CHAPTER 7

Shock and Rarefaction Waves

In the previous chapter, the method of characteristics was discussed as a means to solve the continuity equation (i.e., conservation law) with an initial condition:

$$\begin{cases} k_t + q_x = 0, \\ k(0,x) = k_0(x), \\ -\infty < x < \infty,\ 0 < t, \end{cases}$$

where $q = Q(k)$ is a function of k. To be consistent with the notation in the previous chapter, the following connection needs to be made:

$$q_x = \frac{\partial q}{\partial x} = \frac{\partial Q(k)}{\partial x} = \frac{dQ}{dk}\frac{\partial k}{\partial x} = Q'(k)k_x = ck_x$$

To find the solution of k at an arbitrary time-space point (t^*, x^*), $k(t^*, x^*)$, one simply constructs a characteristic $x = ct + x_0$ which starts from (t^*, x^*) and extends back to the x-axis at intercept $(0, x^* - ct^*)$. Since $k((0, x^* - ct^*)) = k_0(x^* - ct^*)$ is given in the initial condition and k remains constant along the characteristic, the solution is

$$k(t^*, x^*) = k_0(x^* - ct^*).$$

7.1 GRADIENT CATASTROPHES

In the above discussion, if c is a constant, characteristics drawn from two different time-space points are straight, parallel lines. Hence, any time-space point lies on one and only one characteristic, and the solution at this point is single valued. However, if $c = c(k)$ is a function of k and not explicitly dependent on x or t, two different characteristics drawn from two time-space points are still straight lines but they may not necessarily be parallel, in which case they may intersect and the solution at this intersection may be multivalued. For example, Figure 7.1 illustrates two such characteristics A_0A_4 and B_0B_4. As the two characteristics become closer and closer, the gradient (i.e., slope) of the solution profile (represented by the red curves 0, 1, 2, 3, and 4 above the two characteristics) becomes increasingly steep.

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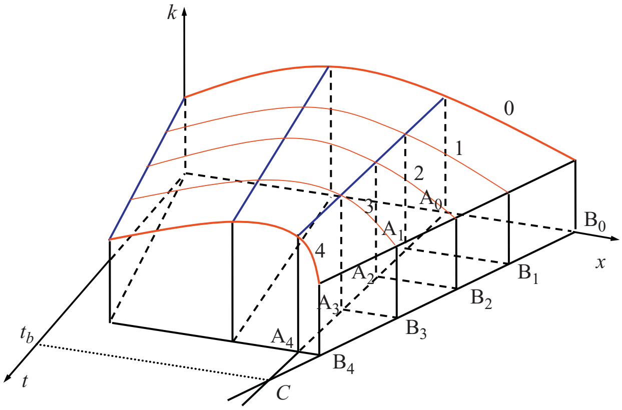

Figure 7.1 A gradient catastrophe.

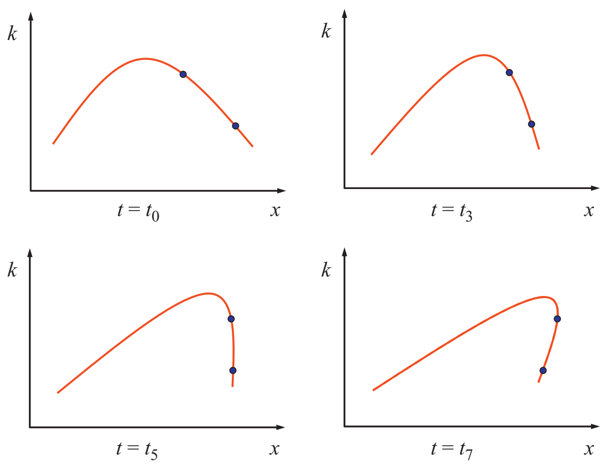

Figure 7.2 Top of profile overtakes bottom of profile.

When the two characteristics intersect at point C, the solution profile will have an infinite gradient at this point. The formation of such an infinite gradient is called a *gradient catastrophe*, and the time when infinite gradient occurs is called the *break time* t_b. Figure 7.2 presents a few frames of time development of the solution profile. Notice that the top dot of the profile moves faster than the bottom dot. Sooner or later, the top dot will catch up with the bottom dot at the break time, creating a gradient catastrophe. After this, the top dot runs over the bottom dot, and the profile ceases to be a valid function. Consequently, the solution beyond the break time will be problematic. The purpose of this chapter is to address such an issue.

