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Primitive Abundant Numbers[1]

Brief introduction and definitions: the sigma function is the sum of the divisors of a number (including itself). An abundant number is a number with $\sigma(n) > 2n$; a perfect number has $\sigma(n) = 2n$.

After an interesting daydream in which I pretended to taste numbers, it occurred to me that all proper multiples of perfect numbers and all multiples of abundant numbers are abundant. This was already known to society, but I quickly found a brief proof:

Theorem: All proper multiples of perfect numbers and all multiples of abundant numbers are abundant.

Proof: Let n be such that $\sigma(n) \geq 2n$, and $n = p_1^{k1}p_2^{k2}...p_t^{kx}$ and let $m = p_1^{j1}p_2^{j2}...p_t^{jx}$, where $j_i \geq k_i$ for all i and > for at least 1. Some of the k's may be 0. Then,

$$\sigma(m) = \frac{p_1^{j1+1}-1}{p_1 - 1} \cdot \frac{p_2^{j2+1}-1}{p_2 - 1} \cdots \frac{p_x^{jx+1}-1}{p_x - 1} > \frac{p_1^{j1+1}-p_1^{j1-k1}}{p_1 - 1} \cdot \frac{p_2^{j2+1}-p_2^{j2-k2}}{p_2 - 1} \cdots \frac{p_x^{jx+1}-p_x^{jx-kx}}{p_x - 1} =$$

$$(p_1^{j1-k1}p_2^{j2-k2} \cdots p_x^{jx-kx}) \frac{p_1^{k1+1}-1}{p_1 - 1} \cdot \frac{p_2^{k2+1}-1}{p_2 - 1} \cdots \frac{p_{xk}^{x+1}-1}{p_x - 1} = (m/n)\,\sigma(n) \geq (m/n)2n = 2m$$

Or, to put it all together,

$\sigma(m) > 2m$, QED.

Main proof: Allowing $k_i = 0$ for some i in the above proof completes it for all multiples of n.

This property allows an easy proof that a great quantity of numbers is abundant (for instance, all multiples of 6). This proof does not apply to those abundant or perfect numbers without abundant or perfect divisors, however. Thus, this property suggests the importance of primitive abundant numbers, or numbers without abundant or perfect proper divisors.

Definition: A number is primitive abundant if and only if it is abundant or perfect, but has no abundant or perfect proper divisors.

As an example, let's examine all primitive abundant numbers of the form $2^a p^b$ (for prime p and positive integers a and b). Some obvious examples appear after analyzing just a few integers: 2×3, $2^2 \times 5$, $2^2 \times 7$, $2^3 \times 11$, $2^3 \times 13$, ... This pattern suggests that $2^a p$ is primitive abundant if and only if $2^a < p < 2^{a+1}$. This

1 According to http://mathworld.wolfram.com/, Someone named Guy discovered these in 1994 and had them on page 46 of some work of his. For a long time, I thought I was the first. Oh well.

statement is relatively simple to prove:

$n=2^a p$
$\sigma(n)=(2^{a+1}-1)(p+1)$

$\sigma(n)\geq 2n$
$(2^{a+1}-1)(p+1)\geq 2^{a+1}p$
$(p+1)/p \geq 2^{a+1}/(2^{a+1}-1)$

Using the properties of the function $f(x) = x/(x-1)$,

$p+1\leq 2^{a+1}$
$p\leq 2^{a+1}-1$
$p<2^{a+1}$

Thus, the first part of the inequality is obtained. The second part comes from the fact that, if $p<2^a$, $2^{a-1}p$ is abundant and therefore $2^a p$ is not primitive abundant. Thus, if $2^a p$ is primitive abundant, $2^a<p<2^{a+1}$, QED.

