CHAPTER 8

LWR Model

In previous chapters, we temporarily left traffic flow and concentrated on the conservation law (Chapter 5), waves (Chapter 6), solutions to the conservation law (Chapter 6), and shock waves (Chapter 7). The purpose of these chapters was to pave the road to addressing traffic dynamics and unveiling traffic evolution on highways.

8.1 THE LWR MODEL

At the end of Chapter 5, a dynamic traffic flow model was formulated on the basis of the conservation law:

$$\begin{cases} k_t + q_x = 0, \\ q = kv, \\ v = V(k), \end{cases} \tag{8.1}$$

where $q = q(t, x)$ is flow, $k = k(t, x)$ is density, and $v = v(t, x)$ is mean traffic speed. If one combines the second and third equations by eliminating v, one obtains a flow-density relationship $q = Q(k)$, and the dynamic model becomes:

$$\begin{cases} k_t + q_x = 0, \\ q = Q(k), \end{cases} \tag{8.2}$$

or further

$$k_t + Q'(k)k_x = 0,$$

where $Q'(k) = \frac{\mathrm{d}Q(k)}{\mathrm{d}k}$. This is the so-called LWR model [24, 25] to honor the three pioneers, Lighthill, Whitham, and Richards, who originally studied this problem. The LWR model is essentially a first-order, homogeneous, quasi-linear partial differential equation.

If we apply the results in the previous chapters, the LWR model with initial condition $k(0, x) = k_0(x)$ can be solved as follows:

1. Construct a time-space diagram (i.e., the t-x plane) with initial condition $k_0(x)$ labeled on the x-axis.

Traffic Flow Theory
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2. Start with an arbitrary point on the x-axis $(0, x^*)$, and determine the k value at this point $k_0(x^*)$ and the value of $c(x^*) = Q'(k_0(x^*))$.
3. Draw a straight line s from point $(0, x^*)$ with slope $c(x^*)$. The line equation is $x_s = c(x^*)t + x^*$, which represents a characteristic along which the k value is constant $k(t, x_s) = k_0(x^*)$.
4. Apply the previous two steps to other points on the x-axis and construct their corresponding characteristics.
5. If two characteristics intersect, terminate both characteristics at their intersection and note the intersection as a point on a shock path. If a characteristic has multiple intersections, use the Rankine-Hugonoit jump condition to determine the right intersection. Repeat this step and find adjacent intersections. Connect these intersections to form a shock path. The solution at both sides of the shock path should be piecewise smooth with a jump along the shock path which forms a shock wave.
6. If two families of characteristics diverge and, hence, leave a wedge-shaped area in between, fill this area with a fan of characteristics and construct a rarefaction wave solution in this area.
7. If an area has multiple rarefaction solutions, apply the entropy condition to select a solution that makes the most physical sense.
8. After the above steps have been followed, the solution space should be filled with characteristics. Each point in the solution space should be swept by one and only one characteristic.
9. If an arbitrary point (t, x) is of interest, one simply follows its characteristic all the way back to the x-axis and reads $k_0(x)$ off the initial condition. This $k_0(x)$ is the k value at the time-space point in question. Consequently, one finds the corresponding $q(x, t) = Q(k(t, x))$ and $v(t, x) = \frac{q(t,x)}{k(t,x)}$. Hence, the solution $k(t, x)$, $q(t, x)$, and $v(t, x)$ of any time-space point (t, x) can be determined.

Note that the conservation law (and consequently the LWR model) involves three dependent variables: flow (flux) q, density (concentration) k, and speed v. One might be curious about why density k is always chosen as the target variable to work on. This is because density k is unique in that, by knowing k, one is able to unambiguously determine flow q and speed v on the basis of equilibrium traffic flow models, while flow q and speed u do not have such a property. Readers should be cautioned again that equilibrium traffic flow models are only of statistical significance, and their use as a lookup table is the last resort when no better choice is available.

8.2 EXAMPLE: LWR WITH GREENSHIELDS MODEL

The Greenshields model [9] assumes the following linear v-k relationship:

$$v = v_{\mathrm{f}}(1 - \frac{k}{k_{\mathrm{j}}}),$$

where v_{f} is free-flow speed and k_{j} is jam density. This model implies the following quadratic q-k relationship:

$$q = Q(k) = v_{\mathrm{f}}\left(k - \frac{k^2}{k_{\mathrm{j}}}\right).$$

Hence

$$c(k) = Q'(k) = v_{\mathrm{f}} - 2\frac{v_{\mathrm{f}}}{k_{\mathrm{j}}}k.$$

If the parameters are traffic speed $v_{\mathrm{f}} = 60$ miles per hour and density $k_{\mathrm{j}} = 240$ vehicles per mile, the explicit form of the LWR model becomes

$$k_t + (60 - \frac{k}{2})k_x = 0.$$

Find solutions at points ($t = \frac{1}{2}\mathrm{h}, x = 25\mathrm{miles}$) and ($t = 1\mathrm{h}, x = 65\mathrm{miles}$) with use of the following initial condition:

$$k(0, x) = k_0(x)\begin{cases} 40 \text{ vehicles per mile} & \text{if } 0 < x \leq 10 \text{ miles}, \\ 20 \text{ vehicles per mile} & \text{if } x > 10 \text{ miles}. \end{cases}$$