The above example illustrates a family of characteristics moving closer and closer over time, so they form a *compression wave*. The opposite case is a family of characteristics moving farther and farther apart without any intersection (see Figure 7.3); such a wave is called an *expansion wave*. The corresponding time development of the solution profile is shown in Figure 7.4. It can be seen that the bottom dot moves faster than the top dot in this case, and the solution profile becomes thinned out or rarefied.

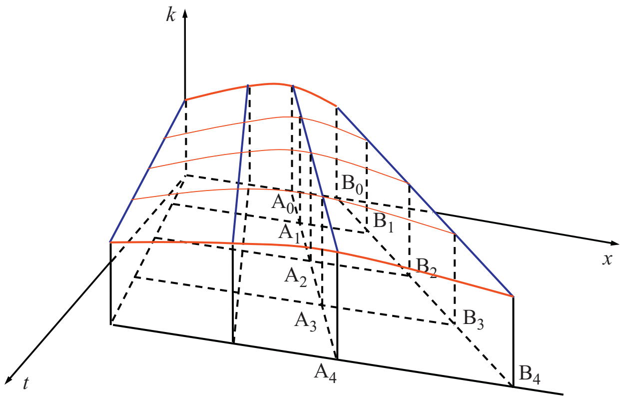

Figure 7.3 Characteristics farther and farther apart.

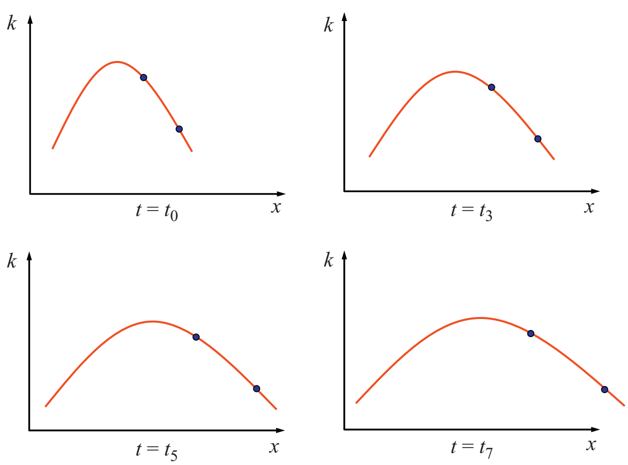

Figure 7.4 Solution profile thinned out.

7.2 SHOCK WAVES

As a continuation of the above discussion, if two characteristics intersect, the solution at the intersection will be multivalued. However, if one allows discontinuity at the intersection, it is possible to construct a piecewise smooth solution. For example, Figure 7.5 illustrates such a solution where curve $x_s(t)$ in the x-t plane is a collection of characteristic intersections. The solution remains constant along each characteristic and terminates at their intersection. Therefore, the curve partitions the solution space into two parts R^- and R^+ and, consequently, separates the solution into two smooth pieces S^- and S^+. The drop or discontinuity of k at the curve denotes an abrupt change of k which creates a *shock wave*. Such a piecewise smooth solution of the partial differential equation (PDE) is called a shock wave solution.

A critical step in the shock wave solution is to find the curve $x_s(t)$ which connects the intersections of characteristics. Since the curve represents the locations at which a shock wave forms, such a curve is called a *shock path*. In Figure 7.6, two families of characteristics are illustrated where a characteristic may have multiple intersections. Hence, many curves can be drawn by connecting different sets of intersections and, hence, the shock wave may take different paths. Fortunately, the underlying conservation law ensures that only one shock path is valid, and such a shock path must satisfy a physical condition called the *Rankine-Hugoniot jump condition*:

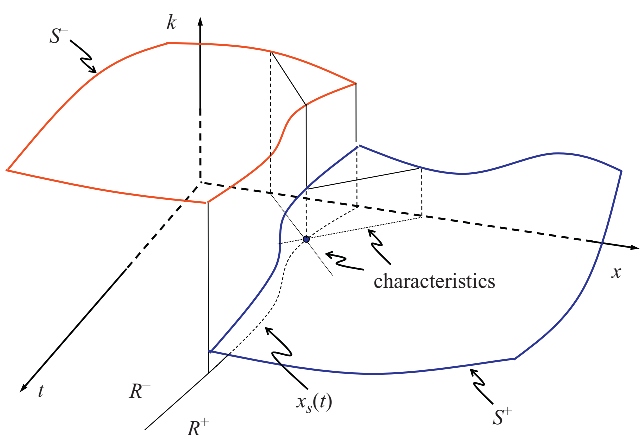

Figure 7.5 Piecewise solution—shock wave.

