CHAPTER 13

Pipes and Forbes Models

As the beginning discussion on car-following models, this chapter introduces two simple models—that is, the Pipes model and the Forbes model, both of which are derived from drivers' daily driving experiences.

13.1 PIPES MODEL

The Pipes model [52] is based on a safe driving rule coined in the California Vehicle Code:

> *A good rule for following another vehicle at a safe distance is to allow yourself at least the length of a car between your vehicle and the vehicle ahead of you for every ten mile per hour of speed at which you are traveling.*

Referring to Figure 13.1 and putting the safety rule in mathematical language, we get

$$g_i^x(t)_{\min} = [(x_{i-1}(t) - x_i(t)) - l_{i-1}]_{\min} = (s_i(t) - l_{i-1})_{\min} = \frac{\dot{x}_i(t)}{0.447 \times 10} l_i, \tag{13.1}$$

where $\dot{x}_i(t)$ is in meters per second (1 mile per hour is approximately 0.447 m/s), and $g_i^x(t)$, $x_{i-1}(t)$, and $x_i(t)$ are measured in meters. The Pipes model is formulated as

$$s_i(t)_{\min} = \frac{l_i}{4.47}\dot{x}_i(t) + l_{i-1}. \tag{13.2}$$

If we assume a vehicle length of 6 m, the model reduces to

$$s_i(t)_{\min} = 1.34\dot{x}_i(t) + 6 \tag{13.3}$$

or

$$h_i(t)_{\min} = 1.34 + \frac{6}{\dot{x}_i(t)}. \tag{13.4}$$

13.1.1 Applications of the Pipes Model

The Pipes model can be applied in many ways, the two foremost of which are automatic driving and computer simulation.

Traffic Flow Theory
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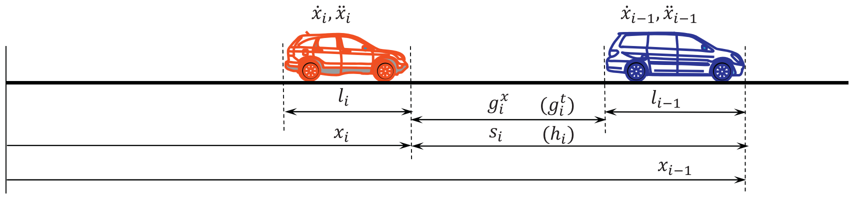

Figure 13.1 A car-following scenario.

Automatic Driving

Perhaps the simplest form of automatic driving is cruise control. As an in-vehicle system, cruise control automatically controls the speed of a motor vehicle (by taking over the control of the throttle) so that the vehicle maintains a constant speed set by its driver. Cruise control makes it easier to drive on long road trips, and hence is a popular car feature. As more and more vehicles join the traffic and the road becomes crowded, the driver has to switch cruise control on and off so frequently that cruise control becomes less useful. To adapt to the dynamics of the vehicle in front, it is desirable that the cruise control system be able to adjust speed accordingly (rather than cruising at a preset speed) to maintain a safe car-following distance. Hence,an adaptive or autonomous cruise control system has been developed. With the aid of distance sensors such as radar or laser sensors, autonomous cruise control allows the vehicle to slow down when approaching another vehicle and accelerate to the preset speed when traffic conditions permit. To make this happen, the system requires an internal logic which relates the vehicle speed to the distance to the vehicle in front. Simple car-following models such as the Pipes model can be employed as the basis of such an internal logic. More specifically, Equation 13.1 can be rearranged as follows:

$$\dot{x}_i(t) \leq \frac{0.447 \times 10}{l_i} g_i^x(t). \tag{13.5}$$

For a vehicle length of 6 m, the above control logic becomes $\dot{x}_i(t) \leq 0.745 g_i^x(t)$. Therefore, the autonomous cruise control works as follows. At any moment t, the distance sensor measures the gap between the two vehicles $g_i^x(t)$. Then, the target speed that the vehicle needs to adapt to is set as $0.745 g_i^x(t)$ or less.

