CHAPTER 5

Conservation Law

In previous chapters, two types of relationships among traffic flow characteristics were discussed

1. The flow-speed-density relationship or the identity,

$$q = k \times v.$$

Note that (1) it is an identity—that is, it is self-guaranteed by the generalized definition of traffic flow characteristics; (2) it is location specific and time specific—$q(t,x) = k(t,x) \times v(t,x)$—that is, flow, speed, and density must refer to the same location and time.

2. Pairwise relationships or equilibrium models,

$$v = V(k),$$
$$q = Q(k),$$
$$v = U(q).$$

Note that (1) they define the fundamental diagram and hence differentiate vehicular traffic flow from other kinds of flows; (2) they are location specific—that is, different locations and roads may have different underlying fundamental diagrams; (3) they are equilibrium models—that is, they describe a steady-state behavior in the long run, and hence are not specific to a particular time; (4) such relationships are only of statistical significance—that is, the equal signs do not strictly hold in the real world. On the basis of points (2) and (3), these relationships may also be expressed as follows:

$$v(x) = V(k(x)),$$
$$q(x) = Q(k(x)),$$
$$v(x) = U(q(x)).$$

The main purpose of formulating a traffic flow theory is to help better understand traffic flow and, by the application of such knowledge, to control traffic for safer and more efficient operations. Hence, a good theory should be able to help answer the following questions:

Traffic Flow Theory
http://dx.doi.org/10.1016/B978-0-12-804134-5.00005-2
Copyright © 2016 Elsevier Inc.
All rights reserved.

- Given existing traffic conditions on a road and upstream arrivals in the near future, how do road traffic conditions change over time?
- Where are the bottlenecks, if any?
- In the case of congestion, how long does it last and how far do queues spill back?
- If an incident occurs, what is the best strategy for cleanup so that the impact on traffic is minimized?

Answers to these questions involve the analysis of dynamic change of traffic states over time and space. Unfortunately, the above relationships or models are capable only of describing traffic states. They do not provide a mechanism to analyze how such states evolve. Starting from this chapter, dynamic models will be introduced to address these questions.

The derivation of a dynamic equation starts with the examination of a small volume of roadway traffic as a continuum. Here traffic flow is treated as a one-dimensional compressible fluid like a gas. Conservation laws apply to this kind of fluid, and the first-order form of conservation is mass conservation, also known as the continuity equation.

5.1 THE CONTINUITY EQUATION

There are several ways to derive the continuity equation, each takes a different perspective on the small volume of roadway traffic (see Figure 5.1).

Derivation I: Finite Difference

The following derivation is found in Ref. [3]. Suppose a highway section is delineated by two observation stations at x_1 and x_2. Let $\Delta x = x_2 - x_1$ denote the section length. During time interval $\Delta t = t_2 - t_1$, N_1 vehicles passed x_1 and N_2 vehicles passed x_2. Therefore, the flow rates at these locations are

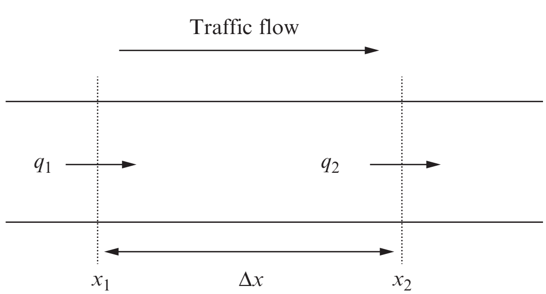


Figure 5.1 Deriving the continuity equation I.

$$q_1 = \frac{N_1}{\Delta t} \quad \text{and} \quad q_2 = \frac{N_2}{\Delta t}.$$

The change in the number of vehicles in the section is

$$\Delta N = N_2 - N_1 = (q_2 - q_1)\Delta t = \Delta q \Delta t.$$

Assume the traffic densities in the section at t_1 and t_2 are k_1 and k_2, respectively. Therefore, there are $M_1 = k_1 \Delta x$ vehicles in the section at time t_1 and $M_2 = k_2 \Delta x$ vehicles in the section at time t_2. Alternatively, the change in the number of vehicles in the section can be expressed as

$$\Delta M = k_1 \Delta x - k_2 \Delta x = (k_1 - k_2)\Delta x = -\Delta k \Delta x.$$

