

## Conic Section :- 2 Ellipse

Ellipse is the locus of a point in a plane which moves in such a way that the ratio of the distance from a fixed point (focus) in the same plane to its distance from a fixed straight line (directrix) is always constant, which is always less than unity.

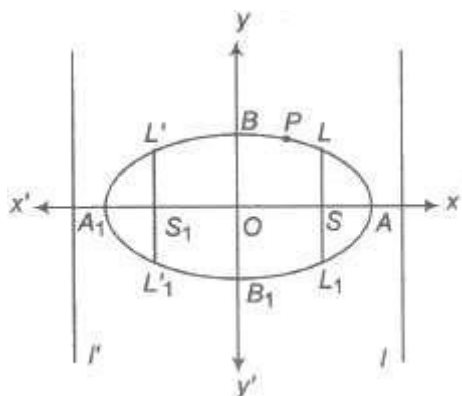
### Major and Minor Axes

The line segment through the foci of the ellipse with its end points on the ellipse, is called its major axis.

The line segment through the centre and perpendicular to the major axis with its end points on the ellipse, is called its minor axis.

**Horizontal Ellipse i.e.,  $x^2 / a^2 + y^2 / b^2 = 1$ ,  $0 < b < a$**

If the coefficient of  $x^2$  has the larger denominator, then its major axis lies along the x-axis, then it is said to be horizontal ellipse.



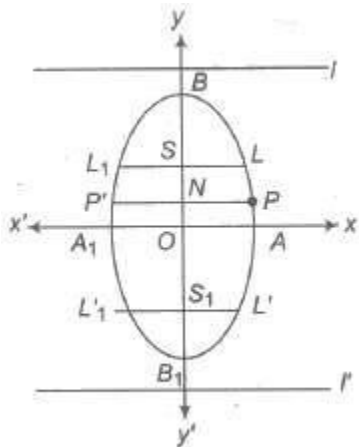
- (i) Vertices  $A(a, 0)$ ,  $A_1(-a, 0)$
- (ii) Centre  $(0, 0)$
- (iii) Major axis,  $AA_1 = 2a$ ; Minor axis,  $BB_1 = 2b$
- (iv) Foci are  $S(ae, 0)$  and  $S_1(-ae, 0)$
- (v) Directrices are  $l : x = a / e$ ,  $l' : x = -a / e$
- (vi) Latusrectum,  $LL_1 = L' L_1' = 2b^2 / a$
- (vii) Eccentricity,  $e = \sqrt{1 - b^2 / a^2} < 1$
- (viii) Focal distances are  $SP$  and  $S_1P$  i.e.,  $a - ex$  and  $a + ex$ . Also,  $SP + S_1P = 2a = \text{major axis}$ .

(ix) Distance between foci =  $2ae$

(x) Distance between directrices =  $2a / e$

**Vertical Ellipse i.e.,  $x^2 / a^2 + y^2 / b^2 = 1$ ,  $0 < a < b$**

If the coefficient of  $x^2$  has the smaller denominator, then its major axis lies along the y-axis, then it is said to be vertical ellipse.



(i) Vertices B(0, b), B<sub>1</sub>(0, -b)

(ii) Centre O(0,0)

(iii) Major axis BB<sub>1</sub> = 2b; Minor axis AA<sub>1</sub> = 2a

(iv) Foci are S(0, ae) and S<sub>1</sub>(0, -ae)

(v) Directrices are l : y = b / e ; l' : y = - b / e

(vi) Latusrectum LL<sub>1</sub> = L'L<sub>1</sub>' =  $2a^2 / b$

(vii) Eccentricity  $e = \sqrt{1 - a^2 / b^2} < 1$

(viii) Focal distances are SP and S<sub>1</sub>P.

i.e.,  $b - ex$  and  $b + ex$  axis.

Also,  $SP + S_1P = 2b = \text{major axis.}$

(ix) Distance between foci =  $2be$

(x) Distance between directrices =  $2b / e$

**Ordinate and Double Ordinate**

Let P be any point on the ellipse and PN be perpendicular to the major axis AA', such that PN produced meets the ellipse at P'. Then, PN is called the ordinate of P and PNP' is the double ordinate of P.

### Special Form of Ellipse

If centre of the ellipse is (h, k) and the direction of the axes are parallel to the coordinate axes, then its equation is  $(x - h)^2 / a^2 + (y - k)^2 / b^2 = 1$

### Position of a Point with Respect to an Ellipse

The point  $(x_1, y_1)$  lies outside, on or inside the ellipse  $x^2 / a^2 + y^2 / b^2 = 1$  according as  $x_1^2 / a^2 + y_1^2 / b^2 - 1 > 0, =$  or  $< 0$ .

