CHAPTER 3

Traffic Flow Characteristics II

3.1 GENERALIZED DEFINITION

In previous chapters, flow q and time mean speed v_t were defined with the help of Figure 2.3 on the basis of point sensor data:

$$q = \frac{N}{T},$$

$$v_t = \frac{1}{N}\sum_{i=1}^{N}\dot{x}_i.$$

Similarly, density k and space mean speed v_s were defined with the help of Figure 2.5 on the basis of space sensor data:

$$k = \frac{N}{L},$$

$$v_s = \frac{1}{N}\sum_{i=1}^{N}\dot{x}_i.$$

However, there is no common ground between the two sets of definitions, and the issue becomes more evident in Figure 2.6, where both cases are illustrated in the same figure. For example, the total number of vehicles N in the point sensor case is not necessarily the same as the total number N in the space sensor case. Similarly, vehicle speeds $\dot{x}_i$ are not necessarily the same in both cases. The two sets of data are simply independent, though we adopted the same notation in both sets of definitions. As such, one is unable to conclude that the identity

$$q = k \times v_s$$

is guaranteed by definition. Therefore, the key to addressing this issue is to provide a common ground such that both sets of definition can be related in a single setting. For the convenience of further discussion, the above definition of flow, density, and mean speeds is referred to as *the Highway Capacity Manual (HCM) definition* hereafter since the definition is formally given in the HCM.

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To find the common ground, let us rearrange the definition above as follows:

$$q = \frac{N}{T} = \frac{N \times \mathrm{d}x}{T \times \mathrm{d}x},$$

where dx denotes an infinitesimal distance (see Figure 3.1). If one ignores the slight error introduced by (possibly) incomplete trajectories of the first and last few vehicles, the physical meaning of the numerator is the sum of the distances traversed by all vehicles in area A during time period T:

$$\mathrm{d}(A) = N \times \mathrm{d}x = \sum_{i=1}^{N} \Delta x_i.$$

The denominator simply means the area of the time-space rectangle A bounded by T and dx, $|A|$. Hence, the definition of q can alternatively be expressed as the total distance traversed by all vehicles within A divided by the area of A:

$$q = \frac{\mathrm{d}(A)}{|A|}.$$

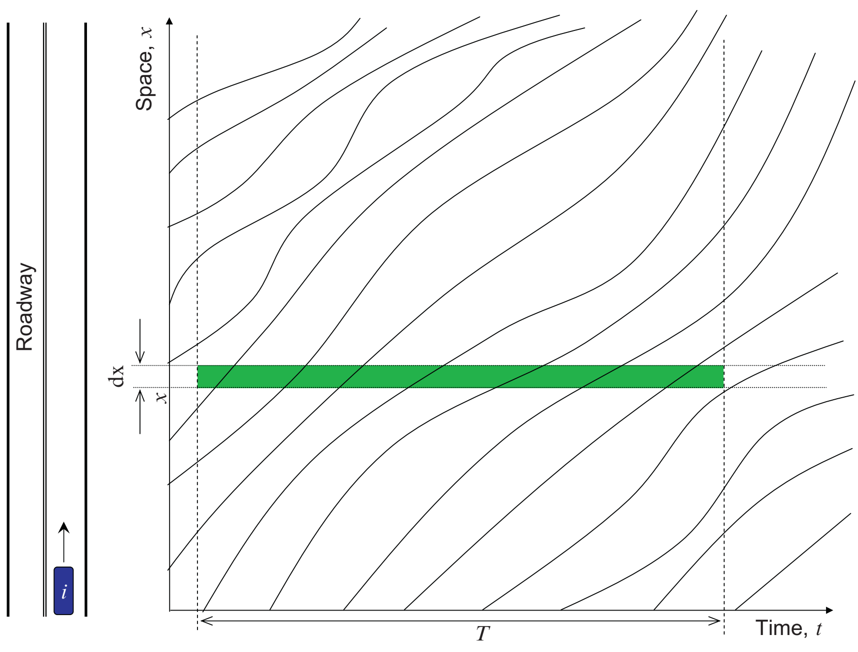


Figure 3.1 Time-space diagram with infinitesimal distance.