Obviously, the target speed can easily go out of bound as the gap $g_i^x(t)$ becomes sufficiently large. Therefore, it is necessary to set an upper bound to the target speed, which is usually referred to as the desirable speed v_i. Therefore, the target speed is actually the minimum of (1) the desirable speed v_i and (2) the speed constrained by the vehicle in front $\dot{x}_i(t)$:

$$\dot{x}_i(t) \leq \min\{v_i, \frac{0.447 \times 10}{l_i} g_i^x(t)\}. \tag{13.6}$$

Computer Simulation

The Pipes model can also be used to simulate a platoon of vehicles moving on a one-lane highway. Before the simulation starts, the following variables need to be initialized—that is, a value needs to be assigned to each of them:

l_i length of vehicle $i \in \{1, 2, \ldots, I\}$
τ_i perception-reaction time of driver i
v_i desired speed of driver i
ΔA_i maximum acceleration of vehicle i
ΔB_i maximum deceleration of vehicle i
Δt simulation time step

At time step j, the displacement x and speed v of each vehicle are updated:

```
FOR i = 1:I
  s(j,i) = x(j-1,i-1) - x(j-1,i);
  s_min(j,i) = l(i) * (v(j-1,i)/(0.447 * 10) + 1);
  IF s(j,i) < s_min(j,i)
     v(j,i) = MAX([0, v(j-1,i) - dB_i]);
  ELSE
     v(j,i) = MIN([v_i, v(j-1,i) + dA_i]);
  END
  x(j,i) = x(j-1,i) + v(j,i) * dt;
END
```

In the above code segment, the actual spacing between vehicle i and its leading vehicle, $s(j, i)$, is computed as the difference of their locations in the previous time step. The minimum safe spacing, $s_{\min}(j, i)$ is determined according to the California Vehicle Code. Then, $s(j, i)$ is compared against $s_{\min}(j, i)$. If $s(j, i)$ is less than $s_{\min}(j, i)$, one should reduce the speed of the vehicle by ΔB_i, but should not go beyond 0. Otherwise, one should increase the speed of the vehicle by ΔA_i without exceeding its desired speed v_i.

Then, one should update the position of the vehicle, advance time by one step, and continue with the next vehicle.

Note that car-following models used for automatic control and computer simulation have different objectives. The objective of automatic control is to guarantee safety but achieve mobility (e.g., arrive at the destination without delay). As such, automatic control calls for "an ideal (or the best) driver/model" that is able to operate the vehicle in the best way. In contrast, the purpose of computer simulation is to reproduce part of the real world as realistically as possible. Consequently, computer simulation necessities "a representative driver/model" that is able to mimic the behavior of day-to-day driving, which is usually not perfect.

13.1.2 Properties of the Pipes Model

In mathematical modeling, it is always interesting to understand how a system's microscopic behavior relates to its macroscopic behavior, or alternatively to interpret the microscopic basis of a macroscopic phenomenon. In traffic flow theory, microscopic car-following models are typically related to macroscopic speed-density relationships and further the fundamental diagram.

Typically, the linkage between microscopic and macroscopic models can be addressed in two ways. One approach is to run a simulation based on the microscopic model. Such a microscopic simulation typically involves random variables such as perception-reaction time, desired speed, and acceleration rate. As a result, simulation results vary in different runs. Hence, the macroscopic behavior implied by the microscopic model can be obtained by a statistical analysis of these simulation results.

The other approach is analytical—that is, one tries to aggregate or integrate the microscopic model (which typically involves ordinary differential equations) under some equilibrium or steady-state assumptions. If a system is in the steady state, any property of the system is unchanging in time. More specifically, a traffic system in the steady state would consist of homogeneous vehicles which exhibit unform behavior over time and space. Therefore, under steady-state conditions, vehicles lose their identities (e.g., $\tau_i \to \tau$ and $l_i \to l$), vehicles travel at uniform speed (i.e., $\dot{x}_i = \dot{x}_j \to v$ and $\ddot{x}_i \to 0$), drivers' desired speeds converge to the free-flow speed (i.e., $v_i \to v_{\mathrm{f}}$), and the vehicle spacing $s_i(t)$ reduces to s, which, in turn, is replaced by the reciprocal of traffic density $\frac{1}{k}$. Uniform vehicle length l is equivalent to the reciprocal of jam density k_{j}—that is, $\frac{1}{k_{\mathrm{j}}}$. Hence, the Pipes model reduces to

$$\frac{k_\mathrm{j}}{k} = \frac{v}{4.47} + 1 \text{ or } v = 4.47\left(\frac{k_\mathrm{j}}{k} - 1\right), \tag{13.7}$$

where k is measured in vehicles per meter and v is measured in meters per second. With $q = k \times v$, the above speed-density relationship gives rise to the following flow-density and speed-flow relationships:

$$q = 4.47(k_\mathrm{j} - k) \tag{13.8}$$

and

$$v = \frac{q}{k_\mathrm{j} - 0.22q}. \tag{13.9}$$

Equations 13.7–13.9 constitute the mathematical representation of the fundamental diagram implied by the Pipes model.