The question that now arises is whether or not there are primitive abundant numbers $2^a p^b$ where $b>1$. The answer is no:

$n=2^a p^b$
$\sigma(n)=(2^{a+1}-1)(p^{b+1}-1)/(p-1)$
$b>1$

$\sigma(n)\geq 2n$
$(2^{a+1}-1)(p^{b+1}-1)/(p-1)\geq 2^{a+1}p^b$
$(p^{b+1}-1)/(p-1)p^b \geq 2^{a+1}/(2^{a+1}-1)$
$p^{b+1}/(p-1)p^b > 2^{a+1}/(2^{a+1}-1)$
$p/(p-1) > 2^{a+1}/(2^{a+1}-1)$

$p<2^{a+1}$
$2^{a+1}p$ is abundant
$n=2^a p^b$ is not primitive abundant, QED.

We have just proved the following result:

Theorem: All primitive abundant numbers of the form $2^a p^b$ have $b=1$ and $2^a<p<2^{a+1}$.

As a matter of fact, all primitive abundant numbers with just 2 distinct prime divisors are of that form, as will be shown later. Notice also, if $p=2^{a+1}-1$, our primitive abundant number is the classic Mersenne perfect number, the only type known to man.

Unfortunately, before we proceed further into odd primitive abundant numbers and even primitive abundant numbers with more prime divisors, it is necessary to diverge a little into more theory.

Definition: $\eta(n)$. $\eta(n)=\sigma(n)/n$

Rather trivially, η is multiplicative, as both σ and $f(n)=n$ are.

Lemma: For any prime p, we have

i) $\eta(p^a)<\eta(p^{a+1})$,

ii) $(p+1)/p\leq\eta(p^a)<p/(p-1)$, and

iii) $\lim_{a \to \text{infinity}} \eta(p^a)=p/(p-1)$.

proof(i):

$$\eta(p^a)=\frac{(p^{a+1}-1)}{p^a(p-1)}=\frac{p(p^{a+1}-1)}{p^{a+1}(p-1)}=\frac{p}{(p-1)}\cdot\frac{(p^{a+1}-1)}{p^{a+1}}$$

$$\eta(p^{a+1})=\frac{p}{(p-1)}\cdot\frac{(p^{a+2}-1)}{p^{a+2}}$$

$$p^{a+2} > p^{a+1}$$

$$\frac{(p^{a+2}-1)}{p^{a+2}} > \frac{(p^{a+1}-1)}{p^{a+1}}$$

$\eta(p^a)<\eta(p^{a+1})$, QED.

proof(ii): The first part of the inequality comes from $\eta(p)=(p+1)/p$ and repeated application of i. As for the second,

$$\frac{(p^{a+1}-1)}{p^{a+1}} < 1$$

$$\frac{p}{(p-1)}\cdot\frac{(p^{a+1}-1)}{p^{a+1}} < \frac{p}{(p-1)}$$

$$\eta(p^a) < p/(p-1)$$

$(p+1)/p\leq\eta(p^a)<p/(p-1)$, QED.

proof(iii):

$$\lim_{a \to \text{infinity}} \frac{(p^{a+1}-1)}{p^{a+1}} = 1$$

$$\lim_{a \to \text{infinity}} \frac{p}{(p-1)}\cdot\frac{(p^{a+1}-1)}{p^{a+1}} = \frac{p}{(p-1)}$$

$\lim_{a \to \text{infinity}} \eta(p^a)=p/(p-1)$, QED.

This lemma is not particularly fascinating in and of itself, although it does provide some interesting results. For instance (using limits), we have $\varphi(n)=n/\eta(n^{infinity})$, or, after rearranging and multiplying both sides by a constant, we have $\eta(n)=(n/\varphi(n))\Pi_{i=1}^{k}(p_i^{ai+1}-1)/p_i^{ai+1}$. Thus, by breaking η into its component functions and cross multiplying, we have $\lim_{a\text{-->inf.}}(n^a)^2/\varphi(n^a)\sigma(n^a)=1$. Curiously, this limit also holds for primes: if p_a is the ath prime, we have $\lim_{a\text{-->inf.}}p_a^2/\varphi(p_a)\sigma(p_a)=1$. Unfortunately, the equation $\lim_{n\text{-->inf}}n^2/\varphi(n)\sigma(n)=1$ does not hold, as there is an unbounded sequence of integers for which this is not true: the product of the first x primes, without exponents for any (the reader may evaluate this on his own).

