Relations and Functions

Relation

If A and B are two non-empty sets, then a relation R from A to B is a subset of A x B.

If $R \subseteq A \times B$ and $(a, b) \in R$, then we say that a is related to b by the relation R, written as aRb.

Domain and Range of a Relation

Let R be a relation from a set A to set B. Then, set of all first components or coordinates of the ordered pairs belonging to R is called : the domain of R, while the set of all second components or coordinates = of the ordered pairs belonging to R is called the range of R.

Thus, domain of $R = \{a : (a , b) \in R\}$ and range of $R = \{b : (a, b) \in R\}$

Types of Relations

(i) Void Relation As $\Phi \subset A \times A$, for any set A, so Φ is a relation on A, called the empty or void relation.

(ii) Universal Relation Since, $A \times A \subseteq A \times A$, so A x A is a relation on A, called the universal relation.

(iii) Identity Relation The relation $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.

(iv) Reflexive Relation A relation R is said to be reflexive relation, if every element of A is related to itself.

Thus, $(a, a) \in R, \forall a \in A = R$ is reflexive.

(v) Symmetric Relation A relation R is said to be symmetric relation, iff

$(a, b) \in R$ $(b, a) \in R, \forall a, b \in A$

i.e., $a R b \Rightarrow b R a, \forall a, b \in A$

$\Rightarrow$ R is symmetric.

(vi) Anti-Symmetric Relation A relation R is said to be anti-symmetric relation, iff

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b, \forall a, b \in A$

(vii) Transitive Relation A relation R is said to be transitive relation, iff $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow (a, c) \in R, \forall a, b, c \in A$

(viii) Equivalence Relation A relation R is said to be an equivalence relation, if it is simultaneously reflexive, symmetric and transitive on A.

(ix) Partial Order Relation A relation R is said to be a partial order relation, if it is simultaneously reflexive, symmetric and anti-symmetric on A.

(x) Total Order Relation A relation R on a set A is said to be a total order relation on A, if R is a partial order relation on A.

Inverse Relation

If A and B are two non-empty sets and R be a relation from A to B, such that $R = \{(a, b) : a \in A, b \in B\}$, then the inverse of R, denoted by R^{-1} , i a relation from B to A and is defined by

$R^{-1} = \{(b, a) : (a, b) \in R\}$

Equivalence Classes of an Equivalence Relation

Let R be equivalence relation in A ($\neq \Phi$). Let $a \in A$.

Then, the equivalence class of a denoted by [a] or {a} is defined as the set of all those points of A which are related to a under the relation R.

Composition of Relation

Let R and S be two relations from sets A to B and B to C respectively, then we can define relation SoR from A to C such that $(a, c) \in$ So R $\Leftrightarrow \exists\, b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

This relation SoR is called the composition of R and S.

(i) RoS $\neq$ SoR
(ii) $(SoR)^{-1} = R^{-1}oS^{-1}$

known as **reversal rule.**

Congruence Modulo m

Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m, if $a - b$ is divisible by m and we write $a \equiv b \pmod{m}$.

i.e., $a \equiv b \pmod{m} \Leftrightarrow a - b$ is divisible by m.

Important Results on Relation

- If R and S are two equivalence relations on a set A, then R ∩ S is also on 'equivalence relation on A.
- The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
- If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.
- If a set A has n elements, then number of reflexive relations from A to A is 2^{n2-2}
- Let A and B be two non-empty finite sets consisting of m and n elements, respectively. Then, A x B consists of mn ordered pairs. So, total number of relations from A to B is 2^{nm}.

Binary Operations

Closure Property

An operation * on a non-empty set S is said to satisfy the closure ' property, if

$a \in S, b \in S \Rightarrow a * b \in S, \forall a, b \in S$

Also, in this case we say that S is closed for *.

An operation * on a non-empty set S, satisfying the closure property is known as a binary operation.

or

Let S be a non-empty set. A function f from S x S to S is called a binary operation on S i.e., f : S x S → S is a binary operation on set S.