Since vehicles cannot be created or destroyed inside the section, the change in the number of vehicles should be the same in the same section during the same time interval. Therefore, $\Delta N = \Delta M$—that is,

$$\Delta q \Delta t = -\Delta k \Delta x,$$

$$\Delta q \Delta t + \Delta k \Delta x = 0.$$

Dividing both sides by $\Delta x \Delta t$, we get

$$\frac{\Delta q}{\Delta x} + \frac{\Delta k}{\Delta t} = 0.$$

If we let $\Delta x \to 0$ and $\Delta t \to 0$, the above difference equation becomes a partial differential equation:

$$\frac{\partial q}{\partial x} + \frac{\partial k}{\partial t} = 0.$$

The above equation can be abbreviated as

$$q_x + k_t = 0,$$

where $q_x = \frac{\partial q}{\partial x}$ and $k_t = \frac{\partial k}{\partial t}$.

Derivation II: Finite Difference

The derivation is basically the same as above, but is presented in a slightly different way. Figure 5.2 sketches a highway section $\Delta x = x_2 - x_1$ during time interval $\Delta t = t_2 - t_1$. At time t_1, there are N_1 vehicles in the section and at time t_2, there are N_2 vehicles in the section. During the period, traffic keeps flowing into the section at rate q_1 and flowing out at rate q_2. On the basis of vehicle conservation, the following relationship holds:

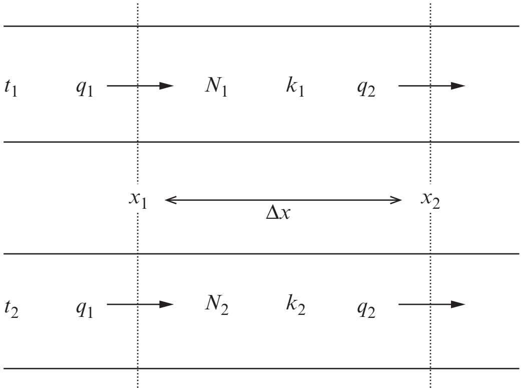


Figure 5.2 Deriving the continuity equation II.

Vehicles at t_2 = vehicles at t_1 + inflow during Δt − outflow during Δt. This is

$$N_2 = N_1 + q_1 \Delta t - q_2 \Delta t.$$

Note that $N = k\Delta x$, so the above becomes

$$k_2 \Delta x = k_1 \Delta x + q_1 \Delta t - q_2 \Delta t.$$

After arranging terms and dividing both sides by $\Delta x \Delta t$, we get

$$\frac{k_2 - k_1}{\Delta t} = -\frac{q_2 - q_1}{\Delta x}.$$

If we let $\Delta x \to 0$ and $\Delta t \to 0$,

$$q_x + k_t = 0$$

Derivation III: Fluid Dynamics

Figure 5.3 illustrates a small fluid cube of size $\delta x \times \delta y \times \delta z$. The fluid velocity v and density k at two sides of the cube also are shown.

The mass flow into the cube is $vk\delta y\delta z$. The mass flow out of the cube is as follows:

$$\begin{aligned}(v + \delta v)(k + \delta k)\delta y \delta z &= \left(v + \frac{\partial v}{\partial x}\delta x\right)\left(k + \frac{\partial k}{\partial x}\delta x\right)\delta y \delta z \\ &= \left(vk + v\frac{\partial k}{\partial x}\delta x + k\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial x}\frac{\partial k}{\partial x}\delta x \delta x\right)\delta y \delta z.\end{aligned}$$

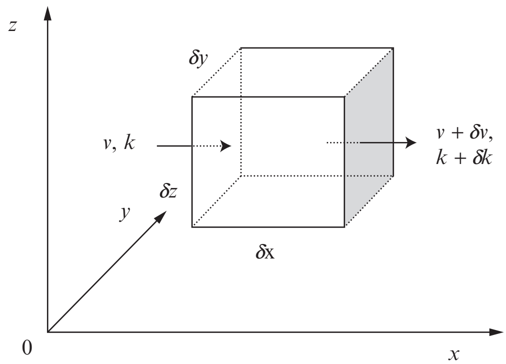


Figure 5.3 Deriving the continuity equation III.