13.2 FORBES MODEL

Rather than ensuring a safe distance between vehicles as the Pipes model does, Forbes [53, 54] stipulates that

> *To ensure safety, the time gap between a vehicle and the vehicle in front of it should be always greater than or equal to reaction time.*

This safety rule can be formulated as

$$g_i^t(t) = h_i(t) - \frac{l_i}{\dot{x}_i} \geq \tau_i. \tag{13.10}$$

For a reaction time of 1.5 s and a vehicle length 6 m, the model becomes

$$h_i(t) \geq 1.5 + \frac{6}{\dot{x}_i} \tag{13.11}$$

or

$$s_i(t) \geq 1.5\dot{x}_i + 6. \tag{13.12}$$

This is very similar to the Pipes model except for a slight difference in the coefficient of the speed term, which is interpreted as perception-reaction time τ_i. Therefore, the Pipes model and the Forbes model are essentially equivalent and can be generically expressed as

$$s_i(t) \geq \tau_i\dot{x}_i + l_i, \tag{13.13}$$

where τ_i and vehicle length $ł_i$ are model parameters. Note that applications and properties of the Pipes model discussed above apply to the Forbes model. In addition, the fundamental diagram implied by the Pipes and Forbes models can be generically expressed as

$$v = \frac{1}{\tau k} - \frac{l}{\tau}, \tag{13.14}$$

$$q = \frac{1}{\tau} - \frac{l}{\tau}k, \tag{13.15}$$

$$v = \frac{ql}{1 - \tau q}, \tag{13.16}$$

where τ is the average perception-reaction time and l is the average vehicle length.

13.3 BENCHMARKING

Since the Pipes and Forbes models are essentially equivalent, the following discussion addresses only the Pipes model with the understanding that the result applies to the Forbes model as well. Microscopic benchmarking refers to the scenario presented in Section 12.3.1 and macroscopic benchmarking refers to the scenario presented in Section 12.3.2.

13.3.1 Microscopic Benchmarking

For convenience, the Pipes model is rearranged as

$$\dot{x}_i(t + \Delta t) = \frac{s_i(t) - l_i}{\alpha}, \tag{13.17}$$

where Δt is the simulation time step and α is a constant resulting from unit conversion ($\alpha = 1.34$ if speed is in meters per second and $l_i = 6$ m).

First, the model has a problem with vehicle acceleration. We refer to the microscopic benchmarking scenario presented in Section 12.3.1, and suppose that initially the leading vehicle is located at $x_{i-1}(0) = 5000$ m and the subject vehicle is at $x_i(0) = -102$ m and both vehicles are standing still. When the simulation begins, vehicle i starts to move according to the Pipes model. A spacing of $s_i(0) = 5102$ m results in a speed of about 3800 m/s at the next time step (assuming $\Delta t = 1$ s), which requires an acceleration of 3800 m/s^2. It follows that an infinite speed and acceleration would result if there is no leading vehicle in front. Therefore, the following external logic has to be imposed on the Pipes model in order to limit its maximum acceleration:

$$\ddot{x}_i(t) = \frac{\dot{x}_i(t + \Delta t) - \dot{x}_i(t)}{\Delta t} \leq A_i, \tag{13.18}$$

where A_i is the maximum acceleration of vehicle i—for example, $A_i = 4\,\text{m/s}^2$. With this addition, the Pipes model loses its mathematical elegance which favors a one-equation-for-all formulation. Even though an external logic is added, the Pipes model still has a problem with the maximum speed. For example, it is true that the acceleration no longer exceeds A_i, but the vehicle can still reach unrealistically high speeds—for example, $\dot{x}_i = 196\,\text{m/s}$ when $s_i = 590\,\text{m}$. Therefore, another external logic has to be imposed to limit the speed:

$$\dot{x}_i \leq v_i, \tag{13.19}$$

where v_i is driver i's desired speed. The third problem is unrealistic deceleration. For example, at time $t = 424$, vehicle i is located at about $x_i = 8734\,\text{m}$ moving at speed $\dot{x}_i = 30\,\text{m/s}$, while vehicle $i - 1$ stops at $x_{i-1} = 8762\,\text{m}$. According to the Pipes model, vehicle i's speed at the next step would be $\dot{x}_i \approx 16.42\,\text{m/s}$. As such, the deceleration rate is $\ddot{x}_i = -13.58\,\text{m/s}^2$. Hence, a third external logic has to be imposed to limit maximum deceleration B_i (e.g., $-6\,\text{m/s}^2$):

$$\ddot{x}_i(t) = \frac{\dot{x}_i(t + \Delta t) - \dot{x}_i(t)}{\Delta t} \geq B_i. \tag{13.20}$$

However, this addition introduces a new problem. For example, vehicle i's speed at the next step becomes $\dot{x}_i = 30 - 6 = 24\,\text{m/s}^2$ and its location is $x_i = 8758\,\text{m}$. This would leave a spacing of $s_i = 4\,\text{m}$, which is less than a vehicle length $l_{i-1} = 6\,\text{m}$—that is, vehicle i has collided with vehicle $i - 1$. Unfortunately, there is no easy remedy to the problem except for accepting the unrealistic deceleration behavior.

The benchmarking result of the Pipes model with the constraints in Equations 13.18-13.20 is plotted in Figure 13.2. The performance of the constrained Pipes model is summarized as follows, and the discussion is based on the benchmarking scenario:

- Start-up: the model is able to start the vehicle up from standstill. See Figure 13.2 when $t > 0\,\text{s}$.

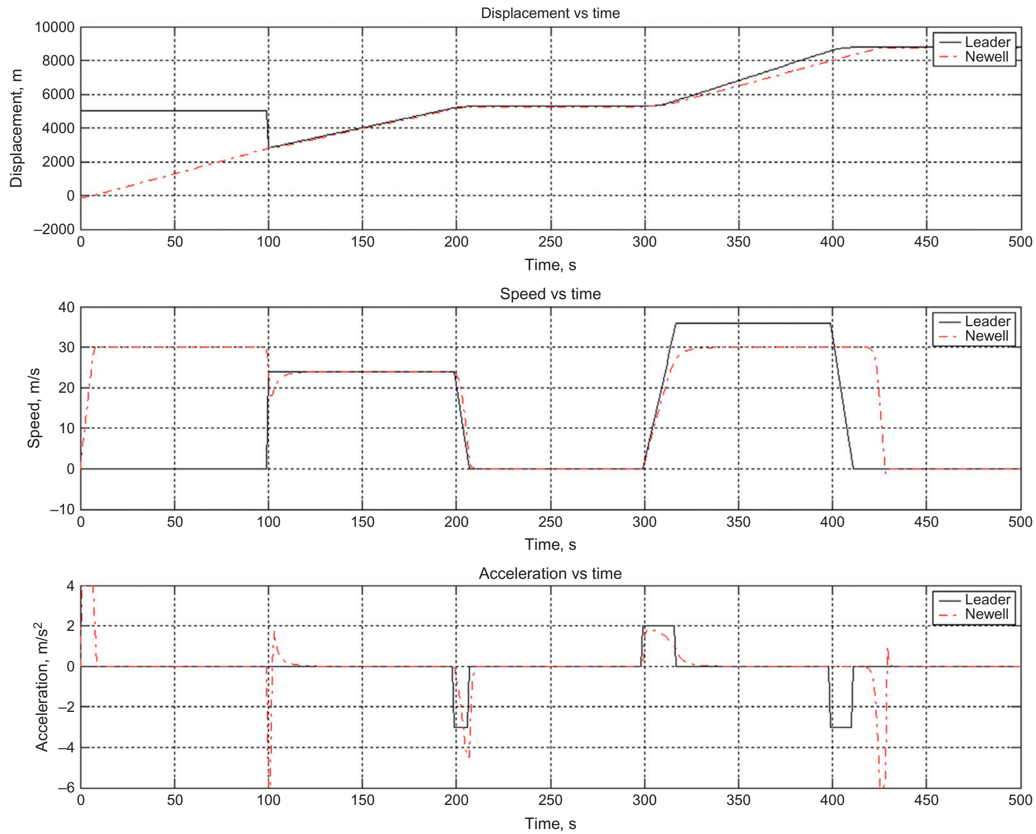

Figure 13.2 Microscopic benchmarking of the Pipes model.

