Homework: Equilibrium Traffic Flow

4.1, 4.4, 4.8

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4.1

Given a speed-density relationship of

$$v = v_f \left(1 - \frac{k}{k_j}\right), \tag{1}$$

and the relationship $q = k \times v$, we can derive flow-density and speed-flow relationships as follows:

$$q = kv \tag{2}$$
$$q = kv_f \left(1 - \frac{k}{k_j}\right) \tag{3}$$
$$q = kv_f - \frac{k^2 v_f}{k_j} \tag{4}$$

and

$$k = \frac{q}{v}, \quad v = v_f \left(1 - \frac{k}{k_j}\right) \tag{5}$$
$$\implies v = v_f \left(1 - \frac{q/v}{k_j}\right) \tag{6}$$
$$\implies v^2 = v_f v - \frac{q v_f}{k_j}. \tag{7}$$

From (4) we can find q_m (the capacity) and k_m (the density at capacity) by determining the maximum of the flow-density relationship (where $\frac{dq}{dk} = 0$):

$$q = kv_f - \frac{k^2 v_f}{k_j} \tag{8}$$

$$\frac{dq}{dk} = 0 = v_f - 2\frac{k_m v_f}{k_j} \tag{9}$$

$$0 = v_f \left(1 - 2\frac{k_m}{k_j}\right) \tag{10}$$

$$0 = 1 - 2\frac{k_m}{k_j} \quad \text{if } v_f \neq 0 \tag{11}$$

$$k_m = \frac{k_j}{2} \tag{12}$$

$$q_m = k_m v_f \left(1 - \frac{k_m}{k_j}\right) \tag{13}$$

$$q_m = \frac{k_j v_f}{2} \left(1 - \frac{k_j/2}{k_j}\right) \tag{14}$$

$$q_m = \frac{k_j v_f}{4}. \tag{15}$$

v_m (the speed at capacity) is then determined from (15) and the original relationship (1) by:

$$v_m = v_f \left(1 - \frac{k_m}{k_j}\right) \tag{16}$$

$$v_m = v_f \left(1 - \frac{k_j/2}{k_j}\right) \tag{17}$$

$$v_m = \frac{v_f}{2}. \tag{18}$$

4.4

The Greenberg model is given by

$$v = v_m \ln \frac{k_j}{k}. \tag{19}$$

The capacity is again determined by setting $\frac{dq}{dk} = 0$, so a flow-density relationship is first determined:

$$q = kv = kv_m(\ln k - \ln k_j). \tag{20}$$

Then:

$$\begin{align} \frac{dq}{dk} = 0 &= \frac{d}{dk} kv_m(\ln k - \ln k_j) \tag{21} \\ 0 &= k_m v_m \left(\frac{1}{k_m}\right) + v_m(\ln k_m - \ln k_j) \tag{22} \\ 0 &= v_m \left(1 + \ln \frac{k_m}{k_j}\right) \tag{23} \\ \ln \frac{k_m}{k_j} &= -1 \quad \text{if } v_m \neq 0 \tag{24} \\ \frac{k_m}{k_j} &= \frac{1}{e} \tag{25} \\ k_m &= \frac{k_j}{e}. \tag{26} \end{align}$$

q_m is then given from (20) and (26):

$$\begin{align} q_m = k_m v_m(\ln k_m - \ln k_j) &= k_m v_m \left(\ln \frac{k_m}{k_j}\right) \tag{27} \\ q_m &= \frac{k_j v_m}{e} \left(\ln \frac{k_j}{ek_j}\right) \tag{28} \\ q_m &= -\frac{k_j v_m}{e} \tag{29} \end{align}$$

4.8