# CHAPTER 9

# Numerical Solutions

Chapter 8 presented the LWR model and the procedure for its solution. In addition, a few concrete examples were provided to show how to apply the procedure. These problems were solved graphically by manually working on a time-space diagram using the method of characteristics. Though illustrative, the graphical approach has limitations since it is capable of dealing only with simple problems which involve only one homogeneous highway section and simple initial conditions. In the real world, a traffic system may consist of a network where multiple segments (links) or highways are considered with traffic flowing in and out via ramps. In addition, the initial and boundary conditions may be more complicated and time-varying. In these cases, the graphical approach is insufficient and sometimes infeasible. Moreover, the purpose of solving LWR problems is to predict traffic dynamics so that traffic engineers are able to anticipate congestion and to develop strategies to alleviate congestion. In such applications, timing is a critical issue, and solving these problems in real time is desirable. Moreover, the wide deployment of intelligent transportation systems makes it possible to provide real-time traffic conditions and allow online prediction. Therefore, a computerized solution to the LWR model is essential to cope with more complicated real-world problems, to enable real-time prediction, and to automate such predictions by the development of online applications.

## 9.1 DISCRETIZATION SCHEME

Computers are digital machines which can work only in a discrete fashion, so computerized solutions to the LWR model have to be numerical and discrete. The first step to develop a computerized solution is to discretize time and space. Figure 9.1 illustrates a time-space diagram where time $t$ is the horizontal axis and space $x$ is the vertical axis with a roadway drawn at the side. The roadway is partitioned into a series of segments labeled as $j \in (0, 1, \ldots, J)$. If $x_0$ is chosen as the reference point and segment length $\Delta x$ is uniform, the location of the end of segment $j$ is

$$x_j = x_0 + j\Delta x.$$

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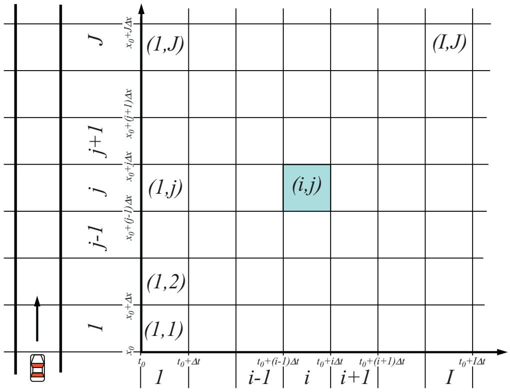



**Figure 9.1** Discretization scheme.

Similarly, the time is divided into a series of durations $i \in (0, 1, \ldots, I)$ with step size $\Delta t$. If the reference point of time is $t_0$, the end of duration $i$ is at time

$$t_i = t_0 + i\Delta t.$$

In general, the following relationship is required in a discretization scheme:

$$\frac{\Delta x}{\Delta t} > v_{\mathrm{f}},$$

where $v_{\mathrm{f}}$ is the free-flow speed. This requirement basically says that a vehicle should not traverse more than one segment $\Delta x$ within a time step $\Delta t$.

A typical numerical solution to the LWR problem starts with initial conditions by determining the number of vehicles contained in each roadway segment one by one from the upstream end to the downstream end:

```
when i = 1
  determine storage in j = 1
  determine storage in j = 2
  ...
  determine storage in j = J
end
```

For easy reference, the time-space region bounded within duration $i$ and segment $j$ is referred to as a *cell* and is denoted as $(i, j)$ and the number of vehicles contained in segment $j$ at the end of duration $i$ is denoted as $n(t_i, x_j)$. The above listing can be rewritten as

```
when i = 1
   determine n(t_1,x_1)
   determine n(t_1,x_2)
   ...
   determine n(t_1,x_J)
end
```

After this, time advances one step, and the above process starts over again.

```
when i = 2
   determine n(t_2,x_1)
   determine n(t_2,x_2)
   ...
   determine n(t_2,x_J)
end
```

Hence, the numerical solution consists of two loops: time $t_i$ as the outer loop and space $x_j$ as the inner loop:

```
Numerical solution procedure:

for i = 1 to I
   for j = 1 to J
     determine n(t_i,x_j)
   end
end
```

The process finishes when all cells have been traversed, and the solution is given as cell storage $[n(t_i, x_j)|i \in (1, 2, \ldots, I), j \in (1, 2, \ldots, J)]$ or, alternatively, traffic condition $k(t_i, x_j)$, $q(t_i, x_j)$, and $v(t_i, x_j)$.

