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Extraction of the muon beam frequency distribution via the Fourier analysis of the fast rotation signal

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1 Introduction

The Fermilab E989 Muon g-2 experiments aims to measure the anomalous part of the magnetic moment of the muon a_μ. The muon acquires a magnetic moment

$$\vec{\mu} = g\frac{Q}{2m}\vec{s}, \tag{1}$$

in presence of an external magnetic field B, where Q is the muon electric charge, $\vec{s}$ is the muon spin vector and g the gyromagnetic ratio. The anomalous part of the magnetic moment is defined via the deviation of the gyromagnetic ratio: $g = 2(1 + a_\mu)$.

The Muon g-2 experiment relies on the storage of muons inside a weak focusing ring. A continuous C-shape dipole magnet occupies the entirety of the storage ring (44.7 m circumference). It provies the 1.45 T inward radial focusing to store the muons in the ring. The field intensity corresponds to storing muons with the so-called magic momentum of 3.09 GeV/c onto the magic orbit with a 7.112 m radius. The muons undergo a cyclotron motion in the ring with a revolution frequency of about 149 ns.

The weak focusing ring does not provide vertical focusing that is required to store the muons. The vertical focusing is provide by four electrostatic quadrupole (ESQ) located at four location around the ring. In their rest frame, the muons see the electric field generated by the ESQs as a magnetic field.

The measurement of a_μ is performed via the measurement of the intensity of the magnetic field in term of the Larmor precession frequency of a free proton

$$\hbar\omega_p = 2\mu_p|\overrightarrow{B}| \tag{2}$$

and the intrinsic spin precession frequency of the muon ω_a. The intrinsic muon spin precession frequency is obtained by subtracting the cyclotron frequency ω_C to the total spin precession frequency of the muon ω_S:

$$\omega_a = \omega_S - \omega_C. \tag{3}$$

Equation 4 shows the most general vectoriel relation between a_μ and ω_a, B:

$$\vec{\omega}_a = \vec{\omega}_S - \vec{\omega}_C = -\frac{Qe}{m}\left[a_\mu \vec{B} - a_\mu\left(\frac{\gamma}{\gamma+1}\right)(\vec{\beta}\cdot\vec{B})\vec{\beta} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right] \tag{4}$$

The term proportional to $\vec{\beta}\times\vec{E}$ in Eq 4 corresponds to the electric field contribution to ω_a. One can see that if $a_\mu = \frac{1}{\gamma^2-1}$ the contribution disappears. This is the approach followed by the previous CERN and BNL E821 experiments. The magic momentum of 3.09 GeV/c allows the electric field contribution to vanish in first order.

The electric field non-vanishing contribution arises from the fact that the stored muon beam has a momentum spread of about 0.1%. This effect is non-negligible and needs to be taken care of. The estimated correction to ω_a due to the electric field was estimated by E821 to be 0.47 ± 0.05 ppm for the 2001 data set for the low n-value data set. The final E821 result for a_μ has a 0.54 ppm precision, which is the order of the electric field correction.

The term proportional to $\vec{\beta}\cdot\vec{B})$ in Eq 4 corresponds to the so-called pitch correction due to the muon velocity not being purely contain in the horizontal plane. This is an other correction that needs to be addressed.

This note presents an attempt at estimating the electric field correction using the Fourier analysis applied to the Fast Rotation signal by E821 and presented in [1].

2 Fast Rotation

Imagine that the initial distribution has zero emittance, zero energy spread and zero bunch length. The particles share a common revolution period T and the time dependence of the intensity signal at a fixed point is

$$I(t) = \delta\left(t - \left(n + \frac{\theta}{2\pi}\right)T\right) \tag{5}$$

where n is any non negative integer and θ is the azimuthal position of the point in the ring. A particle with energy offset Δ will have revolution period $T(1+\Delta)$, so that

$$I(t,\Delta) = \delta\left(t - \left(n + \frac{\theta}{2\pi}\right)T\,(1+\Delta)\right) \tag{6}$$

