Binomial Theorem

Binomial Theorem for Positive Integer

If n is any positive integer, then

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_n\, a^n.$$

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

This is called binomial theorem.

Here, ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, … , ${}^nn_o$ are called binomial coefficients and

${}^nC_r = n! / r!(n-r)!$ for $0 \le r \le n$.

Properties of Binomial Theorem for Positive Integer

(i) Total number of terms in the expansion of $(x + a)^n$ is $(n + 1)$.

(ii) The sum of the indices of x and a in each term is n.

(iii) The above expansion is also true when x and a are complex numbers.

(iv) The coefficient of terms equidistant from the beginning and the end are equal. These coefficients are known as the binomial coefficients and

${}^nC_r = {}^nC_{n-r}$, $r = 0,1,2,\ldots,n$.

(v) General term in the expansion of $(x + c)^n$ is given by

$T_{r+1} = {}^nC_r x^{n-r} a^r$.

(vi) The values of the binomial coefficients steadily increase to maximum and then steadily decrease .

(vii)(vii)

$$(x - a)^n = {}^nC_0 - {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 - {}^nC_3 x^{n-3} a^3 + \ldots + (-1)^n \, {}^nC_n \, a^n$$

$$\text{i.e., } (x - a)^n = \sum_{r=0}^{n} (-1)^r \, {}^nC_r \cdot x^{n-r} \cdot a^r$$

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n \, x^n$$

$$\text{i.e., } \quad (1 + x)^n = \sum_{r=0}^{n} {}^nC_r \cdot x^r$$

(viii)

(ix) The coefficient of x^r in the expansion of $(1+ x)^n$ is ${}^nC_r$.

(x)

(xi) (a) $(x + a)^n + (x - a)^n = 2\,({}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + \ldots)$

(b) $(x + a)^n - (x - a)^n = 2\,({}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \ldots)$

(xii) (a) If n is odd, then $(x + a)^n + (x - a)^n$ and $(x + a)^n - (x - a)^n$ both have the same number of terms equal to (n +1 / 2).

(b) If n is even, then $(x + a)^n + (x - a)^n$ has (n +1 / 2) terms. and $(x + a)^n - (x - a)^n$ has (n / 2) terms.

(xiii) In the binomial expansion of $(x + a)^n$, the r th term from the end is (n – r + 2)th term

$$(1 - x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots + (-1)^r \, {}^nC_r x^r + \ldots + (-1)^n \, {}^nC_n x^n$$

$$\text{i.e., } \quad (1 - x)^n = \sum_{r=0}^{n} (-1)^r \, {}^nC_r \cdot x^r$$

from the beginning.

(xiv) If n is a positive integer, then number of terms in $(x + y + z)^n$ is (n + l)(n + 2) / 2.

Middle term in the Expansion of $(1 + x)^n$

(i) It n is even, then in the expansion of $(x + a)^n$, the middle term is $(n/2 + 1)^{th}$ terms.

(ii) If n is odd, then in the expansion of $(x + a)^n$, the middle terms are $(n + 1) / 2$ th term and $(n + 3) / 2$ th term.

Greatest Coefficient

(i) If n is even, then in $(x + a)^n$, the greatest coefficient is ${}^nC_{n / 2}$

(ii) Ifn is odd, then in $(x + a)^n$, the greatest coefficient is ${}^nC_{n - 1 / 2}$ or ${}^nC_{n + 1 / 2}$ both being equal.

Greatest Term

In the expansion of $(x + a)^n$

(i) If $n + 1 / x/a + 1$ is an integer = p (say), then greatest term is $T_p == T_{p + 1}$.

(ii) If $n + 1 / x/a + 1$ is not an integer with m as integral part of $n + 1 / x/a + 1$, then $T_{m + 1}$. is the greatest term.