Building on the above procedure, we discuss a few numerical solutions to traffic dynamic problems in the following subsections.

## 9.2 FREFLO

FREFLO is an early (if not the earliest) computerized macroscopic traffic simulation model, developed by Payne [21] in the late 1970s. Like the LWR model, FREFLO consists of three equations with a discretization scheme, shown in Figure 9.2. The first equation is the conservation law:

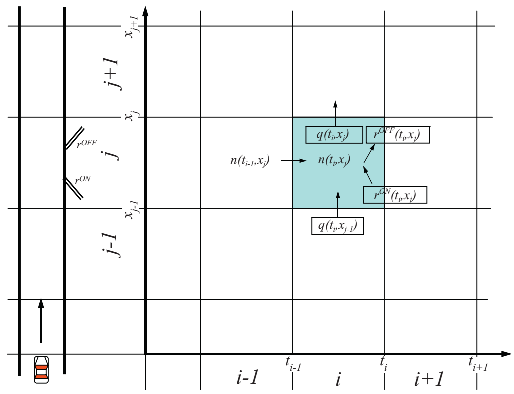


**Figure 9.2** Discretization in FREFLO.

```
storage in the current cell = storage at previous step +
vehicles arrived from upstream - vehicles departed to downstream +
vehicles entered via on-ramp - vehicles exited via off-ramp
```

Mathematically, this can be expressed as

$$n(t_i, x_j) = n(t_{i-1}, x_j) + \Delta t q(t_i, x_{j-1}) - \Delta t q(t_i, x_j) + \Delta t g(t_i, x_j),$$

where $g(t_i, x_j)$ is the net inflow via ramps—that is, $g(t_i, x_j) = r^{\mathrm{on}}(t_i, x_j) - r^{\mathrm{off}}(t_i, x_j)$. Note that $n = k\Delta x$, and the above equation becomes

$$k(t_i, x_j)\Delta x = k(t_{i-1}, x_j)\Delta x + \Delta t q(t_i, x_{j-1}) - \Delta t q(t_i, x_j) + \Delta t g(t_i, x_j),$$

$$k(t_i, x_j) = k(t_{i-1}, x_j) + \frac{\Delta t}{\Delta x}[q(t_i, x_{j-1}) - q(t_i, x_j) + g(t_i, x_j)].$$

The second equation of FREFLO is the identity in discrete form:

$$q(t_i, x_j) = k(t_i, x_j)v(t_i, x_j).$$

Different from most first-order models, which adopt a equilibrium speed-density relationship, FREFLO uses a dynamic speed-density relationship as the third equation:

```
speed in current cell = speed in previous step - convection +
relaxation + anticipation

where:
convection - vehicles tend to continue their speeds when they travel
in the upstream section,
```

```
relaxation - vehicles tend to adopt the equilibrium velocity-
density relationship,
anticipation - vehicles tend to adjust to downstream condition,
i.e. slow down if congested.
```

Mathematically, this can be expressed as

$$\begin{aligned} v(t_i, x_j) = v(t_{i-1}, x_j) - \Delta t\{v(t_{i-1}, x_j)\frac{v(t_{i-1}, x_j) - v(t_{i-1}, x_{j-1})}{\Delta x_i} \\ + \frac{1}{T_j}[v(t_{i-1}, x_j) - V(k(t_{i-1}, x_j)) \\ + \frac{b_j}{k(t_{i-1}, x_j)}\frac{k(t_{i-1}, x_{j+1}) - k(t_{i-1}, x_j)}{\Delta x_j}]\}, \end{aligned}$$

where $T_j = c_T \Delta x_j$ and $b_j = c_b \Delta x_j$. $c_T$ and $c_b$ are relaxation time and anticipation coefficients, respectively. The equilibrium speed-density relationship $V(k)$ takes the following form:

$$v = V(k) = \min\{88.5, (172 - 3.72k + 0.0346k^2 - 0.00119k^3)\},$$

which was an empirical speed-density relationship obtained by least-squares fitting of observed data.

With the above equations, one is able to determine the state $(q, k, v)$ of each cell by starting from initial conditions and following the numerical solution procedure.

## 9.3 FREQ

FREQ is a computerized macroscopic traffic simulation model developed by May [26] in the early 1980s. Its underlying algorithm is not publicly available.