Table 13.1 Microscopic benchmarking parameters of the Pipes model

l_i	v_i	τ_i	α	–
6 m	30 m/s	1.0 s	1.34	–
A_i	B_i	$x_i(0)$	$\dot{x}_i(0)$	$\ddot{x}_i(0)$
4.0 m/s^2	6.0 m/s^2	-120 m	0 m/s	0 m/s^2

- Speedup: the model is able to speed up the vehicle. However, its acceleration profile (i.e., acceleration as a function of speed) is unrealistic because the vehicle is able to retain maximum acceleration at high speeds. Normally, maximum acceleration is available only when a vehicle starts up. As the vehicle speeds up, acceleration decreases and eventually varnishes when the vehicle achieves its desired/cruising speed. See Figure 13.2 when $0\,\text{s} < t < 100\,\text{s}$.
- Free flow: an external logic has to be imposed to limit the maximum speed under the free-flow condition. See Figure 13.2 when $0\,\text{s} < t < 100\,\text{s}$.
- Cutoff: the model retains control and responds reasonably when a vehicle cuts in in front. See Figure 13.2 around $t = 100\,\text{s}$.
- Following: the model is able to adopt the leader's speed and follow the leader by a reasonable distance. See Figure 13.2 when $100\,\text{s} < t < 200\,\text{s}$.
- Stop and go: the model is able to stop the vehicle safely behind its leader and start it moving when the leader departs. See Figure 13.2 when $200\,\text{s} \geq t \leq 300\,\text{s}$.
- Trailing: the model is able to speed up normally without being tempted to speed up by its speeding leader. See Figure 13.2 when $300\,\text{s} < t < 400\,\text{s}$.
- Approaching: the model is unable to decelerate properly when approaching a stationary vehicle at a distance. The vehicle might collide with its leader when maximum deceleration is imposed. See Figure 13.2 when $400\,\text{s} \geq t < 420\,\text{s}$.
- Stopping: this portion is invalid since approaching fails. See Figure 13.2 when $t \geq 420\,\text{s}$.

The above benchmarking is based on the set of parameters in Table 13.1, and the outcome may differ for a different set of parameters.

13.3.2 Macroscopic Benchmarking

The fundamental diagram implied by the Pipes model is plotted in Figure 13.3 against empirical observations. The "cloud" contains 5 min observations of flow, speed, and density, the circles are empirical observa-