By definition in the HCM, the mean speed of vehicles, v, is the total distance traveled by all vehicles divided by the total travel time of these vehicles. The total distance traveled by all vehicles within rectangle A is $d(A) = N \times dx$. The total time spent by all vehicles within A is

$$t(A) = \sum_{i=1}^{N} \frac{dx}{\dot{x}_i}.$$

Therefore,

$$v = \frac{d(A)}{t(A)} = \frac{N \times dx}{\sum_{i=1}^{N} \frac{dx}{\dot{x}_i}} = \frac{N \times dx}{dx \times \sum_{i=1}^{N} \frac{1}{\dot{x}_i}} = \frac{1}{\frac{1}{N}\sum_{i=1}^{N} \frac{1}{\dot{x}_i}}.$$

This is the harmonic mean, which corresponds to the space mean speed presented in the point sensor scenario.

Similarly, density k can be represented as

$$k = \frac{N}{L} = \frac{N \times dt}{L \times dt},$$

where dt denotes an infinitesimal duration (see Figure 3.2). Following the same argument as above, L and dt define a time-space rectangle A. The numerator is the sum of the times spent by all vehicles within A, $t(A)$, and the denominator is the area of the rectangle, $|A|$:

$$t(A) = N \times dt = \sum_{i=1}^{N} dt_i,$$

$$|A| = L \times dt.$$

Hence,

$$k = \frac{t(A)}{|A|}.$$

The total distance traveled by all vehicles within A is $d(A) = \sum_{i=1}^{N} dt \times \dot{x}_i$. Hence, the mean speed of these vehicles is

$$v = \frac{\sum_{i=1}^{N} dt \times v_i}{N \times dt} = \frac{1}{N}\sum_{i=1}^{N} \dot{x}_i.$$

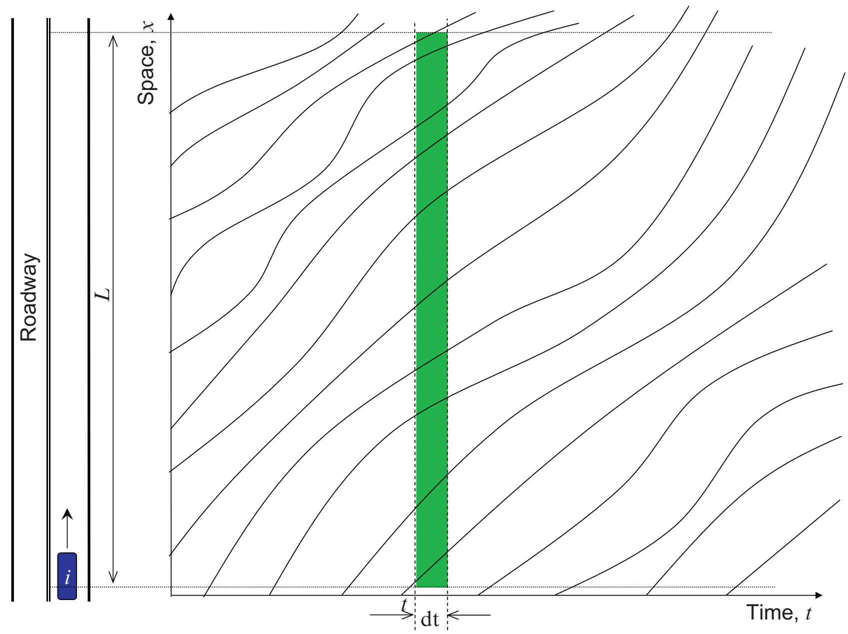


Figure 3.2 Time-space diagram with infinitesimal duration.

This is the arithmetic mean, which corresponds to the space mean speed determined in the space sensor scenario.

