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Extraction of the muon beam frequency distribution via the Fourier analysis of the fast rotation signal

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1 Introduction

The Fermilab E989 Muon g-2 experiments aims to measure the anomalous part a_μ of the magnetic moment of the muon. The muon acquires a magnetic moment

$$\vec{\mu} = g\frac{Q}{2m}\vec{s}, \tag{1}$$

in presence of an external magnetic field B, where Q is the muon electric charge, $\vec{s}$ the muon spin vector and g the gyromagnetic ratio. The anomalous part of the magnetic moment is defined via the deviation of the gyromagnetic ratio: $g = 2(1 + a_\mu)$.

The Muon g-2 experiment relies on the storage of muons inside a weak focusing ring. A continuous C-shape dipole magnet occupies the entirety of the storage ring (44.7 m circumference). It provides the 1.45 T inward radial focusing to store the muons in the ring. The field intensity corresponds to storing muons with the so-called magic momentum of 3.09 GeV/c onto the magic orbit (7.112 m radius). The muons undergo a cyclotron motion in the ring with a revolution frequency of 149 ns.

The weak focusing ring does not provide vertical focusing which is essential to store the muons. The vertical focusing is provided by four electrostatic quadrupoles (ESQ) located around the ring. In their rest frame, the muons see the electric field generated by the ESQs as a magnetic field.

The measurement of a_μ is performed via the measurement of the intensity of the magnetic field in term of the Larmor precession frequency of a free proton

$$\hbar\omega_p = 2\mu_p|\overrightarrow{B}|, \tag{2}$$

and the intrinsic spin precession frequency of the muon ω_a. The intrinsic muon spin precession frequency is obtained by subtracting the cyclotron frequency ω_C to the total spin precession frequency of the muon ω_S:

$$\omega_a = \omega_S - \omega_C. \tag{3}$$

Equation (4) shows the most general vectorial relation between a_μ and ω_a, B:

$$\vec{\omega}_a = \vec{\omega}_S - \vec{\omega}_C = -\frac{Qe}{m}\left[a_\mu\vec{B} - a_\mu\left(\frac{\gamma}{\gamma+1}\right)(\vec{\beta}\cdot\vec{B})\vec{\beta} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right] \tag{4}$$

The term proportional to $\vec{\beta}\times\vec{E}$ in Eq. (4) corresponds to the electric field contribution to ω_a. One can see that if $a_\mu = \frac{1}{\gamma^2-1}$ the contribution disappears. This is the approach followed by the previous CERN and BNL E821 experiments. The magic momentum of 3.09 GeV/c allows the electric field contribution to vanish in first order.

The non-vanishing electric field contribution arises from the fact that the stored muon beam has a momentum spread of about 0.1%. This effect is non-negligible and needs to be taken care of. The estimated correction to ω_a due to the electric field was estimated by E821 to be 0.47 ± 0.05 ppm for the 2001 data set for the low n-value data set. The final E821 result for a_μ has a 0.54 ppm precision, which is of the order of the electric field correction.

The term proportional to $\vec{\beta} \cdot \vec{B}$ in Eq. (4) corresponds to the so-called pitch correction due to the muon velocity not being purely contain in the horizontal plane. This is an other correction that needs to be addressed.

This note presents an attempt at estimating the electric field correction using the Fourier analysis technique applied to the Fast Rotation signal. This technique was developed by E821 and detailed in [1].

2 Fast Rotation

2.1 Generality

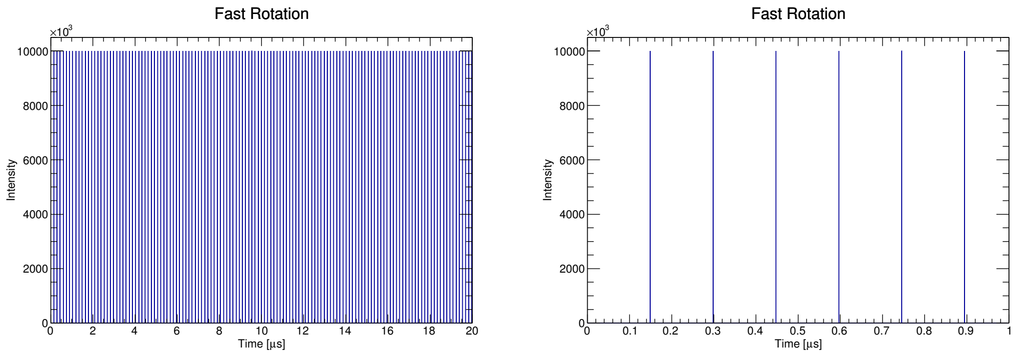
Fast rotation refers to the cyclotron motion in the ring, which has a revolution frequency of 6.71 MHz (period of 149 ns) for muons propagating on the magic orbit with the magic momentum. The fast rotation signal corresponds to observing the evolution of the beam as it rotates around the ring. The transverse emittance of the beam and its momentum spread make the beam decohere (or debunch). To understand how the momentum spread leads to decoherence, one has to remember that the orbit radius for a fixed magnetic field is set by the momentum of the particle. Two particles injected on the magic orbit but with different momenta will have different radii. Given than the speed of the muons almost does not depend on the momentum given they are relativistic, the different momenta lead to different revolution frequencies. The higher momentum muon will complete a revolution slower (larger radius) than the lower momentum muon (smaller radius). The small difference in revolution frequency leads to the beam having decohered after about 30 μs. From an initial beam with a limited longitudinal extension we end up with a beam filling the entirety of the ring.

2.2 Analytical description

Imagine that the initial muon distribution has zero emittance (no transverse size), zero momentum spread and zero bunch length (no longitudinal extension). The particles share a common revolution period T and the time dependence of the intensity signal at a fixed point (signal measured with a fiber harp for example), is

$$I(t) = \sum_{n=0}^{\infty} \delta \left(t - \left(n + \frac{\theta}{2\pi} \right) T \right) , \tag{5}$$

where n corresponds to the number of revolutions in the ring, and θ is the azimuthal position of the point in the ring. Figure 1 shows what the fast rotation signal looks like for a beam



(a) (b)

Figure 1: Fast rotation signal as a function of time for a muon beam with zero emittance, zero momentum spread and zero bunch length for two time windows: (a) 0-20 μs, (b) 0-1 μs.

with the characteristics described above. The beam remains unchanged as a function of time given its perfectness.

