Complex Numbers

Imaginary Quantity

The square root of a negative real number is called an imaginary quantity or imaginary number. e.g., $\sqrt{-3}$, $\sqrt{-7/2}$

The quantity $\sqrt{-1}$ is an imaginary number, denoted by 'i', called iota.

Integral Powers of Iota (i)

$i=\sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4=1$

So, $i^{4n+1}= i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, $i^{4n+4} = i^{4n} = 1$

In other words,

$i^n = (-1)^{n/2}$, if n is an even integer
$i^n = (-1)^{(n-1)/2}.i$, if is an odd integer

Complex Number

A number of the form $z = x + iy$, where $x, y \in R$, is called a complex number

The numbers x and y are called respectively real and imaginary parts of complex number z.

i.e., $x = Re(z)$ and $y = Im(z)$

Purely Real and Purely Imaginary Complex Number

A complex number z is a purely real if its imaginary part is 0.

i.e., $Im(z) = 0$. And purely imaginary if its real part is 0 i.e., $Re(z)= 0$.

Equality of Complex Numbers

Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal, if $a_2= a_2$ and $b_1 = b_2$ i.e., $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

Algebra of Complex Numbers

1. Addition of Complex Numbers

Let $z_1 = (x_1 + iy_i)$ and $z_2 = (x_2 + iy_2)$ be any two complex numbers, then their sum defined as

$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Properties of Addition

(i) Commutative $z_1 + z_2 = z_2 + z_1$

(ii) Associative $(z_1 + z_2) + z_3 = + (z_2 + z_3)$

(iii) Additive Identity $z + 0 = z = 0 + z$

Here, 0 is additive identity.

2. Subtraction of Complex Numbers

Let $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$ be any two complex numbers, then their difference is defined as

$z_1 – z_2 = (x_1 + iy_1) – (x_2 + iy_2)$
$= (x_1 – x_2) + i(y_1 – y_2)$

3. Multiplication of Complex Numbers

Let $z_1 = (x_1 + iy_i)$ and $z_2 = (x_2 + iy_2)$ be any two complex numbers, then their multiplication is defined as

$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 – y_1y_2) + i(x_1y_2 + x_2y_1)$

Properties of Multiplication

(i) Commutative $z_1z_2 = z_2z_1$

(ii) Associative $(z_1\ z_2)\ z_3 = z_1(z_2\ z_3)$

(iii) Multiplicative Identity $z \bullet 1 = z = 1 \bullet z$

Here, 1 is multiplicative identity of an element z.

(iv) Multiplicative Inverse Every non-zero complex number z there exists a complex number z_1 such that $z.z_1 = 1 = z_1 \bullet z$

(v) Distributive Law

(a) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ (left distribution)

(b) $(z_2 + z_3)z_1 = z_2z_1 + z_3z_1$ (right distribution)

4. Division of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers, then their division is defined as

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)}$$
$$= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

where z_2 # 0.

Conjugate of a Complex Number

If z = x + iy is a complex number, then conjugate of z is denoted by z

i.e., z = x – iy

Properties of Conjugate

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} \Leftrightarrow z$ is purely real

(iii) $z - \bar{z} \Leftrightarrow z$ is purely imaginary

(iv) $\text{Re}(z) = \frac{z + \bar{z}}{2}$

(v) $\text{Im}(z) = \frac{z - \bar{z}}{2i}$

(vi) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(vii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(viii) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

(ix) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

(x) $z_1 \bar{z}_2 \pm \bar{z}_1 z_2 = 2\,\text{Re}(\bar{z}_1 z_2) = 2\,\text{Re}(z_1 \bar{z}_2)$

(xi) $(\bar{z})^n = \overline{(z^n)}$

(xii) If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

(xiii) If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\bar{z} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{vmatrix}$

where $a_i, b_i, c_i; (i = 1, 2, 3)$ are complex numbers.

(xiv) $z\bar{z} = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2$

Modulus of a Complex Number

If z = x + iy, , then modulus or magnitude of z is denoted by |z| and is given by

|z| = $x^2 + y^2$.

