

Homework: Equilibrium Traffic Flow

4.1, 4.4, 4.8

Hayden Atchley

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4.1

Given a speed-density relationship of

$$v = v_f \left(1 - \frac{k}{k_j} \right), \quad (1)$$

and the relationship $q = k \times v$, we can derive flow-density and speed-flow relationships as follows:

$$q = kv \quad (2)$$

$$q = kv_f \left(1 - \frac{k}{k_j} \right) \quad (3)$$

$$q = kv_f - \frac{k^2 v_f}{k_j} \quad (4)$$

and

$$k = \frac{q}{v}, \quad v = v_f \left(1 - \frac{k}{k_j} \right) \quad (5)$$

$$\implies v = v_f \left(1 - \frac{q/v}{k_j} \right) \quad (6)$$

$$\implies v^2 = v_f v - \frac{qv_f}{k_j}. \quad (7)$$

From (4) we can find q_m (the capacity) and k_m (the density at capacity) by determining the

maximum of the flow-density relationship (where $\frac{dq}{dk} = 0$):

$$q = kv_f - \frac{k^2 v_f}{k_j} \quad (8)$$

$$\frac{dq}{dk} = 0 = v_f - 2 \frac{k_m v_f}{k_j} \quad (9)$$

$$0 = v_f \left(1 - 2 \frac{k_m}{k_j} \right) \quad (10)$$

$$0 = 1 - 2 \frac{k_m}{k_j} \quad \text{if } v_f \neq 0 \quad (11)$$

$$k_m = \frac{k_j}{2} \quad (12)$$

$$q_m = k_m v_f \left(1 - \frac{k_m}{k_j} \right) \quad (13)$$

$$q_m = \frac{k_j v_f}{2} \left(1 - \frac{k_j/2}{k_j} \right) \quad (14)$$

$$q_m = \frac{k_j v_f}{4}. \quad (15)$$

v_m (the speed at capacity) is then determined from (15) and the original relationship (1) by:

$$v_m = v_f \left(1 - \frac{k_m}{k_j} \right) \quad (16)$$

$$v_m = v_f \left(1 - \frac{k_j/2}{k_j} \right) \quad (17)$$

$$v_m = \frac{v_f}{2}. \quad (18)$$

4.4

The Greenberg model is given by

$$v = v_m \ln \frac{k_j}{k}. \quad (19)$$

The capacity is again determined by setting $\frac{dq}{dk} = 0$, so a flow-density relationship is first determined:

$$q = kv = kv_m(\ln k_j - \ln k). \quad (20)$$

Then:

$$\frac{dq}{dk} = 0 = \frac{d}{dk} kv_m(\ln k_j - \ln k) \quad (21)$$

$$0 = k_m v_m \left(-\frac{1}{k_m} \right) + v_m(\ln k_j - \ln k_m) \quad (22)$$

$$0 = v_m \left(\ln \frac{k_j}{k_m} - 1 \right) \quad (23)$$

$$1 = \ln \frac{k_j}{k_m} \quad \text{if } v_m \neq 0 \quad (24)$$

$$e = \frac{k_j}{k_m} \quad (25)$$

$$k_m = \frac{k_j}{e}. \quad (26)$$

q_m is then given from (20) and (26):

$$q_m = k_m v_m(\ln k_m - \ln k_j) = k_m v_m \left(\ln \frac{k_m}{k_j} \right) \quad (27)$$

$$q_m = \frac{k_j v_m}{e} \left(\ln \frac{k_j}{k_j/e} \right) \quad (28)$$

$$q_m = \frac{k_j v_m}{e} \quad (29)$$

4.8