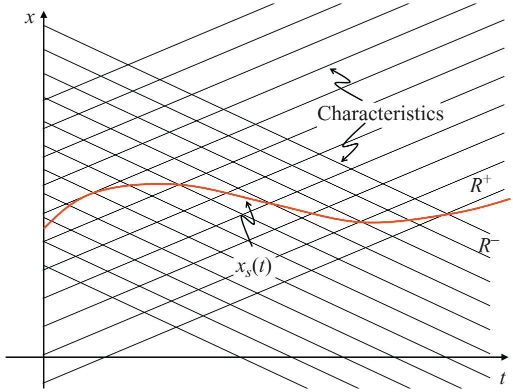

Figure 7.6 Shock path.

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{q(t, x_s^+) - q(t, x_s^-)}{k(t, x_s^+) - k(t, x_s^-)},$$

where $\frac{\mathrm{d}x_s}{\mathrm{d}t}$ is the slope of the shock path, $q = Q(k)$ as defined in the conservation law, $k(t, x_s^-)$ takes the k value on the R^- side, $k(t, x_s^+)$ takes the k value on the R^+ side, and similar notation applies to $q(t, x_s^-)$ and $q(t, x_s^+)$.

Therefore, if one or more intersections on curve $x_s(t)$ are known, one can construct the shock path by starting from the known points and following the slope defined above.

As an example, solve the conservation law with the following initial conditions:

$$\begin{cases} k_t + q_x = 0, \\ q = \frac{1}{2}k^2, \\ k(0, x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 0 & \text{if } x > 0, \end{cases} \\ -\infty < x < \infty, \\ t > 0. \end{cases}$$

The slope of the characteristics is $c = \frac{\mathrm{d}q}{\mathrm{d}k} = k$. Obviously, characteristics drawn below $x = 0$ are straight, parallel lines with slope $c = 1$. These characteristics carry the same constant value of $k = 1$, and hence $q = 1/2k^2 = 1/2$. Similarly, characteristics drawn above $x = 0$ are horizontal lines with slope $c = 0$. They carry $k = 0$, and hence $q = 0$. The origin is a known point on the shock path. According to the Rankine-Hugonoit jump condition, the slope of the shock path is

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{q(t, x_s^+) - q(t, x_s^-)}{k(t, x_s^+) - k(t, x_s^-)} = \frac{0 - 1/2}{0 - 1} = \frac{1}{2}.$$

Therefore, the shock path is a straight line which starts from the origin with constant slope $\frac{1}{2}$—that is,

$$x_s(t) = \frac{1}{2}t.$$

Therefore, the solution is

$$k(t, x) = \begin{cases} 1 & \text{if } x \leq \frac{1}{2}t, \\ 0 & \text{if } x > \frac{1}{2}t. \end{cases}$$

The solution is illustrated in Figure 7.7. Also illustrated are a few concepts discussed before: a characteristic is a line along which the solution k remains constant; a kinematic wave is a family of straight, parallel characteristics, and a shock wave separates two kinematic waves with an abrupt change of the k value; a shock path is the projection of shock locations onto the x–t plane.

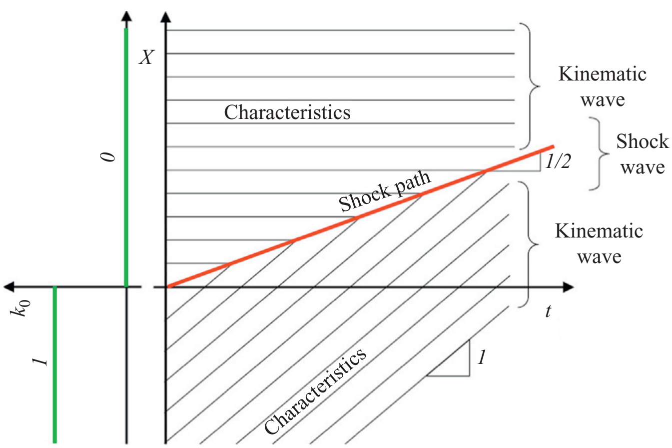

Figure 7.7 An example of a shock path.