It represents a distance of z from origin.

In the set of complex number C, the order relation is not defined i.e., $z_1 > z_2$ or $z_i < z_2$ has no meaning but $|z_1|>|z_2|$ or $|z_1|<|z_2|$ has got its meaning, since $|z|$ and $|z_2|$ are real numbers.

Properties of Modulus

(i) $|z| \geq 0$

(ii) If $|z| = 0$, then $z = 0$ *i.e.*, $\operatorname{Re}(z) = 0 = \operatorname{Im}(z)$

(iii) $-|z| \leq \operatorname{Re}(z) \leq |z|$ and $-|z| \leq \operatorname{Im}(z) \leq |z|$

(iv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(v) $z\bar{z} = |z|^2$

(vi) $|z_1 z_2| = |z_1||z_2|$

In general

$|z_1 z_2 z_3 \ldots z_n| = |z_1||z_2||z_3|\ldots|z_n|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, provided $z_2 \neq 0$

(viii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

In general

$$|z_1 \pm z_2 \pm z_3 \pm \ldots \pm z_n| \leq |z_1| + |z_2| + |z_3| + \ldots + |z_n|$$

(ix) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

(x) $|z^n| = |z|^n$

(xi) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ greatest possible value of $|z_1 + z_2|$ is $|z_1| + |z_2|$ and least possible value of $|z_1 + z_2|$ is

$$||z_1| - |z_2||$$

(xii) $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$$

$$= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

(xiii) $|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$$= |z_1|^2 + |z_2|^2 - (z_1\bar{z}_2 + \bar{z}_1 z_2)$$

$$= |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

(xiv) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(xv) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

(xvi) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.

(xvii) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

where $a, b \in R$.

(xviii) z is **unimodulus**, if $|z| = 1$

Reciprocal/Multiplicative Inverse of a Complex Number

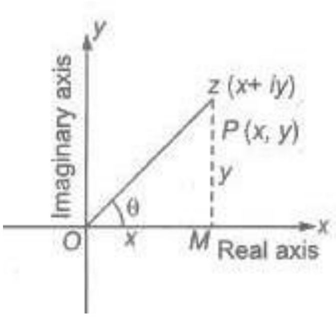
Let z = x + iy be a non-zero complex number, then

$$z^{-1} = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$
$$= \frac{x-iy}{x^2+y^2}$$
$$= \frac{x}{x^2+y^2} + \frac{i(-y)}{x^2+y^2}$$

Here, z^{-1} is called multiplicative inverse of z.

Argument of a Complex Number

Any complex number z=x+iy can be represented geometrically by a point (x, y) in a plane, called Argand plane or Gaussian plane. The angle made by the line joining point z to the origin, with the x-axis is called argument of that complex number. It is denoted by the symbol arg (z) or amp (z).



Argument (z) = θ = $\tan^{-1}(y/x)$

Argument of z is not unique, general value of the argument of z is 2nπ + θ. But arg (0) is not defined.

A purely real number is represented by a point on x-axis.

A purely imaginary number is represented by a point on y-axis.

There exists a one-one correspondence between the points of the plane and the members of the set C of all complex numbers.

The length of the line segment OP is called the modulus of z and is denoted by |z|.

i.e., length of OP = $\sqrt{x^2 + y^2}$.

Principal Value of Argument

The value of the argument which lies in the interval $(-\pi, \pi]$ is called principal value of argument.

