CHAPTER 11

High-Order Models

The macroscopic traffic flow models discussed so far, including both analytical and numerical models, have been focused on the LWR model [24, 25] and its variants. At the center of these models is mass or vehicle conservation, which can be mathematically expressed as a first-order partial differential equation:

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0,$$

where k and q are density and flow, which depend on time t and space x. Hence, these models are referred to as *first-order* models.

Common to first-order models is their prediction of a shock wave when two kinematic waves meet. Consequently, a vehicle crossing the shock wave has to change its speed abruptly, which is physically impossible. This limitation, together with other undesirable features, has led many researchers to seek more realistic models to represent traffic dynamics. Naturally, these efforts gave rise to *high-order* dynamic traffic flow models.

11.1 HIGH-ORDER MODELS

In essence, the conservation law takes several forms, among which mass or vehicle conservation is perhaps the simplest. Other forms of the law are conservation of linear momentum and conservation of energy, which involve high-order partial differential equations. If a model involves such equations, it is classified as a high-order model, a few examples of which are described below.

11.1.1 PW Model (1971)

Proposed by Payne [37] and independently by Whitham [38], the PW model consists of a system of two equations: the first is the conservation of mass given in the LWR model, and the second equation is derived from the Navier-Stokes equation of motion for a one-dimensional compressible flow with a pressure and a relaxation term.

Traffic Flow Theory
http://dx.doi.org/10.1016/B978-0-12-804134-5.00011-8

$$\begin{cases} \dfrac{\partial k}{\partial t} + \dfrac{\partial q}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} = -v\dfrac{\partial v}{\partial x} - \lambda(v - V_{\mathrm{d}}(k)) - \dfrac{1}{k}\dfrac{\mathrm{d}P}{\mathrm{d}k}\dfrac{\partial k}{\partial x}, \end{cases}$$

where v is traffic speed, $V_{\mathrm{e}}(k)$ is the equilibrium speed-density relationship, $P(k)$ is traffic pressure, and λ is a coefficient. Note that FREFLO presented in Chapter 9 is a numerical solution to the PW model.

11.1.2 Phillips's Model (1979)

On the basis of kinetic theory, Phillips [39] developed a model which incorporates mass conservation, momentum conservation, and energy conservation:

$$\begin{cases} \dfrac{\partial k}{\partial t} + \dfrac{\partial q}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = \lambda(V_{\mathrm{e}}(k) - v) - \dfrac{1}{k}\dfrac{\partial P}{\partial x}, \\ \dfrac{\partial P}{\partial t} + u\dfrac{\partial P}{\partial x} = \lambda[k(V_{\mathrm{e}}(k) - v)^2 + (P_{\mathrm{e}} - P)] - 3P\dfrac{\partial v}{\partial x}, \end{cases}$$

where P_{e} is the equilibrium traffic pressure, and everything else is as defined above.

11.1.3 Kühne's Model (1984)

Kühne [40, 41] also proposed a model by considering sound speed and viscosity:

$$\begin{cases} \dfrac{\partial k}{\partial t} + \dfrac{\partial q}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} = \lambda(V_{\mathrm{e}}(k) - v) - \dfrac{c_0^2}{k}\dfrac{\partial k}{\partial x} + \eta\dfrac{\partial^2 v}{\partial x^2}, \end{cases}$$

where c_0 is sound speed and η is a viscosity constant.

11.1.4 Kerner and Konhäuser's Model (1993)

Kerner and Konhäuser [42] showed that given an initially homogeneous traffic flow, regions of high density and low average speed (clusters of cars) can spontaneously appear. These high-density regions can move either with or against the flow of traffic, and two clusters with different speeds, widths, and amplitudes merge when they meet, resulting in a single cluster. The continuum flow model adopted is in the following form:

$$
\begin{cases}
\dfrac{\partial k}{\partial t} + \dfrac{\partial q}{\partial x} = 0, \\
\dfrac{\partial v}{\partial t} = \lambda(V_{\mathrm{e}}(k) - v) - \dfrac{c_0^2}{k}\dfrac{\partial k}{\partial x} + \dfrac{1}{k}\dfrac{\frac{\eta \partial v}{\partial x}}{\partial x}.
\end{cases}
$$