These ruminations, however, are little more than curiosities; the lemma is far more valuable in the proof of the following theorem:

Theorem: $p_1, p_2, p_3, ..., p_k$ is the distinct prime factorization of an abundant number if and only if $\Pi_{i=1}^{k}\ p_i/(p_i-1) > 2$.

Proof: The "only if"condition is obvious from the second half of lemma ii. The "if" is harder to prove.

We will refer to the calculus definition of limit to infinity: "If $\lim_{x \text{ --> infinity}} f(x) = L$, then for any ε, no matter how small, there is an N such that, for all x≥N, |L-f(x)|≤ε."

Because we used a strict inequality, we now that there is some ε (possibly very small) such that

$\Pi_{i=1}^{k}\ (p_i/(p_i-1)-\varepsilon) > 2$

Now, let N_i be the N from the definition for $f(a) = \eta(p_i^a)$, $L = p_i/(p_i-1)$, and ε. Then, we have

$\Pi_{i=1}^{k}\ \eta(p_i^{N_i}) > 2$

Now, let $n= \Pi_{i=1}^{k}\ p_i^{Ni}$. Because η is multiplicative, we have

$\eta(n) > 2$
$\sigma(n)>2n$

n is an abundant number.

There is an abundant number (n) with the distinct prime factorization $p_1, p_2, p_3, ..., p_k$, QED.

Unfortunately, while this test is necessary and sufficient for abundant numbers, it is necessary but not sufficient for primitive abundant numbers, as there is no way to know if any of the abundant numbers it generates will be primitive (a primitive abundant number might have have zero exponents for some of the prime factors, and thus not have that exact prime factorization, eg {2,3,5} or {2,5,7,11}). There are one necessary condition and two sufficient conditions that are variants on the above method, however, that come in quite handy:

> Corollary 1: If none of the primes p_1, p_2, p_3, ..., p_k appear in the prime factorization of n, then there is an abundant number of the form $np_1^{a1}p_2^{a2}p_3^{a3}...p_k^{ak}$ if and only if $\eta(n)\Pi_{i=1}^{k}\ p_i/(p_i-1) > 2$. (The proof is the same as above, just with a constant that stays on the left until near the last).
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> Corollary 2: If $\Pi_{i=1}^{k}\ p_i/(p_i-1) > 2$, but this is not the case for any proper subset of those primes, then there is a primitive abundant number with that distinct prime factorization. (Proof: Keep looking at the factors of your abundant number until you get a primitive abundant number; that number has non-zero exponents for all the primes, or the "only if" condition of the theorem is violated.) NOTE: the simplest test is to test all proper subsets of size k-1; adding more primes will never hurt, and this way you only have to check k sets.
>
> Corollary 3: Given that $\Pi_{i=1}^{k}\ p_i/(p_i-1) > 2$, find the smallest possible exponent of p_1 such that $\eta(p_1^{a1})\Pi_{i=2}^{k}\ p_i/(p_i-1) > 2$. Next, find the smallest exponent of p_2 such that $\eta(p_1^{a1})\eta(p_2^{a2})\Pi_{i=3}^{k}\ p_i/(p_i-1) > 2$. Repeat until the exponents of all primes are determined (unfortunately, some may be 0). Those primes with those exponents will yield a primitive abundant number. NOTE: there are many permutations of k primes; often, the order in which the exponents are determined will yield distinct primitive abundant numbers (eg (3,7,5) yields 3^2x5^2x7, whereas (5,7,3) yields 3^3x5x7). In fact, this is the mode.