7.3 RAREFACTION WAVES

If the initial condition in the above example is reversed[1]—that is,

$$k(0,x) = k_0 = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0, \end{cases}$$

characteristics of this PDE should be drawn as in Figure 7.8. In this case, the two families of characteristics go farther and farther apart, leaving an empty wedge-shaped area in between. Since a characteristic carries a constant k solution, areas swept by characteristics will have solutions. An empty area in the solution space means there is no solution in this area. To resolve this issue, there should be a means to fill the empty area with characteristics.

If one relaxes the step function of the initial condition by assuming that k_0 varies smoothly from 0 to 1 over a small distance Δx (see Figure 7.9), the slopes of characteristics drawn in Δx will gradually increase from 0 to 1 so that any point in the solution space is swept by one and only one characteristic.

To return to the step function of the initial condition, one takes the limit $\Delta x \to 0$, so Figure 7.9 reduces to Figure 7.10. Now the empty area is filled with a fan of characteristics drawn from the origin. If one cuts the solution space with a few planes $t = t_0, t_1, t_2, \ldots$, with t_0 passing the origin and other planes at consequently later times, one obtains a time development of the solution as shown in Figure 7.11. Notice that the profile of the solution is thinned out or rarefied as time moves on. Hence, this fan of characteristics represents a rarefaction wave.

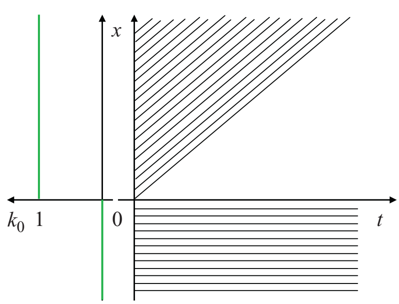

Figure 7.8 Characteristics without an intersection.

[1] The following discussion is derived from Ref. [23] with modifications.

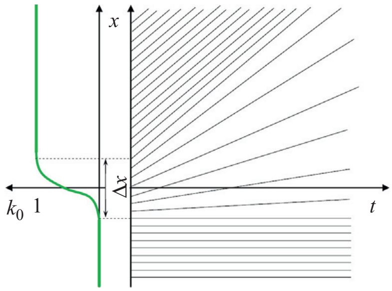

Figure 7.9 Filling an empty area with characteristics.

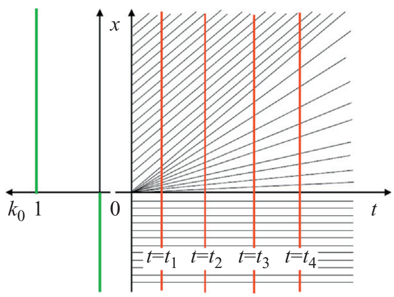

Figure 7.10 A rarefaction wave.

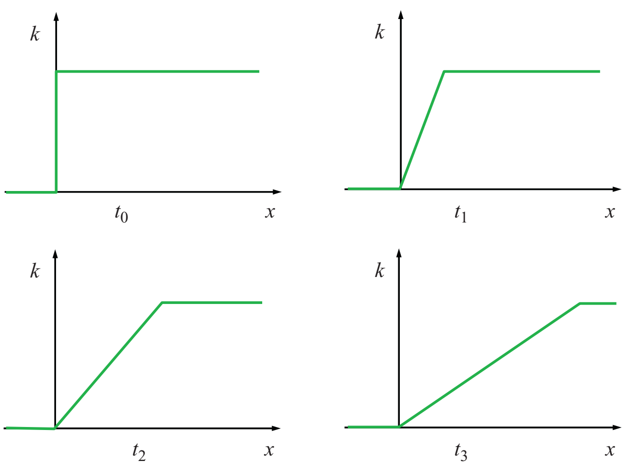

Figure 7.11 Time development of the rarefaction wave.