Important Results on Binomial Coefficients

(i) ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

(ii) $\dfrac{{}^{n}C_r}{{}^{n-1}C_{r-1}} = \dfrac{n}{r}$

(iii) $\dfrac{{}^{n}C_r}{{}^{n}C_{r-1}} = \dfrac{n-r+1}{r}$

(iv) $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$

(v) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

(vi) $C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n C_n = 0$

(vii) $C_0C_r + C_1C_{r+1} + \ldots + C_{n-r}C_n = {}^{2n}C_{n+r} = \dfrac{(2n)!}{(n-r)!(n+r)!}$

(viii) $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = {}^{2n}C_n = \dfrac{(2n)!}{(n!)^2}$

(ix) $C_0 - C_2 + C_4 - C_6 + \ldots = (\sqrt{2})^n \cos\frac{n\pi}{4}$

(x) $C_1 - C_3 + C_5 - C_7 + \ldots = (\sqrt{2})^n \sin\frac{n\pi}{4}$

(xi) $C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^r C_r = (-1)^r\ {}^{n-1}C_r, r < n$

(xii) $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots = \begin{cases} 0, \text{if } n \text{ is odd.} \\ (-1)^{n/2} \cdot {}^nC_{n/2}, \text{if } n \text{ is even.} \end{cases}$

(xiii) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{(n+1)}$

(xiv) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \ldots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

(xv) $C_0 + \frac{C_1}{2} + \frac{C_2}{2^2} + \frac{C_3}{2^3} + \ldots + \frac{C_n}{2^n} = \left(\frac{3}{2}\right)^n$

(xvi) $\sum_{r=0}^{n} (-1)^r\ {}^nC_r \left\{\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \ldots \text{upto } m \text{ terms}\right\}$

$$= \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

Divisibility Problems

From the expansion, $(1+x)^n = 1 + {}^nC_1x + {}^nC_1x^2 + \ldots + {}^nC_nx^n$

We can conclude that,

(i) $(1+x)^n - 1 = {}^nC_1x + {}^nC_1x^2 + \ldots + {}^nC_nx^n$ is divisible by x i.e., it is multiple of x.

$(1+x)^n - 1 = M(x)$

(ii) $(1+x)^n - 1 - nx = {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_nx^n = M(x^2)$

$$(1+x)^n - 1 - nx - \frac{n(n-1)}{2}x^2 = {}^nC_3x^3 + {}^nC_4x^4 + \ldots + {}^nC_nx^n$$
$$= M(x^3)$$

(iii)

Multinomial theorem

For any n ∈ N,

$$(x_1 + x_2)^n = \sum_{r_1 + r_2 = n} \frac{n!}{r_1!r_2!} x_1^{r_1} x_2^{r_2}$$

(i)

$$(x_1 + x_2 + \ldots + x_n)^n = \sum_{r_1 + r_2 + \ldots + r_k = n} \frac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$$

(ii)

(iii) The general term in the above expansion is

$$\frac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$$

(iv)The greatest coefficient in the expansion of $(x_1 + x_2 + \ldots + x_m)^n$ is $\frac{n!}{(q!)^{m-r}[(q+1)!]^r}$ where q and r are the quotient and remainder respectively, when n is divided by m.

(v) Number of non-negative integral solutions of $x_1 + x_2 + \ldots + x_n = n$ is ${}^{n+r-1}C_{r-1}$

R-f Factor Relations

Here, we are going to discuss problem involving (√A + B)sup>n = I + f, Where I and n are positive integers.

0 le; f le; 1, $|A - B^2| = k$ and $|\sqrt{A} - B| < 1$

Binomial Theorem for any Index

If n is any rational number, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \dots, |x| < 1$$

(i) If in the above expansion, n is any positive integer, then the series in RHS is finite otherwise infinite.