(i) If $x > 0$ and $y > 0$, then $\arg(z) = 0$
(ii) If $x < 0$ and $y > 0$, then $\arg(z) = \pi - 0$
(iii) If $x < 0$ and $y < 0$, then $\arg(z) = -(\pi - \theta)$
(iv) If $x > 0$ and $y < 0$, then $\arg(z) = -\theta$

Properties of Argument

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(i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi \ (k = 0, 1 \text{ or } -1)$

In general,

$$\arg(z_1 z_2 z_3 \dots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \dots + \arg(z_n) + 2k\pi \ (k = 0, 1 \text{ or } -1)$$

(iii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi \ (k = 0, 1 \text{ or } = -1)$

(iv) $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$

(v) $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi \ (k = 0, 1 \text{ or } -1)$

(vi) $\arg(z^n) = n \arg(z) + 2k\pi \ (k = 0, 1 \text{ or } -1)$

(vii) If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta, k \in I$

(viii) If $\arg(z) = 0 \Rightarrow z$ is real

(ix) $\arg(z) - \arg(-z) = \begin{cases} \pi, \text{ if } \arg(z) > 0 \\ -\pi, \text{ if } \arg(z) < 0 \end{cases}$

(x) If $|z_1 + z_2| = |z_1 - z_2|$, then

$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$

(xi) If $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) = \arg(z_2)$

(xii) If $|z - 1| = |z + 1|$, then $\arg(z) = \pm\frac{\pi}{2}$

(xiii) If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, then $(z) = 1$

(xiv) If $\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{2}$, then z lies on circle of radius unity and centre at origin.

(xv) (a) If $z = 1 + \cos\theta + i\sin\theta$, then

$\arg(z) = \frac{\theta}{2}$ and $|z| = 2\cos\frac{\theta}{2}$

(b) If $z = 1 + \cos\theta - i\sin\theta$, then

$\arg(z) = -\frac{\theta}{2}$ and $|z| = 2\cos\frac{\theta}{2}$

(c) If $z = 1 - \cos\theta + i\sin\theta$, then

$\arg(z) = \frac{\pi}{2} - \frac{\theta}{2}$ and $|z| = 2\sin\frac{\theta}{2}$

(d) If $z = 1 - \cos\theta - i\sin\theta$, then

$\arg(z) = \frac{\pi}{4} - \frac{\theta}{2}$ and $|z| = \sqrt{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

(xvi) If $|z_1| \le 1, |z_2| \le 1$, then

(a) $|z_1 - z_2|^2 \le (|z_1| - |z_2|)^2 + [\arg(z_1) - \arg(z_2)]^2$

(b) $|z_1 + z_2|^2 \ge (|z_1| + |z_2|)^2 - [\arg(z_1) - \arg(z_2)]^2$

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Square Root of a Complex Number

If z = x + iy, then

$$\sqrt{z} = \sqrt{x+iy} = \pm\left[\sqrt{\frac{|z|+x}{2}} + i\sqrt{\frac{|z|-x}{2}}\right], \text{ for } y > 0$$

$$= \pm\left[\sqrt{\frac{|z|+x}{2}} - i\sqrt{\frac{|z|-x}{2}}\right], \text{ for } y < 0$$

Polar Form

If z = x + iy is a complex number, then z can be written as

$z = |z| (\cos\theta + i \sin\theta)$ where, $\theta = \arg(z)$

this is called polar form.

If the general value of the argument is 0, then the polar form of z is

$z = |z| [\cos(2n\pi + \theta) + i \sin(2n\pi + \theta)]$, where n is an integer.

Eulerian Form of a Complex Number

If z = x + iy is a complex number, then it can be written as

$z = re^{i0}$, where

$r = |z|$ and $\theta = \arg(z)$

This is called Eulerian form and $e^{i0} = \cos\theta + i \sin\theta$ and $e^{-i0} = \cos\theta - i \sin\theta$.

De-Moivre's Theorem

A simplest formula for calculating powers of complex number known as De-Moivre's theorem.

If $n \in I$ (set of integers), then $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$ and if $n \in Q$ (set of rational numbers), then $\cos n\theta + i \sin n\theta$ is one of the values of $(\cos\theta + i \sin\theta)^n$.

