Homework: Equilibrium Traffic Flow

4.1, 4.4, 4.8

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4.1

Given a speed-density relationship of

$$v = v_f \left(1 - \frac{k}{k_j}\right), \tag{1}$$

and the relationship $q = k \times v$, we can derive flow-density and speed-flow relationships as follows:

$$q = kv \tag{2}$$
$$q = kv_f \left(1 - \frac{k}{k_j}\right) \tag{3}$$
$$q = kv_f - \frac{k^2 v_f}{k_j} \tag{4}$$

and

$$k = \frac{q}{v}, \quad v = v_f \left(1 - \frac{k}{k_j}\right) \tag{5}$$
$$\implies v = v_f \left(1 - \frac{q/v}{k_j}\right) \tag{6}$$
$$\implies v^2 = v_f v - \frac{q v_f}{k_j}. \tag{7}$$

From (4) we can find q_m (the capacity) and k_m (the density at capacity) by determining the maximum of the flow-density relationship (where $\frac{dq}{dk} = 0$):

$$q = kv_f - \frac{k^2 v_f}{k_j} \tag{8}$$

$$\frac{dq}{dk} = 0 = v_f - 2\frac{k_m v_f}{k_j} \tag{9}$$

$$0 = v_f \left(1 - 2\frac{k_m}{k_j}\right) \tag{10}$$

$$0 = 1 - 2\frac{k_m}{k_j} \quad \text{if } v_f \neq 0 \tag{11}$$

$$k_m = \frac{k_j}{2} \tag{12}$$

$$q_m = k_m v_f \left(1 - \frac{k_m}{k_j}\right) \tag{13}$$

$$q_m = \frac{k_j v_f}{2} \left(1 - \frac{k_j/2}{k_j}\right) \tag{14}$$

$$q_m = \frac{k_j v_f}{4}. \tag{15}$$

v_m (the speed at capacity) is then determined from (15) and the original relationship (1) by:

$$v_m = v_f \left(1 - \frac{k_m}{k_j}\right) \tag{16}$$

$$v_m = v_f \left(1 - \frac{k_j/2}{k_j}\right) \tag{17}$$

$$v_m = \frac{v_f}{2}. \tag{18}$$

4.4

The Greenberg model is given by

$$v = v_m \ln \frac{k_j}{k}. \tag{19}$$

The capacity is again determined by setting $\frac{dq}{dk} = 0$, so a flow-density relationship is first determined:

$$q = kv = kv_m(\ln k_j - \ln k). \tag{20}$$

Then:

$$\begin{aligned}
\frac{dq}{dk} = 0 = \frac{d}{dk} kv_m(\ln k_j - \ln k) && (21)\\
0 = k_m v_m \left(-\frac{1}{k_m}\right) + v_m(\ln k_j - \ln k_m) && (22)\\
0 = v_m \left(\ln \frac{k_j}{k_m} - 1\right) && (23)\\
1 = \ln \frac{k_j}{k_m} \quad \text{if } v_m \neq 0 && (24)\\
e = \frac{k_j}{k_m} && (25)\\
k_m = \frac{k_j}{e}. && (26)
\end{aligned}$$

q_m is then given from (20) and (26):

$$\begin{aligned}
q_m = k_m v_m(\ln k_j - \ln k_m) = k_m v_m \left(\ln \frac{k_j}{k_m}\right) && (27)\\
q_m = \frac{k_j v_m}{e} \left(\ln \frac{k_j}{k_j/e}\right) && (28)\\
q_m = \frac{k_j v_m}{e} && (29)
\end{aligned}$$

4.8

The empirical speed-density relationship given by

$$v = \min\{88.5,\ 172 - 3.72k + 0.0346k^2 - 0.00119k^3\} \tag{30}$$

is shown graphically in Figure 1. The free-flow speed is the speed when the density is 0, or 88.5 km/h. The jam density k_j is the upper bound of the valid range of the model (i.e. where the speed is nonnegative), which is 40.4 veh/km.

