CHAPTER 6

Waves

To solve the set of equations presented at the end of Chapter 5, one has to leave the topic of traffic flow for a moment and study waves first. One would agree that this is necessary when one looks at Figure 6.1, where vehicle trajectories recorded in the field are plotted on a time-space diagram. The horizontal axis is time, with left being earlier and right later. The vertical axis is space, with traffic flowing upward. Three ripples are clearly visible in this picture depicting the propagation of some disturbances in the traffic. This observation suggests that traffic does behave like waves, and solutions to traffic dynamics can be sought on the basis of the knowledge of waves. As such, the purpose of this chapter is to provide a jump-start introduction to waves.

6.1 WAVE PHENOMENA

Waves are everywhere in the real world. When a pebble is thrown into a pond, one sees ripples circling outward. This is a wave (see Figure 6.2). When the audience at a football stadium becomes thrilled and rows of the audience stand up and sit down successively, one sees a "signal" bouncing. This is also a wave. When shaking a rope at one end with the other end fixed, one sees a "hump" moving away. This is yet another wave. Basically, *a **wave** is the propagation of a disturbance in a medium over time and space*. In the above examples, the ripples, signal, and hump are disturbances, while the water, audience, and rope are media. If we apply the notion to a platoon of vehicles on a highway, when one of the vehicles brakes suddenly and then resumes its original speed, subsequent vehicles will be affected successively. The propagation of such a "jerking" effect is a wave, with the jerk being the disturbance and the traffic being the medium. The ripples in Figure 6.1 are examples of such a wave.

6.2 MATHEMATICAL REPRESENTATION

The mathematical language to describe wave phenomena is the partial differential equation (PDE).

Traffic Flow Theory
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Figure 6.1 Traffic waves observed on a highway.

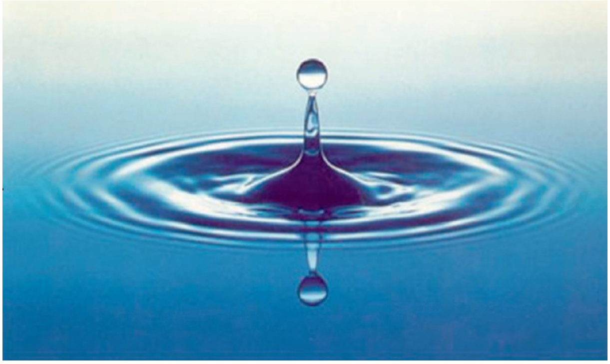

Figure 6.2 Surface waves.

6.2.1 Notation

If a dependent variable k is a function of independent variables t and x, we write $k = k(t, x)$ and we denote its partial derivatives with respect to x and t as follows:

$$k_x = \frac{\partial k}{\partial x}, k_t = \frac{\partial k}{\partial t}, k_{xt} = \frac{\partial^2 k}{\partial x \partial t}, k_{tx} = \frac{\partial^2 k}{\partial t \partial x}, k_{xx} = \frac{\partial^2 k}{\partial x^2}, k_{tt} = \frac{\partial^2 k}{\partial t^2}.$$

A PDE for $k(t, x)$ is an equation that involves one or more partial derivatives of k with respect to t and x, For example,

$$k_t = k_x + k, k_t = k_{xx} + k_x + 5, k_t = k_{xxx} + 4k + \cos x.$$

6.2.2 Terminology

PDEs can be classified on the basis of their order, homogeneity, and linearity.

Order

The order of a PDE is the order of the highest partial derivative in the equation. For example,

- *first-order* PDE: $k_t = k_x + k$;
- *second-order* PDE: $k_t = k_{xx} + k_x + 5$;
- *third-order* PDE: $k_t = k_{xxx} + 4k + \cos x$.

A first-order PDE can be expressed in the following general form:

$$P(t, x, k)k_t + Q(t, x, k)k_x = R(t, x, k),$$

where P, Q, and R are coefficients, and they may be functions of t, x, and k.

Homogeneity

A first-order PDE $P(t, x, k)k_t + Q(t, x, k)k_x = R(t, x, k)$ may be

- *homogeneous* if $R(t, x, k) = 0$;
- *nonhomogeneous* if $R(t, x, k) \neq 0$.