I still cannot figure out whether there are any primitive abundant numbers not generated in this way, but it would seem natural that there would be at least some, as there may be some situations in which you may be able to take away from some exponents and get more of others, but that those exponent patterns had not occurred if the exponents are minimized one at a time. However, I am having difficulty finding these primitive abundant numbers, and I am also having difficulty finding numbers that work in the same way for something other than 2 to the right hand of the equation. For the remainder of the paper we will assume that this method does generate all possible primitive abundant numbers; thus, all computational results should be taken with a grain of salt.

Whether or not there are in fact any primitive abundant numbers not generated by the above method, we have already learned enough about primitive abundant numbers to say some interesting things about them.

For instance, we can prove our claim made earlier about the lack of odd primitive abundant numbers with just two positive factors rather trivially:

$(3/2)*(5/4) = 15/8 < 2$ --> There is no primitive abundant number divisible only by 3 and 5.

If either or both of the primes are replaced by larger primes, this will still not work, as the two numbers will only decrease from what they are now.

Thus, there are no odd primitive abundant numbers divisible by only 2 distinct primes.

Thus, all primitive abundant numbers with just 2 distinct prime divisors are even, and therefore have the form described earlier, QED.

Or, by a similar method, we can prove that all odd primitive abundant numbers with just 3 positive divisors are divisible by 3 and 5 ({3,7,11} does not pass the first test) and by 7, 11, or 13 ({3,5,17} does not pass either). Hence, we rather simply deduce the finite quantity of odd primitive abundant numbers divisible by just 3 distinct primes. As a matter of fact, there are exactly 8: 945, 1575, 2205, 7425, 78975, 131625, 570375 and 342225.

We can also deduce, rather simply, the existence of an infinite quantity of even primitive abundant numbers with exactly 3 distinct prime divisors. The simplest form (this time not exhaustive) of such numbers would be $n=2^a p_1 p_2$, where $2^{a+1}<p_1<p_2<2^{a+2}$. That these numbers have no other abundant or perfect divisors is obvious. That they are abundant themselves may be obtained from the following logic:

Let p_1 and p_2 be the largest that they possibly can (anything smaller will be "more abundant", or have a higher η). Then, we have
$p_1=2^{a+2}-3$
$p_2=2^{a+2}-1$

$$\eta(n)=\frac{2^{a+1}-1}{2^a}\cdot\frac{2^{a+2}-2}{2^{a+2}-3}\cdot\frac{2^{a+2}}{2^{a+2}-1}=4*\frac{2^{a+1}-1}{2^{a+2}-1}\cdot\frac{2^{a+2}-2}{2^{a+2}-3}=\frac{2*(2^{a+2}-2)^2}{(2^{a+2}-1)(2^{a+2}-3)}=\frac{2*(2^{a+2}-2)^2}{(2^{a+2}-2)^2-1}>2\text{, QED.}$$

This is leading me into the strong suspicion that, for every number x, there is a finite quantity of odd primitive abundant numbers with exactly x prime factors, but an infinite quantity of even primitive abundant numbers with just x prime factors. I will return to this (unanswered) query later in my paper, after I develop a more sophisticated primitive abundant number sieve.

Meanwhile, we can return to our less perfect detection methods to prove a couple more interesting properties of primitive abundant numbers and of abundant numbers. I will start with something known and elaborate into something yet more interesting..

Theorem: There is always a primitive abundant number not divisible by the first x prime factors.

Proof: Let us evaluate $\Pi_{p \text{ prime and } p>px} p/(p-1)$, or $\Pi_{p>px} p/(p-1)$ for short.

$$\Pi_{p>p(x)} p/(p-1) = \frac{1}{\Pi_{p>px}(p-1)/p} = \frac{1}{\Pi_{p>px}(1-1/p)} = \Pi_{p>px} \frac{1}{(1-1/p)} = \Pi_{p>px} (\Sigma_{n=0}^{inf.} p^{-n}) = \Sigma_{p(i) \nmid n \text{ for } i \leq x} n^{-1}$$

The final expression is just the harmonic series missing all terms divisible by any of the first x primes, which is trivially proven to be infinite. Thus, we have

$\Pi_{p>px} p/(p-1) = \text{infinity} > 2$

By a ε-N proof similar to the one above, we can stop after some prime p_y and still have the product >2. After we have that, we refer to progressive minimization to find a primitive abundant number, not divisible by the first x primes. Thus, one exists, QED.

