Homework: Numerical Wave Solutions

9.2, 9.3

Hayden Atchley

1 October 2023

9.2

Figure 1 depicts a roadway segment divided into 3 200-meter links ($x_1,\ x_2,\ x_3$). The storage on each link at time step t_1 are 6, 4, and 5 vehicles respectively. In the next time step t_2 (with a step size of $\Delta t = 5$ seconds), 2 vehicles move from link x_1 to x_2, and 2 vehicles move from x_2 to x_3. Additionally, 1 vehicle enters link x_2 via an on-ramp and 2 vehicles exit x_3 via an off-ramp.

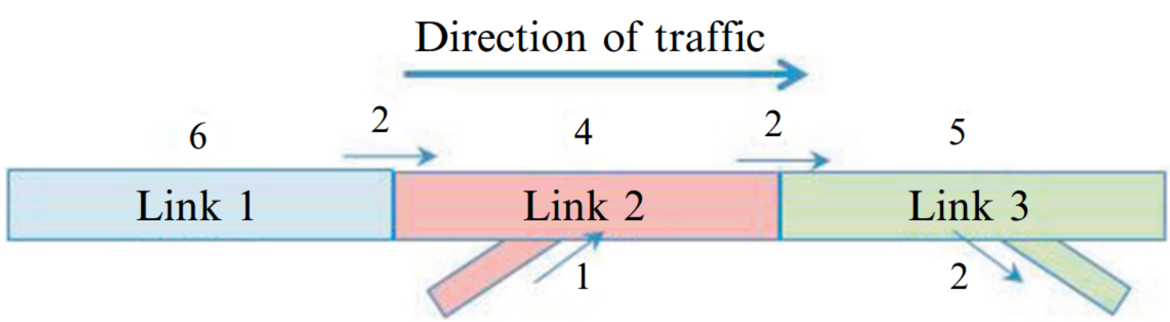

Figure 1: Depiction of road segment with volumes and vehicle movements.

The KRONOS model gives the vehicle storage on each link $n(t_i, x_j)$ for each time step t_i and link x_j as

$$\begin{aligned} n(t_i, x_j) = {} & \frac{n(t_{i-1}, x_{j+1}) + n(t_{i-1}, x_{j-1})}{2} \\ & - \frac{\Delta t q(t_{i-1}, x_{j+1}) + \Delta t q(t_{i-1}, x_{j-1})}{2} \\ & + \frac{\Delta t g(t_{i-1}, x_{j+1}) + \Delta t g(t_{i-1}, x_{j-1})}{2}, \end{aligned}$$

where $q(t_i, x_j)$ is the flow rate and $g(t_i, x_j)$ is the difference in on- and off-ramp flow on link x_j at time t_i.

In our case, the vehicle movement counts are given directly, so the storage on link x_2 at time t_2 ($n(t_2, x_2)$) is:

$$n(t_2, x_2) = \frac{6+5}{2} - \frac{-2+2}{2} + \frac{-2+0}{2},$$

which gives (a) $n(t_2, x_2) = 4.5$ vehicles. The density in link 2 at this point is therefore (b) $k = \frac{n}{\Delta x} = 22.5$ vehicles per kilometer.

The speed on this link is given by a Greenshields model where free-flow speed $v_f = 96$ kph and jam density $K = 120$ vehicles per kilometer, i.e.:

$$\begin{aligned} v(k) &= v_f \left(1 - \frac{k}{K}\right) \\ &= 96 \left(1 - \frac{k}{120}\right). \end{aligned}$$

For a density of 22.5 veh/km, this gives an equilibrium speed of (c) 78 kph. Since $q = k \times v$, this gives a flow of (d) 1755 vph.

9.3

Figure 2 shows a highway segment with 3 150-meter links and their associated storage in time step t_1. In the next time step ($\Delta t = 5$s) 2 vehicles move from link x_2 to x_3. Additionally, Figure 3 shows the flow-density relationship of the segment which applies to each link. This relationship is given by the equation

$$q(k) = \min\left\{w_f \times k, q_m, (K - k) \times w_b\right\}, \tag{1}$$

where q_m, w_f, w_b, and K are given as in Figure 3.

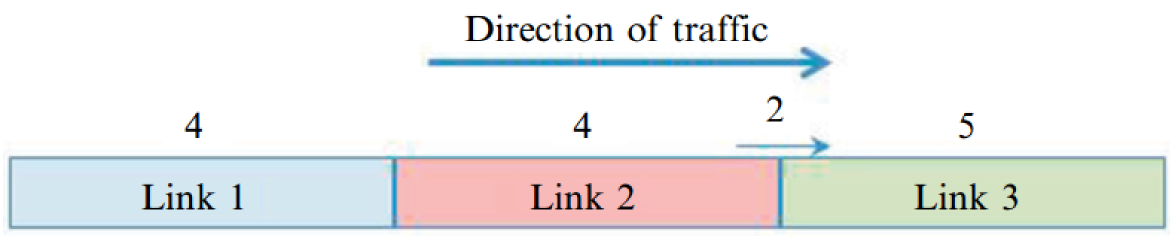

Figure 2: Depiction of highway segment with link storage and movements.