## 9.4 KRONOS

KRONOS is another computerized macroscopic traffic simulation model, developed by Michalopoulos [27] in the mid-1980s. In addition to proposing a numerical solution to the LWR model, Michalopoulos enriched the solution by incorporating ramp flows and lane changes. If net ramp flow $g(t, x)$ is considered, the continuity equation becomes:

$$k_t + q_x = g(t, x).$$

The discrete form of the equation can be stated as

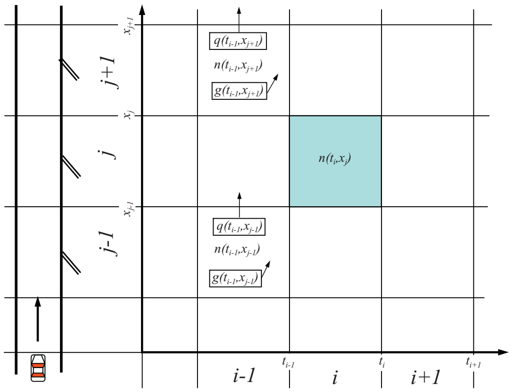



**Figure 9.3** Discretization in KRONOS.

```
Storage in the current cell =
Average of storages in upstream and downstream segments at
previous step -
Average of mainline net outflows in upstream and downstream
segments at previous step +
Average of ramp net inflows in upstream and downstream segments at
previous step
```

See the illustration in Figure 9.3. Mathematically, this is equivalent to

$$n(t_i, x_j) = \frac{n(t_{i-1}, x_{j+1}) + n(t_{i-1}, x_{j-1})}{2} - \frac{\Delta tq(t_{i-1}, x_{j+1}) - \Delta tq(t_{i-1}, x_{j-1})}{2} + \frac{\Delta tg(t_{i-1}, x_{j+1}) + \Delta tg(t_{i-1}, x_{j-1})}{2}.$$

Note that $n = \Delta xk$, and the above equation becomes

$$\Delta xk(t_i, x_j) = \frac{\Delta xk(t_{i-1}, x_{j+1}) + \Delta xk(t_{i-1}, x_{j-1})}{2} - \frac{\Delta tq(t_{i-1}, x_{j+1}) - \Delta tq(t_{i-1}, x_{j-1})}{2} + \frac{\Delta tg(t_{i-1}, x_{j+1}) + \Delta tg(t_{i-1}, x_{j-1})}{2}.$$

Hence,

$$
\begin{aligned}
k(t_i, x_j) = & \frac{k(t_{i-1}, x_{j+1}) + k(t_{i-1}, x_{j-1})}{2} \\
& - \frac{\Delta t}{\Delta x} \frac{q(t_{i-1}, x_{j+1}) - q(t_{i-1}, x_{j-1})}{2} \\
& + \frac{\Delta t}{\Delta x} \frac{g(t_{i-1}, x_{j+1}) + g(t_{i-1}, x_{j-1})}{2}.
\end{aligned}
$$

This equation is supplemented by the identity

$$q(t_i, x_j) = k(t_i, x_j)v(t_i, x_j)$$

and an equilibrium relationship

$$v(t_i, x_j) = V(k(t_i, x_j)),$$

the simplest form of which is the Greenshields model [9]:

$$v(t_i, x_j) = v_f(1 - k(t_i, x_j)/k_j)$$

The initial condition is given as at $t_0$: $k$, $q$, $u$ is known at all locations $x_j$, $j = 0, 1, 2, \ldots, J$.

The boundary condition is given as $q(t_i, x_0)$, $i = 0, 1, 2 \ldots, I$, and $g(t_i, x_j)$, $i = 0, 1, 2 \ldots, I$ and $j = 0, 1, 2 \ldots, J$

Starting from the initial condition and applying the boundary condition, one can work out the numerical solution by following the numerical solution procedure.

## 9.5 CELL TRANSMISSION MODEL

The cell transmission model (CTM) was proposed by Daganzo [28, 29] in the mid-1990s. The model was presented in two papers, with the first addressing mainline traffic and the second addressing network traffic.

### 9.5.1 Minimum Principle

Figure 9.4 shows a triangular flow-density relationship. The relationship consists of three sections: uncongested (left), with free-flow speed $v_f$ equal to forward wave kinematic speed $w_f$, capacity (middle) $q_m$, and congested (right), with backward wave speed $w_b$ and jam density $K$.