Linearity

In the above general first-order PDE, if both P and Q are independent of k—that is, $P = P(t, x)$, $Q = Q(t, x)$—and

- If R is also independent of k—that is, $R = R(t, x)$—then the PDE is *strictly linear*. For example, $2xk_t + 3k_x = 5t$.
- If R is linearly dependent on k—that is, $R = R(t, x, k)$—then the PDE is *linear*. For example, $2xk_t + 3k_x = 5k + 3$.
- If R is dependent on k in a nonlinear manner, then the PDE is *semilinear*. For example, $2xk_t + 3k_x = e^k$.

In particular, if P or Q is dependent on k, or both P and Q are dependent on k—that is, $P = P(t, x, k)$, $Q = Q(t, x, k)$—and $R = R(t, x, k)$, then the PDE is *quasilinear*. For example, $k_t + (3k + 2)k_x = 0$.

A PDE is nonlinear if it involves cross terms of k and its derivatives—for example, $k_t k_x + k = 2$.

Now, test yourself by classifying the following PDEs:

1. $k_t + ck_x = 0$.
2. $k_t + ck_x = e^{-t}$.
3. $k_{tt} = C^2 k_{xx}$, where C is a constant.
4. $k_{tt} - k_x x + k = 0$.
5. $k_{tt} + kk_x + k_{xxx} = 0$.

6.3 TRAVELING WAVES

Many PDEs have solutions in a traveling wave form $k(t, x) = f(x - ct)$.[1] Figure 6.3 illustrates two instants of the traveling wave, $f(x - ct_0)$ and $f(x - ct_1)$. It is easy to find that (1) the traveling wave preserves its shape and (2) the wave at time t_1 is simply a horizontal translation of its initial profile at time

[1] The following discussion is derived from Ref. [23] with modifications.

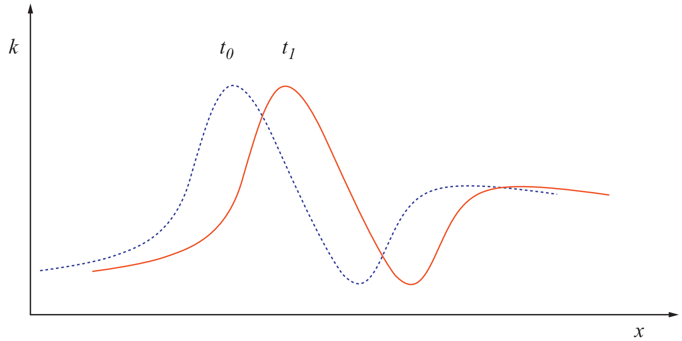

Figure 6.3 A traveling wave.

t_0. If c is a positive constant, wave $k(t, x) = f(x - ct)$ travels to the right over time, while wave $k(t, x) = f(x + ct)$ moves to the left over time.

6.4 TRAVELING WAVE SOLUTIONS

Solve the following wave equation:

$$k_{tt} = ak_{xx},$$

where a is a constant.

Assume that a solution to the above wave equation takes a traveling form $k(t, x) = f(x - ct)$. Let $z = x - ct$. Then

$$k_t = \frac{\partial k}{\partial t} = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\partial z}{\partial t} = f' \times (-c) = -cf'.$$

Similarly, $k_x = f'$, $k_{tt} = c^2 f''$, and $k_{xx} = f''$.

Plugging the above expressions into the wave equation, one obtains

$$(c^2 - a)f'' = 0.$$

There are two ways for the left-hand side to be 0: (1) $c^2 - a = 0$ and (2) $f'' = 0$.

1: If $c^2 - a = 0$, then $k(t, x) = f(x \pm \sqrt{a}t)$, where f can take any functional form.

2: If $f'' = 0$, then $k(t, x) = A + B(x - ct)$, where A and B are arbitrary constants.

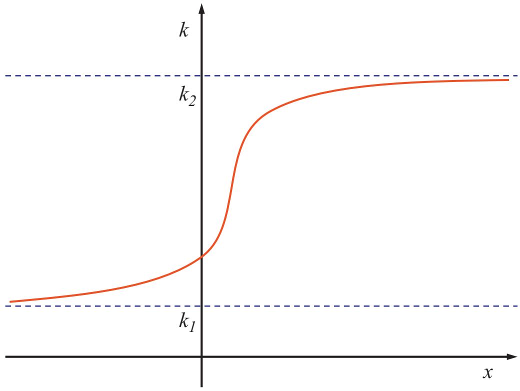

Figure 6.4 Wave front and pulse.