11.1.5 Model of Michalopoulos et al. (1993)

Michalopoulos et al. [43] proposed a model which does not require the use of an equilibrium speed-density relationship. Traffic friction at interrupted flows and changing geometries is also addressed through the use of a viscosity term. Tests with field data and comparison with existing models suggested that the proposed model is more accurate and computationally more efficient than existing models.

$$
\begin{cases}
\dfrac{\partial k}{\partial t} + \dfrac{\partial q}{\partial x} = 0, \\
\dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = \dfrac{1}{\tau}(v_{\mathrm{f}} - v) - G\dfrac{\partial v}{\partial t} - \nu k^{\beta}\dfrac{\partial k}{\partial x},
\end{cases}
$$

where $G = \mu k^{\varepsilon} g$, μ, ν, ε, and β are all constant parameters, and v_{f} is the free-flow speed.

11.1.6 Zhang's Model (1998)

Zhang [44] proposed a nonequilibrium traffic flow model which is based on both empirical evidence of traffic flow behavior and basic assumptions about drivers' reactions to stimuli. By assuming an equilibrium speed-density relationship and introducing a disturbance propagation speed, the model includes the LWR model as a special case and removes some of its deficiencies. Unlike existing high-order continuum models, this model eliminates "wrong-way travel" because in this model traffic disturbances are always propagated against the traffic stream.

$$
\begin{cases}
\dfrac{\partial k}{\partial t} + \dfrac{\partial q}{\partial x} = 0, \\
\dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = \lambda(V_{\mathrm{e}}(k) - v) - k(V_{\mathrm{e}}'(k))^2\dfrac{\partial k}{\partial x}.
\end{cases}
$$

11.1.7 Model of Treiber et al. (1999)

Treiber et al. [45] derived macroscopic traffic equations from specific gas-kinetic equations, and the resulting partial differential equations for vehicle density and average speed contain a nonlocal interaction term which is very favorable for a fast and robust numerical integration, so several thousand freeway kilometers can be simulated in real time.

$$\begin{cases} \dfrac{\partial k}{\partial t} + \dfrac{\partial q}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = \lambda(V_e(k) - v) + \dfrac{1}{k}\dfrac{\partial kAv^2}{\partial x} - \dfrac{V_e A(k)}{\tau A(k_j)}\left[\dfrac{k_\alpha Tv}{1 - \frac{k_\alpha}{k_j}}\right]^2 B(\delta_v), \end{cases}$$

where $A = A(k)$ is a density-dependent function, k_α is the density at point x_α ahead of x, $B(\delta_v)$ is a macroscopic interaction term, and $V_e(k)$ is the normal equilibrium speed-density relationship.

11.2 RELATING CONTINUUM FLOW MODELS

Starting from mass or vehicle conservation, a variety of continuum flow models have been developed by the inclusion of additional assumptions. Generally, these models can be summarized by the following model [46]:

$$\begin{cases} \dfrac{\partial k}{\partial t} + \dfrac{\partial q}{\partial x} = g(t, x), \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = \dfrac{1}{\tau}(V_e(k) - v) + \dfrac{1}{k}\dfrac{\partial P}{\partial x}, \end{cases}$$

where $V_e(k, v)$ is the generalized equilibrium speed-density relationship. $P(k, v)$ is the traffic pressure, and τ is the relaxation time, which is the time constant of regulating the traffic speed v to the equilibrium speed V_e. $g(t, x)$ is the net ramp inflow. For a highway without on-ramps or off-ramps, $g(t, x) = 0$.

Each of the above-mentioned continuum flow models can be viewed as a special case of the general model when different traffic pressure P, relaxation time τ, and generalized equilibrium speed U_e are applied. For example,

- the LWR model results if $\tau = 0$ and $P = 0$;
- the PW model results if $P = -\frac{V_e(k)}{2\tau}$ with $V_e(k, v) = V_e(k)$;
- Phillips's model results if $P = k\Theta$ with $\Theta = \Theta_0(1 - \frac{k}{k_j})$, where k_j is the jam density;
- Kerner and Konhäuser's model results if $P = k\Theta_0 - \eta\frac{\partial v}{\partial x}$;
- the model of Michalopoulos et al results if $P = \frac{\nu}{\beta+2}k^{\beta+2}$, where ν is an anticipation parameter, β is a dimensionless constant, and $V_e(k) = v_f$, where v_f is the free-flow speed;
- Zhang's model resulted if $P = \frac{1}{3}k^3 V_e'^2(k)$, where $V_e'(k) = \frac{\mathrm{d}V_e(k)}{\mathrm{d}k}$;

- the model of Treiber et al. results if $P = Akv^2$, where $A = A(k)$ is a density-dependent function, and $V_e(k, v) = V_e(k)\{1 - \frac{A}{A(k_j)}[\frac{k_\alpha Tv}{1-\frac{k_\alpha}{k_j}}]^2 B(\delta_v)\}$.