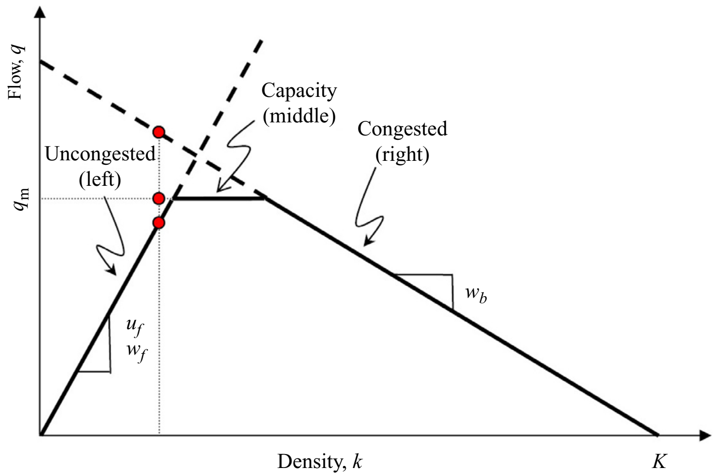



**Figure 9.4** Triangular flow-density relationship.

A vertical line at any density $k$ will intersect the three sections at height $kw_f$, $q_m$, and $(K - k)w_b$. Hence, flow corresponding to this density is found as the minimum of the three intersections:

$$q = \min\{kw_f, q_m, (K - k)w_b\}$$

Physically, if one considers the left section as conditions dictated by arrival traffic, the middle section as local capacity, and the right section as conditions dictated by downstream traffic, the above equation basically says that traffic flowing through a point of highway should not exceed the upstream arrival rate, local capacity, and the rate allowed by downstream conditions.

## 9.5.2 Mainline Scenario

The CTM uses the same discretization scheme presented in Figure 9.1. Everything else remains the same except for one thing: the cell now has a uniform length as the distance traveled by a vehicle at free-flow speed during one time step:

$$\Delta x = v_f \ \Delta t$$

According to the minimum principle, traffic that can flow into segment $j$, $q_j(t_i)$, is constrained by the following:

$$q_j(t_i) = \min\{k_{j-1}(t_{i-1})w_f, q_m, (K - k_j(t_{i-1}))w_b\}.$$

Hence, the number of vehicles that can move into segment $j$, $y_j(t_i)$, is found by multiplying both sides by $\Delta t$:

$$y_j(t_i) = q_j(t_i)\Delta t = \min\{k_{j-1}(t_{i-1})w_\mathrm{f}\Delta t, q_\mathrm{m}\Delta t, (K - k_j(t_{i-1}))w_\mathrm{b}\Delta t\}.$$

Note that $n = k\Delta x$, $\Delta x = v_\mathrm{f}\Delta t$, and $v_\mathrm{f} = w_\mathrm{f}$ owing to the triangular flow-density relationship. The above equation can be transformed to the following form:

$$y_j(t_i) = \min\{k_{j-1}(t_{i-1})\Delta x, q_\mathrm{m}\ \Delta t, \frac{w_\mathrm{b}}{w_\mathrm{f}}(K - k_j(t_{i-1}))\Delta x\};$$

that is,

$$y_j(t_i) = \min\{n_{j-1}(t_{i-1}), q_\mathrm{m}\Delta t, \frac{w_\mathrm{b}}{w_\mathrm{f}}(K\Delta x - n_j(t_{i-1}))\}.$$

The above equation stipulates that the number of vehicles that can move into segment $j$, $y_j(t_i)$, is constrained by

- the number of vehicles in $j - 1$ previously: $n_{j-1}(t_{i-1})$ ,
- the capacity of segment $j$, $q_\mathrm{m}\Delta t$, and
- the empty space in $j$: $\frac{w_\mathrm{b}}{w_\mathrm{f}}(K\Delta x - n_j(t_{i-1}))$.

The equation can be further reduced to

$$y_j(t_i) = \min\{S_{j-1}, R_j\},$$

where $S_{j-1} = \min\{n_{j-1}(t_{i-1}), q_\mathrm{m}\Delta t\}$ represents flow being sent from an upstream position and $R_j = \min\{q_\mathrm{m}\Delta t, \frac{w_\mathrm{b}}{w_\mathrm{f}}(K\Delta x - n_j(t_{i-1}))\}$ is flow ready to be received downstream.