The above discussion suggests that a time-space rectangle may serve as the common ground to unify the definition of flow q, mean speed v, and density k. Figure 3.3 illustrates a general time-space rectangle A covering length L (bounded by upstream location x_{lo} and downstream location x_{hi}) and duration T (bounded by instants t_{lo} and t_{hi}). On the basis of A, the three traffic flow characteristics can be defined as follows:

$$q(A) = \frac{\mathrm{d}(A)}{|A|},$$

$$k(A) = \frac{t(A)}{|A|},$$

$$v(A) = \frac{\mathrm{d}(A)}{t(A)}.$$

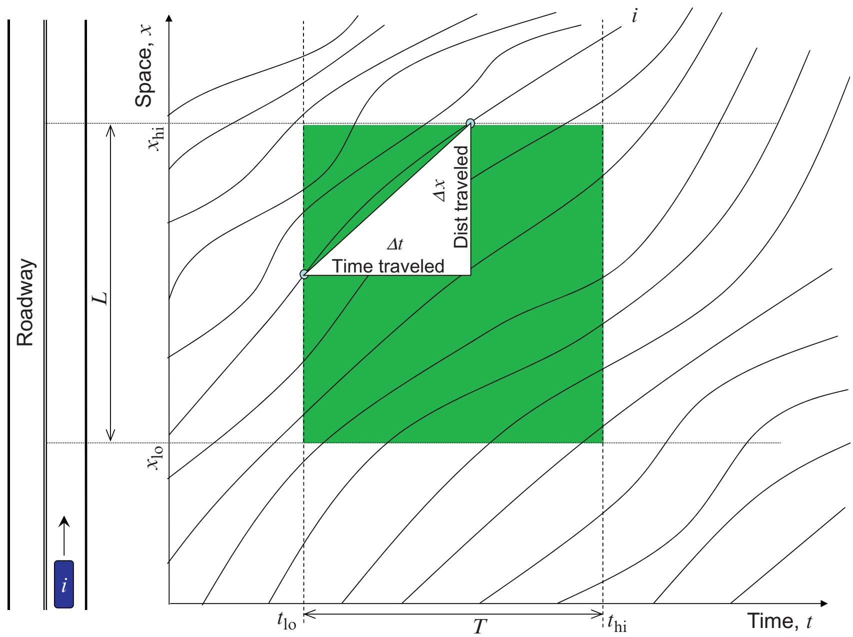


Figure 3.3 Time-space diagram with rectangle.

Therefore, the identity $q = k \times v$ is now guaranteed by definition. The question that remains is how to determine $\mathrm{d}(A)$, $t(A)$, and $|A|$. Take an arbitrary vehicle i. For example, the vehicle enters A at location $x_i(t_{lo})$ or x_{lo} whichever comes later and at time t_{lo} or $t_i(x_{lo})$ whichever comes later; the vehicle exits A at location x_{hi} or $x_i(t_{hi})$ whichever comes earlier and at time t_{hi} or $t_i(x_{hi})$ whichever comes earlier. Hence, the distance traveled by vehicle i in A is

$$\Delta x_i = \min(x_{hi}, x_i(t_{hi})) - \max(x_i(t_{lo}), x_{lo}).$$

Therefore, the total distance traveled by all vehicles in A is

$$\mathrm{d}(A) = \sum_{i=1}^{N} \Delta x_i.$$

The time spent by vehicle i in A is

$$\Delta t_i = \min(t_{hi}, t_i(x_{hi})) - \max(t_{lo}, t_i(x_{lo})).$$

Hence, the total time spent by all vehicles in A is

$$t(A) = \sum_{i=1}^{N} \Delta t_i.$$

The area of A is simply

$$|A| = L \times T.$$

Therefore, all the quantities needed to calculate the flow, mean speed, and density have been determined.

The following question naturally arises: Does the common ground have to be a rectangle? The answer is no. Actually, any time-space region will work as long as the region is closed (see Figure 3.4). The above definition was originally proposed by Edie [6]. Readers are referred to the original paper for an in-depth discussion. For convenience, the above set of definitions of flow, mean speed, and density based on a time-space region is referred to as **the generalized definition**.