11.3 RELATIVE MERITS OF CONTINUUM MODELS

Daganzo [47] noted that, as a first-order continuum flow model, the LWR model is proposed for dense traffic with an equilibrium and it is flawed for light traffic. This is because, when passing is allowed, the LWR model fails to recognize that the preferred speed for each vehicle varies over time and the desired speeds among a group of vehicles vary as well. These variations can cause a platoon to disperse in a way that is not predicted by the LWR model. When passing is allowed, the LWR model produces unsatisfactory results in the following three aspects. First, the LWR model predicts an abrupt speed change when a vehicle passes through a shock wave, an action that is unrealistic in the real world. Second, the LWR model fails to predict instabilities of stop-start traffic. Third, the LWR model assumes zero reaction time, which does not happen in the real world. Readers are referred to Daganzo's original paper for full information.

Given these deficiencies, the continuum flow models developed so far have been trying to fix the deficiencies, and almost all of these models follow the direction of incorporating a momentum conservation equation. An early attempt to fix the deficiencies in the LWR model was made by Prigogine [48], who proposed a kinetic model incorporating a speed distribution to address platoon dispersion. A decade later, Payne [37] and Whitham [38] proposed a dynamic model, the so-called PM model, trying to smooth out the discontinuity in speed change across shock waves. A momentum equation was introduced in this model to describe the structure of a shock wave. This seminal work has inspired many thoughts regarding analytical explanation of shock wave behavior, and thus has spawned several variants, among which are those of Phillips [39], Kühne [40, 41], Kerner and Konhäuser [42], Michalopoulos et al. [43], Zhang [44], and Treiber et al. [45].

Several deficiencies are found in the PW model [47]. First, it does not remove all the shock waves. Second, as reported by del Castillo et al. [49], vehicles in the PW model can adjust their speeds in response to disturbance from behind, while in reality vehicles typically respond to their leaders. Third, the PW model incorporates a momentum equation which is derived from a car-following model. This momentum equation does not consider second-order and higher-order terms of spacings and speeds, which may not be negligible when spacings and speeds are not slowly varying. Fourth, the PM model as well as other high-order models always produces wave speeds that are greater than traffic speeds. This is an unattractive property of macroscopic models because it implies that future conditions of a vehicle are partially decided by what happens behind it. Fifth, the strength that high-order models smooth out shocks turns out to be these models' weakness. This is because any model that attempts to smooth all the discontinuities must sometimes predict negative speeds and such negative speeds observed in computer models cannot be removed by convergent numerical approximation methods. Sixth, but probably not the last, high-order models involve more complex partial differential equations and more variables, which increases computational complexity, and are more difficult to calibrate and implement. Given these limitations, many researchers [47, 50, 51] tend to believe that high-order models, despite their added complexity and additional parameters, might not be superior to the LWR model.

11.4 TAXONOMY OF MACROSCOPIC MODELS

Figure 11.1 shows a rather simple and incomplete taxonomy which relates macroscopic traffic flow models to each other. The figure starts with the basic principle, conservation laws, which takes the forms of mass conservation, momentum conservation, and energy conservation.

Mass conservation and a functional flow-density relationship (typically derived from an equilibrium speed-density relation) constitute the core of the LWR model. This model is classified as a first-order model since it involves a first-order partial differential equation. Numerical models derived from LWR models are indicated as double-line boxes in the left panel. These models include KRONOS, the kinematic waves model (though this is a graphical solution involving discrete space but continuous time), and the cell transmission model. The kinematics waves model was further extended to network traffic by Ni [35] and Ni et al. [36].

Central to high-order models are equations of mass and momentum conservation. These models include the models of Payne [37] and Whitham [38], Phillips [39], Kühne [40, 41], Kerner and Konhäuser[42], Michalopoulos [43], Zhang [44], Treiber et al. [45], etc. FREFLO is a numerical model derived from the model of Payne [37] and Whitham [38].