6.5 WAVE FRONT AND PULSE

A traveling wave is called a wave front if

$$\begin{cases} k(t,x) = k_1 & \text{as } x \to -\infty, \\ k(t,x) = k_2 & \text{as } x \to +\infty. \end{cases}$$

Figure 6.4 illustrates a wave front. A traveling wave is called a pulse if $k_1 = k_2$.

6.6 GENERAL SOLUTION TO WAVE EQUATIONS

Many wave equations have a general solution in the form of superposition of traveling waves:

$$k(t,x) = F(x - ct) + G(x + ct).$$

Note that even though each of the terms on right-hand side is a traveling wave, their superposition may not necessarily be.

Example 1

Solve the following wave equation with initial conditions

$$\begin{cases} k_{tt} = c^2 k_{xx}, \\ k(x,0) = f(x), \\ k_t(x,0) = g(x), \\ -\infty < x < +\infty, t > 0. \end{cases}$$

Solution

Applying the above general solution to the first initial condition, we have

$$F(x) + G(x) = k(x, 0) = f(x).$$

Applying the above general solution to the second initial condition, we have

$$-cF'(x) + cG'(x) = k_t(x, 0) = g(x).$$

Dividing both sides by c and integrating, we obtain

$$-F(x) + G(x) = -F(0) + G(0) + \frac{1}{c}\int_0^x g(s)\mathrm{d}s.$$

Solving for $F(x)$ and $G(x)$, we obtain

$$\begin{cases} F(x) = \frac{1}{2}f(x) - \frac{1}{2}[-F(0) + G(0) + \frac{1}{c}\int_0^x g(s)\mathrm{d}s], \\ G(x) = \frac{1}{2}f(x) + \frac{1}{2}[-F(0) + G(0) + \frac{1}{c}\int_0^x g(s)\mathrm{d}s]. \end{cases}$$

Plugging the result back into the general solution, we obtain

$$\begin{aligned} k(t, x) &= F(x - ct) + G(x + ct) \\ &= \frac{1}{2}f(x - ct) - \frac{1}{2}\left[-F(0) + G(0) + \frac{1}{c}\int_0^{x-ct} g(s)\mathrm{d}s\right] \\ &+ \frac{1}{2}f(x + ct) + \frac{1}{2}\left[-F(0) + G(0) + \frac{1}{c}\int_0^{x+ct} g(s)\mathrm{d}s\right]. \end{aligned}$$

We combine terms and we obtain a specific generic solution:

$$k(t, x) = \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s.$$

This is called the *d'Alembert solution*.

Example 2

Solve the following wave equation with initial conditions

$$\begin{cases} k_{tt} = 4k_{xx}, \\ k(x, 0) = \mathrm{e}^{-x^2}, \\ k_t(x, 0) = 0, \\ -\infty < x < +\infty, t > 0. \end{cases}$$

Solution

Applying the result in Example 6.6 and considering that $k_t(x,0) = g(x) = 0$, one obtains

$$k(t,x) = \frac{1}{2}[f(x-ct) + f(x+ct)].$$

Therefore,

$$k(x,0) = \frac{1}{2}[f(x) + f(x)] = f(x) = \mathrm{e}^{-x^2}$$

Since $f(x) = k(x,0)$, one obtains

$$k(t,x) = \frac{1}{2}[k(x-ct,0) + k(x+ct,0)] = \frac{1}{2}[\mathrm{e}^{-(x-ct)^2} + \mathrm{e}^{-(x+ct)^2}].$$

6.7 CHARACTERISTICS

Consider Example 6.6, since $f(x) = k(x,0)$ and $g(x) = k_t(x,0)$, the solution can be transformed to the following form:

$$k(t,x) = \frac{1}{2}[k(x-ct,0) + k(x+ct,0)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s.$$

6.7.1 Domain of Dependence

Applying the above conclusion, one notices that the solution k at an arbitrary time-space point (t^*, x^*) is

$$k(t^*,x^*) = \frac{1}{2}[k(x^*-ct^*,0) + k(x^*+ct^*,0)] + \frac{1}{2c}\int_{x^*-ct^*}^{x^*+ct^*} g(s)\mathrm{d}s.$$

The above equation suggests that the solution at an arbitrary point (t^*, x^*) can be determined by the initial condition at points $(0, x^*-ct^*)$ and $(0, x^*+ct^*)$ and the interval $\mathbf{I}$ bounded by the two points (inclusive)—that is, $I = [x^*-ct^*, x^*+ct^*]$. This is illustrated in the left part of Figure 6.5. Therefore, the interval $\mathbf{I}$ is called the *domain of dependence* of point (t^*, x^*).