This result is well known and is prominently featured on the Wikipedia page on abundant numbers. The following result is a little more interesting:

Theorem: There is always a squarefree primitive abundant number not divisible by the first x primes.

Proof: From the proof of the previous theorem, we have $\Pi_{p>p(x+1) \text{ and } p<p(y+1)} p/(p-1) > 2$ for some number y.

Now, as $p_i \leq p_{i+1}-1$ for all i, we have $\Pi_{p>p(x) \text{ and } p<p(y)}(p+1)/p > 2$. This means that $p_x p_{x+1} \ldots p_y$ is abundant. We can hack away at primes until we have made this number primitive abundant. Thus, there exists a squarefree primitive abundant number not divisible by the first x primes, QED.

It is noteworthy that a somewhat similar result may be found for deficient numbers: for any number x, it is possible to create a deficient number with exactly x positive divisors. This is found by the following: because $\lim_{p \to \text{infinity}} p/(p-1) = 1$, it is possible to find such a p that $p/p-1 < 2^{1/x}$. We then take the next x-1 primes (larger than p). We have, for those x primes, the product of (p/(p-1)) for each of the primes is < 2. Construct a deficient number by means of any exponents.

Apart from the squarefree primitive abundant numbers, another interesting subclass exists. These are the primitive abundant numbers where the exponent for every prime is ≥ 2. I have, so far, found several (including 570375 and 342225, which we saw earlier as having only 3 prime divisors). The infinitude of this class is an interesting question.

As intriguing as the above discussions are, it would be of more value to redirect the train of though to a sieve for primitive abundant numbers, as well as to the above hinted question of "quantities of odd primitive abundant numbers with exactly x prime factors". Before this happens, however, it is necessary to introduce one more concept. (NOTE: the lemma discussed on page two allows us to define $\eta(p^a)$ even for infinite a. Keep in mind that the exponents of primes may be infinite, and the number more of a "symbol" for a matching of primes to exponents, unless otherwise specified in the discussion below).

Definition: valid deficient number. A valid deficient number n of a certain (non-repeating) set of primes P is defined as any matching of those primes with non-zero exponents where
a) if none of the exponents are infinite, $\eta(n)<2$
b) otherwise, $\eta(n)\leq 2$.

Examples: a valid deficient number of {3,5,7} would be $3^1 5^{\text{infinity}} 7^{\text{infinity}}$ or $3^2 5^1 7^1$. {2,3} has no valid deficient numbers.

Definition: maximum deficiency, maximum deficient number. Given a set of primes P, the maximum deficiency χ of P is the maximum η achieved by any of the valid deficient numbers that P generates. A maximum deficient number m of P is a number with $\eta(m)= \chi$ (it is theoretically possible for more than one valid deficient number to generate the maximum deficiency)

So far, so good. The closer you can get to 2 without reaching it (or reaching it only with infinite exponents), the better. But, given the existence of valid deficient numbers, will there always be a maximum deficiency?

Theorem: P has a maximum deficiency if and only if it has valid deficient numbers, which it has if and only if $\Pi_{p \in P}(p+1)/p<2$

Proof: The second half of the statement is obvious from the lemma on page 2. Now, if P has only a finite number of valid deficient numbers, the first statement is also obvious; just choose the largest χ. If P has an infinite number of valid deficient numbers, this means that, for some primes (under certain situations in terms of exponents for the other primes), the exponents are unbounded; thus, they all have have a η less than the η if the exponents for all of them are made infinite. So, in the end, we are still left comparing only a finite number of valid deficient numbers; some of them fall into the category of infinite exponents; others may not. Choose the largest η value and call it χ.

Now, we can define certain very explicit conditions for whether or not a certain set of prime numbers P is the distinct prime factorization of any primitive abundant number.