The rarefaction wave can be used to construct a solution for the following conservation law problem:

$$\begin{cases} k_t + q_x = 0, \\ q = \dfrac{1}{2}k^2, \\ k(0,x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0, \end{cases} \\ -\infty < x < \infty, \\ t > 0. \end{cases}$$

From the initial condition and with use of the method of characteristics, solutions for the two areas swept by the two parallel characteristics in Figure 7.8 can be easily determined:

$$k(t,x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } t < x. \end{cases}$$

The fan of characteristics in the wedge-shaped area consists of lines $x = ct$, where $0 < c < 1$. Therefore, the solution $k(x,t)$ in this area should have the form $f(x/t)$. Hence,

$$k_t = \left(-\frac{x}{t^2}\right)f', \quad k_x = \frac{1}{t}f'.$$

The conservation law can be rewritten as

$$k_t + kk_x = 0.$$

Plugging f and its partial derivatives back into the above equation $k_t + q_x = 0$, we obtain

$$\frac{1}{t}f'\left(f - \frac{x}{t}\right) = 0.$$

If we solve this equation, we get $f'(\frac{x}{t}) = 0$ or $f = \frac{x}{t}$. If $f'(\frac{x}{t}) = 0$, $f(\frac{x}{t}) = k(t,x) = a$, where a is an integral constant. A simple check along $x = 0$ and $x = t$ reveals that this solution does not satisfy the Rankine-Hugonoit jump condition. For example, if we apply the Rankine-Hugonoit jump condition, a shock wave solution bordering area $x \le 0$ should have a shock path with slope $\frac{a}{2}$. As such, the shock path is a straight line drawn from the origin with a slope of $\frac{a}{2}$. However, none of the characteristics in area $x \leq 0$ touche the shock path, which makes the shock path an invalid one by definition—that is, a shock wave is the intersection of two or more kinematic waves. Similar reasoning applies to the other side of the wedge bordering area $x > t$. Therefore, solution $f(\frac{x}{t}) = k(t, x) = a$ is not a shock wave solution. The smooth solution is given by $k(t, x) = f(\frac{x}{t}) = \frac{x}{t}$. Hence, the rarefaction wave solution of the original problem is

$$k(t, x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{x}{t} & \text{if } 0 < x \leq t, \\ 1 & \text{if } t < x. \end{cases}$$

On can also construct a shock wave solution by choosing a constant a such that $0 < a < 1$ and applying the Rankine-Hugonoit jump condition:

$$k(t, x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2}at, \\ a & \text{if } \frac{1}{2}at < x \leq \frac{1}{2}(a+1)t, \\ 1 & \text{if } \frac{1}{2}(a+1)t < x. \end{cases}$$

Figure 7.12 illustrates the solution space which has two shock waves $x = \frac{1}{2}at$ and $x = \frac{1}{2}(a+1)t$. Since the choice of constant a is arbitrary as long as $0 < a < 1$ is met, the above shock wave solution is multivalued. Combining the above rarefaction wave and shock wave solutions, one concludes that the solution to the original problem is not unique. However, the physical process has only one outcome, and hence the solution must be unique. The question is "which solution makes the most physical sense?"

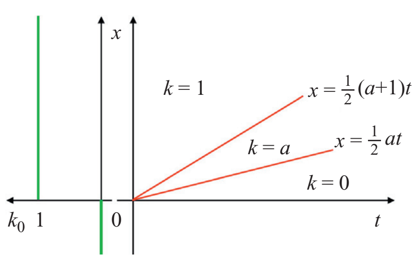

Figure 7.12 A shock wave solution.

7.4 ENTROPY CONDITION

In fluid dynamics, the *entropy condition* is used to select a solution that makes the most physical sense. The entropy condition of a function $k(x, t)$ requires the existence of a positive constant E such that the following inequality is met:

$$\frac{k(t, x + \Delta x) - k(t, x)}{\Delta x} \leq \frac{E}{t}$$

for $\Delta x > 0$ and $t > 0$. Such a condition is shown in Figure 7.13.

Now let us check if a shock wave solution satisfies this condition. We do by slicing the solution space in Figure 7.12 using a plane AA′ and projecting the result onto the k-x plane, as shown in Figure 7.14. The solution profile consists of three discontinuous sections with $k = 0, a, 1$. If one choose two arbitrary points Δx apart on the profile as indicated, the slope is $\frac{a}{\Delta x}$. The slope becomes larger and larger as Δx shrinks and becomes infinity at the jump location. Therefore, one cannot find a positive constant E to satisfy the entropy condition, and hence shock wave solutions do not make physical sense.

