HOMEWORK 1

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1. Problem 1

Many constraint surfaces can be locally linearized. Let us examine the behavior of the dynamical system

$$\dot{\mathbf{x}} = P_A(\mathbf{x}) - P_B(\mathbf{x})$$

for this large class of systems by just looking at a simple, 2 dimensional case. Let constraint A be $x_2 = mx_1$ and constraing B be $x_2 = -mx_1$ for some slope m. (see Fig. 1).

The system is simple enough that we can explicitly solve for P_A and P_B:

$$P_A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{m^2+1} \begin{pmatrix} x_1 + mx_2 \\ mx_1 + m^2 x_2 \end{pmatrix}$$

$$P_B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{m^2+1} \begin{pmatrix} x_1 - mx_2 \\ m^2 x_2 - mx_1 \end{pmatrix}$$

We can ignore the m^2+1 coefficient as we only care about long term behavior. So, we get

$$\dot{\mathbf{x}} = P_A(\mathbf{x}) - P_B(\mathbf{x}) = \begin{pmatrix} 2mx_2 \\ 2mx_1 \end{pmatrix} = \begin{pmatrix} 0 & 2m \\ 2m & 0 \end{pmatrix} \mathbf{x}$$

This is a classic linear hyperbolic system. The two eigenvalues are $\pm 2m$, so all trajectories except a set of measure 0 will flow away.

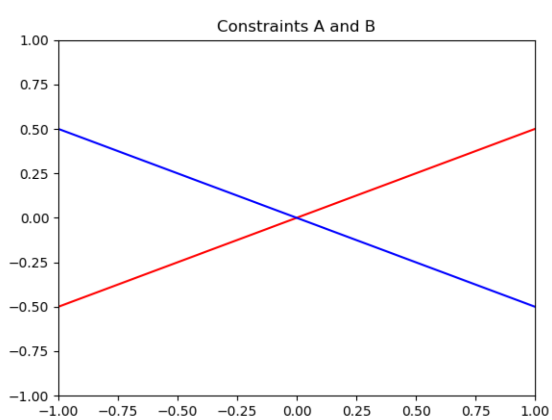

Figure 1. Example constraint surfaces. A is red, B is blue.

2. Problem 2

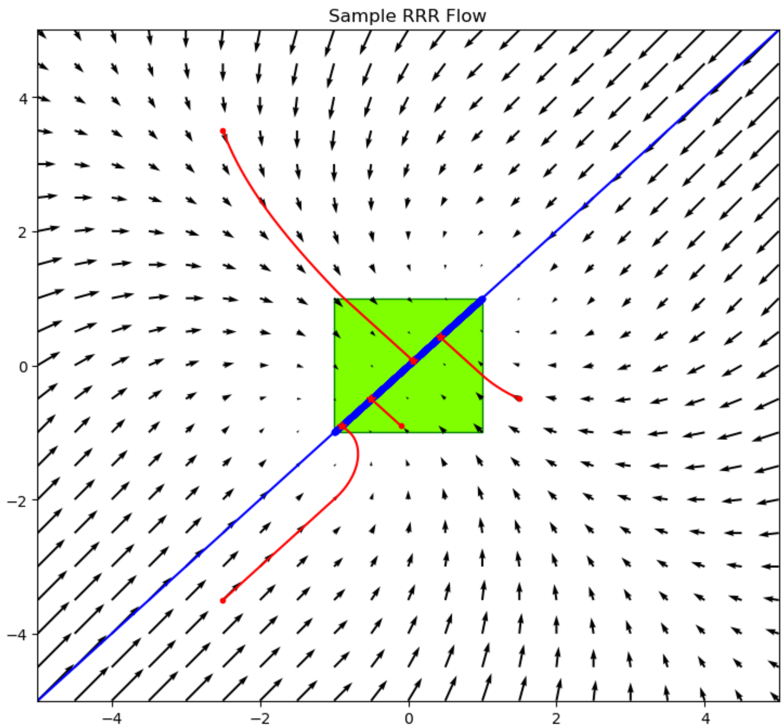

Figure 2. RRR flow, including constraint surfaces (blue and green), the set of fixed points (dark blue), 4 sample trajectories (red), and the vector field for this dynamical system. Notice that since the problem domain $\mathbb{R}^2$ is spanned by the constraint surfaces, we do not need to worry about the distinction between $\mathbf{x}$ and $P_A(\mathbf{x})$.

Fig. 2 shows the RRR flow. Numerical integration does not suggest any cycles, at least not in the smooth limit (I use $\beta = 0.01$); all 4 of the tested trajectories converge, and my sample points represent all 4 important regions:

1. inside A;
2. directly above or to the left of A;
3. above-right / below-left of A;
4. and above-left / below-right of A.

It would be possible to go through and rigorously confirm that trajectories from all 4 of these regions do, in fact, converge, but that= does not seem necessarry - after all, the entire point of the algorithms is to find solutions for problems where the flow is **not** fully understood.

3. Problem 3