It can be seen that the HCM definition is a special case of the generalized definition. For example, if one takes a time-space region like the one in Figure 3.1 and allows d$x \to 0$, a point sensor scenario results, while one obtains the space sensor scenario in Figure 3.2 if one keeps L constant and makes d$t \to 0$.

Figure 3.4 Time-space diagram with general region.

3.2 THREE-DIMENSIONAL REPRESENTATION OF TRAFFIC FLOW

The following discussion is based on Ref. [7]. Interested readers are referred to the original paper for in-depth information.

So far, we have been working with a time-space diagram and vehicle trajectories, on the basis of which a connection is made to traffic flow characteristics. A time-space diagram is a two-dimensional representation, and the discussion can be made more informative if we adopt a three-dimensional perspective. Taking the family of vehicle trajectories in Figure 3.3, for example, we see these trajectories lie on the same plane defined by time (t) and space (x). These vehicles are numbered cumulatively (i.e., ID = $1, 2, 3, \ldots$) in the order they appear on the road, and each vehicle is elevated along the third dimension to the height corresponding to the vehicle's ID (i.e., vehicle 1 raised to height 1, vehicle 2 raised to 2, and so on). Let us call the third dimension the cumulative number of vehicles (N) and denote the surface that passes these elevated vehicle trajectories $N(x, t)$. Figure 3.5 illustrates two examples of such a three-dimensional representation adopted from Ref. [7, 8].[1,2]

What makes this three-dimensional representation interesting is that it can be used to illustrate and relate some key concepts of traffic flow conveniently. For example, if one cuts $N(x, t)$ in the lower part of Figure 3.5 using plane $t = 20$, one obtains the shape PQN and its projection P′Q′N′ on the $N - x$ plane. Curve P′N′ can be interpreted as the snapshot taken at time $t = 20$, which shows the location of each vehicle at this moment. Figure 3.6 illustrates more examples of such curves (they look like stairs before smoothing),where $N(x, t)$ in the upper part of Figure 3.5 is cut at different instants and projected it onto the N-x plane. Each curve represents a snapshot taken at the time instant indicated on that curve. For example, the lowest curve is a snapshot taken at time $t^{(1)}$. If one draws a horizontal line at height $N = 2$, the intersection of this line and the curve labeled $t^{(1)}$ is the location of vehicle with ID 2 at time $t^{(1)}$. Note that this line needs to be slightly lower—say, at height $N = 1.999$—to avoid multiple intersections, and the same applies hereafter. Similarly, the intersection of line $N = 2$ and curve $t^{(2)}$ is the location of vehicle 2 at time $t^{(2)}$. The distance between

[1] $N(x, t)$ in the upper part of Figure 3.5 is not smoothed, while that in the lower part is smoothed. By default, a smoothed surface is assumed in order to take derivatives.

[2] If two trajectories intersect, the surface will be multivalued at a time-space point. Makigami et al. [7] showed how to resolve the problem.