A particle with momentum offset Δ will have revolution period $T(1+\Delta)$, so that

$$I(t,\Delta)=\sum_{n=0}^{\infty}\delta\left(t-\left(n+\frac{\theta}{2\pi}\right)T\left(1+\Delta\right)\right) \tag{6}$$

The fast rotation signal at the point is then [1]

$$S(t)=\sum_{n=0}^{\infty}\int\varrho(\Delta)\delta\left(t-nT\left(1+\Delta\right)\right)d\Delta \tag{7}$$

where $\varrho(\Delta)$ is the distribution of momenta offsets. If the momentum distribution is a Gaussian with width Δ_0, then

$$\begin{aligned}
S(t) &= \sum_{n=0}^{\infty}\int\frac{e^{-\Delta^2/(2\Delta_0^2)}}{\sqrt{2\pi}\Delta_0}\delta\left(t-nT\left(1+\Delta\right)\right)d\Delta && (8)\\
&= \sum_{n=0}^{\infty}\int\frac{e^{-\Delta^2/(2\Delta_0^2)}}{\sqrt{2\pi}\Delta_0}\frac{\delta\left(\Delta-\left(\frac{t}{nT}-1\right)\right)}{nT}d\Delta && (9)\\
&= \sum_{n=0}^{\infty}\frac{e^{-(\frac{t}{nT}-1)^2/2\Delta_0^2}}{\sqrt{2\pi}\Delta_0 nT} && (10)
\end{aligned}$$

Note that there is no muon decay in Eq. (10) on the accompanying plots. Figure 2 shows the fast rotation signal for a beam with zero emittance, zero bunch length but with a 0.112% momentum spread. One can see that the beam decoheres. The decoherence envelope is set

[1] the notation $(n+\frac{\theta}{2\pi})T$ will be noted thereafter nT for improved text clarity

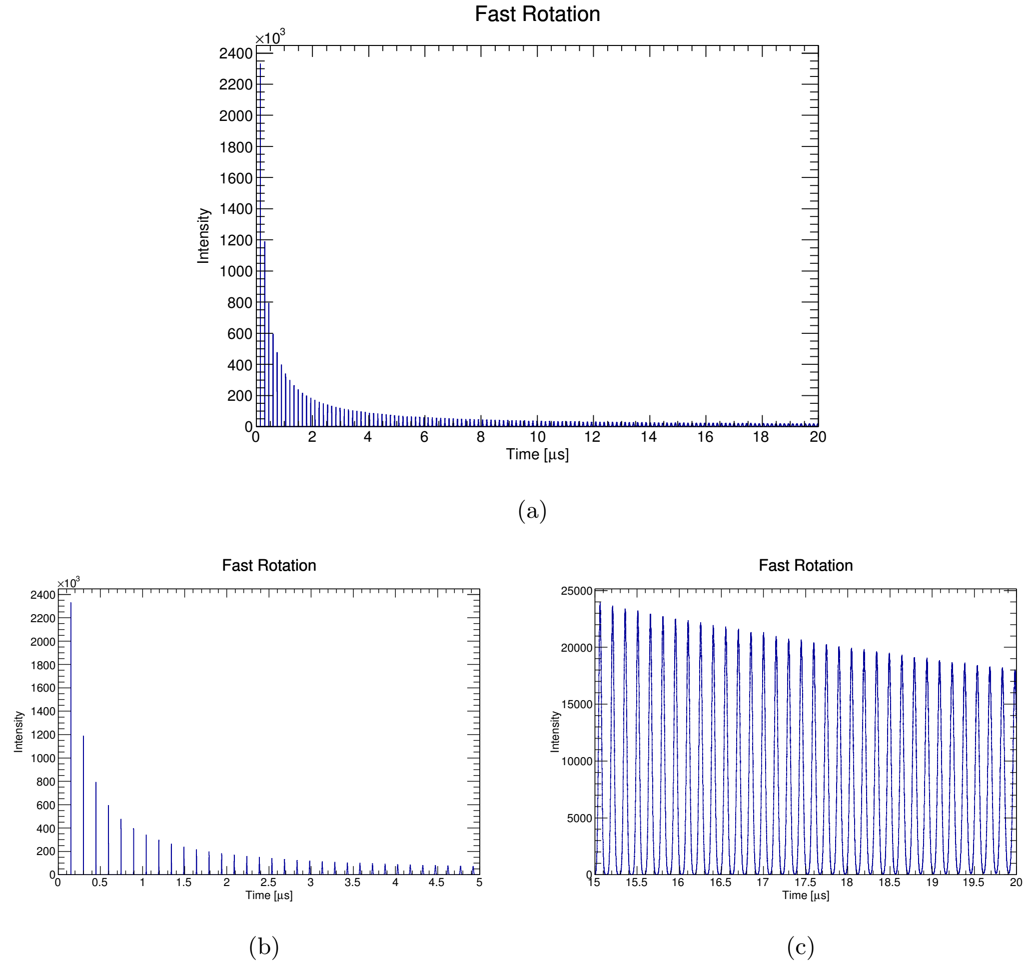


Figure 2: Fast rotation signal as a function of time for a muon beam with zero emittance, zero bunch length but width a 0.112% momentum spread for three time windows: (a) 0-20 μs, (b) 0-5 μs, (c) 15-20 μs.

by the momentum spread, with the greater spread leading to faster the decoherence. The intensity asymptotically approaches that of the average stored current $\langle I \rangle = eN\mu/T$.