### Auxiliary Circle

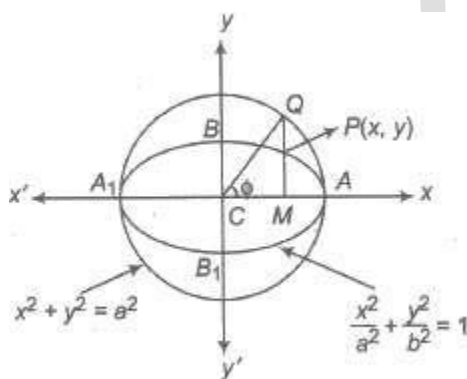
the ellipse  $x^2 / a^2 + y^2 / b^2 = 1$ , becomes the ellipse  $x^2 + y^2 = a^2$ , if  $b = a$ . This is called auxiliary circle of the ellipse. i. e., the circle described on the major axis of an ellipse as diameter is called auxiliary circle.

### Director Circle

The locus of the point of intersection of perpendicular tangents to an ellipse is a director circle. If equation of an ellipse is  $x^2 / a^2 + y^2 / b^2 = 1$ , then equation of director circle is  $x^2 + y^2 = a^2 + b^2$ .

### Eccentric Angle of a Point

Let P be any point on the ellipse  $x^2 / a^2 + y^2 / b^2 = 1$ . Draw PM perpendicular to the major axis from P on the major axis of the ellipse and produce MP to the auxiliary circle in Q. Join CQ. The  $\angle ACQ = \phi$  is called the eccentric angle of the point P on the ellipse.



### Parametric Equation

The equation  $x = a \cos \phi$ ,  $y = b \sin \phi$ , taken together are called the parametric equations of the ellipse  $x^2 / a^2 + y^2 / b^2 = 1$ , where  $\phi$  is any parameter.

## Equation of Chord

Let  $P(a \cos \theta, b \sin \theta)$  and  $Q(a \cos \phi, b \sin \phi)$  be any two points of the ellipse  $x^2/a^2 + y^2/b^2 = 1$ .

(i) The equation of the chord joining these points will be

$$(y - b \sin \theta) = b \sin \phi - b \sin \theta / a \cos \phi - a \sin \theta (x - a \cos \theta)$$

$$\text{or } x/a \cos(\theta + \phi/2) + y/b \sin(\theta + \phi/2) = \cos(\theta - \phi/2)$$

(ii) The equation of the chord of contact of tangents drawn from an point  $(x_1, y_1)$  to the ellipse  $x^2/a^2 + y^2/b^2 = 1$  is  $xx_1/a^2 + yy_1/b^2 = 1$ .

(iii) The equation of the chord of the ellipse  $x^2/a^2 + y^2/b^2 = 1$  bisected at the point  $(x_1, y_1)$  is given by

$$xx_1/a^2 + yy_1/b^2 - 1 = x_1^2/a^2 + y_1^2/b^2 - 1$$

$$\text{or } T = S_1$$

## Equation of Tangent

(i) **Point Form** The equation of the tangent to the ellipse  $x^2/a^2 + y^2/b^2 = 1$  at the point  $(x_1, y_1)$  is  $xx_1/a^2 + yy_1/b^2 = 1$ .

(ii) **Parametric Form** The equation of the tangent to the ellipse at the point  $(a \cos \theta, b \sin \theta)$  is  $x/a \cos \theta + y/b \sin \theta = 1$ .

(iii) **Slope Form** The equation of the tangent of slope  $m$  to the ellipse  $x^2/a^2 + y^2/b^2 = 1$  are  $y = mx \pm \sqrt{a^2m^2 + b^2}$  and the coordinates of the point of contact are

$$\left( \pm \frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \mp \frac{b^2}{\sqrt{a^2m^2 + b^2}} \right)$$

(iv) **Point of Intersection of Two Tangents** The equation of the tangents to the ellipse at points  $P(a \cos \theta_1, b \sin \theta_1)$  and  $Q(a \cos \theta_2, b \sin \theta_2)$  are

$$x/a \cos \theta_1 + y/b \sin \theta_1 = 1 \text{ and } x/a \cos \theta_2 + y/b \sin \theta_2 = 1$$

and these two intersect at the point

$$\left( \frac{a \cos \left( \frac{\theta_1 + \theta_2}{2} \right)}{\cos \left( \frac{\theta_1 - \theta_2}{2} \right)}, \frac{b \sin \left( \frac{\theta_1 + \theta_2}{2} \right)}{\cos \left( \frac{\theta_1 - \theta_2}{2} \right)} \right)$$