Therefore, the evolution of traffic on a freeway mainline can be stated as

```
Storage in current cell =
Storage in the cell previously +
Vehicles flowed in -
vehicles flowed out
```

Mathematically, this can be expressed as

$$n_j(t_i) = n_j(t_{i-1}) + y_j(t_i) - y_{j+1}(t_i).$$

### 9.5.3 Merger Scenario

To be able to address network traffic, a queuing model is needed for a merger where two streams of traffic flow into one. The merger consists of two upstream links (e.g., a mainline link $(j-1)$ and an on-ramp $(j-1)'$) and one downstream link $j$ (see Figure 9.5). Assume that during interval $(t_{i-1}, t_i)$ links $(j-1)$ and $(j-1)'$ have $S_{j-1}$ and $S'_{j-1}$ vehicles to send, respectively, and link $j$ can receive $R_j$ vehicles. Considering that demand (i.e., $S_{j-1} + S'_{j-1}$) and supply (i.e., $R_j$) may not match in this case, link $(j-1)$ actually sends $y_{j-1}$ vehicles into link $j$ and link $(j-1)'$ actually sends $y'_{j-1}$ vehicles, where $y_{j-1} \leq S_{j-1}$, $y'_{j-1} \leq S'_{j-1}$, and $y_{j-1} + y'_{j-1} \leq R_j$. In addition, mainline and on-ramp traffic have their relative priorities $p_{j-1}$ and $p'_{j-1}$, respectively, where $p_{j-1} \geq 0$, $p'_{j-1} \geq 0$, and $p_{j-1} + p'_{j-1} = 1$. The merger queuing model is essentially solving for $y_{j-1}$ and $y'_{j-1}$ given $S_{j-1}$, $S'_{j-1}$, $R_j$, $p_{j-1}$, and $p'_{j-1}$.

Figure 9.6 illustrates how to find the solution. The horizontal and vertical axes are $y_{j-1}$ (mainline outflow) and $y'_{j-1}$ (on-ramp outflow), respectively. A rectangle is constructed as being bounded by the two axes, a horizontal line at $y'_{j-1} = S'_{j-1}$ and a vertical line at $y_{j-1} = S_{j-1}$. The latter two intersect at point $A$. Draw a line from the origin $O$ with slope $\frac{p'_{j-1}}{p_{j-1}}$ and the line intersects the rectangle at point $C$. Curve $ACO$ denotes the collection of solutions and reason is as follows.

Given the sending flows $S_{j-1}$ and $S'_{j-1}$ and receiving flow $R_j$, there are three possibilities:

1. *Supply exceeds demand*: This is to say that $S_{j-1} + S'_{j-1} \leq R_j$. Physically, this means that link $j$ is able to receive more vehicles than the total to be sent from both upstream links. For example, $S_{j-1} = 100$, $S'_{j-1} = 80$, and $R_j = 200$. In this case, vehicles from both upstream links can flow into the downstream link without any problem. Graphically, this situation corresponds to a line (e.g., line 1) which represents the collection of

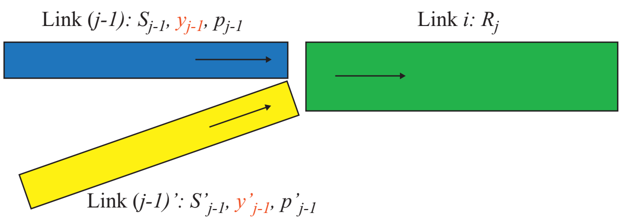



**Figure 9.5** A freeway merge.

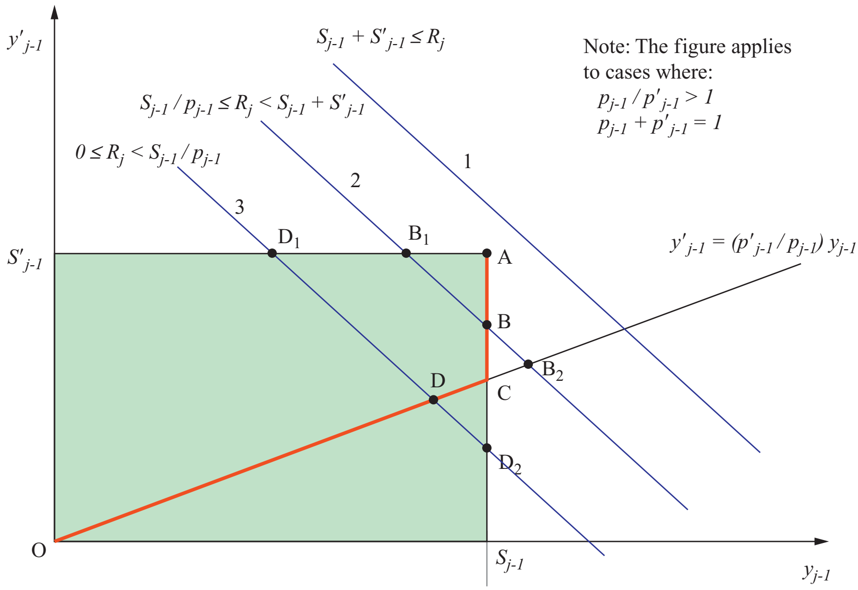



**Figure 9.6** Queuing at a freeway merger.

points whose coordinates sum up to $R_j$. Such a line is always to the right of vertex $A$ without intersecting the rectangle which represents the collection of all feasible solutions. Therefore, the solution is

$$\begin{cases} y_{j-1} = S_{j-1} \\ y'_{j-1} = S'_{j-1} \end{cases} \quad \text{if } R_j \geq S_{j-1} + S'_{j-1}. \tag{9.1}$$