A realistic beam has, on top of the momentum spread, a non-zero bunch length. In this case Eq. (10) is re-expressed introducing a temporal offset t' as

$$S(t, t') = \sum_{n=0}^{\infty} \frac{e^{-(\frac{t-t'}{nT}-1)^2/(2\Delta_0^2)}}{\sqrt{2\pi}\Delta_0 nT}. \tag{11}$$

Suppose the intial temporal (longitudinal) distribution of the muons is Gaussian

$$\xi(t') = \frac{1}{\sqrt{2\pi}\sigma_t} e^{-t'^2/(2\sigma_t^2)}$$

. Then

$$\begin{aligned}
S(t) &= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} dt' \frac{e^{-(\frac{t-t'}{nT}-1)^2/(2\Delta_0^2)}}{\sqrt{2\pi}\Delta_0 nT}\frac{1}{\sqrt{2\pi}\sigma_t}e^{-t'^2/(2\sigma_t^2)} \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi\Delta_0 nT\sigma_t}\int_{-\infty}^{\infty} dt' e^{-(\frac{t-t'}{nT}-1)^2/(2\Delta_0^2)}e^{-t'^2/(2\sigma_t^2)} \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi\Delta_0 nT\sigma_t}e^{-t^2/(2(nT\Delta_0)^2)}\int_{-\infty}^{\infty} dt' e^{-(\frac{t'^2-2tt'}{(nT)^2}+1-\frac{2(t-t')}{nT})/(2\Delta_0^2)}e^{-t'^2/(2\sigma_t^2)} \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi\Delta_0 nT\sigma_t}e^{-t^2/(2(nT\Delta_0)^2)}e^{\frac{t}{nT\Delta_0^2}}\int_{-\infty}^{\infty} dt' \exp(-t'^2(\frac{1}{2(nT)^2\Delta_0^2}+\frac{1}{2\sigma_t^2})+t'(\frac{t}{nT}-1)/(nT\Delta_0^2))e^{-1/(2\Delta_0^2)} \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi\Delta_0 nT\sigma_t}e^{-(\frac{t}{nT}-1)^2/2\Delta_0^2}\int_{-\infty}^{\infty} dt' \exp\left(-\alpha t'^2+\beta t'\right) \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi\Delta_0 nT\sigma_t}e^{-(\frac{t}{nT}-1)^2/(2\Delta_0^2)}\sqrt{\frac{\pi}{\alpha}}e^{\beta^2/4\alpha},
\end{aligned}$$

where $\alpha = \frac{1}{2(nT)^2\Delta_0^2}+\frac{1}{2\sigma_t^2}$ and $\beta = (\frac{t}{nT}-1)/(nT\Delta_0^2)$. Finally

$$\begin{aligned}
S(t) &= \sum_{n=0}^{\infty}\frac{1}{2\pi\Delta_0 nT\sigma_t}e^{-(\frac{t}{nT}-1)^2/(2\Delta_0^2)}\frac{\sqrt{2\pi}nT\Delta_0\sigma_t}{\sqrt{(nT)^2\Delta_0^2+\sigma_t^2}}\exp(\frac{(\frac{t}{nT}-1)^2}{(nT\Delta_0^2)^2}\frac{(nT)^2\Delta_0^2\sigma_t^2}{2(nT)^2\Delta_0^2+2\sigma_t^2}) \\
&= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}e^{-(\frac{t}{nT}-1)^2/(2\Delta_0^2)}\frac{1}{\sqrt{((nT)^2\Delta_0^2+\sigma_t^2}}\exp(\frac{(\frac{t}{nT}-1)^2}{\Delta_0^2}\frac{\sigma_t^2}{2(nT)^2\Delta_0^2+2\sigma_t^2}) \\
&= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{1}{\sqrt{((nT)^2\Delta_0^2+\sigma_t^2}}\exp\left(\frac{-(\frac{t}{nT}-1)^2}{2\Delta_0^2}\right)\left(1-\frac{\sigma_t^2}{(nT)^2\Delta_0^2+\sigma_t^2}\right).
\end{aligned} \tag{12}$$

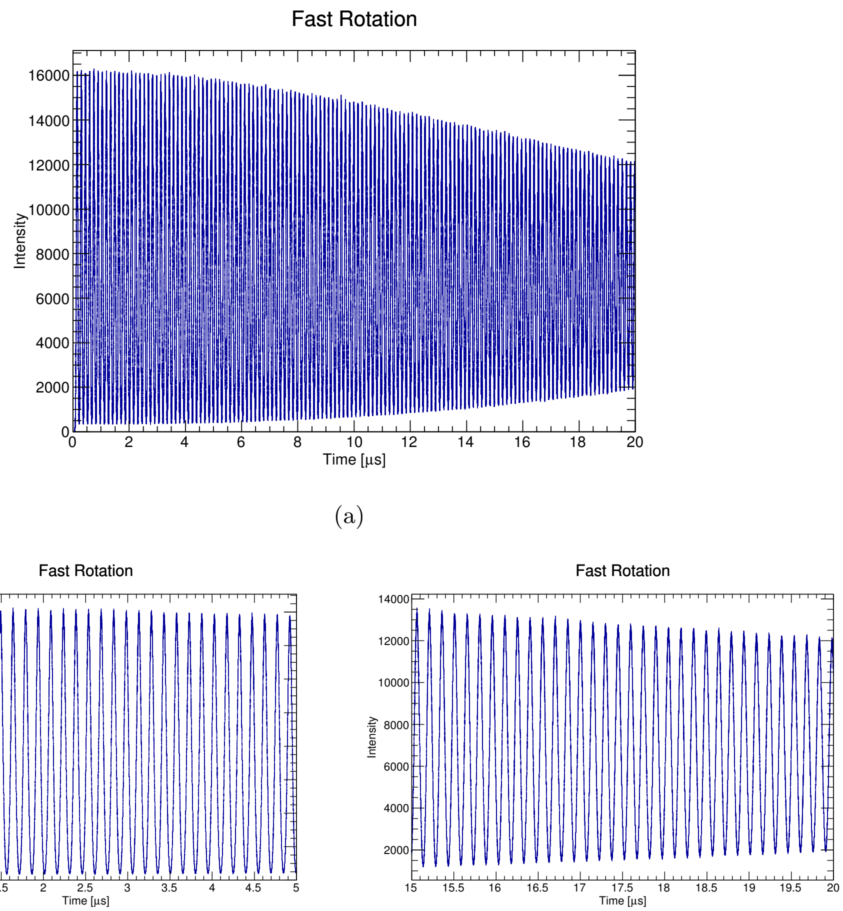
It will turn out to be convenient to rewrite equation (12) as

$$S(t) = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{e^{-(t-nT)^2/2\Delta_0'^2 n^2T^{22}}}{\sqrt{n^2T^2\Delta_0'^2}} \tag{13}$$

where

$$\Delta_0'^2 = \frac{n^2T^2\Delta_0^2+\sigma_t^2}{n^2T^2}$$

Figure 3 shows the fast rotation signal for a beam with zero emittance, non-zero bunch length and non-zero momentum spread. One can see that the beam decoheres faster having a longitudinal extension. It is expected since the longitudinal extension can be seen as created by a beam with a momentum spread having undergone an arbitrary number of revolution.