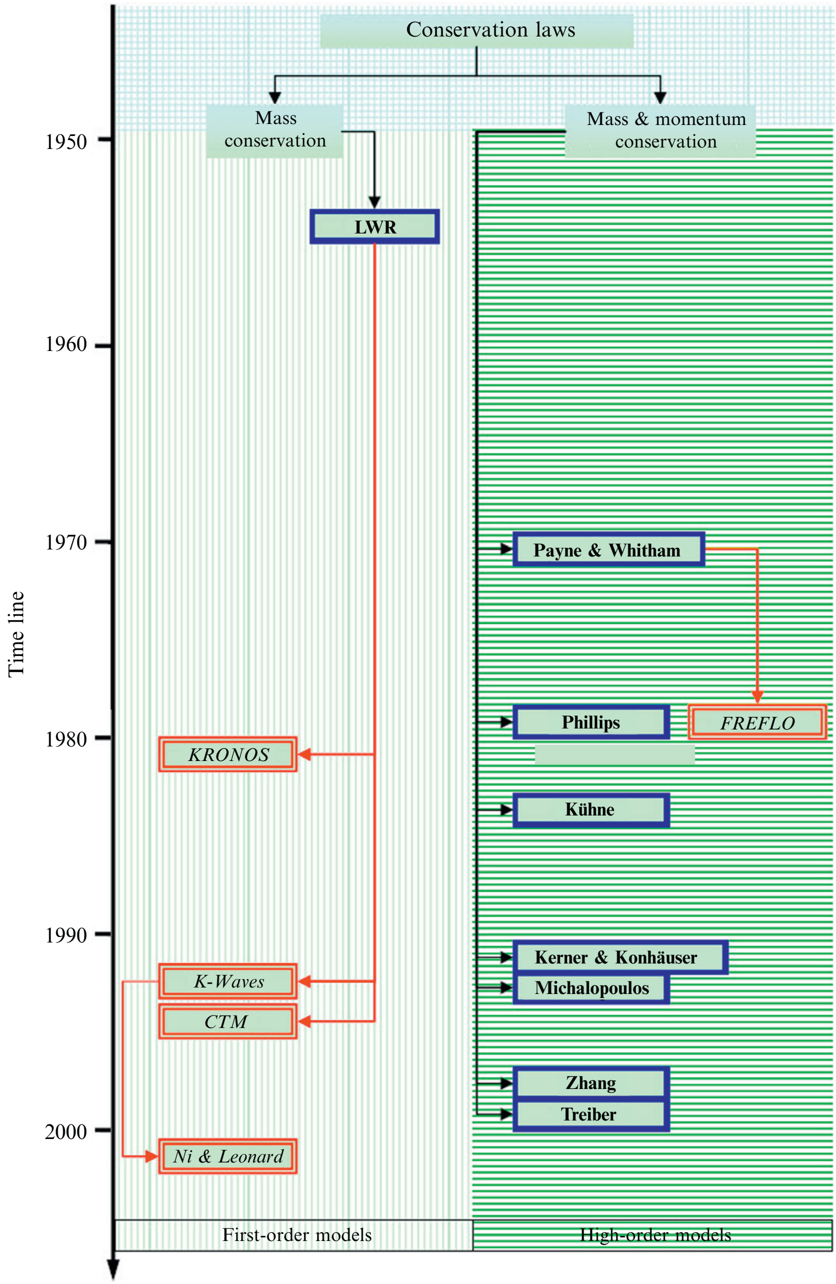

Figure 11.1 Taxonomy of macroscopic models. CTM, cell transmission model; K-waves, kinematic waves.

PROBLEMS

1. Name a few similarities and differences between first-order and high-order models.
2. What are the major reasons that motivated the exploration of high-order models?
3. Highlight a few drawbacks of high-order models.
4. Compare the equation of momentum conservation in the PW model and Phillips's model.
   a. Comment on how they differ.
   b. Which one is more general?
   c. Can one of them be derived from the other?
5. The high-order models introduced in this book include an equation representing the conservation of vehicles. This equation implicitly assumes a uniform freeway segment without on-ramps and off-ramps—that is, traffic flows in and out via mainline lanes but not ramps. How would the equation of vehicle conservation change if traffic from and to ramps is considered?