6.7.2 Range of Influence

The term "*range of influence*" refers to a collection of time-space points whose solutions are influenced either completely or partially by the domain of dependence $\mathbf{I}$; see the shaded area in the right part of Figure 6.5.

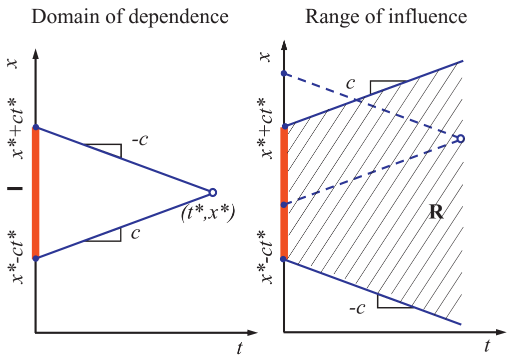

Figure 6.5 Characteristics.

6.7.3 Characteristics

Notice that in the left part of Figure 6.5, the two lines coming from point (t^*, x^*) intersecting the x-axis at $(x^* - ct^*, 0)$ and $(x^* + ct^*, 0)$ have slopes c and $-c$. These two lines are called *characteristic lines* or simply *characteristics* (please do not mix this up with traffic flow characteristics).

6.8 SOLUTION TO THE WAVE EQUATION

In a special case where $k_t(0, x) = 0$, the solution of the wave equation in Example 6.6 reduces to

$$k(t, x) = \frac{1}{2}[k(0, x - ct) + k(0, x + ct)].$$

This shows that the value of k at (t, x) depends only on the initial values of k at two points, $x_1 = x - ct$ and $x_2 = x + ct$. Once the initial values $k(0, x - ct)$ and $k(0, x + ct)$ are known, one constructs the solution k at (t, x) by taking the average of $k(0, x_1)$ and $k(0, x_2)$.

Example 3

Use characteristics to solve the following wave equation:

$$\begin{cases} k_{tt} = 4k_{xx}, \\ k(0, x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ or} \\ 0 & \text{otherwise,} \end{cases} \\ k_t(0, x) = 0, \\ -\infty < x < +\infty, t > 0. \end{cases}$$

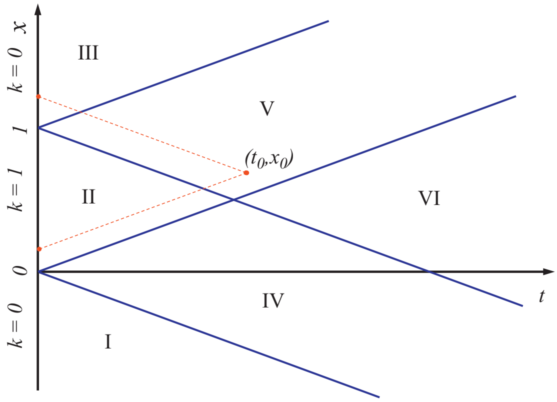

Figure 6.6 Solution to Example 6.8.

In this equation, the traveling wave speed $c = \pm 2$—that is, $k(t, x) = f(x \pm 2t)$. First, one constructs an x–t plane. Locate points 0 and 1 on the x-axis. Then one draws two characteristics (their slopes are ± 2) from each of the two points (see Figure 6.6). The four characteristics partition the x–t plane into six regions as labeled in Figure 6.6. Take an arbitrary point (t_0, x_0), for example. The solution at this point is found by drawing two characteristics from this point. Then find the intersections of the two characteristics on the x-axis. Next, find the k values at the two intersections. In this case the k values are 1 and 0. Then the solution k at point (t_0, x_0) is the average of the k values at the two intersections—that is, $k(t_0, x_0) = \frac{1}{2}$.