(a)

(b) (c)

Figure 3: Fast rotation signal as a function of time for a muon beam with zero emittance, bunch length of 20 ns and momentum spread of 0.112% for three time windows: (a) 0-20 μs, (b) 0-5 μs, (c) 15-20 μs.

3 Theory of the Fourier Transform of the Fast Rotation Signal

3.1 Mathematical Preliminaries

Definition 1 (Fourier's transform and inverse transform) *Given a function f, its Fourier transform is*

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ixt} dt$$

and its inverse Fourier transform is

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ixt} dt$$

If f is absolutely integrable, then the Fourier transforms of f always exist. It is easy to check that $f = \tilde{\hat{f}} = \hat{\tilde{f}}$.

Theorem 1 (Fourier's Integral) *For any piecewise continuous, piecewise differentiable, and absolutely integrable function f, the following equality holds:*

$$f(x) = \frac{1}{\pi} \int_0^\infty d\omega \int_{-\infty}^\infty f(t) \cos \omega(t-x) dt$$

If f is even, then Fourier's integral simplifies to

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x d\omega \int_0^\infty f(t) \cos \omega t dt$$

Suppose f is a piecewise continuous, piecewise differentiable, and absolutely integrable function on $[0, \infty)$. If we extend f to an even function defined on the whole real line, then the formula above is valid for the extension. In particular, it is also valid for $x \geq 0$.

Theorem 2 (Convolution Theorem) *The convolution of two functions f and g is defined to be*

$$(f * g)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(\tau) g(t-\tau) d\tau$$

The convolution theorem states that

$$\widehat{f * g} = \hat{f}\hat{g} \text{ and } \widetilde{f * g} = \tilde{f}\tilde{g}$$

3.2 The case of initially zero bunch length

In this case we have that $\xi(t') = \delta(t')$, so the fast rotation signal becomes

$$S(t) = \sum_{n=0}^\infty \frac{\varrho\left(\frac{t}{(n+\theta/2\pi)T} - 1\right)}{(n+\theta/2\pi)T}$$

The beam first arrives at the detector with azimuthal position θ at time t_0. Given that the frequency distrbution which the muons orbit the ring with is narrow, $\omega - \omega_0/\omega \sim 10^{-3}$, the bunch evolves such that in the first turn the beam spreads negligibly. Hence t_0 is the arrival time of the center of mass of the bunch at the detector. Therefore we have that $\theta/2\pi = t_0/T$, so $S(t)$ may be written

$$S(t) = \sum_{n=0}^\infty \frac{\varrho\left(\frac{t}{nT+t_0} - 1\right)}{nT+t_0}$$

We next assume that $\varrho(\Delta)$ is sufficiently well behaved so as to be absolutely integrable, piecewise continuous and piecewise differentiable. Define $\mathcal{S}(t) \equiv S(t+t_0)$. Extend $\mathcal{S}$ to the entire real line as an even function. This is valid to do, since $S(t)$ is time-reversal symmetric about t_0. By Theorem 1, we may express $\mathcal{S}$ as

$$\mathcal{S}(t) = \frac{2}{\pi}\int_0^\infty \cos\omega t d\omega \int_0^\infty \mathcal{S}(t')\cos\omega t' dt'$$

We then recover $S(t)$:

$$S(t+t_0) = \frac{2}{\pi}\int_0^\infty \cos\omega t d\omega \int_0^\infty S(t'+t_0)\cos\omega t' dt' =$$

$$\frac{2}{\pi}\int_0^\infty \cos\omega t d\omega \int_{t_0}^\infty S(t')\cos\omega(t'-t_0) dt' \Rightarrow$$

$$S(t) = \frac{2}{\pi}\int_0^\infty \cos\omega(t-t_0) d\omega \int_{t_0}^\infty S(t')\cos\omega(t'-t_0) dt'$$

But we also know from Definition 1 that

$$\mathcal{S}(t) = \tilde{\hat{\mathcal{S}}}(t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\omega t} d\omega \int_{-\infty}^\infty \mathcal{S}(t')e^{i\omega t'} dt' =$$

$$\frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\omega t} d\omega \int_{-\infty}^\infty \mathcal{S}(t')(\cos\omega t' + i\sin\omega t') dt' = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\omega t} d\omega \int_{-\infty}^\infty \mathcal{S}(t')\cos\omega t' dt'$$

where we used the fact that $\mathcal{S}$ is an even function, and the symmetric integral of the product of an even and odd function vanishes. From the preceding expression we now see that $\hat{\mathcal{S}}$ is an even function. So we write

$$\frac{1}{2\pi}\int_{-\infty}^\infty \hat{\mathcal{S}}(\omega)\cos\omega t d\omega = \frac{1}{\pi}\int_0^\infty \hat{\mathcal{S}}(\omega)\cos\omega t d\omega = \frac{2}{\pi}\int_0^\infty \cos\omega t d\omega \int_0^\infty \mathcal{S}(t')\cos\omega t' dt'$$

Thus

$$\hat{S}(\omega) = \sqrt{\frac{2}{\pi}}\int_{t_0}^\infty S(t)\cos\omega(t-t_0) dt \tag{14}$$

is the Fourier transform of S. So if we're given $\hat{S}(\omega)$, we may express S as

$$S(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{S}(\omega)\cos\omega(t-t_0) d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{S}(\omega)\cos\omega(t-t_0) d\omega \tag{15}$$

3.3 Symmetric Bunch Length Distribution $\xi(t')$

We call the fast rotation signal with no inital bunch length $S_0(t)$. Given an initial longitudinal distribution $\xi(t')$, the fast rotation signal is described by

$$S(t) = \int \xi(t')S_0(t-t')dt' = \sqrt{2\pi}(\xi * S_0)(t)$$

as was seen in the example above. In the ideal case we would directly apply the convolution theorem to find that

$$\hat{S}(\omega) = \sqrt{2\pi}\hat{\xi}(\omega)\hat{S}_0(\omega) \tag{16}$$

However, we won't have $S_0(t)$ to work with directly, nor $\xi(t)$. We need to use $S(t)$ to get $\hat{S}(\omega)$.