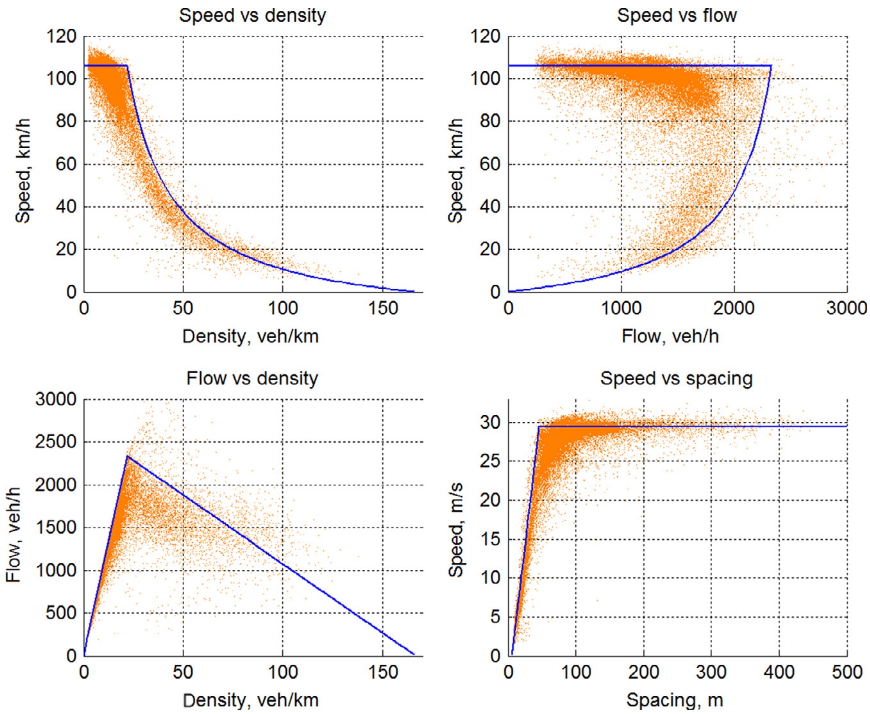

Figure 13.3 Fundamental diagram implied by the Pipes model.

tions aggregated with respect to density, and the curves are the equilibrium relationships implied by the Pipes model.

These curves roughly fit the empirical data in the middle to upper range of density (e.g., $k > 20$ vehicles per kilometer), but do not apply to the low-density range (e.g., $k < 20$ vehicles per kilometer). It appears that the Pipes model is designed to literally describe car-following behavior. Cases when the leading vehicle is absent have to be handled by an external logic. In addition, the Pipes model predicts that traffic speed would increase to infinity as density approaches zero.

The above benchmarking is based on the set of parameters in Table 13.2, and the outcome may differ for a different set of parameters.

Table 13.2 Macroscopic benchmarking parameters of the Pipes model

α (τ)	l
1.34 s	6 m

Since the Forbes model is essentially the same as the Pipes model, the above benchmarking results apply to the Forbes model as well.

PROBLEMS

1. Prove that the Pipes and Forbes models are mathematically equivalent.
2. From the perspective of the spatial domain, the Pipes model suggests that drivers add a space gap of at least one car length for every 10 miles per hour of speed at which they travel. From the perspective of the temporal domain, the Forbes model requires that drivers leave a time gap of at least one perception–reaction time.
   a. Since the two models are mathematically equivalent, what is the equivalent perception–reaction time that the Pipes model implies when the model is translated into the temporal domain? Assume a vehicle length of 6 m.
   b. Assume a perception–reaction time of 1.5 s and a vehicle length of 6 m. Translate the Forbes model into the spatial domain and elaborate the equivalent driving rule—for example, drivers need to add a space gap of at least one car length for every x miles per hour of speed at which they travel.
3. An autonomous cruise control system is designed as follows. At any moment t, the onboard sensor measures the distance from this vehicle to the vehicle in front. Then the target speed of the vehicle is set as the minimum of (1) the distance multiplied by 0.8 and (2) the desired speed of 108 km/h. Assume all vehicles are controlled by this logic and the vehicle length is uniformly 7.5 m. What is the maximum number of vehicles that can pass a point of highway in 1 h?
4. Perform the following analysis based on the Forbes model under the assumption that the desired speeds of all drivers are uniformly 108 km/h, perception–reaction times are uniformly 1.5 s, and vehicle lengths are uniformly 5 m.
   a. Find the capacity condition implied by the Forbes model (capacity, optimal speed, and optimal density).
   b. If the uniform desired speed drops to 96 km/h, how would your answer change?
   c. If the uniform perception–reaction time becomes 1 s but the uniform desired speed is held at 108 km/h, how would your answer change?
   d. Illustrate and indicate the direction of change graphically on the basis of the underlying flow–density relationship.

5. Perform a one-step simulation based on the following conditions: Two cars are traveling in the same lane on a freeway. The length of both vehicles is $l_{i-1} = l_i = 6\,\text{m}$. Lane change is not considered in this problem. At time t, the leading vehicle $i-1$ is traveling at a speed of $\dot{x}_{i-1}(t) = 72\,\text{km/h}$ and the following vehicle i is traveling at a speed of $\dot{x}_i(t) = 108\,\text{km/h}$. The spacing between the two vehicles (measured from front bumper to front bumper) is $s_i(t) = 40\,\text{m}$. The perception-reaction time of the following driver is $\tau_i = 1.5\,\text{s}$. Assume that the acceleration rate and deceleration rate are $\ddot{x}_i = 1\,\text{m/s}^2$.
   - **a.** Use the Pipes model to predict the speed that the following driver will adopt after a perception-reaction time.
   - **b.** Use the Forbes model to predict the speed that the following driver will adopt after a perception-reaction time.