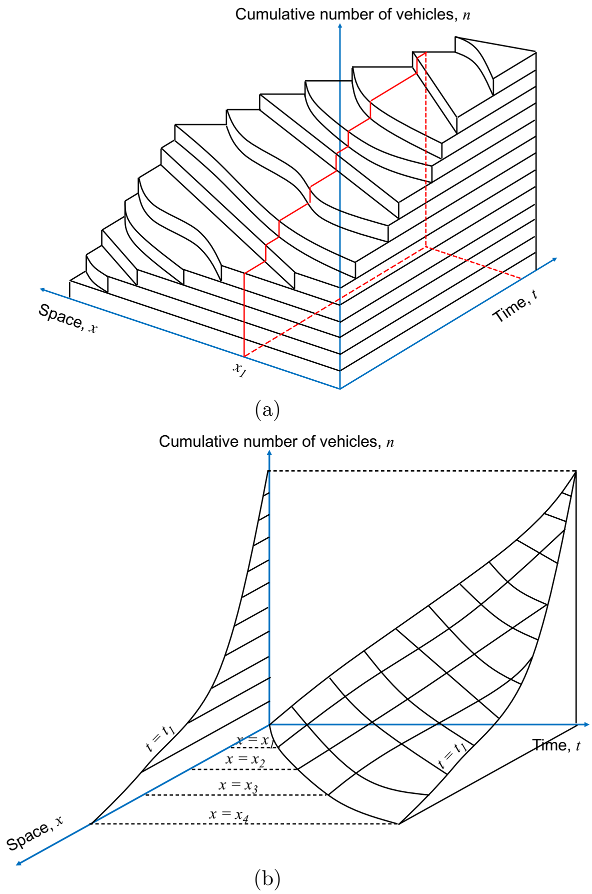


Figure 3.5 Three-dimensional representation examples.

the two intersections is the distance traversed by vehicle 2 from $t^{(1)}$ to $t^{(2)}$. If an N-x curve at time t is smoothed (like curve P′N′ in Figure 3.5), the tangent of the curve denotes the **density** k at this instant. Note that the tangent slants down (because lower-numbered vehicles are in front), so it has a negative value. Hence,

$$k|_t = -\left.\frac{\mathrm{d}N}{\mathrm{d}x}\right|_t .$$

Similarly, if one cuts the three-dimensional model with a plane passing a specific location and parallel to the N-t plane, one obtains a curve representing the cumulative number of vehicles passing this location over

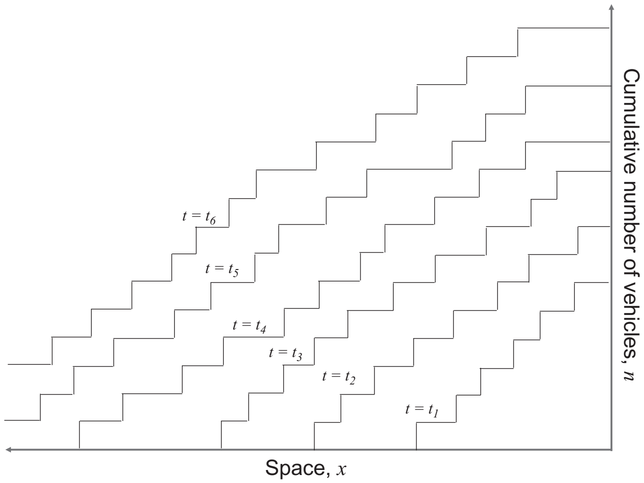


Figure 3.6 The *N*-*x* diagram.

time—for example, the curves in the lower part of Figure 3.5, such as the $x = 0, 2, 4, 6$ curves, and the curves in Figure 3.7. If one draws a horizontal line at height $N = 2$ in Figure 3.7, the intersection of this line and the curve labeled $x^{(2)}$ indicates the time when the vehicle with ID 2 passes location $x^{(2)}$. Similarly, the intersection of line $N = 2$ and curve $x^{(3)}$ is the time when vehicle 2 passes location $x^{(3)}$. The distance between the two intersections is the travel time for vehicle 2 to traverse from location $x^{(2)}$ to location $x^{(3)}$. If an N-t curve at location x is smoothed, the tangent of this curve denotes the **flow** q at this location:

$$q|_x = \frac{\mathrm{d}N}{\mathrm{d}t}|_x.$$

Therefore, flow and density can be expressed as partial differentials of the surface $N(x, t)$:

$$q = \frac{\partial N(x, t)}{\partial t},$$

$$k = -\frac{\partial N(x, t)}{\partial x}.$$

In addition, if one projects a region on the surface $N(x, t)$ (e.g., region A in Figure 3.8) onto the x-t, N-t, and N-x planes, one obtains three projections—A_N, A_x, and A_t, respectively. Makigami et al. [7] demonstrated