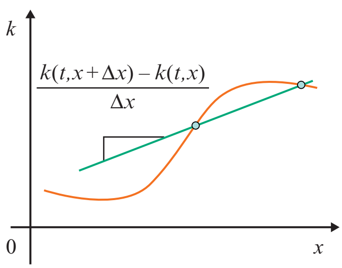

Figure 7.13 Entropy condition.

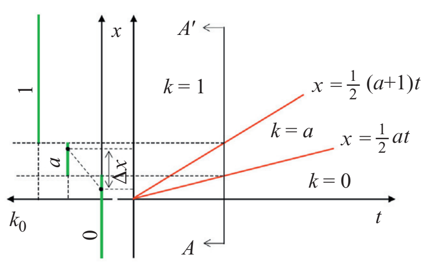

Figure 7.14 Entropy condition in a shock wave solution.

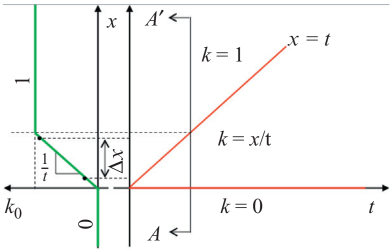

Figure 7.15 Entropy condition in the rarefaction solution.

To apply the same technique to check the rarefaction solution, one slices the solution space with a plane at AA′; the resultant solution profile is illustrated in Figure 7.15. It can be seen that the maximum slope between any two points on the solution profile is $\frac{1}{t}$. Hence, if one chooses $E = 1$, the entropy condition is always met. Therefore, the rarefaction solution is chosen as the (unique) solution that makes the most physical sense:

$$k(t,x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \dfrac{x}{t} & \text{if } 0 < x \leq t, \\ 1 & \text{if } t < x. \end{cases}$$

7.5 SUMMARY OF WAVE TERMINOLOGY

At this point, it is helpful to summarize the definition of a few terms frequently used in the analysis of waves and their solutions:

- A *wave* is the propagation in time t and space x of a disturbance in a medium.
- A *signal* is a physical measure (e.g., traffic density k) that describes the disturbance.
- A *characteristic* is a line in the x–t plane along which the signal remains constant.
- A *kinematic wave* is a family of parallel characteristics in the x–t plane.
- A *compression wave* is a family of characteristics which are closer and closer to each other over time.
- A *shock wave* is the formation of an abrupt change in signal in the medium. A compression wave consists of intersecting characteristics. The intersection of these characteristics causes gradient catastrophe, which, in turn, is a precursor of a shock wave.

- An *expansion wave* is a family of characteristics which are farther and farther apart over time.
- A *rarefaction wave* is the effect that the signal profile thins out over time. An expansion wave consists of diverging characteristics which cause two neighboring signals to move farther and farther apart, which, in turn, causes a rarefaction wave.

PROBLEMS

1. Explain the following concepts, and use examples assisted by sketches if necessary.
   a. Wave
   b. Characteristic
   c. Kinematic wave
   d. Shock wave

2. Find the shock wave solution to the following PDE with initial conditions:

$$\begin{cases} k_t + q_x = 0, \\ q = 2k^2, \\ k(0,x) = \begin{cases} 5 \text{ if } x \leq 0, \\ 2 \text{ if } x > 0, \end{cases} \\ -\infty < x < \infty, \\ t > 0. \end{cases}$$

3. Find the shock wave solution to the following PDE with initial conditions:

$$\begin{cases} k_t + q_x = 0, \\ q = k^3, \\ k(0,x) = \begin{cases} 1 \text{ if } x \leq 2, \\ 0 \text{ if } x > 2, \end{cases} \\ -\infty < x < \infty, \\ t > 0. \end{cases}$$

4. Find the rarefaction wave solution to the following PDE with initial conditions:

$$\begin{cases} k_t + q_x = 0, \\ q = k^2, \\ k(0,x) = \begin{cases} 0 \text{ if } x \leq 0, \\ 1 \text{ if } x > 0, \end{cases} \\ -\infty < x < \infty, \\ t > 0. \end{cases}$$

5. Using the entropy condition to check whether your solution to the above problem is a physically meaningful one.