This solution corresponds to vertex $A$ in Figure 9.6.

2. *Demand exceeds supply and one upstream link is congested*:This is to say that $S_{j-1}+S'_{j-1} > R_j$. In addition, one upstream link fails to send all vehicles that it has. For example, $S_{j-1} = 100$, $S'_{j-1} = 80$, and $R_j = 160$. The priority rules stipulate a split of $\frac{p_{j-1}}{p'_{j-1}} = \frac{3}{1}$, meaning that, among the 160 spaces downstream, the mainline can send $160 \times \frac{3}{4} = 120$ vehicles and the on-ramp can send $160 \times \frac{1}{4} = 40$ vehicles. Since the mainline has only100 vehicles to send, these vehicles are able to enter link $j$ without delay, leaving $160 - 100 = 60$ spaces in link $j$ for traffic from the on-ramp. The on-ramp has 80 vehicles to send, 60 of which are admitted

by link $j$ and the remaining 20 are delayed. Summing up, the solution to this example is

$$y_{j-1} = S_{j-1} = 100, y'_{j-1} = R_j - S_{j-1} = 160 - 100 = 60.$$

Graphically, this situation corresponds to a line (e.g., line 2) which is parallel to line 1 and intersects the rectangle between points $A$ and $C$. Line 2 consists of all points whose coordinates sum up to $R_j$. This line intersects line $y'_{j-1} = S'_{j-1}$ at point $B_1$, line $y_{j-1} = S_{j-1}$ at point $B$, and the priority line $y'_{j-1} = \frac{p'_{j-1}}{p_{j-1}} y_{j-1}$ at point $B_2$. The three intersections are three feasible solutions. With sue of the above example as an illustration, these solutions can be interpreted as follows:

### Point *B*₁

Suggests that $S'_{j-1} = 80$ vehicles from the on-ramp can depart without delay and the remaining $160 - 80 = 80$ spaces in link $j$ can be used to admit 80 of the 100 vehicles from link $j - 1$. This violates the priority rule.

### Point *B*

Suggests that $S_{j-1} = 100$ vehicles from link $j - 1$ can depart without delay and the remaining $160 - 100 = 60$ spaces in link $j$ can be used to admit 60 of the 80 vehicles from the on-ramp. This is the correct solution.

### Point *B*₂

Suggests that link $j$ will admit $160 \times \frac{3}{4} = 120$ vehicles from link $j-1$ and the remaining $160 - 120 = 40$ remaining spaces in link $j$ can be used to admit 40 of the 80 vehicles from the on-ramp. Since more vehicles cannot depart from link $j-1$ than link $j-1$ has, this solution is incorrect.

From the outcome of the example, it is clear that the true solution is point $B$, which is the middle of the three points. Mathematically, this can be expressed as follows:

$$\begin{cases} y_{j-1} = \mathrm{mid}\{S_{j-1}, R_j - S'_{j-1}, p_{j-1}R_j\} \\ y'_{j-1} = \mathrm{mid}\{S'_{j-1}, R_j - S_{j-1}, p'_{j-1}R_j\} \end{cases} \quad \text{if } R_j < S_{j-1} + S'_{j-1}, \quad (9.2)$$

where the *mid* operator takes the middle value of all the members. Line segment $AC$ contains all solutions of this nature.

3. *Demand exceeds supply and both upstream links are congested*: This is to say that $S_{j-1} + S'_{j-1} > R_j$. In addition, both upstream links fail to send all vehicles that they have. We use the above example except that $R_j = 120$. The priority rules stipulate that link $j-1$ can send as a maximum $120 \times \frac{3}{4} = 90$ vehicles to link $j$ and the on-ramp can send $120 \times \frac{1}{4} = 30$ vehicles. Since both upstream links have more vehicles than they are able to send, the priority rule takes control—that is, link $j-1$ will actually send 90 vehicles, with the remaining $100 - 90 = 10$ vehicles delayed, and the on-ramp will send 30 vehicles, leaving 50 vehicles delayed.