The fast rotation signal at the point is then

$$S(t) = \sum_{n=0}^{\infty}\int \varrho(\Delta)\delta\left(t - \left(n + \frac{\theta}{2\pi}\right)T\,(1+\Delta)\right)d\Delta \tag{7}$$

where $\varrho(\Delta)$ is the distribution of momenta offsets. If the energy distribution is a Gaussian with width Δ_0, then

$$S(t) = \sum_{n=0}^{\infty} \int \frac{e^{-\Delta^2/(2\Delta_0^2)}}{\sqrt{2\pi}\Delta_0} \delta\left(t - \left(n + \frac{\theta}{2\pi}\right) T\left(1+\Delta\right)\right) d\Delta =$$

$$\sum_{n=0}^{\infty} \int \frac{e^{-\Delta^2/(2\Delta_0^2)}}{\sqrt{2\pi}\Delta_0} \frac{\delta\left(\Delta - \left(\frac{t}{(n+\theta/2\pi)T} - 1\right)\right)}{(n+\theta/2\pi)T} d\Delta =$$

$$\sum_{n=0}^{\infty} \frac{\exp[-(\frac{t}{(n+\theta/2\pi)T} - 1)^2/2\Delta_0^2]}{\sqrt{2\pi}\Delta_0(n+\theta/2\pi)T} =$$

$$\sum_{n=0}^{\infty} \frac{\exp[-(t-(n+\theta/2\pi)T)^2/2\Delta_0^2(n+\theta/2\pi)^2T^2]}{\sqrt{2\pi}\Delta_0(n+\theta/2\pi)T}$$

Using the fact that $\theta/2\pi = t_0/T$, we rewrite the above expression as

$$\sum_{n=0}^{\infty} \frac{e^{-(t-(nT+t_0))^2/2\Delta_0^2(nT+t_0)^2}}{\sqrt{2\pi}\Delta_0(nT+t_0)} \tag{8}$$

FRS.png

Figure 1: Fast Rotation Signal

3 Fourier Analysis

ADD CASE INTEGRATING FROM 0 $->$ infinite because it helps to get good intuition about the correction for t0 $->$ tS

3.1 Momentum distribution from fast rotation signal

The established method for extracting the energy (or equivalently the frequency distribution) is to take the real part of the Fourier transform of the fast rotation signal. Suppose the center of mass of the muon beam first passes the detector at time t_0, and the detector starts detecting at some time $t_s > t_0$. Let $S(t)$ be the fast rotation signal of a muon beam with Gaussian momentum distribution. Then

$$F(\omega) = \int_{t_s}^{\infty} S(t) \cos\omega(t-t_0)dt \tag{9}$$

$$= \sum_{n=0}^{\infty} \int_{t_s}^{\infty} \frac{e^{-(t-(nT+t_0))^2/2\Delta_0^2(nT+t_0)^2}}{\sqrt{2\pi}\Delta_0(nT+t_0)} \cos\omega(t-t_0)dt$$

$$= \sum_{n=0}^{\infty} \frac{e^{-1/2\Delta_0^2}}{2\sqrt{2\pi}\Delta_0(nT+t_0)} \int_{t_s}^{\infty} e^{-t^2/2\Delta_0^2(nT+t_0)^2} e^{t/(nT+t_0)\Delta_0^2} e^{\pm i\omega t} e^{\mp i\omega t_0} dt$$

From Gradshteyn and Rizhik, we know that

$$\int_u^{\infty} \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) dx = \sqrt{\pi\beta} e^{\beta\gamma^2} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left(\gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}}\right)\right) \tag{10}$$

for $\mathrm{Re}(\beta) > 0$ and $u \geq 0$. Thus the result of using this integral in computing the Fourier transform is $\tilde{F}(\omega) =$

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} - \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) + \\ &\sum_{n=0}^{\infty} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{-i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} + \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) \end{aligned} \tag{11}$$

Something to note about (5) is that if we start the sum from say, index $j-1$, then the first $j-1$ spikes of the signal, corresponding to the detector seeing the beam the first $j-1$ turns, will disappear. So the observed signal may be written as

$$\sum_{n=j}^{\infty} \frac{\exp[-(t-(nT+t_0))^2/2\Delta_0^2(nT+t_0)^2T^2]}{\sqrt{2\pi}\Delta_0(nT+t_0)T}$$

Reindexing the sum gives us

$$\sum_{n=0}^{\infty} \frac{\exp[-(t-((n+j)T+t_0))^2/2\Delta_0^2((n+j)T+t_0)^2T^2]}{\sqrt{2\pi}\Delta_0((n+j)T+t_0)T}$$

Given t_s, t_0, and T, the corresponding starting index in the expression for $S(t)$ is $m = \lceil (t_s - t_0)/T \rceil$. Hence the immediately observed frequency spectrum can also be expressed as $\tilde{F}(\omega) =$

$$\begin{aligned} &\sum_{n=m}^{\infty} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} - \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) + \\ &\sum_{n=m}^{\infty} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{-i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} + \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) \end{aligned} \tag{12}$$