Following the above solution procedure, one constructs a time–space diagram, shows the initial condition at the side of the diagram, and identifies the two points in question (see Figure 8.1). Next, one constructs characteristics. All characteristics drawn between $0 < x \leq 10$ miles will bear a k value of 40 vehicles per mile, which can be read from the initial condition, so the slope of these characteristics is $c = 60 - \frac{k}{2} = 40$ miles

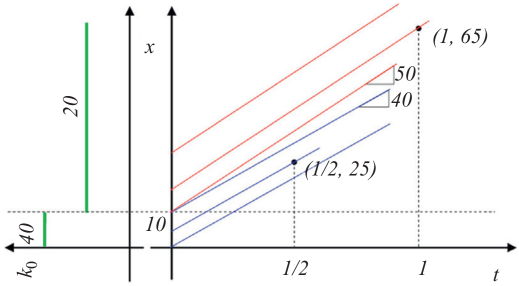


Figure 8.1 Example: LWR with Greenshields model.

per hour. Point ($t = \frac{1}{2}, x = 25$) is within this area, and the characteristic passing this point intercepts the x-axis at $(0, 5)$. Hence, $k(\frac{1}{2}, 25) = k(0, 5) =$ 40 vehicles per mile. Similarly, All characteristics drawn from $x > 10$ miles have slope $c = 50$ miles per hour, and point ($t = 1, x = 65$) is within this area. The characteristic passing this point intercepts the x-axis at $(0, 15)$. Hence, $k(1, 65) = k(0, 15) = 20$ vehicles per mile.

8.3 SHOCK WAVE SOLUTION TO THE LWR MODEL

The above example actually involves two platoons: a fast one running in front and a slow one trailing behind. Each platoon corresponds to a family of characteristics called a *kinematic wave*. The characteristics of the fast platoon have a slope of 50 miles per hour, which is the speed of the fast kinematic wave. Similarly, the speed of the slow kinematic wave is 40 miles per hour. Noticeably, there is a wedge between the two families of characteristics starting from $(0, 10)$, meaning there is an increasing "vacuum" (or gap) between the two platoons.

If the two platoons are reversed—that is, the slow platoon leads the fast platoon, sooner or later the fast platoon will catch up with the slow platoon. When this occurs, the first vehicle in the fast platoon will have to adopt the speed of the last vehicle in the slow platoon. Shortly afterward, the second vehicle in the fast platoon will have to slow down, and so will the third vehicle, the fourth vehicle, and so on. The "slowing down" effect will propagate backward along the fast platoon. The propagation of a sudden change of traffic condition (e.g., speed drop in this example) creates a *shock wave* which delineates regions of different traffic conditions (e.g., slow and fast traffic in this example). The trajectory of the shock wave in the x–t plane is called a *shock path*.

As discussed in the method of characteristics, a characteristic carries a constant k value (i.e., density), and the intersection of two characteristics will inevitably have two k values. This means that at this point two traffic conditions coexist, and after the intersection, the two platoons resume their original conditions along their respective characteristics. This situation does not make any physical sense. To develop a solution that is physically meaningful, one has to make the solution piecewise smooth. This requires that a characteristic carries one and only one traffic condition (e.g., a k value). When two characteristics meet, both characteristics terminate, and there is a jump (or shock) at the intersection.

To illustrate the idea, the previous example is revisited with the fast platoon being behind. In the x-t plane in Figure 8.2, two families of characteristics—that is, two kinematic waves—are drawn, but this time those characteristics drawn between $0 < x < 10$ will have a slope of 50, while those drawn from $x > 10$ have a slope of 40. Since the fast kinematic wave is behind, it will catch up with the slow kinematic wave—that is, the two families of characteristics will intersect. Whenever two characteristics intersect, they terminate at their intersection. A curve that connects these intersections gives a shock path, along which two regions are delineated: one region belongs to the slow platoon—that is, all points in this region carry the condition of the slow platoon—and the other region belongs to the fast platoon—that is, all points in this region carry the condition of the fast platoon. When one moves across the shock path, the traffic condition changes suddenly from one condition to another—that is, experiencing a shock, which is how a shock wave gets its name. Therefore, it is convenient to read from Figure 8.2 that $k(\frac{1}{2}, 25) = k(0, 0) = 20$ vehicles per mile and $k(1, 65) = k(0, 25) = 40$ vehicles per mile.

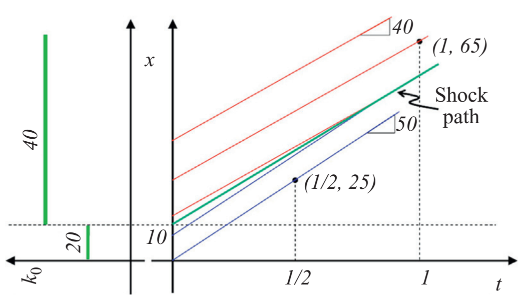


Figure 8.2 Example: LWR with Greenshields model revisited.

8.4 RIEMANN PROBLEM

In the above example, two properties are noticeable:

1. Each of the two kinematic waves consists of a family of straight, parallel characteristics.
2. The shock path is a straight line.

As discussed in Chapter 6, if c is a constant or dependent on k but not explicitly dependent on t or x, the resultant characteristic is a straight line, which is the case in the above example since $c = Q'(k) = v_{\mathrm{f}} - 2\frac{v_{\mathrm{f}}}{k_{\mathrm{j}}}k$.

From Chapter 6, the method of characteristics stipulates that the slope of a characteristic be $\frac{\mathrm{d}x}{\mathrm{d}t} = c$, which depends on the initial condition. If the initial condition consists of piecewise constant k_0, each family of characteristics will have the same slope—that is, they are parallel. The above example is such a case.

