Three Dimensional Geometry

- **Direction cosines of a line** : Direction cosines of a line are the cosines of the angles made by the line with the positive direct ions of the coordinate axes.
- *If* l, m, n are the direct ion cosines of a line, then $l^2 + m^2 + n^2 = 1$
- Direct ion cosines of a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

  $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_2}{PQ}$

· where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- Direction ratios of a line are the numbers which are proportional to the direct ion cosines of a line.
- If l, m, n are the direct ion cosines and a, b, c are the direct ion ratios of a line

Then, $l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

- **Skew lines**: Skew lines are lines in space which are neither parallel nor intersecting. They lie in different planes.
- **Angle between two skew lines**: Angle between skew lines is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then,

  $$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_{12}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

- Vector equation of a line that passes through the given point whose position vector is $\overline{a}$ and parallel to a given vector $\overline{b}$ is $\overline{r} = \overline{a} + \lambda \overline{b}$

- Equation of a line through a point (x_1, y_1, z_1) and having direction cosines l, m, n is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$
- The vector equation of a line which passes through two points whose position vectors are $\bar{a}$ and $\bar{b}$ is $\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$
- Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$
- If θ is the acute angle between $\bar{r} = \bar{a}_1 + \lambda \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda \bar{b}_2$ then,

  $$\cos\theta = \left| \frac{\vec{b}_1 \vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|} \right|$$
- If $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are the equations of two lines, then the acute angle between the two lines is given by $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$
- Shortest distance between two skew lines is the line segment perpendicular to both the lines.
- Shortest distance between $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ is $\left| \frac{\left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_1\right)}{|b_1 \times b_2|} \right|$
- Shortest distance between the lines: $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_1} = \frac{y-y_2}{b_1} = \frac{z-z_2}{c_1}$ is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

- Distance between parallel lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda\vec{b_2}$ is $\left|\frac{\vec{b} \times \left(\vec{a_2} - \vec{a_1}\right)}{|b|}\right|$
- In the vector form, equation of a plane which is at a distance d from the origin, and $\hat{n}$ is the unit vector normal to the plane through the origin is $\bar{r}.\hat{n} = d$
- Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.
- The equation of a plane through a point whose position vector is a and perpendicular to the vector $\vec{N}$ is $\left(\vec{r} - \vec{a}\right).\vec{N} = 0.$
- Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$
- Equation of a plane passing through three non collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$
- Vector equation of a plane that contains three non collinear points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $\left(\vec{r} - \vec{a}\right).\left[\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right)\right] = 0.$
- Equation of a plane that cuts the coordinates axes at $(a, 0, 0), (0, b, 0)$ *and* $(0, 0, c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$
- Vector equation of a plane that passes through the intersection of planes $\vec{r}.\vec{n_1} = d_1$ and $\vec{r}.\vec{n_2} = d_2$ is $\vec{r}\left(\vec{n_1} - \lambda\vec{n_2}\right) = d_1 + \lambda d_2$, where λ is any non-zero constant.
- Cartesian equation of a plane that passes that passes through the intersection of two given planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is $(A_1x + B_1y + C_1z + D_1) + \lambda(A_2x + B_2y + C_2z + D_2 = 0$
- Two lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are coplanar if $\left(\vec{a_2} - \vec{a_1}\right).\left(\vec{b_1} \times \vec{b_2}\right) = 0.$
- Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

coplanar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

- In the vector form, if θ is the angle between the two planes, $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$, then $\theta = \cos^{-1}\left|\frac{\vec{n_1} \cdot \vec{n_2}}{\left|\vec{n_1}\right|\left|\vec{n_2}\right|}\right|$
- The angle ϕ between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r} \cdot \hat{n} = d$ is $\sin\phi = \frac{\vec{b} \cdot \hat{n}}{\left|\vec{b}\right|\left|\hat{n}\right|}$
- The angle θ between the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is given by

$$\cos\theta = \left|\frac{A_1A_2 + +B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\,\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

- The distance of a point whose position vector is $\vec{a}$ from the plane $\vec{r} \cdot \hat{n} = d$ is $\left|d - \vec{a} \cdot \hat{n}\right|$.
- The distance from a point (x_1, y_1, z_1) to the plane Ax + By + Cz + D = 0 is

$$\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

- Equation of any plane that is parallel to a plane that is parallel to a plane Ax + By + Cz + D = 0 is Ax + By + Cz + k = 0, where k is a different constant other than D.