We express $S_0(t-t')$ through Fourier's integral:

$$S_0(t-t') = \frac{2}{\pi}\int_0^\infty \cos\omega(t-t_0-t')d\omega \int_{t_0}^\infty S_0(\bar{t})\cos\omega(\bar{t}-t_0)d\bar{t} =$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{S}_0(\omega)\cos\omega(t-t_0-t')d\omega$$

Substituting this expression into the convolution integral, we get

$$S(t) = (\xi * S_0)(t) = \int_{-\infty}^\infty \xi(t)S_0(t-t')dt' = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{S}_0(\omega)d\omega \int_{-\infty}^\infty \xi(t')\cos\omega(t-t_0-t')dt' =$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{S}_0(\omega)\langle\cos\omega(t-t_0-t')\rangle d\omega =$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{S}_0(\omega)\langle\cos\omega(t-t_0)\cos\omega t' + \sin\omega(t-t_0)\sin\omega t'\rangle d\omega =$$

$$\sqrt{\frac{2}{\pi}}\left(\int_0^\infty \hat{S}_0(\omega)\langle\cos\omega t'\rangle\cos\omega(t-t_0)d\omega + \int_0^\infty \hat{S}_0(\omega)\langle\sin\omega t'\rangle\sin\omega(t-t_0)d\omega\right)$$

where $\langle\ \rangle$ denotes averaging with respect to $\xi(t)$.

If $\xi(t')$ is even, $\langle\sin\omega t'\rangle = 0$. We are then left with

$$S(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{S}_0(\omega)\langle\cos\omega t'\rangle\cos\omega(t-t_0)d\omega \tag{17}$$

suggesting that $\hat{S}(\omega) = \hat{S}_0(\omega)\langle\cos\omega t'\rangle$. We can see that this is true by looking back at (16):

$$\hat{\xi}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \xi(t')e^{i\omega t'}dt' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \xi(t')\cos\omega t' dt' = \frac{\langle\cos\omega t'\rangle}{\sqrt{2\pi}}$$

and therefore $\hat{S}(\omega) = \sqrt{2\pi}\hat{\xi}(\omega)\hat{S}_0(\omega) = \hat{S}_0(\omega)\langle\cos\omega t'\rangle$. Thus (17) is the inverse Fourier transform, implying that

$$\hat{S}(\omega) = \sqrt{\frac{2}{\pi}}\int_{t_0}^\infty S(t)\cos\omega(t-t_0)dt \tag{18}$$

3.4 Asymmetric Longitudinal Bunch Distribution

If $\xi(t')$ is not symmetric, then we are faced with the problem that $\hat{S}(\omega)$ features an imaginary component. Directly applying the convolution theorem yields

$$\hat{S}(\omega) = \hat{S}_0(\omega)\langle\cos\omega t'\rangle + i\hat{S}_0(\omega)\langle\sin\omega t'\rangle$$

Let's take a specific example of an asymmetric distribution to get a feel for what to do later. Suppose ξ is even about some point $\mu \neq 0$ (also the center of mass of the distribution), which is likely to happen in the g-2 experiment (accidental offset from the origin). Let t_0 be the time that the center of mass of the beam first arrives at a given detector if μ were to equal 0. Then the actual arrival time of the center of the bunch at the detector is $t'_0 = t_0 + \mu$. Furthermore, let's define $\xi'(t) = \xi(t + \mu)$. Then the physical situation described by ξ' and t'_0 is the same as that described by ξ and t_0. But if we use ξ' and t'_0 for the Fourier analysis, we no longer have the imaginary term in $\hat{S}(\omega)$, since we're now dealing with a familiar situation: a symmetric longitudinal distribution. Therefore $\hat{S}(\omega) = \hat{S}'_0(\omega)\langle\cos\omega t'\rangle'$ where $\langle\ \rangle'$ denotes averaging with respect to $\xi'(t)$, and $S'_0(t)$ is the fast rotation signal without initial longitudinal spread and with start time t'_0

What happens with an asymmetric distribution $\xi(t)$ defined on the interval $[a, b]$? There is now no unique symmetry point which makes it obvious how to redefine t_0. But using the idea of the last paragraph, let $t'_0 = t_0 + x_0$ where x_0 is such that the for the function defined as

$$f(x) = \int_{a-x}^{b-x} \xi(t + x)\sin\omega t dt$$

$f(x_0) = 0$. This effectively amounts to finding a new origin about which $\langle\sin\omega t'\rangle = 0$. Since the frequency distribution that we deal with is very narrow, we may approximate $f(x)$ by

$$f(x) \approx \int_{a-x}^{b-x} \xi(t)\sin\omega_0 t dt$$

Once we find x_0 and thus t'_0, then once again $\hat{S}(\omega) = \hat{S}'_0(\omega)\langle\cos\omega t'\rangle'$.

3.5 Physical Interpretation of the Fourier Transform of the Fast Rotation Signal

Assuming that the energy offset distribution $\varrho(\Delta)$ is centered with a maximum at $\Delta = 0$, the harmonics of the Fourier transform of the fast rotation signal will be featured at integer multiples of the magic frequency; this is evident from the construction of the general fast rotation signal. The signal is due to the orbits of muons at various revolution frequencies, so $\hat{S}(\omega)$ describes the distribution of revolution frequencies among the muons. However, the muons are restricted to the vacuum chamber, making frequencies corresponding to muons outside the chamber nonphysical. Hence the first harmonic corresponds to the physical revolution frequency distribution of the muons.

3.6 Corrections to the Fourier Transform

If the detector detects the muons at a time $t_s > t_0$, we see a fast rotation signal $S_1(t)$ which vanishes on (t_0, t_s) and equals the full fast rotation signal $S(t)$ on (t_s, ∞). Taking the Fourier transform of $S_1(t)$, we get the frequency spectrum

$$\hat{S}_1(\omega) = \sqrt{\frac{2}{\pi}} \int_{t_s}^{\infty} S_1(t) \cos \omega (t - t_0) dt \tag{19}$$

This is not the complete frequency spectrum as we are missing the following component

$$\Delta(\omega) = \sqrt{\frac{2}{\pi}} \int_{t_0}^{t_s} S(t) \cos \omega (t - t_0) dt \tag{20}$$

The problem now is to somehow obtain $\Delta(\omega)$ using only $S_1(t)$ and $\hat{S}_1(\omega)$.