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(i) If $\frac{p}{q}$ is a rational number, then

$$(\cos\theta + i\sin\theta)^{p/q} = \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)$$

(ii) $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta - i\sin\theta)^n$

(iii) More generally, for a complex number $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$z^n = r^n(\cos\theta + i\sin\theta)^n$$
$$= r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta}$$

(iv) $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)\right]$

(v) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$
$$= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$$

(vi) $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$

(vii) $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$

The nth Roots of Unity

The nth roots of unity, it means any complex number z, which satisfies the equation $z^n = 1$ or $z = (1)^{1/n}$

or $z = \cos(2k\pi/n) + i\sin(2k\pi/n)$, where $k = 0, 1, 2, \ldots, (n — 1)$

Properties of nth Roots of Unity

1. nth roots of unity form a GP with common ratio $e^{(i2\pi/n)}$.
2. Sum of nth roots of unity is always 0.
3. Sum of nth powers of nth roots of unity is zero, if p is a multiple of n
4. Sum of pth powers of nth roots of unity is zero, if p is not a multiple of n.
5. Sum of pth powers of nth roots of unity is n, ifp is a multiple of n.
6. Product of nth roots of unity is $(-1)^{(n-1)}$.
7. The nth roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into n equal parts.

The Cube Roots of Unity

Cube roots of unity are $1, \omega, \omega^2$,

where $\omega = -1/2 + i\sqrt{3}/2 = e^{(i2\pi/3)}$ and $\omega^2 = (-1 - i\sqrt{3})/2$

$\omega^{3r+1} = \omega$, $\omega^{3r+2} = \omega^2$

Properties of Cube Roots of Unity

(i) $1 + \omega + \omega^{2r} =$
0, if r is not a multiple of 3.
3, if r is,a multiple of 3.

(ii) $\omega^3 = \omega^{3r} = 1$

(iii) $\omega^{3r+1} = \omega$, $\omega^{3r+2} = \omega^2$

(iv) Cube roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into 3 equal parts.

(v) It always forms an equilateral triangle.

(vi) Cube roots of -1 are $-1, -\omega, -\omega^2$.

Important Identities

(i) $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

(ii) $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

(iii) $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

(iv) $x^2 - xy + y^2 = (x + \omega y)(x + y\omega^2)$

(v) $x^2 + y^2 = (x + iy)(x - iy)$

(vi) $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

(vii) $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$

(viii) $x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

or $(x\omega + y\omega^2 + z)(x\omega^2 + y\omega + z)$

or $(x\omega + y + z\omega^2)(x\omega^2 + y + z\omega)$

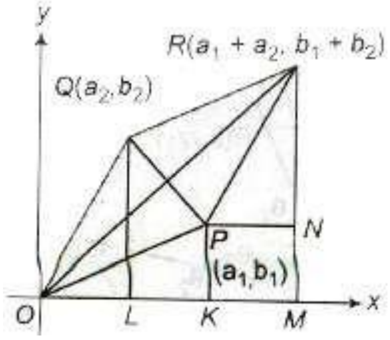
(ix) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

Geometrical Representations of Complex Numbers

1. Geometrical Representation of Addition

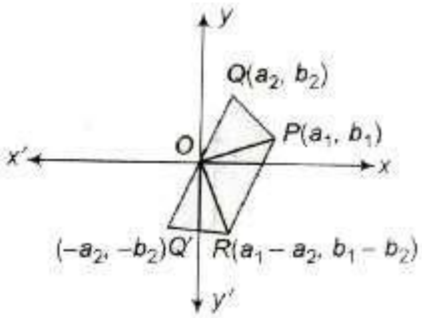
If two points P and Q represent complex numbers z_1 and z_2 respectively, in the Argand plane, then the sum $z_1 + z_2$ is represented

by the extremity R of the diagonal OR of parallelogram OPRQ having OP and OQ as two adjacent sides.