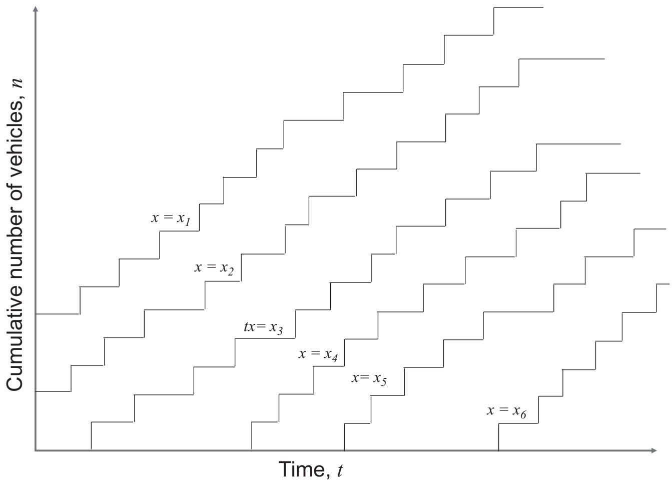


Figure 3.7 The *N*-*t* diagram.

that the following relationships hold:

$$A^2 = A_N^2 + A_x^2, + A_t^2,$$

$$q = \frac{A_t}{A_N},$$

$$k = \frac{A_x}{A_N},$$

$$v = \frac{A_t}{A_x}.$$

Figure 3.9 summarizes the previous graphics in one figure. Plot A shows vehicle trajectories in the x–t plane. Plot D raises the vehicle trajectories to their corresponding height and forms the three-dimensional surface $N(x, t)$. Plot B shows two N–t curves observed at locations $x = x_2$ and $x = x_4$. Plot C depicts two N–x curves resulting from snapshots taken at $t = t_6$ and $t = t_8$.

In addition to deepening the understanding of traffic flow and its characteristics, the three-dimensional model can be used to solve practical problems. For example, as mentioned before, space-based measures such as density and space mean speed are desired. In addition, determination of these traffic flow characteristics based on generalized definition (as opposed

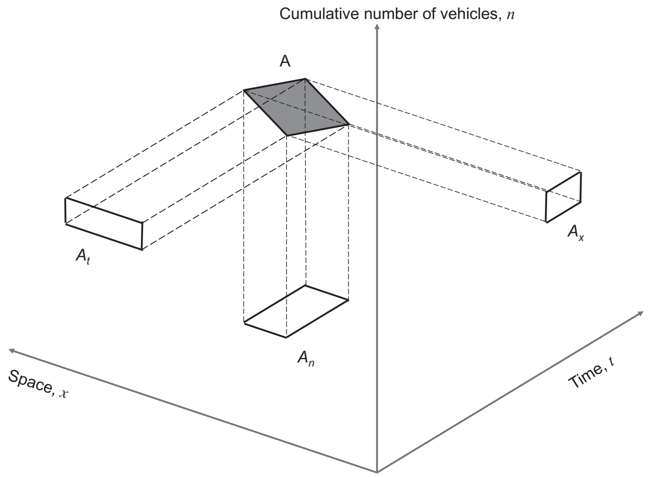


Figure 3.8 Projection of an *N*-*t*-*x* region.