We first obtain an expression for $\Delta(\omega)$ using $\hat{S}(\omega)$:

$$\Delta(\omega) = \sqrt{\frac{2}{\pi}} \int_{t_0}^{t_s} S(t) \cos \omega (t - t_0) dt = \frac{2}{\pi} \int_{t_0}^{t_s} \int_{0}^{\infty} \hat{S}(\omega') \cos \omega' (t - t_0) \cos \omega (t - t_0) d\omega' dt$$

where we used (15). Switching the order of integration to integrate with respect to t first, we get

$$\Delta(\omega) = \frac{1}{\pi} \int_{0}^{\infty} \hat{S}(\omega') \left(\frac{\sin(\omega - \omega')(t_s - t_0)}{\omega - \omega'} + \frac{\sin(\omega + \omega')(t_s - t_0)}{\omega + \omega'} \right) d\omega'$$

Outside the vacuum chamber (given by the interval (ω_1^-, ω_1^+)) and the "harmonics" of the vacuum chamber ($(\omega_n^-, \omega_n^+) = (\omega_1^- + (n-1)\omega_0, n\omega_1^+ + (n-1)\omega_0)$ where ω_0 is the magic frequency), $\hat{S}(\omega)$ vanishes, so we may rewrite the above integral as

$$\sum_{n=1}^{\infty} \frac{1}{\pi} \int_{\omega_n^-}^{\omega_n^+} \hat{S}(\omega') \left(\frac{\sin(\omega - \omega')(t_s - t_0)}{\omega - \omega'} + \frac{\sin(\omega + \omega')(t_s - t_0)}{\omega + \omega'} \right) d\omega'$$

As such, we may neglect the second term in the parentheses as it will be negligibly small:

$$\Delta(\omega) = \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{\omega_n^-}^{\omega_n^+} \hat{S}(\omega') \frac{\sin(\omega - \omega')(t_s - t_0)}{\omega - \omega'} d\omega'$$

To get the correction, we first note that the Fourier transform has associated with it an uncertainty principle, namely that $\Delta\omega\Delta t \sim 2\pi$. If $t_s - t_0$ isn't too great, we can expect that the spread of $\hat{S}_1(\omega)$ about the magic frequency will change negligibly relative to that of $\hat{S}(\omega)$. This means that the frequency content of $\hat{S}_1(\omega)$ is approximately the same as that of $\hat{S}(\omega)$.

With $t_s > t_0$, $\hat{S}_1(\omega)$ features frequencies outside the vacuum chamber boundaries due to aliasing effects. However, these are non-physical as no muons can actually be outside the chamber. Due to the uncertainty principle, the spread of $\hat{S}_1(\omega)$ about the magic frequency remains approximately the same as $\hat{S}(\omega)$. And since the only physical frequencies lie within (ω_n^-, ω_n^+), we may approximate $\Delta(\omega)$ by writing

$$\Delta(\omega) = \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{\omega_n^-}^{\omega_n^+} \hat{S}_1(\omega') \frac{\sin(\omega - \omega')(t_s - t_0)}{\omega - \omega'} d\omega' \tag{21}$$

3.6.1 A Limiting Case: One Muon

Suppose we have a single muon in the ring. The fast rotation signal that describes the muon is

$$S(t) = \sum_{n=0}^{\infty} \delta(t - nT - t_0)$$

By time reversal symmetry, we may rewrite $S(t)$ as $\sum_{n=-\infty}^{\infty} \delta(t - nT - t_0)$, the Dirac comb. Then $\hat{S}(\omega) = \frac{\sqrt{2\pi}}{T}\delta(\omega - 2\pi n/T)$. Now suppose we miss several turns of the signal. Thus

$$\Delta(\omega) = \sqrt{\frac{2}{\pi}} \int_{t_0}^{t_s} S(t) \cos\omega(t - t_0) dt = \sqrt{\frac{2}{\pi}} \sum_{n=0}^{N} \cos\omega nT$$

where $N = \lfloor \frac{t_s - t_0}{T} \rfloor$ so that $\hat{S}_1(\omega) = \sqrt{2/\pi} \sum_{n=N+1}^{\infty} \cos\omega nT$. See Figures INSERT and INSERT. With Figure INSERT we note that as the number of turns approaches infinity, the frequency spectrum becomes the Dirac comb; this is due to the fact

$$\sum_{n=0}^{\infty} \cos nx = \pi\delta(x - 2\pi n) + \frac{1}{2}$$

see http://functions.wolfram.com/ElementaryFunctions/Cos/23/02/0013/. From the figures we see that in fact the approximation for $\Delta(\omega)$ works.

4 Analysis for a Gaussian Energy Offset Distribution and Initially Zero Bunch Length

If the energy offset distribution is a Gaussian with width Δ_0, then

$$S(t) = \sum_{n=0}^{\infty} \frac{e^{-(t-(nT+t_0))^2/2\Delta_0^2(nT+t_0)^2}}{\sqrt{2\pi}\Delta_0(nT + t_0)}$$

Suppose the center of mass of the muon beam first passes the detector at time t_0, and the detector starts detecting at some time $t_s > t_0$. Then the Fourier transform of the fast rotation signal is

$$
\begin{aligned}
\hat{S}(\omega) &= \sqrt{\frac{2}{\pi}} \int_{t_s}^{\infty} S(t) \cos \omega (t - t_0) dt \\
&= \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \int_{t_s}^{\infty} \frac{e^{-(t-(nT+t_0))^2/2\Delta_0^2(nT+t_0)^2}}{\sqrt{2\pi}\Delta_0(nT+t_0)} \cos \omega (t - t_0) dt \\
&= \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{e^{-1/2\Delta_0^2}}{2\sqrt{2\pi}\Delta_0(nT+t_0)} \int_{t_s}^{\infty} e^{-t^2/2\Delta_0^2(nT+t_0)^2} e^{t/(nT+t_0)\Delta_0^2} e^{\pm i\omega t} e^{\mp i\omega t_0} dt. \qquad (22)
\end{aligned}
$$