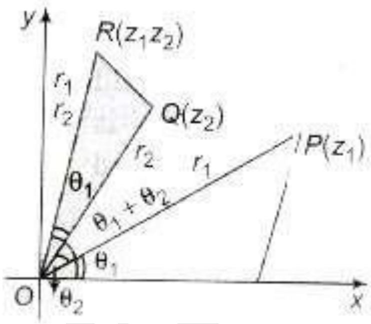
2. Geometrical Representation of Subtraction

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ia_2$ be two complex numbers represented by points P (a_1, b_1) and Q(a_2, b_2) in the Argand plane. Q' represents the complex number ($—z_2$). Complete the parallelogram OPRQ' by taking OP and OQ' as two adjacent sides.



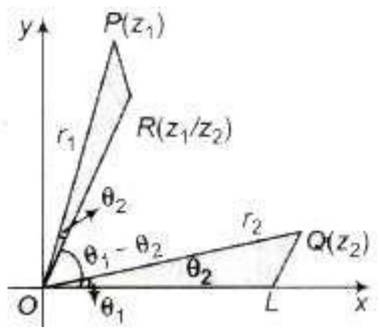
The sum of z_1 and $—z_2$ is represented by the extremity R of the diagonal OR of parallelogram OPRQ'. R represents the complex number $z_1 — z_2$.

3. Geometrical Representation of Multiplication of Complex Numbers



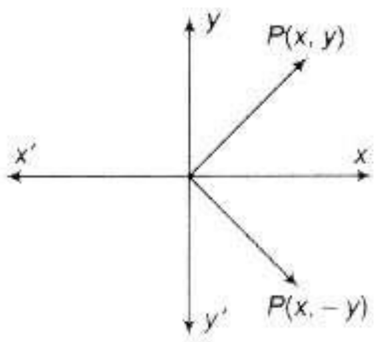
R has the polar coordinates $(r_1r_2, \theta_1 + \theta_2)$ and it represents the complex numbers z_1z_2.

4. Geometrical Representation of the Division of Complex Numbers



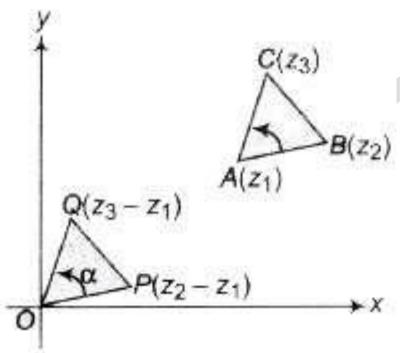
R has the polar coordinates $(r_1/r_2, \theta_1 - \theta_2)$ and it represents the complex number z_1/z_2.
|z|=|z| and arg (z) = – arg (z). The general value of arg (z) is 2nπ – arg (z).

If a point P represents a complex number z, then its conjugate i is represented by the image of P in the real axis.

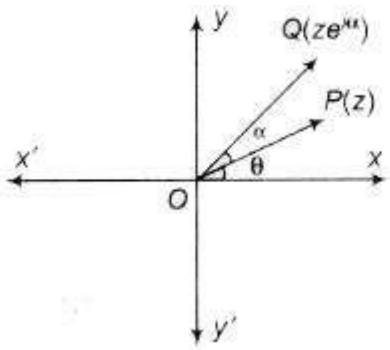


Concept of Rotation

Let z_1, z_2 and z_3 be the vertices of a ΔABC described in anti-clockwise sense. Draw OP and OQ parallel and equal to AB and AC, respectively. Then, point P is $z_2 - z_1$ and Q is $z_3 - z_1$. If OP is rotated through angle a in anti-clockwise, sense it coincides with OQ.



Important Points to be Remembered



(a) $ze^{i\alpha}$ a is the complex number whose modulus is r and argument $\theta + \alpha$.
(b) Multiplication by $e^{-i\alpha}$ to z rotates the vector OP in clockwise sense through an angle α.

(ii) If z_1, z_2, z_3 and z_4 are the affixes of the points A, B,C and D, respectively in the Argand plane.

(a) AB is inclined to CD at the angle arg $[(z_2 - z_1)/(z_4 - z_3)]$.

(b) If CD is inclines at 90° to AB, then arg $[(z_2 - z_1)/(z_4 - z_3)] = \pm(\pi/2)$.