With use of a similar technique, the solution in other regions can be determined. To sum up, the solution to the above wave equation is as follows:

$$k(t,x) = \begin{cases} 0 & \text{if } (t,x) \in \text{region I}, \\ 1 & \text{if } (t,x) \in \text{region II}, \\ 0 & \text{if } (t,x) \in \text{region III}, \\ \frac{1}{2} & \text{if } (t,x) \in \text{region IV}, \\ \frac{1}{2} & \text{if } (t,x) \in \text{region V}, \\ 0 & \text{if } (t,x) \in \text{region VI}. \end{cases}$$

The above discussion presents the following notion:

1. For some wave equations such as that in Example 6.8, solution k at point (t, x) can somehow be related to the initial condition k_0 at point $(0, x_0)$.

2. This is done by drawing two lines, called characteristics, from (t, x) with slopes c and $-c$.
3. These characteristics intersect the x-axis at two points $(0, x_1)$ and $(0, x_2)$, where $x_1 = x - ct$ and $x_2 = x + ct$. Then the solution is $k(t, x) = \frac{1}{2}[k(0, x_1) + k(0, x_2)]$.

6.9 METHOD OF CHARACTERISTICS

Now let us consider a very simple PDE derived from the conservation law with an initial condition. In Chapter 5, the conservation law led to the following continuity equation:

$$k_t + q_x = 0.$$

If one assumes $q = ck$, where c is a constant, then $q_x = ck_x$, and the PDE can be defined as follows (please ignore the physical meaning of k and q for the moment—this issue will be revisited later):

$$\begin{cases} k_t + ck_x = 0, \\ k(0, x) = k_0(x), \\ -\infty < x < \infty,\ 0 < t, \\ c \text{ is a constant.} \end{cases}$$

The goal is to find a solution to this PDE, or equivalently find the value of k at an arbitrary time-space point, $k(t, x)$. Rather than working on an arbitrary point in the entire time-space plane, one starts with a simpler case by working on a point on a specific curve in the time-space plane. To do this, one draws a curve $x = x(t)$ (how to draw this curve will be made clear shortly), and the new goal is to find the value of k at an arbitrary point $(t, x(t))$ on the curve—that is, $k(t, x(t))$. To find the solution, let us examine how k changes along the curve $x = x(t)$. The rate of change of k with time is the first (and total) derivative of k with respect to time t; that is,

$$\frac{\mathrm{d}k(t, x(t))}{\mathrm{d}t} = \frac{\partial k}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} + \frac{\partial k}{\partial x}\frac{\mathrm{d}x(t)}{\mathrm{d}t} = k_t + \frac{\mathrm{d}x}{\mathrm{d}t}k_x.$$

If one compares the right-hand side of this equation with the left-hand side of the original PDE, one recognizes that they are very similar. Actually, they will be identical if one imposes

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = c.$$

Consequently, one obtains

$$\frac{\mathrm{d}k(t,x(t))}{\mathrm{d}t} = k_t + ck_x = 0.$$

This means that the total time derivative of k along the curve $x = x(t)$ is zero—that is, the value of k is constant along the curve. This implies that the curve $x = x(t)$ needs to be drawn such that it is a straight line with slope of c. Therefore, one finds the equation of the line by solving the following ordinary differential equation:

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = c, \\ x(0) = x_0. \end{cases}$$

This yields

$$\begin{cases} x(t) & = ct + x_0, \\ x_0 & = x - ct. \end{cases}$$

At time $t = 0$, this line intersects the x-axis at x_0. Since k remains constant along this line, the solution k at any point on this line, $k(t, x(t))$, is the same as $k(0, x_0) = k_0(x_0)$, which is given in the initial condition. Therefore, we have found the solution for all points on this line. Such a line is called a *characteristic*. Sounds familiar? Yes, it has the same meaning as the characteristic in previous sections, where it is a line drawn from a time-space point with slope c, which is the speed of the traveling wave $f(x - ct)$.

With the above knowledge, it is simple to find the solution at an arbitrary point, $k(t^*, x^*(t^*))$. The procedure is as follows:

1. Construct the equation of the characteristic drawn from this point: $x(t) = ct + x_0$.
2. Find the intercept of this characteristic on the x-axis: $x_0 = x^* - ct^*$.
3. Find the value of k at the intercept from the initial condition: $k(0, x_0) = k_0(x_0) = k_0(x^* - ct^*)$.
4. Apply this value of k to the point of interest: $k(t^*, x^*) = k_0(x^* - ct^*)$.