## Equation of Normal

(i) **Point Form** The equation of the normal at  $(x_1, y_1)$  to the ellipse  $x^2/a^2 + y^2/b^2 = 1$  is

$$a^2x/x_1 + b^2y/y_1 = a^2 - b^2$$

(ii) **Parametric Form** The equation of the normal to the ellipse  $x^2/a^2 + y^2/b^2 = 1$  at  $(a \cos \theta, b \sin \theta)$  is

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

(iii) **Slope Form** The equation of the normal of slope  $m$  to the ellipse  $x^2/a^2 + y^2/b^2 = 1$  are given by  $y = mx - m(a^2 - b^2)/\sqrt{a^2 + b^2m^2}$

and the coordinates of the point of contact are

$$\left( \pm \frac{a^2}{\sqrt{a^2 + b^2m^2}}, \pm \frac{b^2m}{\sqrt{a^2 + b^2m^2}} \right)$$

(iv) **Point of Intersection of Two Normals** Point of intersection of the normal at points  $(a \cos \theta_1, b \sin \theta_1)$  and  $(a \cos \theta_2, b \sin \theta_2)$  are given by

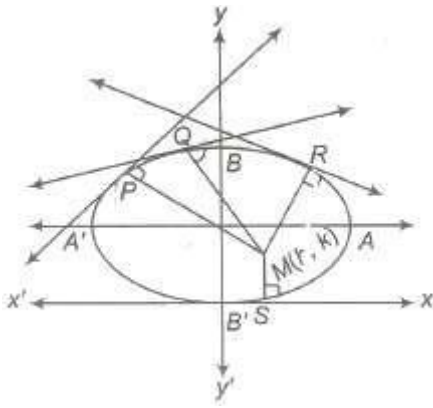
$$\left( \frac{a^2 - b^2}{a} \cos \theta_1 \cos \theta_2 \frac{\cos \left( \frac{\theta_1 + \theta_2}{2} \right)}{\cos \left( \frac{\theta_1 - \theta_2}{2} \right)}, \frac{-(a^2 - b^2)}{b} \sin \theta_1 \sin \theta_2 \frac{\sin \left( \frac{\theta_1 + \theta_2}{2} \right)}{\cos \left( \frac{\theta_1 - \theta_2}{2} \right)} \right)$$

(v) If the line  $y = mx + c$  is a normal to the ellipse  $x^2/a^2 + y^2/b^2 = 1$ ,

$$\text{then } c^2 = m^2(a^2 - b^2)^2 / (a^2 + b^2m^2)$$

## Conormal Points

The points on the ellipse, the normals at which the ellipse passes through a given point are called conormal points.



Here, P, Q, R and S are the conormal points.

(i) The sum of the eccentric angles of the conormal points on the ellipse,  $x^2/a^2 + y^2/b^2 = 1$  is an odd multiple of  $\pi$ .

(ii) If  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$  are eccentric angles of four points on the ellipse, the normals at which are concurrent, then

(a)  $\Sigma \cos (\theta_1 + \theta_2) = 0$

(b)  $\Sigma \sin (\theta_1 + \theta_2) = 0$

(iii) If  $\theta_1, \theta_2$  and  $\theta_3$  are the eccentric angles of three points on the ellipse  $x^2/a^2 + y^2/b^2 = 1$ , such that

$$\sin (\theta_1 + \theta_2) + \sin (\theta_2 + \theta_3) + \sin (\theta_3 + \theta_1) = 0,$$

then the normal at these points are concurrent.

(iv) If the normal at four points  $P(x_1, y_1)$ ,  $Q(x_2, y_2)$ ,  $R(x_3, y_3)$  and  $S(x_4, y_4)$  on the ellipse  $x^2/a^2 + y^2/b^2 = 1$ , are concurrent, then

$$(x_1 + x_2 + x_3 + x_4) (1/x_1 + 1/x_2 + 1/x_3 + 1/x_4) = 4$$

### Diameter and Conjugate Diameter

The locus of the mid-point of a system of parallel chords of an ellipse is called a diameter, whose equation of diameter is

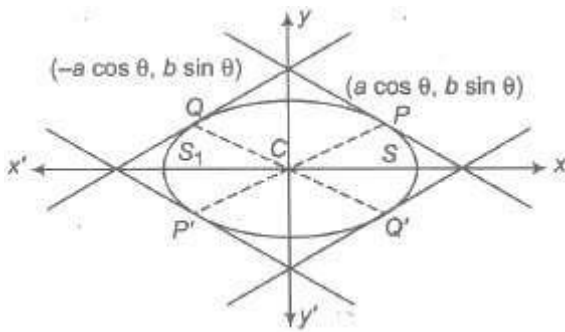
$$y = -(b^2/a^2m)x$$

Two diameters of an ellipse are said to be conjugate diameters, if each bisects the chords parallel to the other.