From Gradshteyn and Rizhik, we know that

$$
\int_u^{\infty} \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) dx = \sqrt{\pi\beta} e^{\beta\gamma^2} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left(\gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}}\right)\right) \qquad (23)
$$

for $\mathrm{Re}(\beta) > 0$ and $u \geq 0$. Thus the result of using this integral in computing the Fourier transform is $\hat{S}(\omega) =$

$$
\begin{aligned}
&\sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} - \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) + \\
&\sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{-i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} + \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right)
\end{aligned} \qquad (24)
$$

4.1 Corrections to the Fourier transform

The detector in the section above started to detect the muon beam at a time $t_s > t_0$. This results in some lost frequency content in the Fourier transform above, creating distortions of the frequency spectrum. The missing part of the frequency spectrum is described by formula (20). Let's take a closer look at Δ.

$$
\Delta(\omega) = \sqrt{\frac{2}{\pi}} \int_{t_0}^{t_s} S(t) \cos \omega (t - t_0) dt
$$

$$
= \sqrt{\frac{2}{\pi}} \int_{t_0}^{\infty} S(t) \cos \omega (t - t_0) dt - \sqrt{\frac{2}{\pi}} \int_{t_s}^{\infty} S(t) \cos \omega (t - t_0) dt
$$

Using (23), we have

$$\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2}e^{i\omega nT}}{4}\left(1-\frac{\sqrt{\pi}}{2}\mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}}-\frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}}+\frac{t_0}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right)+$$
$$\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2}e^{-i\omega nT}}{4}\left(1-\frac{\sqrt{\pi}}{2}\mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}}+\frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}}+\frac{t_0}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right)$$
$$-\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2}e^{i\omega nT}}{4}\left(1-\frac{\sqrt{\pi}}{2}\mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}}-\frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}}+\frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right)+$$
$$-\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2}e^{-i\omega nT}}{4}\left(1-\frac{\sqrt{\pi}}{2}\mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}}+\frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}}+\frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) \tag{25}$$

Figure 4 shows how the Fourier transform and the corresponding corrections are altered by increasing t_s.

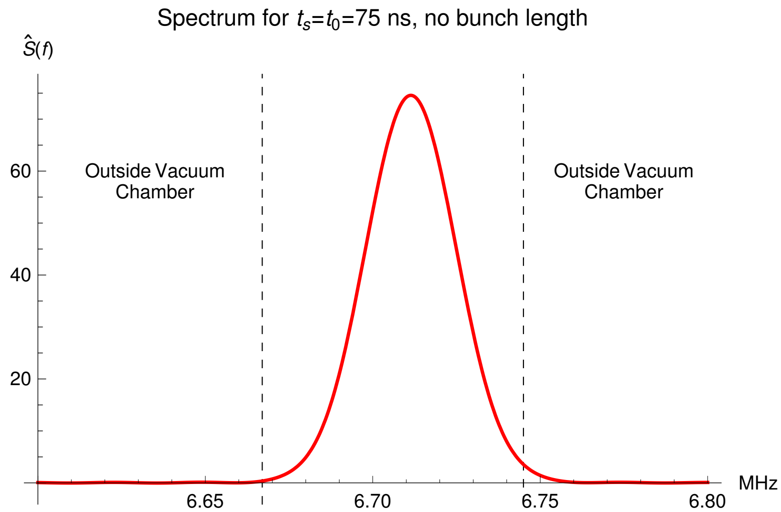
5 Analysis for a Gaussian Energy Offset Distribution and Gaussian Longitudinal Distribution

The fast rotation signal in this case is of the same form as in the case of no longitudinal distribution:

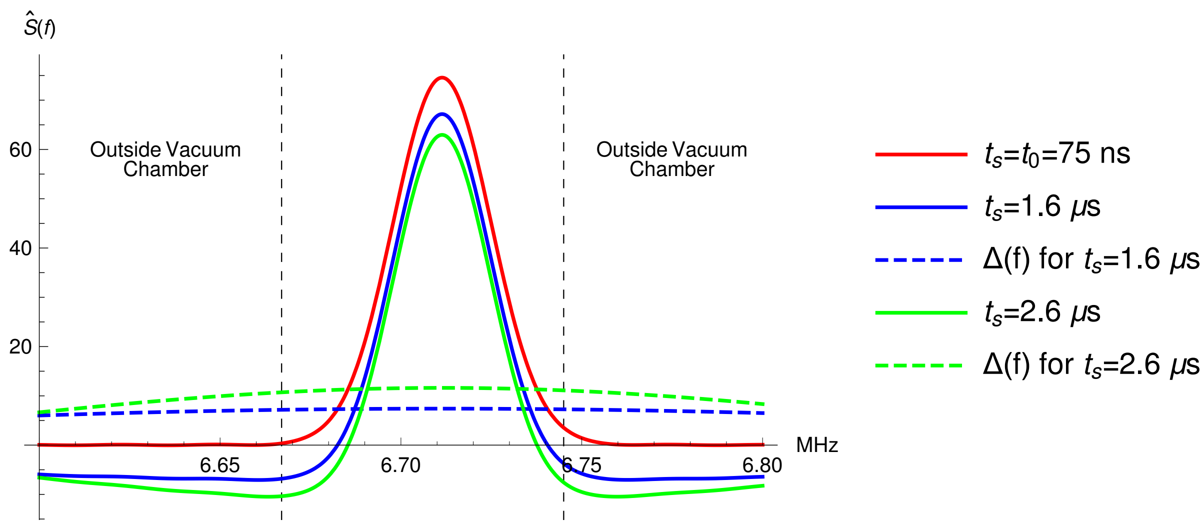
$$S(t)=\sum_{n=0}^{\infty}\frac{e^{-(t-(nT+t_0))^2/2\Delta_0'^2(nT+t_0)^2}}{\sqrt{2\pi}\Delta_0'(nT+t_0)}$$