As discussed in Chapter 7, the slope of a shock path is determined by the Rankine-Hugonoit jump condition. If the initial condition consists of piecewise constant k_0, then solutions k and q on both sides of the shock path are piecewise constant. Consequently, the Rankine-Hugonoit jump condition will result in a shock path with a constant slope—that is, a straight line—which is also the case in the above example.

Hence, it becomes clear that the solution to an LWR model will always have the above two properties as long as the initial data are given as piecewise constant. In general, a conservation law with piecewise constant initial data is referred to as a *Riemann problem*, named after Bernhard Riemann, who was a German mathematician.

8.5 LWR MODEL WITH A GENERAL *Q-K* RELATIONSHIP

In the above examples, the underlying q-k relationship is explicitly given—for example, the Greenshields model. Hence, it is convenient to determine the speed of a kinematic wave (i.e., the slope of a family of straight, parallel characteristics) from the initial condition. However, it is recognized that the Greenshields model suffers from inaccuracy, and often the underlying q-k relationship is graphically given by fitting from empirical data. In this case, the solution to the LWR model with a general q-k relationship is typically determined graphically.

Consider the following LWR model with a general q-k relationship:

$$\begin{cases} k_t + q_x = 0, \\ q = Q(k), \\ k(t,0) = k_0(x) = \begin{cases} A & \text{if } x \leq 0, \\ B & \text{if } x > 0, \end{cases} \end{cases} \tag{8.3}$$

where the underlying q-k relationship is given in Figure 8.3, where A denotes an operating point characterized by flow q_A, density k_A, and speed v_A, and similar notation applies to point B. A time-space diagram is constructed below the q-k relationship with the initial condition at the side. Since this is a Riemann problem, each kinematic wave has a constant slope, and the shock path will be a straight line. From the initial condition, there are two kinematic waves: kinematic wave A emitted from $x \leq 0$, and kinematic wave B emitted from $x > 0$. The speed of kinematic wave A is

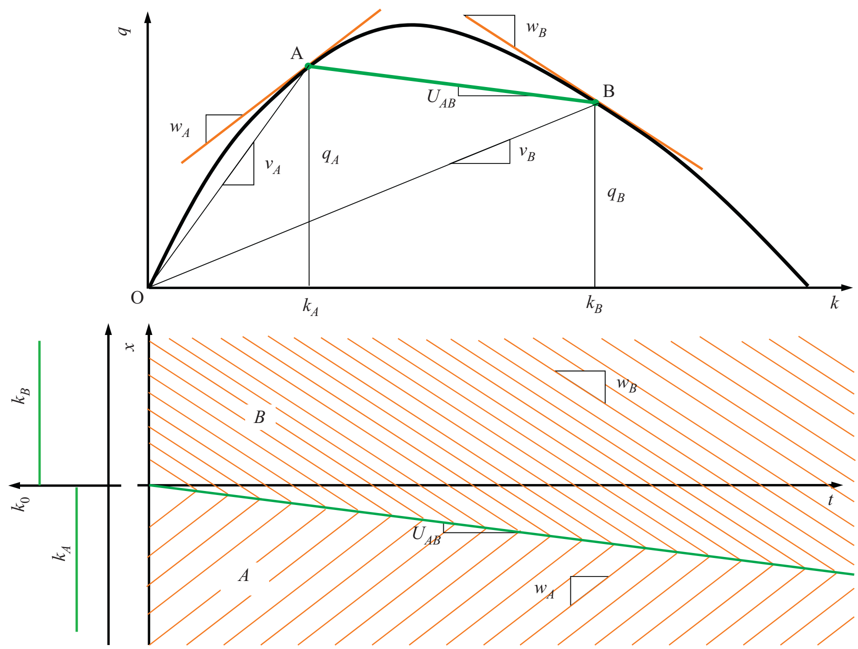


Figure 8.3 Example: LWR model with a general q-k relationship.

$$w_A = \frac{\mathrm{d}Q}{\mathrm{d}k} = Q'(k)|_{k=k_A};$$

that is, the derivative of the q-k relationship evaluated at operating point A. This is the tangent to the q-k curve at point A. Therefore, one constructs kinematic wave A by drawing a family of straight, parallel lines drawn from $x \leq 0$ with slope w_A. Similarly, the speed of kinematic wave B, w_B, is the tangent to the q-k curve at point B, and the wave can be constructed accordingly. Since kinematic wave B represents a heavy, slow platoon in front and kinematic wave A represents a light, fast platoon behind, kinematic wave A will catch up with kinematic wave B, creating a shock wave. Again, since this is a Riemann problem, the shock path is a straight line. The slope of this line (i.e., the speed of the shock wave) is determined by the Rankine-Hugonoit jump condition:

$$U_{AB} = \frac{q_B - q_A}{k_B - k_A}.$$

This happens to be the slope of the chord connecting points A and B in the q-k curve. In addition, one already knows from the initial condition that the shock path starts at the origin in the time-space diagram. Therefore, one can determine the shock path by drawing a line from the origin with slope U_{AB}. Characteristics in the two kinematic waves will terminate once they meet the shock path. Hence, the shock wave solution is graphically constructed, and consists of two piecewise smooth solutions: the region above the shock path has a uniform traffic condition B (q_B, k_B, u_B) and the region below the shock path has condition A (q_A, k_A, u_A).