The mass stored in the cube is equivalent to the mass that flows in minus mass that flows out:

$$\left(v\frac{\partial k}{\partial x}\delta x+k\frac{\partial v}{\partial x}\delta x+\frac{\partial v}{\partial x}\frac{\partial k}{\partial x}\delta x\delta x\right)\delta y\delta z=\left(v\frac{\partial k}{\partial x}+k\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial k}{\partial x}\delta x\right)\delta x\delta y\delta z.$$

If we ignore the higher-order term, we have

$$\left(v\frac{\partial k}{\partial x}+k\frac{\partial v}{\partial x}\right)\delta x\delta y\delta z=\frac{\partial(kv)}{\partial x}\delta x\delta y\delta z.$$

Similar treatment applies to the other two directions of the cube, so the total mass stored in the cube is

$$\left(\frac{\partial(kv)}{\partial x}+\frac{\partial(ku)}{\partial y}+\frac{\partial(kw)}{\partial z}\right)\delta x\delta y\delta z.$$

The mass stored in the cube must be balanced by the change of mass in the cube:

$$\frac{\partial k}{\partial t}\delta x\delta y\delta z.$$

The law of mass conservation requires that

$$\left(\frac{\partial(kv)}{\partial x}+\frac{\partial(ku)}{\partial y}+\frac{\partial(kw)}{\partial z}\right)\delta x\delta y\delta z+\frac{\partial k}{\partial t}\delta x\delta y\delta z=0.$$

Therefore,

$$\frac{\partial k}{\partial t}+\left(\frac{\partial(kv)}{\partial x}+\frac{\partial(ku)}{\partial y}+\frac{\partial(kw)}{\partial z}\right)=0.$$

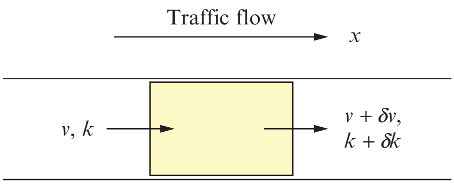


Figure 5.4 Reducing three dimensions to one dimension.

Highway traffic constitutes a special case of the above situation with only one dimension (see Figure 5.4). Using the result derived above, one obtains

$$\frac{\partial(kv)}{\partial x} + \frac{\partial k}{\partial t} = 0.$$

Note that $q = kv$. Therefore,

$$q_x + k_t = 0.$$

Derivation IV: Scalar Conservation Law

This derivation is adopted from [22]. Consider a cell in the time-space domain bounded by $(t_1, t_2) \times (x_1, x_2)$ (see Figure 5.5). Let traffic flow, speed, and density be functions of time and space—that is, $q = q(t, x)$, $v = v(t, x)$, and $k = k(t, x)$. Obviously, the conservation of vehicles in the cell requires the following:

$$\int_{x_1}^{x_2} k(t_2, x)\mathrm{d}x - \int_{x_1}^{x_2} k(t_1, x)\mathrm{d}x = \int_{t_1}^{t_2} q(t, x_1)\mathrm{d}t - \int_{t_1}^{t_2} q(t, x_2)\mathrm{d}t,$$

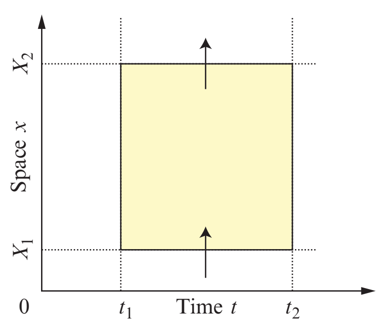


Figure 5.5 Deriving the continuity equation IV.

$$\int_{x_1}^{x_2} [k(t_2, x) - k(t_1, x)]\mathrm{d}x = \int_{t_1}^{t_2} [q(t, x_1) - q(t, x_2)]\mathrm{d}t.$$