### Properties of Conjugate Diameters

- (i) The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle.
- (ii) The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi-axis of the ellipse i. e.,  $CP^2 + CD^2 = a^2 + b^2$ .
- (iii) If CP, CQ are two conjugate semi-diameters of an ellipse  $x^2/a^2 + y^2/b^2 = 1$  and S, S<sub>1</sub> be two foci of an ellipse, then

$$SP \cdot S_1P = CQ^2$$



- (iv) The tangent at the ends of a pair of conjugate diameters of an ellipse form a parallelogram.
- (v) The area of the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant and is equal to the product of the axis.

### Important Points

1. The point P(x<sub>1</sub> y<sub>1</sub>) lies outside, on or inside the ellipse  $x^2/a^2 + y^2/b^2 = 1$  according as  $x_1^2/a^2 + y_1^2/b^2 - 1 > 0$ , or  $< 0$ .

2. The line  $y = mx + c$  touches the ellipse

$$x^2/a^2 + y^2/b^2 = 1, \text{ if } c^2 = a^2m^2 + b^2$$

3. The combined equation of the pair of tangents drawn from a point (x<sub>1</sub> y<sub>1</sub>) to the ellipse  $x^2/a^2 + y^2/b^2 = 1$  is

$$(x^2/a^2 + y^2/b^2 - 1)(x_1^2/a^2 + y_1^2/b^2 - 1) = (xx_1/a^2 + yy_1/b^2 - 1)^2$$

$$\text{i.e., } SS_1 = T^2$$

4. The tangent and normal at any point of an ellipse bisect the external and internal angles between the focal radii to the point.

5. If SM and S' M' are perpendiculars from the foci upon the tangent at any point of the ellipse, then  $SM \times S' M' = b^2$  and M, M' lie on the auxiliary circle.

6. If the tangent at any point P on the ellipse  $x^2/a^2 + y^2/b^2 = 1$  meets the major axis in T and minor axis in T', then  $CN * CT = a^2$ ,  $CN' * Ct' = b^2$ , where N and N' are the foot of the perpendiculars from P on the respective axis.
7. The common chords of an ellipse and a circle are equally inclined to the axes of the ellipse.
8. The four normals can be drawn from a point on an ellipse.
9. Polar of the point  $(x_1, y_1)$  with respect to the ellipse  $x^2/a^2 + y^2/b^2 = 1$  is  $xx_1/a^2 + yy_1/b^2 = 1$ .
- Here, point  $(x_1, y_1)$  is the pole of  $xx_1/a^2 + yy_1/b^2 = 1$ .
10. The pole of the line  $lx + my + n = 0$  with respect to ellipse  $x^2/a^2 + y^2/b^2 = 1$  is  $p(-a^2l/n, -b^2m/n)$ .
11. Two tangents can be drawn from a point P to an ellipse. These tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.
12. Tangents at the extremities of latusrectum of an ellipse intersect on the corresponding direction.
13. Locus of mid-point of focal chords of an ellipse  $x^2/a^2 + y^2/b^2 = 1$  is  $x^2/a^2 + y^2/b^2 = ex/a^2$ .
14. Point of intersection of the tangents at two points on the ellipse  $x^2/a^2 + y^2/b^2 = 1$ , whose eccentric angles differ by a right angles lies on the ellipse  $x^2/a^2 + y^2/b^2 = 2$ .
15. Locus of mid – point of normal chords of an ellipse  $x^2/a^2 + y^2/b^2 = 1$  is  $(x^2/a^2 + y^2/b^2)^2 (a^6/x^2 + b^6/y^2) = (a^2 - b^2)^2$ .
16. Eccentric angles of the extremities of latusrectum of an ellipse  $x^2/a^2 + y^2/b^2 = 1$  are  $\tan^{-1} (\pm b/ae)$ .
17. The straight lines  $y = m_1x$  and  $y = m_2x$  are conjugate diameters of an ellipse  $x^2/a^2 + y^2/b^2 = 1$ , if  $m_1m_2 = -b^2/a^2$ .
18. The normal at point P on an ellipse with foci S, S<sub>1</sub> is the internal bisector of  $\angle SPS_1$ .