Thus $\hat{S}(\omega)=$

$$\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{e^{-\omega^2(nT+t_0)^2\Delta_0'^2/2}e^{i\omega nT}}{4}\left(1-\frac{\sqrt{\pi}}{2}\mathrm{Erf}\left[\frac{-1}{\Delta_0'\sqrt{2}}-\frac{i\omega(nT+t_0)\Delta_0'}{\sqrt{2}}+\frac{t_s}{\Delta_0'(nT+t_0)\sqrt{2}}\right]\right)+$$
$$\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{e^{-\omega^2(nT+t_0)^2\Delta_0'^2/2}e^{-i\omega nT}}{4}\left(1-\frac{\sqrt{\pi}}{2}\mathrm{Erf}\left[\frac{-1}{\Delta_0'\sqrt{2}}+\frac{i\omega(nT+t_0)\Delta_0'}{\sqrt{2}}+\frac{t_s}{\Delta_0'(nT+t_0)\sqrt{2}}\right]\right) \tag{26}$$



(a)

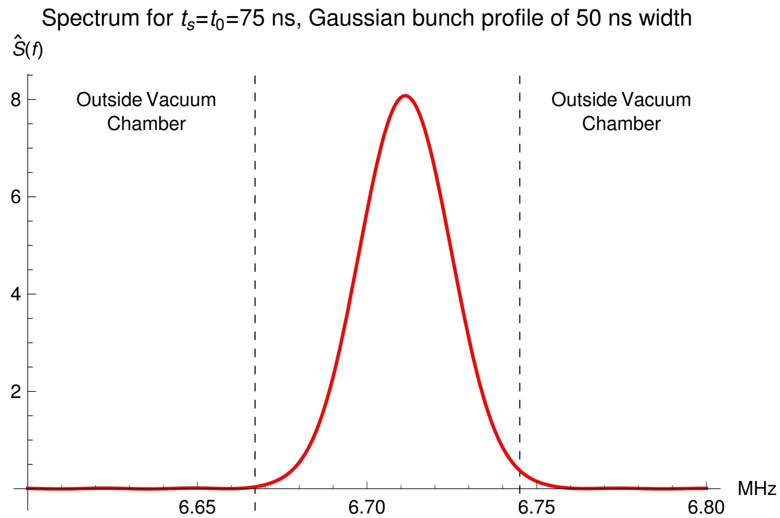


(b)

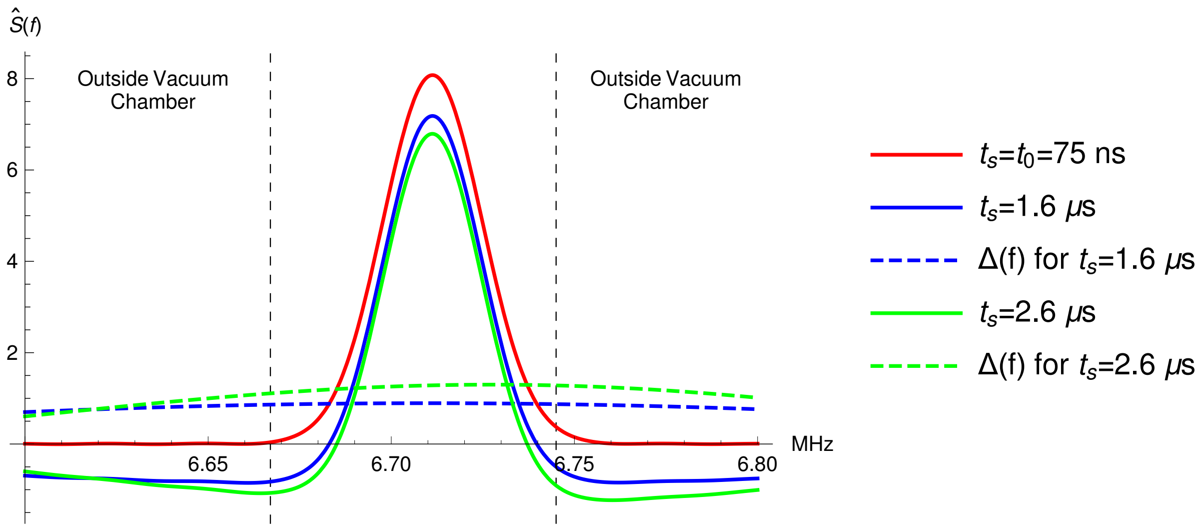
Figure 4: Frequency spectra for the case of a Gaussian energy offset distribution and no initial bunch length. Spectra for different t_s are displayed, as well the corresponding $\Delta(f)$ functions

and $\Delta(\omega) =$

$$
\begin{aligned}
&\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{e^{-\omega^2(nT+t_0)^2\Delta_0'^2/2}e^{i\omega nT}}{4}\left(1-\frac{\sqrt{\pi}}{2}\mathrm{Erf}\left[\frac{-1}{\Delta_0'\sqrt{2}}-\frac{i\omega(nT+t_0)\Delta_0'}{\sqrt{2}}+\frac{t_0}{\Delta_0'(nT+t_0)\sqrt{2}}\right]\right)+\\
&\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{e^{-\omega^2(nT+t_0)^2\Delta_0'^2/2}e^{-i\omega nT}}{4}\left(1-\frac{\sqrt{\pi}}{2}\mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}}+\frac{i\omega(nT+t_0)\Delta_0'}{\sqrt{2}}+\frac{t_0}{\Delta_0'(nT+t_0)\sqrt{2}}\right]\right)\\
&-\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{e^{-\omega^2(nT+t_0)^2\Delta_0'^2/2}e^{i\omega nT}}{4}\left(1-\frac{\sqrt{\pi}}{2}\mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}}-\frac{i\omega(nT+t_0)\Delta_0'}{\sqrt{2}}+\frac{t_s}{\Delta_0'(nT+t_0)\sqrt{2}}\right]\right)+\\
&-\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{e^{-\omega^2(nT+t_0)^2\Delta_0'^2/2}e^{-i\omega nT}}{4}\left(1-\frac{\sqrt{\pi}}{2}\mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}}+\frac{i\omega(nT+t_0)\Delta_0'}{\sqrt{2}}+\frac{t_s}{\Delta_0'(nT+t_0)\sqrt{2}}\right]\right)
\end{aligned}
\tag{27}
$$



(a)



(b)

Figure 5: Frequency spectra for the case of a Gaussian energy offset distribution and Gaussian bunch length distribution. Spectra for different t_s are displayed, as well the corresponding $\Delta(f)$ functions

Figure 5 shows how the Fourier transform and the corresponding corrections are altered by increasing t_s.

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