If $k(t, x)$ and $q(t, x)$ are differentiable in x and t, one obtains

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \frac{\partial k(t, x)}{\partial t}\mathrm{d}t\mathrm{d}x = -\int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{\partial q(t, x)}{\partial x}\mathrm{d}x\mathrm{d}t,$$

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \left[\frac{\partial k(t, x)}{\partial t} + \frac{\partial q(t, x)}{\partial x}\right]\mathrm{d}x\mathrm{d}t = 0.$$

According to the fundamental theorem of calculus of variables, one obtains

$$\frac{\partial k(t, x)}{\partial t} + \frac{\partial q(t, x)}{\partial x} = 0;$$

that is,

$$q_x + k_t = 0.$$

Derivation V: Three-Dimensional Representation of Traffic Flow

As discussed in Chapter 3, the surface which represents the cumulative number of vehicles, N, can be expressed as a function of time t and space x—that is, $N = N(t, x)$. The density at time-space point (t, x) is the first partial derivative of $N(t, x)$ with respect to x, but takes a negative value:

$$k(t, x) = -\frac{\partial N(t, x)}{\partial x}.$$

The flow at (t, x) is the first partial derivative of $N(t, x)$ with respect to t:

$$q(t, x) = \frac{\partial N(t, x)}{\partial t}.$$

If both the flow and the density have first-order derivatives,

$$\frac{\partial q(t, x)}{\partial x} = \frac{\partial N(t, x)/\partial t}{\partial x} = \frac{\partial N^2(t, x)}{\partial x \partial t}$$

and

$$\frac{\partial k(t, x)}{\partial t} = \frac{-\partial N(t, x)/\partial x}{\partial t} = -\frac{\partial N^2(t, x)}{\partial x \partial t},$$

then

$$\frac{\partial q(t, x)}{\partial x} = -\frac{\partial k(t, x)}{\partial t};$$

that is,

$$q_x + k_t = 0.$$

5.2 FIRST-ORDER DYNAMIC MODEL

Traffic evolution is the process of how traffic states (e.g., flow q, speed v, and density k) evolve over time t and space x given some initial conditions (e.g., $k_0 = k(0, x)$)and boundary conditions (e.g., $q(t) = q(t, x_0)$). One recognizes that time t and space x are independent variables and traffic states are dependent variables—that is, they are functions of time and space ($q = q(t, x)$, $v = v(t, x)$, $k = k(t, x)$). The continuity equations derived above are able to dynamically relate the change of flow q_x to the change of density k_t:

$$q_x + k_t = 0.$$

This equation contains two unknown variables $q(t, x)$ and $k(t, x)$. Since the number of unknown variables is greater than the number of equations, the problem is underspecified. Because of this, another simultaneous equation is needed. Hopefully, the identity comes handy:

$$q(t, x) = k(t, x)v(t, x).$$

By adding a new equation, we introduce a third unknown variable—that is, speed $v(t, x)$. Therefore, a third simultaneous equation is called for. Unfortunately, we are running out of options now since we are unable to find a third governing equation that will definitely to hold for any time and space. Consequently, we have to accept the less-than-ideal option by looking at equilibrium traffic flow models (e.g., the Greenshields model), which are known to hold only statistically. Such a model takes the form of

$$v = V(k).$$

Putting everything together, one obtains a system of three equations involving three unknown variables:

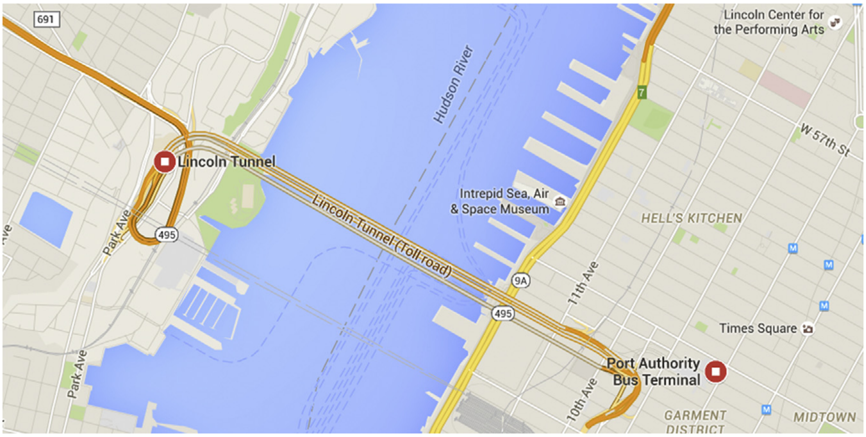
$$\begin{cases} q_x + k_t = 0, \\ q = kv, \\ v = V(k). \end{cases} \tag{5.1}$$