8.6 SHOCK PATH AND QUEUE TAIL

In Figure 8.3, the shock path actually represents the time-varying location which separates the fast platoon and the slow platoon—that is, the tail of a moving queue. As the leading vehicle of the fast platoon catches up with the tail of the slow platoon, that vehicle joins the slow platoon and becomes its new tail. Since the slow platoons is still moving, the location of its tail changes dynamically depending on how quick the fast platoon arrives. Figure 8.4 shows a few snapshots to illustrate such a dynamic process.

One may have recognized that although characteristics are used to illustrate how to find the shock path, they are actually unnecessary. With a known point on the shock path and known shock speed, the shock path can be determined directly without characteristics being drawn. In the above

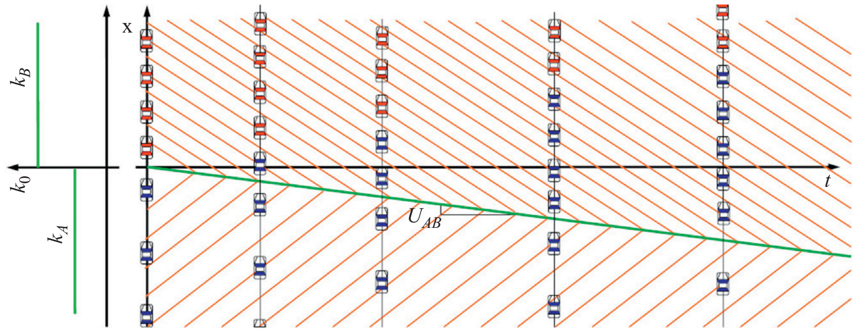


Figure 8.4 Shock path and queue tail.

example, one can construct the solution directly by drawing a line from the origin with slope U_{AB}. This line is the shock path and also the queue tail which separates regions with conditions A and B.

8.7 PROPERTIES OF THE FLOW-DENSITY RELATIONSHIP

It can be seen from the above example that the flow-density (q–k) relationship is very illustrative to show various speeds. Figure 8.5 gives the full picture.

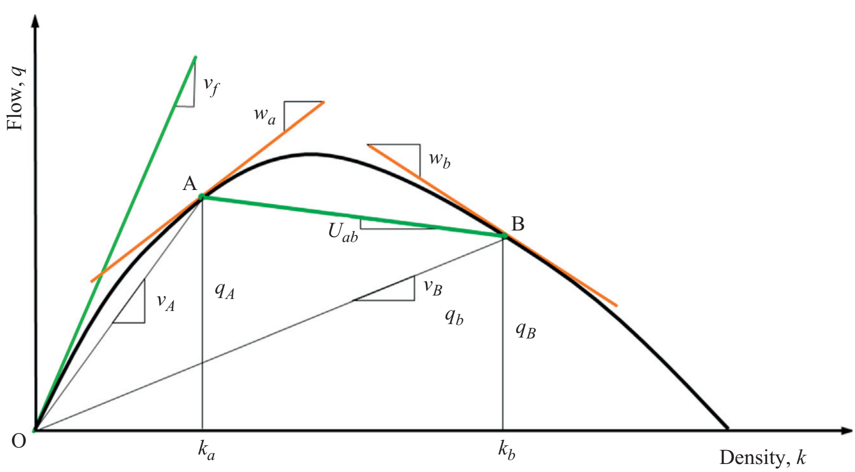


Figure 8.5 Speeds in a flow-density relationship.

8.7.1 Flow-Density Relationship and Speeds

Traffic speed v

If operating point A, which represents a traffic condition with flow q_A and density k_A, is known, the corresponding traffic speed under condition A, by definition, is

$$v_A = \frac{q_A}{k_A}.$$

Graphically, this can be represented as the slope of the line connecting the origin O and operating point A.

Free-flow speed v_f

If k_A decreases, point A will move along the curve toward the origin O. In the limiting case where $k_A \to 0$, line OA becomes the tangent to the curve at the origin. The slope of this tangent denotes the traffic speed when the density is close to zero. By definition, the slope represents the free-flow speed v_{f}:

$$v_{\mathrm{f}} = \lim_{A \to O} v_A = \lim_{k_A \to 0} \frac{q_A}{k_A}.$$

Kinematic wave speed w

If one draws a line tangent to the curve at point A, as discussed above, the slope of this tangent is the speed of a kinematic wave carrying traffic condition A:

$$w_A = Q'(k)|_{k=k_A}.$$

Shock wave speed U

If A and B represent two different traffic conditions, as discussed above, the slope of chord AB is the speed of the shock wave should traffic with condition A catch up with traffic with condition B.

$$U_{AB} = \frac{q_B - q_A}{k_B - k_A}.$$

8.7.2 Flow-Density Relationship Observed by a Moving Observer

Note that the above discussion is based on the observation from the perspective of a stationary observer—that is, everything is relative to an observer standing stationary at the roadside. Now what happens if the observer is moving? For example, what if the observer is riding on the kinematic wave carrying traffic condition A? The moving observer will now observe less flow than he or she would have observed if he or she were stationary. The following equation quantifies the relative flow that the moving observer sees:

$$\tilde{q}_A = q_A - w_A k_A.$$

This is equivalent to drawing a line from the origin with w_A as its slope. Then run a vertical line through point A intersecting the drawn line at A'' and the horizontal axis at A'. The length of AA' is q_A, the segment of $A'A''$ is $w_A k_A$, and the segment of AA'' is the relative flow, $\tilde{q}_A$, observed by the moving observer. As another example, suppose traffic is operating at condition B which is on the congested side of the q–k curve. The kinematic wave speed is now w_B, which is negative. What happens if an observer is moving along with wave w_B? With the same treatment, one obtains

$$\tilde{q}_B = q_B - w_B k_B.$$

This is equivalent to drawing a line from the origin O with slope w_B which slants downward. Run a vertical line through point B intersecting the drawn line at B'' and the horizontal axis at B'. The absolute value of relative flow (i.e., the length of BB'') in this case is the sum of BB' and $B'B''$ because w_B takes a negative value.