   Graphically, this situation corresponds to a line (e.g., line 3) which is parallel to line 1 and intersects the priority line between points $C$ and $O$. Again, line 3 consists of all points whose coordinates sum up to $R_j$. From the above example, line 3 intersects line $y'_{j-1} = S'_{j-1}$ at point $D_1$, the priority line $y'_{j-1} = \frac{p'_{j-1}}{p_{j-1}} y_{j-1}$ at point $D$, and line $y_{j-1} = S_{j-1}$ at point $D_2$. The three intersections are three feasible solutions, and their physical meaning is as follows:

#### Point $D_1$

Suggests that $S'_{j-1} = 80$ vehicles from the on-ramp can depart without delay and the remaining $120 - 80 = 40$ spaces in link $j$ can be used to admit 40 of 100 vehicles from link $j-1$. This violates the priority rule.

#### Point $D$

Follows the priority rule by allowing 90 of the 100 vehicles from link $j-1$ to enter link $j$ and using the remaining $120 - 90 = 30$ spaces to admit 30 of the 80 vehicles from the on-ramp. This is the correct solution.

#### Point $D_2$

Suggests that link $j-1$ can actually send $S_{j-1} = 100$ vehicles to link $j$ and the remaining $120 - 100 = 20$ spaces are used to admit 20 of the 80 vehicles from the on-ramp. This, again, violates the priority rule.

Therefore, the true solution is, again, the middle of the three points, and the mathematical notation is the same as above. In addition, line segment $CO$ contains all solutions of this nature.

In summary, the merger model is as follows:

$$\begin{cases} \begin{cases} y_{j-1} = S_{j-1} \\ y'_{j-1} = S'_{j-1} \end{cases} & \text{if } R_j \geq S_{j-1} + S'_{j-1}, \\ \begin{cases} y_{j-1} = \text{mid}\{S_{j-1}, R_j - S'_{j-1}, p_{j-1}R_j\} \\ y'_{j-1} = \text{mid}\{S'_{j-1}, R_j - S_{j-1}, p'_{j-1}R_j\} \end{cases} & \text{if } R_j < S_{j-1} + S'_{j-1}, \end{cases} \tag{9.3}$$

## 9.5.4 Divergence Scenario

A queuing model is also needed for a divergence where one stream of traffic splits into two. The divergence consists of one upstream link $j-1$ and two downstream links (e.g., a mainline link $j$ and an off-ramp $j'$) (see Figure 9.7). Assume that during interval $(t_{i-1}, t_i)$ link $j-1$ has $S_{j-1}$ vehicles to send, link $j$ is able to receive $R_j$ vehicles, and the off-ramp $j'$ can receive $R'_j$ vehicles. In addition, the turning movements are predetermined: $\beta$ (e.g., 80%) traffic from link $j-1$ goes to link $j$ and $\beta'$ (e.g., 20%) link $j'$, where $0 \leq \beta \leq 1$, $0 \leq \beta' \leq 1$, and $\beta + \beta' = 1$. Further assume that vehicles depart following a first in-first out queuing discipline, and if a vehicles fails to depart, it holds up all vehicles behind it. The question here is to determine the actual outflow of link $j-1$, $y_{j-1}$, among which how many vehicles are destined for link $j$, $y_j$ and how are many are destined for the off-ramp, $y'_j$.

With these assumptions, the divergence queuing model is quite simple. First, the following relationships must hold:

$$\begin{cases} y_{j-1} = y_j + y'_j \leq S_{j-1}, \\ y_j = \beta y_{j-1} \leq R_j, \\ y'_j = \beta' y_{j-1} \leq R'_j. \end{cases} \tag{9.4}$$

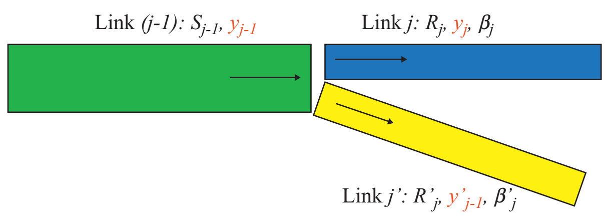



**Figure 9.7** A freeway divergence.

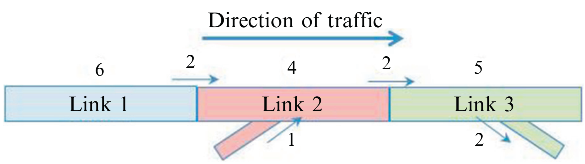
Hence,

$$\gamma_{j-1} = \min\{S_{j-1}, \frac{R_j}{\beta}, \frac{R'_j}{\beta'}\}.$$

Consequently, one obtains $\gamma_j = \beta\gamma_{j-1}$ and $\gamma'_j = \beta'\gamma_{j-1}$.