Properties

- Generally binary operations are represented by the symbols * , +, … etc., instead of letters figure etc.
- Addition is a binary operation on each one of the sets N, Z, Q, R and C of natural numbers, integers, rationals, real and complex numbers, respectively. While addition on the set S of all irrationals is not a binary operation.
- Multiplication is a binary operation on each one of the sets N, Z, Q, R and C of natural numbers, integers, rationals, real and complex numbers, respectively. While multiplication on the set S of all irrationals is not a binary operation.
- Subtraction is a binary operation on each one of the sets Z, Q, R and C of integers, rationals, real and complex numbers, respectively. While subtraction on the set of natural numbers is not a binary operation.
- Let S be a non-empty set and P(S) be its power set. Then, the union and intersection on P(S) is a binary operation.

- Division is not a binary operation on any of the sets N, Z, Q, R and C. However, it is not a binary operation on the sets of all non-zero rational (real or complex) numbers.
- Exponential operation $(a, b) \rightarrow a^b$ is a binary operation on set N of natural numbers while it is not a binary operation on set Z of integers.

Types of Binary Operations

(i) Associative Law A binary operation * on a non-empty set S is said to be associative, if $(a * b) * c = a * (b * c)$, $\forall$ a, b, c $\in$ S.

Let R be the set of real numbers, then addition and multiplication on R satisfies the associative law.

(ii) Commutative Law A binary operation * on a non-empty set S is said to be commutative, if

$a * b = b * a$, $\forall$ a, b $\in$ S.

Addition and multiplication are commutative binary operations on Z but subtraction not a commutative binary operation, since

$2 — 3 \neq 3— 2$.

Union and intersection are commutative binary operations on the power P(S) of all subsets of set S. But difference of sets is not a commutative binary operation on P(S).

(iii) Distributive Law Let * and o be two binary operations on a non-empty sets. We say that * is distributed over o., if

$a * (b \text{ o } c)= (a * b) \text{ o } (a * c)$, $\forall$ a, b, c $\in$ S also called (left distribution) and $(b \text{ o } c) * a = (b * a) \text{ o } (c * a)$, $\forall$ a, b, c $\in$ S also called (right distribution).

Let R be the set of all real numbers, then multiplication distributes addition on R.

Since, $a.(b + c) = a.b + a.c$,$\forall$ a, b, c $\in$ R.

(iv) Identity Element Let * be a binary operation on a non-empty set S. An element e a S, if it exist such that

$a * e = e * a = a$, $\forall$ a $\in$ S.

is called an identity elements of S, with respect to *.

For addition on R, zero is the identity elements in R.

Since, $a + 0 = 0 + a = a$, $\forall$ a $\in$ R

For multiplication on R, 1 is the identity element in R.

Since, a x 1 =1 x a = a,$\forall$ a $\in$ R

Let P (S) be the power set of a non-empty set S. Then, Φ is the identity element for union on P (S) as

A $\cup$ Φ =Φ $\cup$ A = A, $\forall$ A $\in$ P(S)

Also, S is the identity element for intersection on P(S).

Since, A $\cap$ S=A $\cap$ S=A, $\forall$ A $\in$ P(S).

For addition on N the identity element does not exist. But for multiplication on N the idenitity element is 1.

(v) Inverse of an Element Let * be a binary operation on a non-empty set 'S' and let 'e' be the identity element.

Let a $\in$ S. we say that a^{-1} is invertible, if there exists an element b $\in$ S such that a * b = b * a = e

Also, in this case, b is called the inverse of a and we write, a^{-1} = b

Addition on N has no identity element and accordingly N has no invertible element.

Multiplication on N has 1 as the identity element and no element other than 1 is invertible.

Let S be a finite set containing n elements. Then, the total number of binary operations on S in n^{n2}

Let S be a finite set containing n elements. Then, the total number of commutative binary operation on S is n [n(n+1)/2].