8.8 EXAMPLE LWR MODEL PROBLEMS

The above discussion focused on the LWR model and its solutions using the method of characteristics and shock waves. It is time to apply this method to solve some concrete traffic flow problems.

8.8.1 A Bottleneck with Varying Traffic Demand

Traffic arriving at the upstream point of a highway was initially under condition A (see Table 8.1 and Figure 8.7). At 9:00 a.m., the arriving traffic switches to condition B. After 1 h, the arriving traffic switches back to condition A. The capacity at the bottleneck is 1400 vehicles per hour. Find how far the queue extends back and how long the queue persists.

Table 8.1 Traffic data: a bottleneck with varying traffic demand

Condition	*q* (vehicles/h)	*k* (vehicles/km)	*v* (km/h)
A	600	8.57	70
B	2000	40	50
D	1400	21.5	65
D'	1400	130	10.8

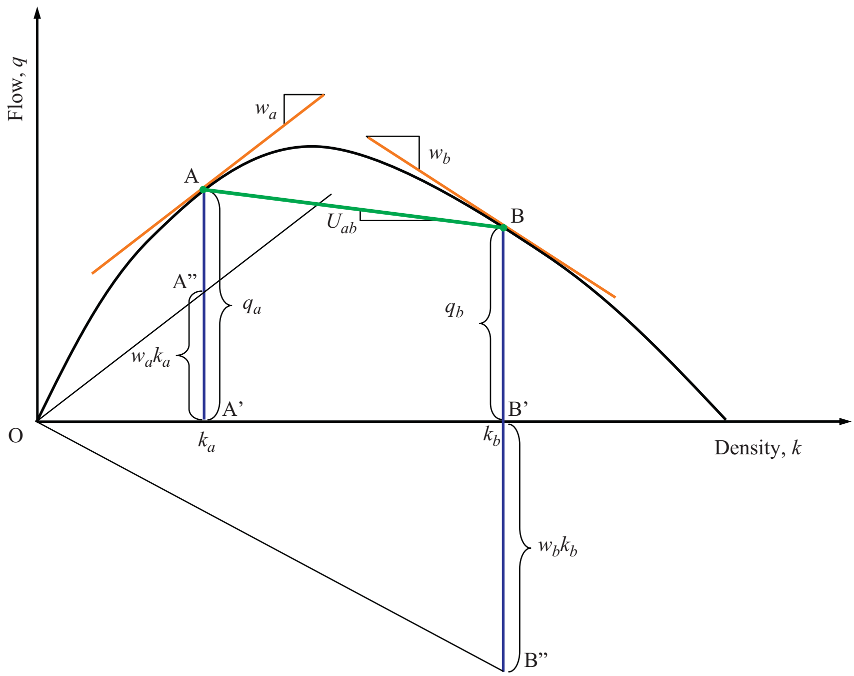


Figure 8.6 Traffic flow observed by a moving observer.

Solution. With the aid of the graphical construction in Figure 8.7, the rate at which the queue grows is:

$$U_{BD'} = \frac{q_{D'} - q_B}{k_{D'} - k_B} = \frac{1400 - 2000}{130 - 40} = -\frac{600}{90} = -6.67\,\text{km/h}.$$

The queue tail extends back at this rate for 1 h, so the farthest point it reaches is 6.67 km upstream of the bottleneck. The rate at which the queue dissipates is