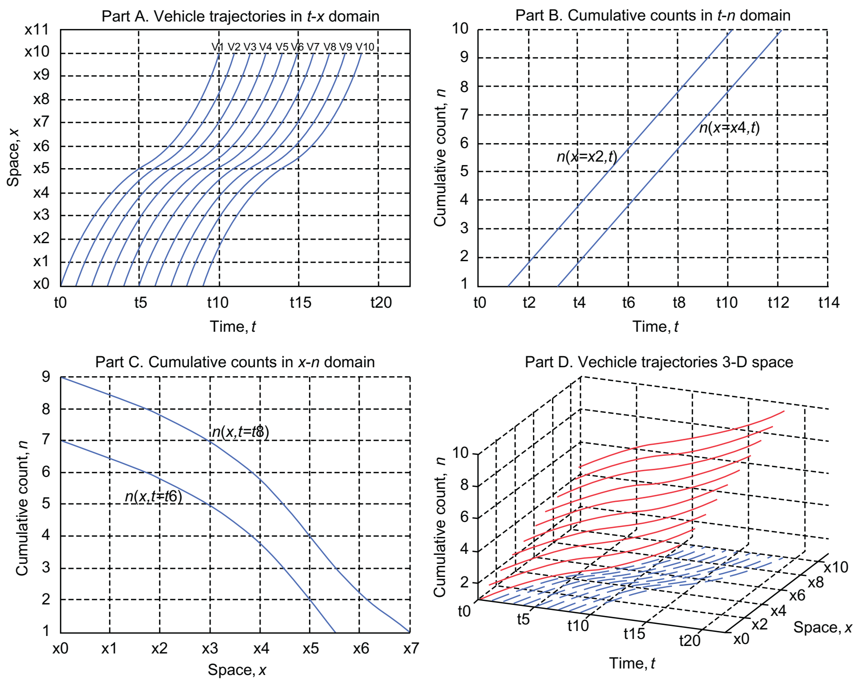
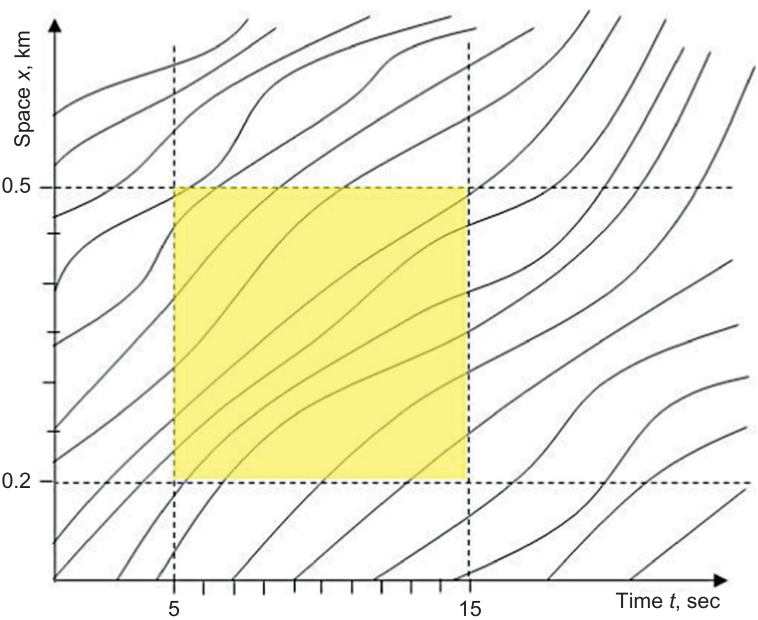


Figure 3.9 Three-dimensional representation of traffic flow.

to the HCM definition) is also preferred. However, the widely deployed intelligent transportation systems consist mostly of point sensors, which are generally unable to report space-based traffic flow characteristics. Interested readers are referred to [4] to learn how the three-dimensional representation helps address the problem by computing the desired traffic flow characteristics based on the preferred definition from intelligent transportation system data.

PROBLEMS

1. The figure below shows a set of vehicle trajectories in the time-space plane. An analysis of traffic flow characteristics is performed on the basis of a bounding box. As shaded in the figure, the box is confined in time between $t = 5\,\mathrm{s}$ and $t = 15\,\mathrm{s}$ and in space between $x = 0.2\,\mathrm{km}$ and $x = 0.5\,\mathrm{km}$.



   a. Identify vehicles that traverse the bounding box.
   b. Transcribe the time and location when each of the above-mentioned vehicles enter and exit the bounding box.

c. Find the duration and distance that each vehicle traveled in the bounding box.

d. Find the travel speed for each vehicle when it traveled in the bounding box.

e. Find the average speed of all vehicles that traveled in the bounding box.

f. With use of the generalized definition, find the flow, space mean speed, and density based on conditions in the bounding box.

g. State the general relationship among flow, speed, and density. Then verify the relationship using the result in (f). Does this relationship hold? Does it hold strictly? Is it guaranteed by definition?

h. Now, focus on observations made at location $x = 0.2\,\text{km}$ during time $t = 5\,\text{s}$ to $t = 15\,\text{s}$. Find number of vehicles passing this location during this time period. Calculate the equivalent hourly flow rate.