Example 4

Use the method of characteristics to find the solution to the following PDE at point $(t^* = 3, x^* = 10)$:

$$\begin{cases} k_t + 2k_x = 0, \\ k(0,x) = 2x^2 + 5, \\ -\infty < x < \infty,\ 0 < t. \end{cases}$$

Solution

Following the above procedure, one obtains the following:

1. The characteristic drawn from this point is $x(t) = 2t + x_0$.
2. The intercept of this characteristic on the x-axis is $x_0 = 10 - 2 \times 3 = 4$.
3. The value of k at the intercept is $k(0, 4) = 2 \times 4^2 + 5 = 37$.
4. Therefore, $k(3, 10) = 37$.

6.10 SOME PROPERTIES

The above discussion is based on a very simple first-order, linear, homogeneous PDE. It is informative to examine further the method of characteristics and note some of its properties.

6.10.1 Properties of Characteristics

In the above example, the characteristic is a straight line, and this is so because c is a constant. Similarly, another characteristic drawn from another time-space point is also a straight line. In addition, the two straight lines are parallel since they have the same slope c. Figure 6.7 illustrates a family of characteristics (in the x-t plane) which are straight and parallel. Each characteristic carries a constant k value denoted by a line above which is labeled as the characteristic curve. Different characteristics may carry different k values, so the surface $k(t, x)$ is not necessarily flat. A *kinematic wave* is a family of characteristics which carry and propagate signals, such as those characteristics illustrated in Figure 6.7.

Now, what if c is not a constant? The following are two examples.

Example 5

In this example, c depends on k but not explicitly on t and x—that is, $c = c(k(t, x))$. In this case, the characteristic equation needs to be derived from

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = c(k(t, x)).$$

Hence. the characteristic equation is

$$x = c(k_0(x_0))t + x_0.$$

Therefore, the characteristic is still a straight line. However, the slope of the line may take different values at different intercepts x_0. Consequently, two characteristics may intersect. See Figure 6.8 for an illustration.

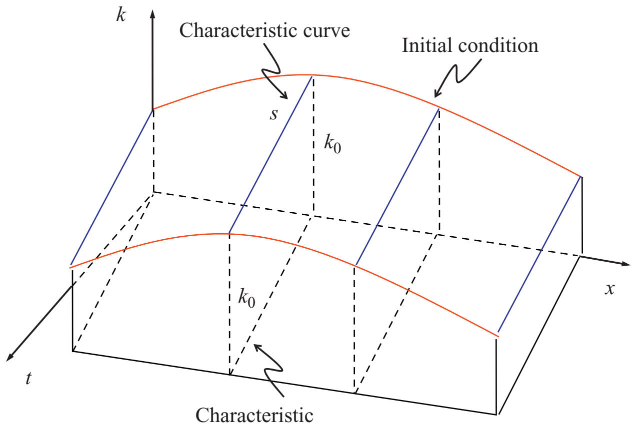

Figure 6.7 Illustration of parallel characteristics.

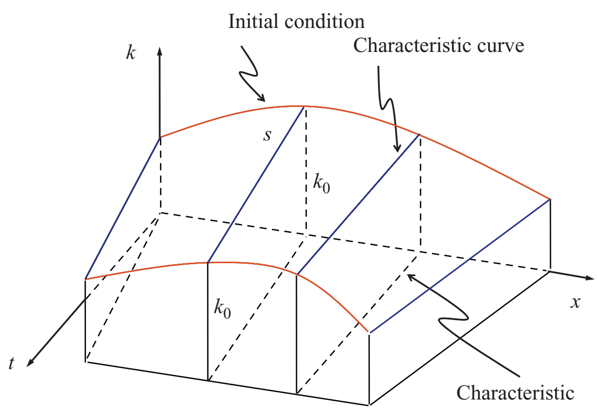

Figure 6.8 Illustration of nonparallel characteristics.

Example 6

In this example, c explicitly depends on x and or t—for example, $c = t$. The equation of the characteristic is derived from the following ordinary differential equation:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = c = t.$$

After integration, one obtains $x = \frac{1}{2}t^2 + A$, where A is an integral constant. In this case, the characteristic is no longer a straight line, but is a parabola. In addition, characteristics drawn from different time-space points are no longer parallel. Instead, they may intersect.