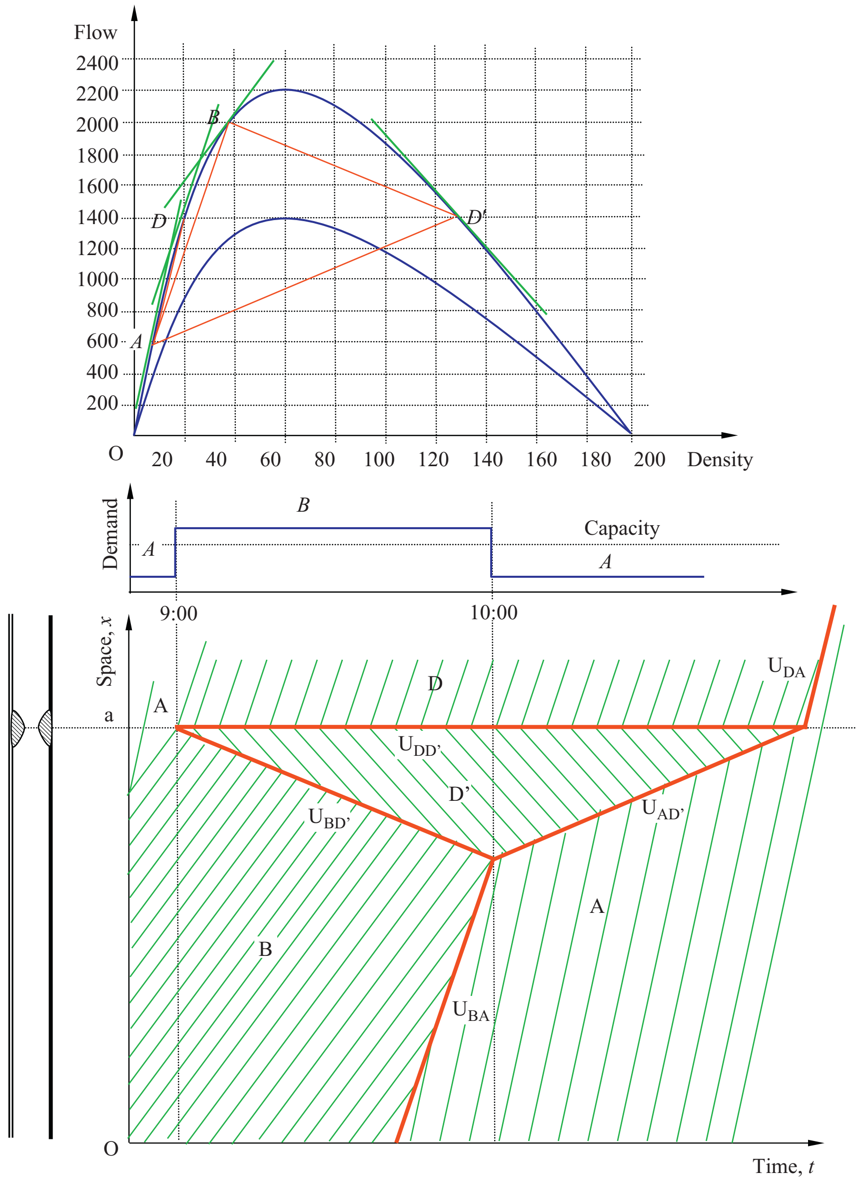


Figure 8.7 A highway bottleneck with varying traffic demand.

Table 8.2 Traffic data: a moving bottleneck

Condition	*q* (vehicles/h)	*k* (vehicles/km)	*v* (km/h)
A	700	10	70
B	1600	120	13.3
C	2200	60	36.7
O	0	0	75

$$U_{AD'} = \frac{q_{D'} - q_A}{k_{D'} - k_A} = \frac{1400 - 600}{130 - 8.57} = 6.60\,\text{km/h}.$$

So the time needed to dissipate the queue is $\frac{6.67}{6.60} = 1.01$ h, and the total time for which the queue persists is 2.01 h.

8.8.2 A Moving Bottleneck

A freeway was initially operating under condition A (see Table 8.2). At 2:30 p.m., a sluggish truck entered the freeway traveling at a speed of 13.3 km/h. The truck turned off the freeway at the next exit 6.67 km away. Find when the impact of the truck will disappear.

Solution. With the aid of the graphical construction in Figure 8.8, the following can be calculated:

$$U_{OB} = \frac{q_B - q_O}{k_B - k_O} = \frac{1600 - 0}{120 - 0} = 13.30\,\text{km/h},$$

$$U_{AB} = \frac{q_B - q_A}{k_B - k_A} = \frac{1600 - 700}{120 - 10} = 8.18\,\text{km/h},$$

$$U_{CB} = \frac{q_B - q_C}{k_B - k_C} = \frac{1600 - 2200}{120 - 60} = -10.00\text{km/h},$$

$$\frac{be}{ae} = U_{OB} \rightarrow ae = \frac{be}{U_{OB}} = \frac{6.67}{13.3} = 0.5\,\text{h},$$