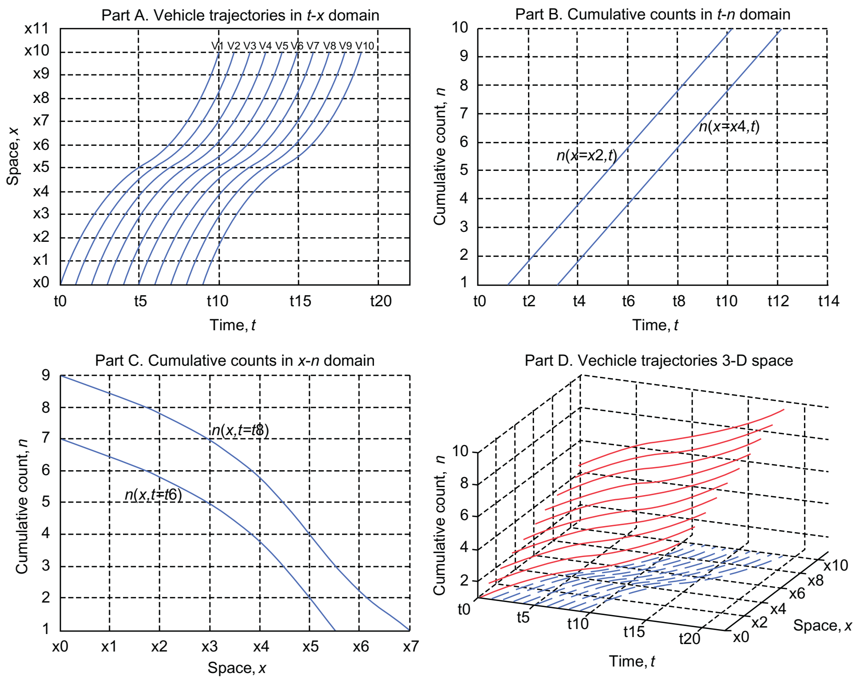
i. Next, focus on observations made at instant $t = 5\,\text{s}$ on a road section between $x = 0.2\,\text{km}$ and $x = 0.5\,\text{km}$. Find the number of vehicles within this section of road at that moment. Calculate the equivalent density.

j. With the average speed found in (e), flow in (h), and density in (i), another set of speed, flow, and density results. Redo (g) and comment on your findings.

2. Part D in the figure below illustrates a stream of traffic in a three-dimensional representation which is then projected onto three planes: the time-space plane (t–x plane), the cumulative number-time plane (n–t plane), and the cumulative number-space plane (n–x plane). Time is labeled in minutes—for example, "t5" means $t = 5\,\text{min}$—and space is labeled in kilometers—for example "x2" means $x = 2\,\text{km}$.

a. Redo the set of problems in 1 based on bounding box with time between $t = 5$ and $t = 10\,\text{min}$ and space between $x = 2\,\text{km}$ and $x = 4\,\text{km}$, as shaded. When you answer question (h), assume the location is $x = 2\,\text{km}$ during time $t = 5$ to $t = 10\,\text{min}$. When you answer (i), assume the instant is $t = 5\,\text{min}$ on the road section between $x = 2\,\text{km}$ and $x = 4\,\text{km}$.

b. With use of part B in the figure, find the instantaneous flow observed at time $t = 6\,\text{min}$ and location $x = 2\,\text{km}$.



c. With use of part C in the figure, find the instantaneous density observed at time $t = 6$ min and location $x = 2$ km.

d. With use of the space mean speed found in (a), flow found in (b), and density found in (c), verify the fundamental relationship among flow, space mean speed, and density. Comment on your result.