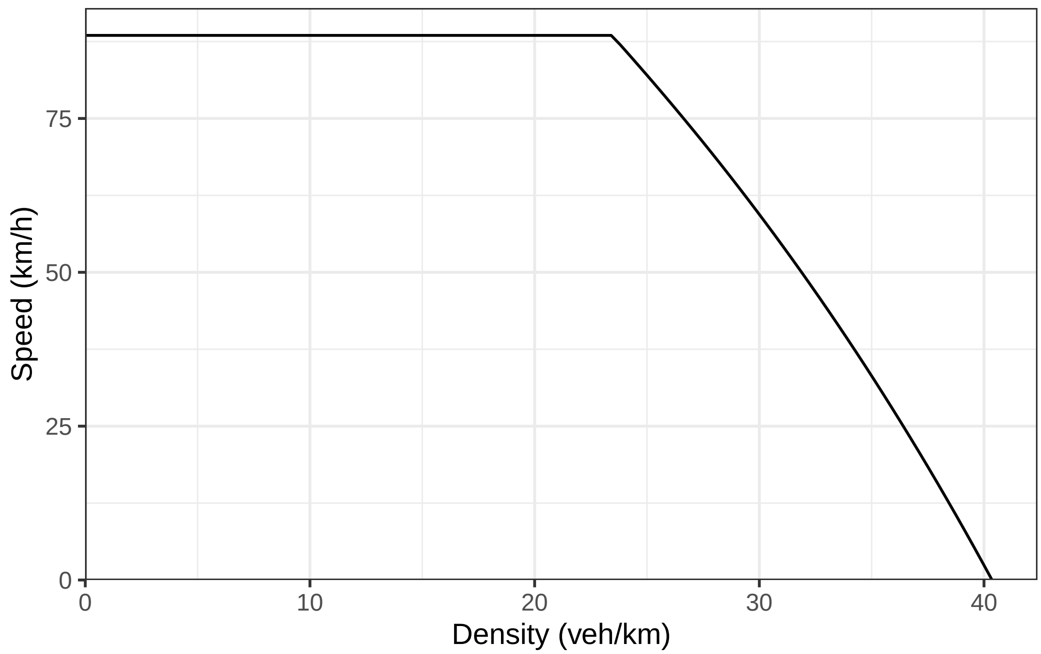

Figure 1: Empirical speed-density relationship.

To find the capacity q_m we need a flow-density plot. Since $q = k \times v$, from (30) we get

$$q = k \times \min\{88.5,\ 172 - 3.72k + 0.0346k^2 - 0.00119k^3\}. \tag{31}$$

Figure 2 shows this graphically. q_m is the maximum of this function, which is 2074 veh/h. k_m is the density at capacity (q_m), or 23.4 veh/km. v_m is the speed at this density, or 88.5 km/h.

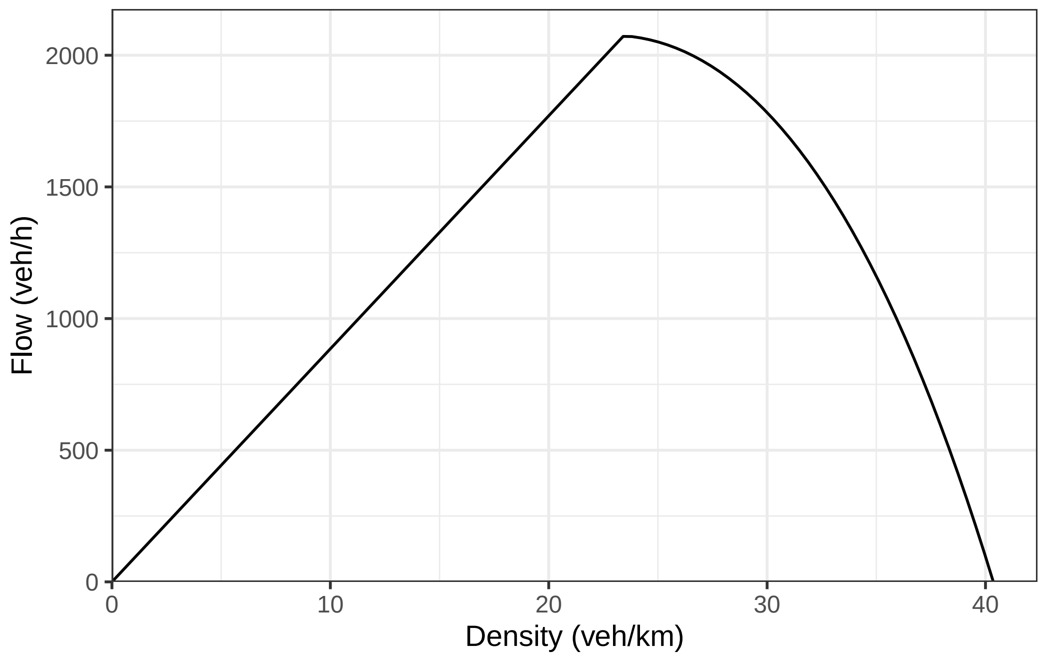

Figure 2: Empirical flow-density relationship.