$$\frac{cd}{bc} = U_{CB} \rightarrow cd = U_{CB} \times bc = 10bc,$$

$$\frac{df}{af} = U_{AB} \rightarrow df = U_{AB} \times af = 8.18af,$$

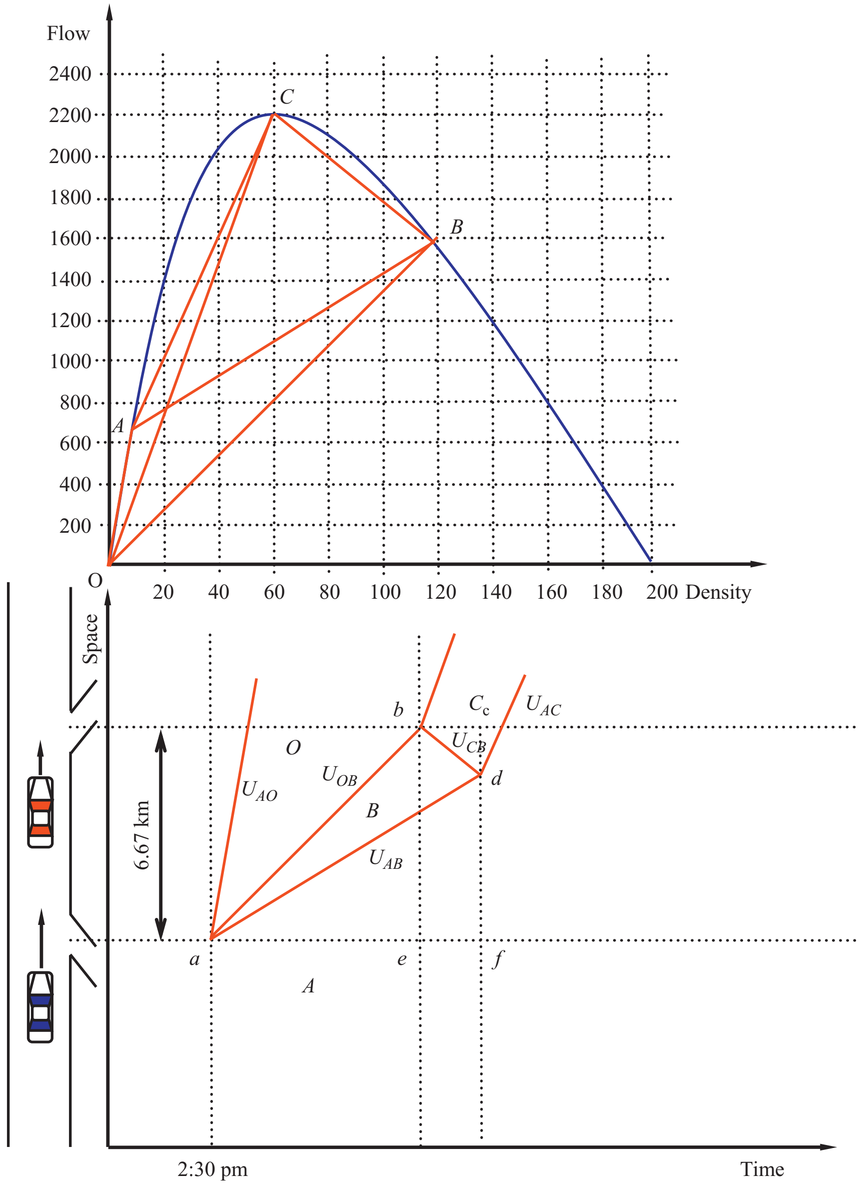
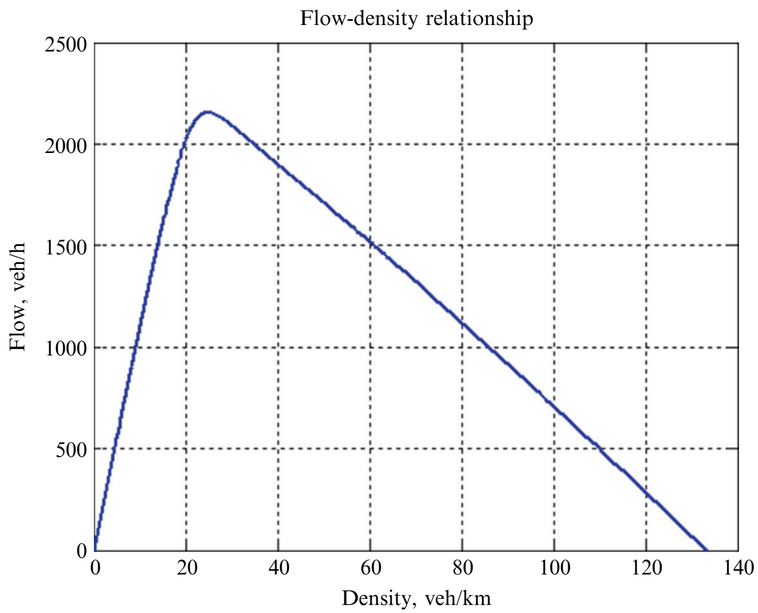


Figure 8.8 A moving bottleneck with constant demand.



$$\begin{cases} 10bc + 8.18af = 6.67, \\ af - bc = 0.5, \end{cases}$$

$$af = 0.64\,\text{h}.$$

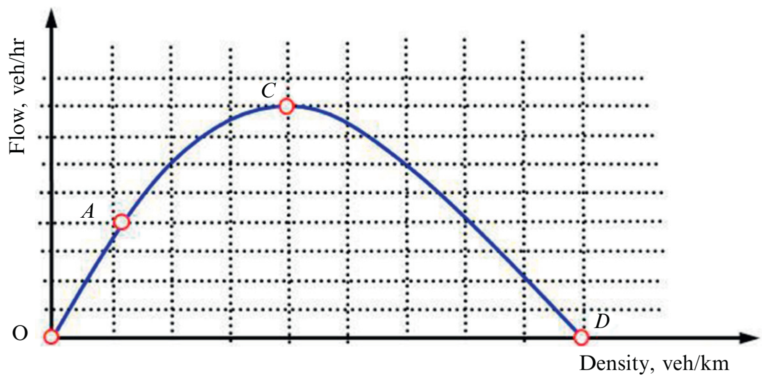
So the impact of the truck lasts for 0.64 h.

PROBLEMS

1. The figure below illustrates a hypothetical flow-density relationship. Identify the following from the figure (mark them on the figure if necessary to show your answers):
   - **a.** Free-flow speed v_f
   - **b.** Jam density k_j
   - **c.** Capacity condition $(q_\mathrm{m}, k_\mathrm{m}, v_\mathrm{m})$
   - **d.** Traffic condition at point A (q_A, k_A, v_A)
   - **e.** Traffic condition at point B (q_B, k_B, v_B)
   - **f.** Kinematic wave speeds (i) when the density is zero, (ii) at jam density, (iii) at capacity, (iv) at condition A, and (v) at condition B
   - **g.** Shock wave speed when a platoon of vehicles at condition A catches up with a platoon of vehicles at condition B