(c) If z_1 and z_2 are fixed complex numbers, then the locus of a point z satisfying arg [([(z – z_1)/(z – z_2)] = $\pm(\pi/2)$.

Logarithm of a Complex Number

Let z = x + iy be a complex number and in polar form of z is $re^{i\theta}$, then

$\log(x + iy) = \log (re^{i\theta}) = \log (r) + i\theta$

$\log(\sqrt{x^2 + y^2}) + i\tan^{-1} (y/x)$

or $\log(z) = \log (|z|)+ i\text{amp} (z)$,

In general,

$z = re^{i(\theta + 2n\pi)}$

$\log z = \log|z| + i\arg z + 2n\pi i$

Applications of Complex Numbers in Coordinate Geometry

Distance between complex Points

(i) Distance between $A(z_1)$ and $B(_1)$ is given by

$AB = |z_2 — z_1| = \sqrt{(x_2 + x_1)^2 + (y_2 + y_1)^2}$

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where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

(ii) The point P (z) which divides the join of segment AB in the ratio m : n is given by

$z = (mz_2 + nz_1)/(m + n)$

If P divides the line externally in the ratio m : n, then

$z = (mz_2 – nz_1)/(m – n)$

Triangle in Complex Plane

(i) Let ABC be a triangle with vertices A (z_1), B(z_2) and C(z_3) then

(a) Centroid of the ΔABC is given by

$z = 1/3(z_1 + z_2 + z_3)$

(b) Incentre of the AABC is given by

$z = (az_1 + bz_2 + cz_3)/(a + b + c)$

(ii) Area of the triangle with vertices A(z_1), B(z_2) and C(z_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$$

For an equilateral triangle,

$z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2$

(iii) The triangle whose vertices are the points represented by complex numbers z_1, z_2 and z_3 is equilateral, if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

i.e., $$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Straight Line in Complex Plane

(i) The general equation of a straight line is az + az + b = 0, where a is a complex number and b is a real number.

(ii) The complex and real slopes of the line az + az are -a/a and – i[(a + a)/(a – a)].

(iii) The equation of straight line through z_1 and z_2 is $z = tz_1 + (1 — t)z_2$, where t is real.

(iv) If z_1 and z_2 are two fixed points, then $|z — z_1| = z — z_2|$ represents perpendicular bisector of the line segment joining z1 and z2.

(v) Three points z_1, z_2 and z_3 are collinear, if

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

This is also, the equation of the line passing through $_1$, z_2 and z_3 and slope is defined to be $(z_1 – z_2)/z_1 – z_2$

(vi) **Length of Perpendicular** The length of perpendicular from a point z_1 to az + az + b = 0 is given by $|az_1 + az_1 + b|/2|a|$

(vii) arg $(z – z_1)/(z – z_2) = \beta$

Locus is the arc of a circle which the segment joining z_1 and z_2 as a chord.

(viii) The equation of a line parallel to the line az + az + b = 0 is az + az + λ = 0, where $\lambda \in R$.

(ix) The equation of a line parallel to the line az + az + b = 0 is az + az + iλ = 0, where $\lambda \in R$.

(x) If z_1 and z_2 are two fixed points, then I z — z11 =I z z21 represents perpendicular bisector of the segment joining A(z1) and B(z2).

(xi) The equation of a line perpendicular to the plane $z(z_1 – z_2) + z(z_1 – z_2) = |z_1|^2 – |z_2|^2$.

(xii) If z_1, z_2 and z_3 are the affixes of the points A, B and C in the Argand plane, then

(a) $\angle BAC = \arg[(z_3 – z_1/z_2 – z_1)]$

(b) $[(z_3 – z_1)/(z_2 – z_1)] = |z_3 – z_1|/|z_2 – z_1| (\cos \alpha + i\sin \alpha)$, where $\alpha = \angle BAC$.

(xiii) If z is a variable point in the argand plane such that arg (z) = θ, then locus of z is a straight line through the origin inclined at an angle θ with X-axis.

