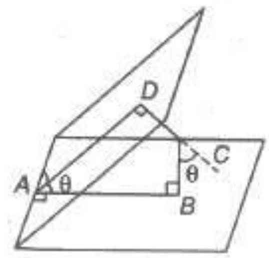
Angle between Two Planes

The angle between two planes is defined as the angle between the normal to them from any point.

Thus, the angle between the two planes

$a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$



is equal to the angle between the normals with direction cosines

$\pm a_1 / \sqrt{\Sigma a^2_1}, \pm b_1 / \sqrt{\Sigma a^2_1}, \pm c_1 / \sqrt{\Sigma a^2_1}$

and $\pm a_2 / \sqrt{\Sigma a^2_2}, \pm b_2 / \sqrt{\Sigma a^2_2}, \pm c_2 / \sqrt{\Sigma a^2_2}$

If θ is the angle between the normals, then

$\cos \theta = \pm a_1a_2 + b_1b_2 + c_1c_2 / \sqrt{a^2_1 + b^2_1 + c^2_1} \sqrt{a^2_2 + b^2_2 + c^2_2}$

Parallelism and Perpendicularity of Two Planes

Two planes are parallel or perpendicular according as the normals to them are parallel or perpendicular.

Hence, the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

are parallel, if $a_1 / a_2 = b_1 / b_2 = c_1 / c_2$ and perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Note The equation of plane parallel to a given plane $ax + by + cz + d = 0$ is given by $ax + by + cz + k = 0$, where k may be determined from given conditions.

Angle between a Line and a Plane

In Vector Form The angle between a line $r = a + \lambda b$ and plane $r *\bullet n = d$, is defined as the complement of the angle between the line and normal to the plane:

$\sin \theta = n * b / |n||b|$

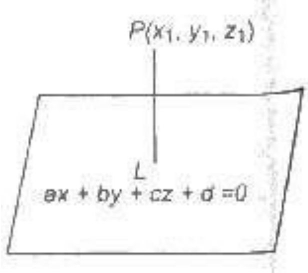
In Cartesian Form The angle between a line $x - x_1 / a_1 = y - y_1 / b_1 = z - z_1 / c_1$

and plane $a_2x + b_2y + c_2z + d_2 = 0$ is $\sin \theta = a_1a_2 + b_1b_2 + c_1c_2 / \sqrt{a^2_1 + b^2_1 + c^2_1} \sqrt{a^2_2 + b^2_2 + c^2_2}$

Distance of a Point from a Plane

Let the plane in the general form be $ax + by + cz + d = 0$. The distance of the point $P(x_1, y_1, z_1)$ from the plane is equal to

$$\left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$



If the plane is given in, normal form $lx + my + nz = p$. Then, the distance of the point $P(x_1, y_1, z_1)$ from the plane is $|lx_1 + my_1 + nz_1 - p|$.

Distance between Two Parallel Planes

If $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ be equation of two parallel planes. Then, the distance between them is

$$\left|\frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Bisectors of Angles between Two Planes

The bisector planes of the angles between the planes

$a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ is

$a_1x + b_1y + c_1z + d_1 / \sqrt{\Sigma a^2{}_1} = \pm\, a_2x + b_2y + c_2z + d_2 / \sqrt{\Sigma a^2{}_2}$

One of these planes will bisect the acute angle and the other obtuse angle between the given plane.

Sphere

A sphere is the locus of a point which moves in a space in such a way that its distance from a fixed point always remains constant.

General Equation of the Sphere

In Cartesian Form The equation of the sphere with centre (a, b, c) and radius r is

$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ …….(i)

In generally, we can write

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Here, its centre is (-u, v, w) and radius $= \sqrt{u^2 + v^2 + w^2 - d}$

In Vector Form The vector equation of a sphere of radius a and Centre having position vector c is $|r - c| = a$

Important Points to be Remembered

(i) The general equation of second degree in x, y, z is $ax^2 + by^2 + cz^2 + 2hxy + 2kyz + 2lzx + 2ux + 2vy + 2wz + d = 0$

represents a sphere, if

(a) $a = b = c\ (\neq 0)$

(b) $h = k = 1 = 0$

The equation becomes

$ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d - 0$ …(A)

To find its centre and radius first we make the coefficients of x^2, y^2 and z^2 each unity by dividing throughout by a.

Thus, we have

$x^2+y^2+z^2 + (2u / a)\, x + (2v / a)\, y + (2w / a)\, z + d / a = 0$ …..(B)

∴ Centre is $(- u / a, - v / a, - w / a)$

and radius $= \sqrt{u^2 / a^2 + v^2 / a^2 + w^2 / a^2 - d / a}$

$= \sqrt{u^2 + v^2 + w^2 - ad} / |a|$.

(ii) Any sphere concentric with the sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + k = 0$

(iii) Since, $r^2 = u^2 + v^2 + w^2 - d$, therefore, the Eq. (B) represents a real sphere, if $u^2 + v^2 + w^2 - d > 0$

(iv) The equation of a sphere on the line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) as a diameter is

$(x - x_1)(x - x_1) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$

(v) The equation of a sphere passing through four non-coplanar points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Tangency of a Plane to a Sphere

The plane $lx + my + nz = p$ will touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, if length of the perpendicular from the centre $(-u, -v, -w)$= radius,

i.e., $|lu - mv - nw - p| / \sqrt{l^2 + m^2 + n^2} = \sqrt{u^2 + v^2 + w^2 - d}$

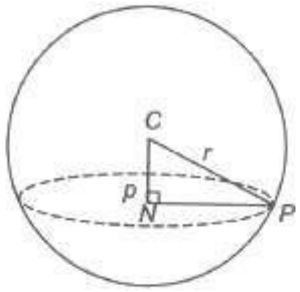
$(lu - mv - nw - p)^2 = (u^2 + v^2 + w^2 - d)(l^2 + m^2 + n^2)$

Plane Section of a Sphere

Consider a sphere intersected by a plane. The set of points common to both sphere and plane is called a plane section of a sphere.

In ΔCNP, $NP^2 = CP^2 - CN^2 = r^2 - p^2$

$\therefore NP = \sqrt{r^2 - p^2}$



Hence, the locus of P is a circle whose centre is at the point N, the foot of the perpendicular from the centre of the sphere to the plane.

The section of sphere by a plane through its centre is called a great circle. The centre and radius of a great circle are the same as those of the sphere.