2. Assume that traffic on a uniform freeway section follows the above flow-density relationship. At one time, traffic is operating at condition A.
   a. Find the relative flow observed by a moving observer who is traveling with the traffic at 25 km/h (assume no interaction between the observer and the traffic, e.g., the observer flies over the traffic).
   b. Similarly, find the relative flow when the traffic condition is B and the observer is riding on the kinematic wave carrying condition B.
3. Traffic on a 16 km uniform segment of Interstate 90 was initially operating at condition B, as illustrated in the figure for problem 1. Starting at 7:00 p.m. and upstream of the midpoint of the uniform section, demand drops and traffic begins to operate at condition A. Assume that the flow-density relationship in the figure applies.
   a. Determine the traffic condition at a location 2 km downstream of the midpoint at 7:30 p.m.
   b. Determine the traffic condition at a location 2 km upstream of the midpoint at 7:30 p.m.
   c. When will the end of the queue reach the upstream end of the uniform section?
4. Traffic on a 16 km uniform segment of Interstate 90 was initially operating at condition A, as illustrated in the figure for problem 1. Starting at 7:00 a.m. and upstream of the midpoint of the uniform section, demand increases and traffic begins to operate at condition B. Assume that the flow-density relationship in the figure applies.
   a. Determine the traffic condition at a location 2 km downstream of the midpoint at 7:30 a.m.
   b. Determine the traffic condition at a location 2 km upstream of the midpoint at 7:30 a.m.
5. An intersection with constant demand. Traffic arrives at an approach of a signalized intersection at a constant rate of 800 vehicles per hour. All conditions are given in the table and the flow-density relationship below. The intersection is under pretimed signal control with a cycle length of 90 s and a split of effective green/red of 0.5/0.5. Determine the farthest point of the queue.
6. An intelligent transportation system problem. On Wednesday at 9:00 a.m., there is an accident on northbound Interstate 91. The traffic operation center (TOC) has to decide how to clean up the accident. After collecting information and communicating with highway patrol and emergency operator, the TOC determines that there are two alternatives:

Condition	*q* (vehicles/h)	*k* (vehicles/km)	*v* (km/h)
A	800	25	32
C	1600	80	20
D	0	180	0
O	0	0	40

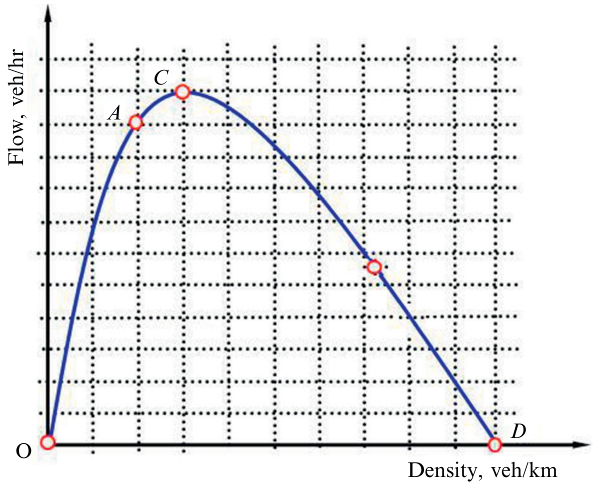


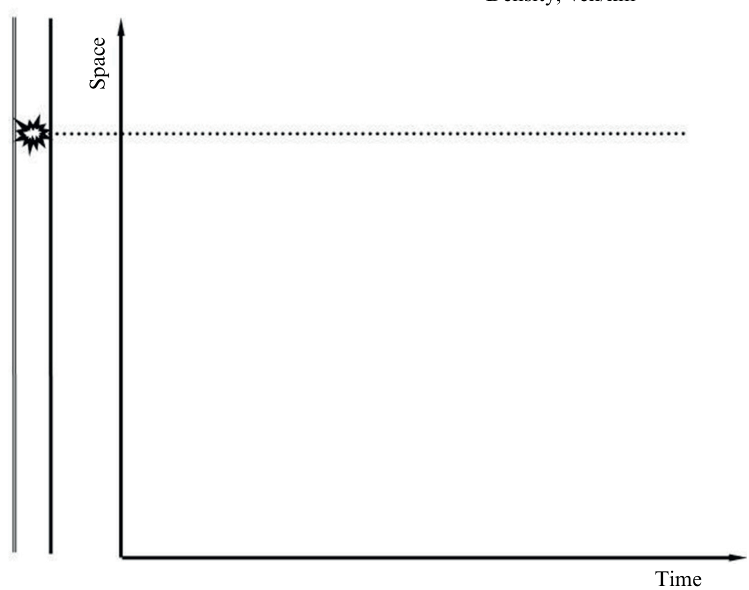
a. Completely shut Interstate 91 for 10 min, clean it up, and then reopen Interstate 91 for normal operation.

b. Partially open Interstate 91 at reduced capacity, but the cleanup requires longer—about 30 min—before normal operation can resume.

One of the concerns at the TOC is how far the queue will spill back because the queue on Interstate 91 will overflow via ramps and further block upstream surface streets. As a transportation engineering student, you are asked to offer your knowledge to help the TOC make a decision. More details are given in the table and the flow-density relationship below. Find which alternative creates the longer queue.

Condition	Description	*q* (vehicles/h)	*k* (vehicles/km)	*v* (km/h)
A	Arrival flow	2000	40	50
D	Queued flow	0	200	0
C	Capacity flow	2200	60	36.7
E	Reduced capacity flow	1100	50	22


Flow, veh/hr
C
A
O
D
Density, veh/km


Space
Time