Summing up the above discussion on characteristics, we have the following:

- If c is a constant, characteristics are straight, parallel lines.
- If c depends on k but not explicitly on t and x, characteristics are still straight lines, but different characteristics may have different slopes and hence these characteristics may intersect.
- If c explicitly depends on x and or t, characteristics are neither straight nor parallel. Consequently, these characteristics may intersect.
- Since a characteristic denotes a set of time–space points on which the solution of k remains constant, k may be multivalued at the intersection of two characteristics. Such an occurrence is called a *gradient catastrophe*.

6.10.2 Properties of the Solution

If one imposes $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = c$, one obtains

$$\frac{\mathrm{d}k}{\mathrm{d}t} = 0.$$

This implies that the solution of k remains constant on a characteristic $x = x(t)$. This conclusion holds *only* if the underlying PDE is homogeneous—that is,

$$k_t + ck_x = 0.$$

What if the PDE is not homogeneous? For example,

$$k_t + ck_x = -1.$$

In this case, the total derivative of k with respect to t becomes

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -1.$$

This implies that k is no longer constant along characteristic $x = x(t)$, but rather linearly decreases at the rate of 1—that is, $k = k_0 - t$, where k_0 is found in the initial conditions. Figure 6.9 illustrate such a case.

PROBLEMS

1. Classify the following partial differential equations:

a. $k_{txt} = 3xk_{tt} + 4tkk_xk_t - 8xt$.

b. $k_t = 9k_x$.

c. $k_{xx} + \frac{1}{5}k_x + \frac{1}{25}k_{tt} = 0$.

d. $kk_x + ak_t + bk = 0$, where a and b are constants.

e. $5k_t + 9k_x = 3k^3$.

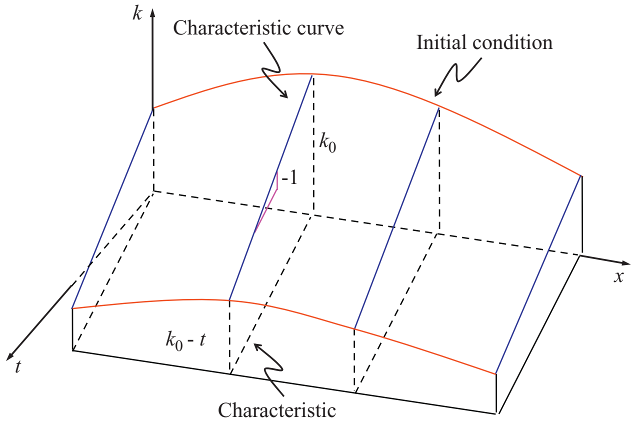

Figure 6.9 Solution of a nonhomogeneous PDE.

2. Use characteristics to find a solution to the following PDE with initial conditions:

$$k_{tt} - k_{xx} = 0,$$
$$\text{where } k(0,x) = \begin{cases} 2 \text{ when } & x > 1, \\ 1 \text{ when } & -1 \le x \le 1 \\ 0 \text{ when } & x < -1. \end{cases} \text{ and } k_t(0,x) = 0,$$

3. Find the d'Alembert solution to the following PDE with initial conditions

$$k_{tt} - \frac{1}{9}k_{xx} = 0,$$
$$\text{where } k(0,x) = 0 \text{ and } k_t(0,x) = 2.$$

4. Find the d'Alembert solution to the following PDE with initial conditions

$$k_{tt} = 4k_{xx},$$
$$\text{where } k(0,x) = 2x \text{ and } k_t(0,x) = \mathrm{e}^{-x}.$$

5. Use the method of characteristics to solve the following first-order homogeneous linear PDE with an initial condition at time-space point $(t,x) = (4,5)$:

$$\begin{cases} k_t + \frac{1}{2}k_x = 0, \\ k(0,x) = 4x + \ \ln x^2, \\ -\infty < x < \infty, \\ t > 0. \end{cases}$$

6. Use the method of characteristics to solve the following first-order homogeneous quasi-linear PDE with an initial condition at time-space point $(t, x) = (2, 20)$:

$$\begin{cases} k_t + (2k+1)k_x = 0, \\ k(0,x) = x + 10, \\ -\infty < x < \infty. \\ t > 0. \end{cases}$$

7. Use the method of characteristics to solve the following first-order nonhomogeneous quasi-linear PDE with an initial condition at time-space point $(t, x) = (5, 9)$:

$$\begin{cases} k_t + 2tk_x = 2, \\ k(0,x) = 2x + 1, \\ -\infty < x < \infty, \\ t > 0. \end{cases}$$