If initial and boundary conditions are provided, the above system of equations may be solvable. If that is the case, one is able to determine the traffic state at an arbitrary time-space point (t, x)—that is, $q(t, x)$, $v(t, x)$, and $k(t, x)$. With such information, one is able to answer the questions posed at the beginning of this chapter.

However, solving such a system of equations is not easy. To make this book self-contained, the following three chapters are designed to help readers ramp up their mathematical knowledge in terms of addressing partial differential equations.

PROBLEMS

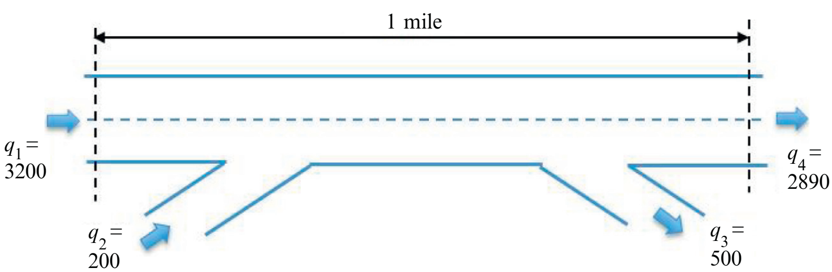
1. The Lincoln Tunnel is an integral conduit within the New York Metropolitan Area (see the figure below). The tunnel is approximately 1.5 miles (2.4 km) long and consists of three tunnels (north, center, and south) under the Hudson River. A civil engineering consulting firm was contracted to carry out a traffic engineering study on the tunnel, and automatic data collection devices were set up at both ends of the tunnel. The following data were recorded for the south tunnel (number of passenger cars in one direction over two lanes).



Find the level of service in each hour with use of the criteria specified by the Transportation Research Board

2. A freeway junction includes an on-ramp followed by an off-ramp over a section of 1 mile. At 10:00 a.m., there are 20 vehicles in this section. Assume traffic flows in and out at a rate in vehicles per hour as indicated in the figure below, and calculate the number of vehicles in this section 2 h later.

Time	No. of vehicles that entered	No. of vehicles that exited
00:00	0	0
01:00	90	80
02:00	400	390
03:00	900	874
04:00	1860	1870
05:00	2060	2028
06:00	2200	2210
07:00	3000	2978
08:00	4060	4026
09:00	4200	4154
10:00	3207	3223
11:00	3386	3424
12:00	2810	2832
13:00	3019	3029
14:00	3880	3838
15:00	3665	3637
16:00	4020	3980
17:00	4600	4634
18:00	4282	4316
19:00	3740	3772
20:00	3120	3138
21:00	1680	1706
22:00	408	438
23:00	0	10



3. An accident on Interstate 91 occurred at 8:00 a.m. which blocked one lane, resulting in the remaining lane being capable of discharging traffic at a rate of only 1800 vehicles per hour. Assume that there was no initial queue at the accident location and that traffic keeps arriving from the upstream mainline at a rate of 2600 vehicles per hour and from an on-ramp at a rate of 400 vehicles per hour. The on-ramp is 2 miles from the accident location. Also assume that vehicles maintain a spacing (i.e., front bumper-to-front bumper distance) of 29.3 feet when they are in a queue. Massachusetts Department of Transportation's goal is to avoid vehicles backing up to the on-ramp. Otherwise, the queue may spill over onto location intersections via the on-ramp, further worsening the situation. Calculate when vehicles will back up to the on-ramp so that the Massachusetts Department of Transportation has a sense of urgency in dispatching a rescue team to clean up the accident.