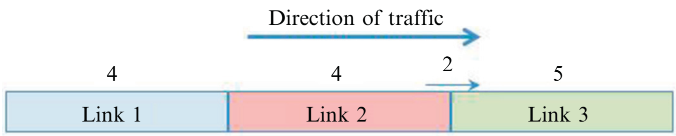
## PROBLEMS

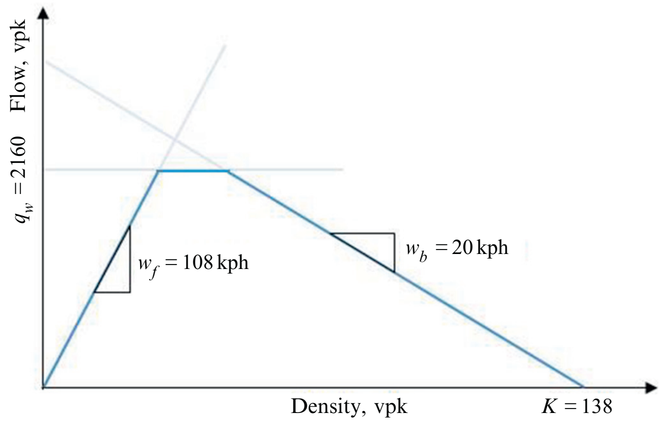
1. A small roadway system is illustrated in the figure below. The system consists of three links, each of which is 200 m long. In addition, there is an on-ramp at link 2 and an off-ramp at link 3. Currently, the storage in each link is indicated above the link. In the next time step (step size of 5 s), vehicles moving on are indicated at the borders, with arrows indicating where they go. Use the FLEFLO model to answer the following questions involved in a one-step simulation:




   a. What is the storage in link 2 at the end of the next step?
   b. What is the density in link 2 at the end of the next step?
   c. What is the equilibrium speed corresponding to this density?
2. Assume that the above road system and conditions remain the same. Use the KRONOS model to answer the following questions involved in a one-step simulation:
   a. What is the storage in link 2 at the end of the next step?
   b. What is the density in link 2 at the end of the next step?
   c. What is the equilibrium speed corresponding to this density if there is a Greenshields speed-density relationship with free-flow speed $v_f$ = 96 km/h and jam density $k_j$ = 120 vehicles per kilometer?
   d. On the basis of your answer to (c), what is the corresponding equilibrium flow?
3. Assume that the road system and conditions are given in the figure below. The system consists of three links, each of which is 150 m long. Currently, the storage in each link is indicated above the link. In the next time step (step size of 5 s), two vehicles will move from link 2 to

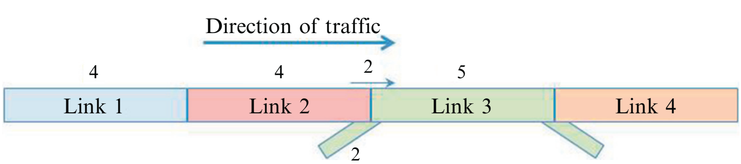
link 3. Moreover, the triangular flow-density relationship illustrated in the figure below applies to each mainline link. Use the CTM to answer the following questions involved in a one-step simulation:




**a.** What is the storage in link 2 at the end of the next step?
**b.** What is the density in link 2 at the end of the next step?
**c.** What is the equilibrium speed corresponding to this density?
**d.** On the basis of your answer to (c), what is the equilibrium flow?

**4.** Now more details of the road system become known. In addition to the conditions given above, the system includes a fourth link (link 4), an on-ramp at the upstream end of link 3, and an off-ramp at the downstream end of link 3. The on-ramp has two vehicles waiting to enter the mainline. Assume that the priority of the mainline versus the on-ramp is 2:1. At the off-ramp, 25% of vehicles in link 3 plan to exit.



**a.** How many on-ramp vehicles may enter link 3 if it is able to receive five vehicles in the next step?
**b.** How would your answer change if link 3 is able to receive three vehicles in the next step?
**c.** If link 3 has room to accept only two vehicles, how many vehicles can actually enter link 3 from the mainline and the on-ramp?
**d.** If the condition in (b) is true, how many vehicles want to enter link 4 in the next step?