(xiv) If z is a variable point and z_1 is fixed point in the argand plane such that $(z — z_1)= \theta$, then locus of z is a straight line passing through the point z_1 and inclined at an angle θ with the X-axis.

(xv) If z is a variable point and z_1, z_2 are two fixed points in the Argand plane, then

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

Locus of z is the line segment joining z_1 and z_2.

(b) $|z - z_1| - |z - z_2| = |z_1 - z_2|$

Locus of z is a straight line joining z_1 and z_2 but z does not lie between z1 and z_2.

(c) $\arg[(z - z_1)/(z - z_{2)}] = 0$ or π

Locus z is a straight line passing through z_1 and z_2.

(d) $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

Locus of z is a circle with z_1 and z_2 as the extremities of diameter.

Circle in Complete Plane

(i) An equation of the circle with centre at z_0 and radius r is

$|z - z_0| = r$

or $zz - z_0z - z_0z + z_0$

- $|z — z_0| < r$, represents interior of the circle.
- $|z — z_0| > r$, represents exterior of the circle.
- $|z — z_0| \leq r$ is the set of points lying inside and on the circle $|z — z_0| = r$. Similarly, $|z — z_0| \geq r$ is the set of points lying outside and on the circle $|z — z_0| = r$.
- **General equation of a circle is**

$zz - az - az + b = 0$

where a is a complex number and b is a real number. Centre of the circle = – a

Radius of the circle = $\sqrt{aa} - b$ or $\sqrt{|a|^2} - b$

(a) Four points z_1, z_2, z_3 and z_4 are concyclic, if

$[(z_4 — z_1)(z_2 — z_3)]/[(z_4 - z_3)(z_2 - z_1)]$ is purely real.

(ii) $|z — z_1|/|z - z_2| = k \Rightarrow$ Circle, if $k \neq 1$ or Perpendicular bisector, if $k = 1$

(iii) The equation of a circle described on the line segment joining z_1 and $_1$ as diameter is $(z- z_1)(z - z_2) + (z - z_2)(z — z_1) = 0$

(iv) If z_1, and z_2 are the fixed complex numbers, then the locus of a point z satisfying arg $[(z - z_1)/(z - z_2)] = \pm \pi / 2$ is a circle having z_1 and z_2 at the end points of a diameter.

Conic in Complex plane

(i) Let z_1 and z2 be two fixed points, and k be a positive real number.

If $k > |z_1 - z_2|$, then $|z - z_1| + |z - z_2| = k$ represents an ellipse with foci at $A(z_1)$ and $B(z_2)$ and length of the major axis is k.

(ii) Let z_1 and z2 be two fixed points and k be a positive real number.

If $k \neq |z_1 - z_2|$, then $|z - z_1| - |z - z_2| = k$ represents hyperbola with foci at $A(z_1)$ and $B(z_2)$.

Important Points to be Remembered

- $\sqrt{-a} \times \sqrt{-b} \neq \sqrt{ab}$

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is possible only, if both a and b are non-negative.

So, $i^2 = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{1}$

- is neither positive, zero nor negative.
- Argument of 0 is not defined.
- Argument of purely imaginary number is $\pi/2$
- Argument of purely real number is 0 or π.
- If $|z + 1/z| = a$ then the greatest value of $|z| = a + \sqrt{a^2} + 4/2$ and the least value of $|z| = -a + \sqrt{a^2} + 4/2$
- The value of $i^i = e^{-\pi 2}$
- The complex number do not possess the property of order, i.e., $x + iy < (\text{or}) > c + id$ is not defined.
- The area of the triangle on the Argand plane formed by the complex numbers z, iz and z + iz is $1/2|z|^2$.
- (x) If ω_1 and ω_2 are the complex slope of two lines on the Argand plane, then the lines are

(a) perpendicular, if $\omega_1 + \omega_2 = 0$.
(b) parallel, if $\omega_1 = \omega_2$.