Theorem: Let P = $\{p_1, p_2, ..., p_k\}$ be a set of prime numbers, with $p_1<p_2<...<p_k$. Now, Let

$Q = \{p_1, p_2, ..., p_{k-1}\}$, let $\chi(Q)$ be the maximum deficiency of Q, and let m(Q) be a maximum deficient number of Q. P is the distinct prime factorization of some primitive abundant number n if and only if $\chi(Q)$ exists and $\chi(Q)\, p_k/(p_k-1)>2$.

Proof: That, if $\chi(Q)$ does not exist, P cannot be the distinct prime factorization of any primitive abundant numbers is obvious; no matter what the exponent of p_k is, the rest of the number is already abundant, so including p_k makes the number not primitive abundant.

That, if $\chi(Q)\, p_k/(p_k-1)>2$, P *is* the distinct prime factorization of some primitive abundant number is also obvious; start with an exponent of infinity for p_k and whatever is prescribed by m for all the other primes, and then, starting with p_k, lower the exponents until they are finite and can be lowered no more. Of course, the exponent of p_k will not be 0; if it is, then $\chi(Q)>2$, which we know to be false. Nor is the exponent for any other prime going to be 0:

$\eta(p_i^{a_i})\geq(p_i+1)/p_i$, if a_i is the exponent prescribed to p_i.

$(p_i+1)<p_k$ for all i, so $(p_i+1)/p_i \geq p_k/(p_k-1)$

$$\frac{\chi(Q)\, p_k/(p_k-1)}{\eta(p_i^{a_i})} \leq \frac{\chi(Q)\, p_k/(p_k-1)}{(p_i+1)/p_i} < \frac{\chi(Q)\, p_k/(p_k-1)}{p_k/(p_k-1)} = \chi(Q) \leq 2$$

So, at least some exponent of p_i will be preserved for all i.

The hardest to prove is that $\chi(Q)\, p_k/(p_k-1)\leq 2$, P will not be the distinct prime factorization of any primitive abundant number. This will be a proof by contradiction.

Let n be that primitive abundant number, and let $x=n/p_k$. Then, we have

$\eta(x)<2$ (or n is not *primitive* abundant)

x is a valid deficient number of Q.
$\eta(x)\leq\chi(Q)$

$\eta(p_i^{a_i})<p_k/(p_k-1)$
$\eta(n)=\eta(x)\eta(p_i^{a_i})<\chi(Q)\, p_k/(p_k-1)\leq 2$

or, $\eta(n)<2$, so it is not abundant or perfect. But we said it was primitive abundant. Contradiction. Reject initial proposition. P is not the distinct prime factorization of any primitive abundant number, QED.

To illustrate the power of this analysis, let us examine all primitive abundant numbers with distinct prime factorization {3, 5, 7, p}. Only a little guess and check (checking only 2 "numbers", $3^1 5^{\text{infinity}} 7^{\text{infinity}}$ and $3^2 5^1 7^1$) yields $\chi(\{3,5,7\})=\eta(3^2 5^1 7^1)=(13*6*8)/(9*5*7) \sim 1.98095$. $1/(2/\chi(\{3,5,7\})-1) = 104$, so 103 is the largest prime such that {3,5,7,p} is the distinct prime factorization of some primitive abundant number. In fact, at least in this case, we can rather simply deduce the form of these primitive abundant numbers: $3^2*5*7*p$, always (3*5*7*11 is deficient, and so

3*5*7*p will not be for any greater p).

Thus, we have developed a sieve for primitive abundant numbers based on the quantity of distinct prime factors that it has. Each time that we find a set of primes that yields primitive abundant numbers, we find that set's maximum deficiency (we can create “candidate” valid deficient numbers by “progressive maximization” starting with the formula $\Pi_{p \in P}(p+1)/p<2$ and increasing the exponents until they are maximal or infinite for each prime at a time, then test those against each other to see which has the largest η). Now, beginning with primitive abundant numbers with just two factors, and continuing into those with, 3, 4, and so on, we create an “exhaustive” (for all primes below some point) list of sets that work, sets that don't, and their maximum deficiencies. Thus, if confronted with a set P, we first check if its “Q set” (missing the largest prime) is valid simply by looking at the list, then checking if its maximum deficiency is large enough. After that, we can place this set on the table. By this method, the set builds on itself continuously, making it a “sieve”, of sorts.

