Evolution of the Muon Distribution in the g-2 Ring

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1 Momentum spread

The energy dependence of the betatron tunes and the revolution frequency, contribute to the evolution and decoherence of betatron motion due to the finite energy spread in the beam. In a cyclotron with electrostatic focusing distributed uniformly around the ring, the tune depends on the focusing index n according to

$$\begin{aligned} Q_x &= \sqrt{1-n} \\ Q_y &= \sqrt{n} \end{aligned}$$

where

$$n = \left(\frac{r}{v_s B}\right)\frac{\partial E_r}{\partial r}$$

r is the radius of curvature of the on momentum muon in magnetic field B and v_s is the azimuthal velocity, that is $r = \frac{\gamma m v_s}{qB} = \frac{p}{qB}$. The dependence of betatron tune on energy, (chromaticity) follows from

$$n(\delta) = \frac{p(1+\delta)}{q v_s B^2}\frac{\partial E_r}{\partial r}$$

so that

$$\begin{aligned} \frac{\partial Q_x}{\partial \delta} &= -\frac{1}{2}\frac{1}{\sqrt{1-n}}\frac{\partial n}{\partial p} = -\frac{n}{2\sqrt{1-n}} = \frac{Q_y^2}{2Q_x} = -0.103 \\ \frac{\partial Q_y}{\partial \delta} &= \frac{n}{2\sqrt{n}} = \frac{1}{2}Q_y = 0.215 \end{aligned}$$

evaluated for $n = 0.185$. An important assumption in the above is that the electrostatic quadrupoles extend continuously around the circumference of the ring which is not the case in the g-2 ring. To understand the implications for chromaticity it is useful to distiguish three contributions; energy dependence of quadrupole focal length, energy dependence of pathlength and effect of quad curvature.

The horizontal and vertical tunes are written in terms β-function

$$Q_{h/v} = \frac{1}{2\pi}\oint \frac{1}{\beta_{h/v}} ds \tag{1}$$

In a ring with constant β ($\frac{d\beta}{ds} = 0$), we have that $K = 1/\beta^2$ and the tunes are

$$\begin{aligned} Q_{h/v} &= \frac{1}{2\pi}\oint \sqrt{K_{x/y}} ds \\ Q_{h/v} &= \sqrt{K_{x/y}} R \end{aligned} \tag{2}$$

where

$$
\begin{aligned}
K_x &= \frac{1}{\rho^2} - \frac{q}{mv^2}\frac{\partial E}{\partial r} = \frac{1}{\rho^2} - J \\
K_y &= \frac{q}{mv^2}\frac{\partial E}{\partial y}
\end{aligned}
$$

where for convenience we define $J \equiv \frac{q}{mv^2}\frac{\partial E}{\partial r} = n$ and note that $Q_y = \sqrt{J}$.

The three distinct contributions to the chromaticity are enumerated.

1. The effective gradient (K) decreases with energy, increasing horizontal and decreasing the vertical tune.

$$
\begin{aligned}
\frac{\partial K_x^{1/2}}{\partial \delta} &= \frac{1}{2}K^{-1/2}\frac{\partial K}{\partial \delta} = -\frac{(\rho^{-2} - \frac{1}{2}J)}{\sqrt{K}} = -\sqrt{K} - \frac{J}{\sqrt{K}} = -Q_x - \frac{Q_y^2}{2Q_x} \\
\frac{\partial K_y^{1/2}}{\partial \delta} &= -\frac{1}{2}\sqrt{K} = -\frac{1}{2}Q_y
\end{aligned}
$$

   where we have assumed rectangular coordinates so that $\frac{\partial E_r}{\partial r} = \frac{\partial E_x}{\partial x} = -\frac{\partial E_y}{\partial y}$

2. The path length increases with energy, $\Delta P = 2\pi\eta\delta$. If the inner and outer quad plates have equal angular length, then a longer path corresponds to longer quads and more focusing.

$$
\frac{\partial Q_{h/v}}{\partial \delta} = \sqrt{K_{x/y}}\eta = \frac{\eta}{R}Q_{h,v}
$$

   where R is the magic radius. If the quad plates have equal linear length (rather than equal angular length), then there is no pathlength dependent focusing. The effect of pathlength will depend on the details of the fringe field at the ends of the quads.