3.2 Corrections to the Fourier transform

The detector in the section above started to detect the muon beam at a time $t_s > t_0$. In the time that the detector does not detect the beam, some decoherence of the fast rotation signal occurs, resulting in lost frequency content in the Fourier transform above. This results in distortions of the frequency spectrum. Consider the integral of $S(t)$ between times t_0 and t_s:

$$\Delta(\omega) = \int_{t_0}^{t_s} S(t) \cos\omega(t - t_0) dt \tag{13}$$

This is the part of the frequency spectrum that the detector fails to account for. Let's take a closer look at Δ.

$$\Delta(\omega) = \int_{t_0}^{t_s} S(t) \cos\omega(t - t_0) dt$$
$$= \int_{t_0}^{\infty} S(t) \cos\omega(t - t_0) dt - \int_{t_s}^{\infty} S(t) \cos\omega(t - t_0) dt$$

Using the integral given in Gradshteyn and Ryzhik, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} - \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_0}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) + \\
&\sum_{n=0}^{\infty} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{-i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} + \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_0}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) \\
-&\sum_{n=0}^{\infty} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} - \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) + \\
-&\sum_{n=0}^{\infty} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{-i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} + \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right)
\end{aligned} \tag{14}$$

But keeping in mind the reasoning leading to (9), we may instead write $\Delta(\omega) =$

$$\begin{aligned}
&\sum_{n=0}^{m-1} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} - \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_0}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) \\
+&\sum_{n=0}^{m-1} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{-i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} + \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_0}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) \\
-&\sum_{n=0}^{m-1} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} - \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right) \\
-&\sum_{n=0}^{m-1} \frac{e^{-\omega^2(nT+t_0)^2\Delta_0^2/2} e^{-i\omega nT}}{4} \left(1 - \frac{\sqrt{\pi}}{2} \mathrm{Erf}\left[\frac{-1}{\Delta_0\sqrt{2}} + \frac{i\omega(nT+t_0)\Delta_0}{\sqrt{2}} + \frac{t_s}{\Delta_0(nT+t_0)\sqrt{2}}\right]\right)
\end{aligned} \tag{15}$$

In the vicinity of the magic frequency, Δ is expected to be parabolic. Looking at the figures below, we can see that this is indeed true.

3.3 Investigation of the Established Method for Extracting Δ

The established method for extracting Δ asserts that we may extract it by using $\tilde{F}(\omega)$. We have that

$$\Delta(\omega) = \int_{t_0}^{t_s} S(t) \cos \omega (t - t_0) dt$$

The idea is that although we do not observe $S(t)$ for times $t_0 < t < t_s$, we may reasonably approximate $S(t)$ on this interval of time:

$$S(t) = \int \tilde{F}(f') \cos 2\pi f'(t - t_0) df'$$

Substituting this expression for $S(t)$ into the integral for Δ, we have

$$\Delta(f) = \int\int_{t_0}^{t_s} \tilde{F}(f') \cos 2\pi f'(t - t_0) \cos 2\pi f(t - t_0) dt df' =$$

$$\frac{A}{2\pi^2} \int \tilde{F}(f') \left(\frac{\sin 2\pi(f - f')(t_s - t_0)}{f - f'} + \frac{\sin 2\pi(f + f')(t_s - t_0)}{f + f'} \right) df'$$

We may ignore the second term in the parentheses as their contribution is negligible. Thus

$$\Delta(f, t_s) = \frac{A}{2\pi^2} \int \tilde{F}(f') \frac{\sin 2\pi(f - f')(t_s - t_0)}{f - f'} df'$$

The established method for extracting Δ asserts that we may get

$$\Delta(\omega) = 2\pi(t_s - t_0) \int_{-\infty}^{\infty} F(\omega') \frac{\sin(\omega - \omega')(t_s - t_0)}{(\omega - \omega')(t_s - t_0)} d\omega'$$

where $F(\omega')$ is the frequency distribution extracted from the immediate detector data, with no corrections applied. For small $\omega - \omega')(t_s - t_0)$, we may approximate the integral using the fact that $\mathrm{sinc}(x) = 1 - x^2/6$ for $x << 1$. Then

$$\Delta(\omega) \approx 2\pi(t_s - t_0) \int_{-\infty}^{\infty} F(\omega') \left(1 - \frac{(\omega - \omega')^2 (t_s - t_0)^2}{6} \right) d\omega'$$

$$= 2\pi(t_s - t_0) \int_{-\infty}^{\infty} F(\omega') d\omega' - \frac{2\pi(t_s - t_0)^3}{6} \int_{-\infty}^{\infty} F(\omega')(\omega^2 - 2\omega\omega' + \omega'^2) d\omega'$$

References

[1] Y. Orlov et al., NIM A 482 (2002) 767-755.