I will finally discuss one more topic, the one hinted at earlier about the “quantities of odd and even primitive abundant numbers with exactly x distinct prime factors”. Let us first introduce this conjecture, which I am almost certain is correct (and will treat as a theorem in further discussion).

Conjecture: If the set of prime numbers Q= $\{p_1, p_2, ..., p_{k-1}\}$ has $\Pi_{p \in Q}(p+1)/p<2$ and $\Pi_{p \in Q}p/(p-1)>2$, then there is at least one prime $p_k>p_{k-1}$ for which $\chi(Q)\ p_k/(p_k-1)>2$.

If the above is true, then it is trivial to prove that there is an infinite amount of even primitive abundant numbers with any number x of distinct prime factors: if the primitive abundant number is divisible by a greater power of 2 than 1 (that case has a finite and thus negligible amount of elements), its distinct prime factorization will have $\Pi_{p \in P}(p+1)/p<2$ and $\Pi_{p \in P}p/(p-1)>2$, and thus there will be another prime you can throw in there and still get a distinct prime factorization that creates a primitive abundant number; this is the step of the induction. The base is that no power of two is abundant or perfect, but $\eta(2^{\text{infinity}}) = 2$, so $\chi(\{2\}) = 2$, so every single prime will, with 2, create a primitive abundant number.

By means of this analysis, and combining the progressive minimization results of enough sets of

3 primes to cover all sets of 4 primes, making sure there are no repeats, it is possible to show that there are exactly 565 odd primitive abundant numbers with exactly 4 distinct primes factors.

Now, let us imagine that there is some set Q of odd primes, say with k-1 of them, for which $\chi(Q) = 2$ (naturally, for infinite exponents somewhere). Then, any prime $p_k>p_{k-1}$ will have $\chi(Q)\, p_k/(p_k-1)>2$, so there is an infinite amount of odd primitive abundant numbers with exactly k distinct prime factors. By an induction step similar to the one for the discussion of even primitive abundant numbers, the same is true for all $x\geq k$.

I personally doubt the existence of odd perfect numbers, even if the primes are allotted infinite exponents. If the above anomaly does not occur, then my hypothesis is correct: for any set Q, there will be a maximum prime p_k, because $2/\chi(Q)>1$, and, by a N-ε proof, the value of $p_k/(p_k-1)$ (which approaches 1 as the prime grows to infinity) will eventually dive below this bound. The finite quantity may be proved by induction, with the already discussed case of x=3 as the base, and the step a combination of the arguments above and that the sum of a finite quantity of finite numbers is always finite. Unfortunately, I do not pretend to be able to list off the "odd infinite perfect number" anomaly.

Apart from providing interesting questions, primitive abundant numbers do have more direct applications in the world of number theory. Because perfect numbers are a subset of primitive abundant numbers, the search for odd perfect numbers may be simplified with my sieve for odd primitive abundant numbers, or the lack of their existence may be proven by means of properties of odd primitive abundant numbers.

Further, a list of a large quantity of primitive abundant numbers may be used to evaluate increasingly better lower bounds on the natural density of abundant numbers, as each primitive abundant number n takes out 1/nth of the number line (though overlaps would have to be accounted for). Unfortunately, this method is very computationally intensive. As yet, a method that uses only the first 35 primitive abundant numbers (and several days) returns a value of a little more than 0.2427; I do not see any way to increase accuracy without taking many lifetimes. (See attached paper specifically on this topic).

This concludes my essay. I hope it was illuminating in this as yet (hopefully) ill-explored corner of number theory.