(ii) General term in the expansion of $(1 + x)^n$ is $T_{r+1} = n(n-1)(n-2)\dots[n-(r-1)] / r! * x^r$

(iii) Expansion of $(x + a)^n$ for any rational index

***Case* I.** When $x > a$ i.e., $\frac{a}{x} < 1$

In this case, $(x+a)^n = \left\{x\left(1+\frac{a}{x}\right)\right\}^n = x^n\left(1+\frac{a}{x}\right)^n$

$$= x^n\left\{1 + n \cdot \frac{a}{x} + \frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{a}{x}\right)^3 + \dots\right\}$$

***Case* II.** When $x < a$ i.e., $\frac{x}{a} < 1$

In this case, $(x+a)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n = a^n\left(1+\frac{x}{a}\right)^n$

$$= a^n\left\{1 + n \cdot \frac{x}{a} + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \dots\right\}$$

(iv) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1 \cdot 2}x^2 + \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}x^3 + \ldots$

$$= 1 + {}^nC_1 x + {}^{(n+1)}C_2 x^2 + {}^{(n+2)}C_3 x^3 + \ldots$$

(v) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$

$$+ (-1)^r \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$$

(vi) $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

$$+ (-1)^r \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + ..$$

(vii) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\infty$

(viii) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\infty$

(ix) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\infty$

(x) $(1 - x)^{-2} = 1 + 2x + 3x^2 - 4x^3 + \ldots\infty$

(xi) $(1 + x)^{-3} = 1 - 3x + 6x^2 - \ldots\infty$

(xii) $(1 - x)^{-3} = 1 + 3x + 6x^2 - \ldots\infty$

(xiii) $(1 + x)^n = 1 + nx$, if $x^2, x^3,\ldots$ are all very small as compared to x.

Important Results

(i) Coefficient of x^m in the expansion of $(ax^p + b / x^q)^n$ is the coefficient of T_{r+1} where r = np – m / p + q

(ii) The term independent of x in the expansion of $ax^p + b / x^q)^n$ is the coefficient of T_{r+1} where r = np / p + q

(iii) If the coefficient of rth, (r + l)th and (r + 2)th term of $(1 + x)^n$ are in AP, then $n^2 - (4r+1)n + 4r^2 = 2$

(iv) In the expansion of $(x + a)^n$

$T_{r+1} / T_r = n - r + 1 / r * a / x$

(v) (a) The coefficient of x^{n-1} in the expansion of

$(x - l)(x - 2) \ldots.(x - n) = - n\ (n + l) / 2$

(b) The coefficient of x^{n-1} in the expansion of

$(x + l)(x + 2) \ldots.(x + n) = n\ (n + l) / 2$

(vi) If the coefficient of pth and qth terms in the expansion of $(1 + x)^n$ are equal, then $p + q = n + 2$

(vii) If the coefficients of x^r and x^{r+1} in the expansion of $a + x / b)^n$ are equal, then

$n = (r + 1)(ab + 1) - 1$

(viii) The number of term in the expansion of $(x_1 + x_2 + \ldots + x_r)_{\text{n is n + r – 1C r – 1.}}$

(ix) If n is a positive integer and $a_1, a_2, \ldots, a_m \in C$, then the coefficient of x^r in the expansion of $(a_1 + a_2x + a_3x^2 + \ldots + a_mx^{m-1})^n$ is

$$\sum \frac{n!}{n_1!n_2!\ldots n_m!} a_1^{n_1} x\, a_2^{n_2} \ldots a_m^{n_m}$$

(x) For $|x| < 1$,

(a) $1 + x + x^2 + x^3 + \ldots + \infty = 1 / 1 - x$

(b) $1 + 2x + 3x^2 + \ldots + \infty = 1 / (1 - x)^2$

(xi) Total number of terms in the expansion of $(a + b + c + d)^n$ is $(n + l)(n + 2)(n + 3) / 6$.

Important Points to be Remembered

(i) If n is a positive integer, then $(1 + x)^n$ contains $(n + 1)$ terms i.e., a finite number of terms. When n is general exponent, then the expansion of $(1 + x)^n$ contains infinitely many terms.

(ii) When n is a positive integer, the expansion of $(l + x)^n$ is valid for all values of x. If n is general exponent, the expansion of $(i + x)^n$ is valid for the values of x satisfying the condition $|x| < 1$.