3. The sextupole component of the quad fields couples to the tune via the dispersion. If the quad plates have no curvature, then the quadrupole symmetry precludes a sextupole moment. But there is curvature in the g-2 quads, and solutions to Laplace's equation in cylindrical coordinates are guaranteed a sextupole component. In cartesian coordinates the quadrupole potential

$$
V(x,y) = \frac{1}{2}k(x^2 - y^2)
$$

   gives

$$
\mathbf{E} = -\nabla V = -kx\hat{\mathbf{x}} + ky\hat{\mathbf{y}}
$$

   and $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$. In cylindrical coordinates, a solution to the Laplace equation with lowest order term linear in displacement, all of the nonlinearity in the radial coordinate, and a strictly linear vertical dependence is

$$
V = k\left(\frac{1}{2}(r^2 - 1) - r_0 \ln\frac{r}{r_0} - y^2\right)
$$

   and

$$
\mathbf{E} = \frac{1}{2}k\left((r - \frac{r_0^2}{r})\hat{\mathbf{r}} - 2y\hat{\mathbf{y}}\right)
$$

With the substitution $r = r_0 + x$, where r_0 is the magic radius,

$$\begin{aligned}
\mathbf{E} &= \frac{1}{2}k\left((r_0 + x - \frac{r_0^2}{r_0}[1 - \frac{x}{r_0} + \frac{1}{2}\left(\frac{x}{r_0}\right)^2 + \ldots])\hat{\mathbf{r}} - 2y\hat{\mathbf{y}}\right) \\
&\sim k\left(x - \frac{x^2}{2r_0} + \ldots)\hat{\mathbf{r}} - y\hat{\mathbf{y}}\right)
\end{aligned}$$

The closed orbit for an off energy particle is shifted to $x \rightarrow \eta\delta + x$ and the radial component of the field for an off energy muon becomes

$$\begin{aligned}
E_r &= k\left(x + \eta\delta - \frac{1}{2r_0}(x + \eta\delta)^2\right) \\
\frac{\partial E_r}{\partial x} &\rightarrow k(1 - \frac{1}{r_0}\eta\delta) \\
\frac{\partial\sqrt{K_x}}{\partial\delta} &= -\sqrt{K_x}\frac{\eta}{2r_0}
\end{aligned}$$

The contribution to the chromaticity due to the quadratic component of the electric field is

$$\frac{\partial Q_h}{\partial\delta} = -\frac{\eta}{2r_0}\sqrt{K_x} = \frac{\eta}{2r_0}Q_x$$

In summary, contributions to chromaticity include

1. the energy dependence of the quad gradient
2. energy dependence of path length
3. The nonlinearity associated with the curvature.

Assuming equal angular length of inner and outer plates and strictly linear vertical quad focusing (so that all of the nonlinearity due to curvature appears in the horizontal)

$$\frac{\partial Q_h}{\partial\delta} = -Q_x - \frac{Q_y^2}{2Q_x} + Q_x\frac{\eta}{r_0} - Q_x\frac{\eta}{2r_0}$$

$$\frac{\partial Q_v}{\partial\delta} = -Q_y\left(\frac{1}{2} - \frac{\eta}{r_0}\right)$$

It turns out that for continuous focusing,

$$\eta = \frac{1}{Kr_0} = \frac{r_0}{Q_x^2}$$

Then

$$\frac{\partial Q_h}{\partial\delta} = -Q_x - \frac{Q_y^2}{2Q_x} + \frac{1}{Q_x}(1 - \frac{1}{2}) = \frac{-1}{2Q_x} + \frac{1}{Q_x}(1 - \frac{1}{2})$$

$$\frac{\partial Q_v}{\partial\delta} = -Q_y\left(\frac{1}{2} - \frac{1}{Q_x^2}\right)$$

In order to get the chromaticity right, we need a 3D map of the quad fields that includes end effects as well as curvature.

Finally, suppose that all of the particles in the initial distribution appear in the ring at the same point in space and time but with a spread in energy. The particles will execute betatron oscillations with a frequency that depends on the energy, namely $Q_x(\delta) = Q'_x\delta$ and $Q_y(\delta) = Q'_y\delta$, where δ is the fractional energy offset. The particles will circulate with cyclotron frequencies $\frac{1}{\omega(\delta)} = \frac{1+\delta}{\omega_0}$. The betatron frequency

$$
\begin{aligned}
\omega_\beta &= Q\omega \quad \text{becomes} \\
\omega_\beta &\rightarrow (Q + Q'\Delta)\frac{\omega_0}{1+\Delta}
\end{aligned}
$$