The Cell Transmission Model gives the following equation for storage in a cell:

$$n_j(t_i) = n_j(t_{i-1}) + \gamma_j(t_i) - \gamma_{j+1}(t_i),$$

where $n_j(t_i)$ is the number of vehicles on link x_j at time t_i, and

$$\gamma_j(t_i) = \min\left\{n_{j-1}(t_{i-1}), q_m \Delta t, \frac{w_b}{w_f}(K\Delta x - n_j(t_i - 1))\right\},$$

where q_m, w_f, and w_b are given as shown in Figure 3.

Because $\gamma_{j+1}(t_i)$ is the flow out of x_j and into x_{j+1} for time step t_i, and 2 vehicles travel from x_2 to x_3 in time step t_2, $\gamma_3(t_2)$ must equal 2. Therefore,

$$\begin{aligned} & n_2(t_2) = n_2(t_1) + \gamma_2(t_2) - 2 \\ & n_2(t_2) = 4 + 4 - 2 \\ \text{(a)} \qquad & n_2(t_2) = 6. \end{aligned}$$

The density $k_2(t_2)$ is then (b) $k_2(t_2) = \frac{n_2(t_2)}{\Delta x} = 40$ vehicles per kilometer. From Equation 1 the flow is (d) $q_2(t_2) = 1960$ vph. The speed on this link is then:

$$\text{(c)} \qquad v_2(t_2) = \frac{q_2(t_2)}{k_2(t_2)} = \frac{1960}{40} = 49\,\text{kph}.$$

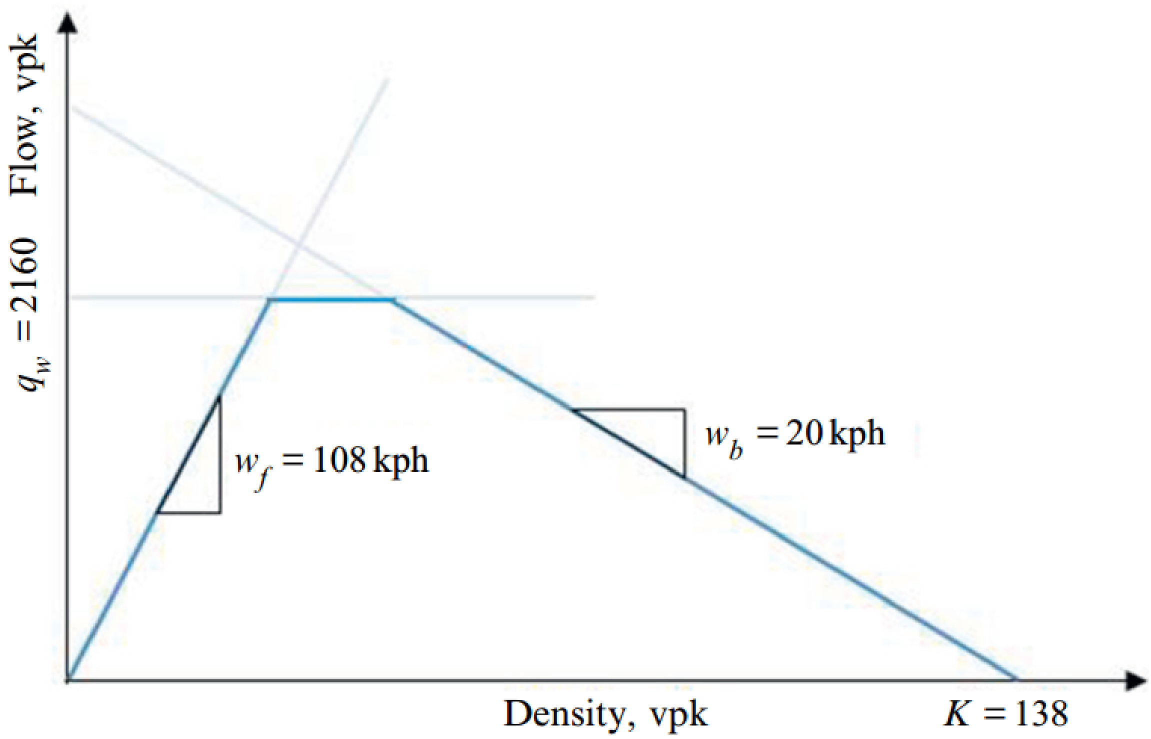

Figure 3: Flow-density relationship of links in Figure 2.