CHAPTER 10

Simplified Theory of Kinematic Waves

The previous chapters have focused on analyzing the dynamic change of traffic states over time and space using the theory of waves. In particular, the method of characteristics applied to the LWR problem resulted in the identification of shock waves as a means to solve the continuity equation (i.e., the conservation law). In addition, numerical methods were discussed to approximate the solution.

Alternatively, Derivation V in Chapter 5 seems to suggest that the conservation law is self-guaranteed if a three-dimensional representation of traffic flow is used—that is,

$$q_x + k_t = \frac{\partial N^2(t,x)}{\partial x \partial t} - \frac{\partial N^2(t,x)}{\partial x \partial t} = 0.$$

The significance of the above equation is that there is no need to solve partial differential equations and find shock waves in order to analyze traffic dynamics. Instead, one only needs to count cars over time and space. As such, traffic dynamics is contained in these cumulative counts and can be extracted as the need arises.

This idea has been explored by Gorden F. Newell, who creatively integrated D/D/1 queuing theory into this idea to allow prediction from boundary and initial conditions. The result is known as the simplified theory of kinematic waves, published in a trio of papers [30–32] in the early 1990s. The first paper addresses the general theory, the second paper focuses on queuing at a freeway bottleneck, and the third paper deals with multidestination flows. Below I present the main points of the first and second papers interpreted from my own perspective. Interested readers are encouraged to use this chapter as a key to unlock the original papers for an enriched learning experience.

Traffic Flow Theory
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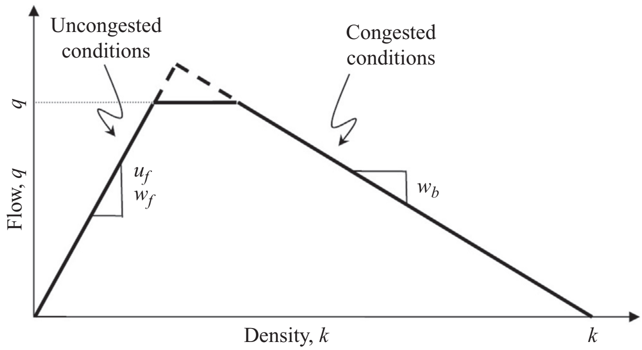


Figure 10.1 Triangular flow-density relationship.

10.1 TRIANGULAR FLOW-DENSITY RELATIONSHIP

The kinematic waves model was proposed as a (graphical) solution to the LWR model under a special condition: the underlying flow-density relationship is a triangular one with jam density K and capacity Q (see Figure 10.1).

From Figure 8.5, a point on the flow-density curve uniquely defines the operating condition of a stream of traffic. The speed of a kinematic wave carried by the traffic, w, is the tangent to the curve at this point. If the underlying flow-density relationship is triangular, finding kinematic wave speeds is greatly simplified. Actually, there are only two kinematic wave speeds: a forward wave speed w_{f} for all uncongested conditions (the left branch of the triangle) and a backward wave speed w_{b} for all congested conditions (the right branch). In addition, w_{f} happens to be the same as the free-flow speed v_{f}. As a special property of the triangular flow-density relationship, v_{f} applies to all uncongested conditions.

10.2 FORWARD WAVE PROPAGATION

Unlike conventional numerical models such as FREFLO, KRONOS, and the cell transmission model which keep track of cell storages $n(t_i, x_j)$ or equivalently cell densities $k(t_i, x_j)$, simplified kinematic waves model just counts vehicles at some predetermined locations. The outcome of the model is a set of cumulative flows representing the number of vehicles counted at these locations over time, $N(t, x_j), j \in (1, 2, \ldots, J)$. These cumulative flows contain all the information that is needed to determine traffic dynamics over time and space.

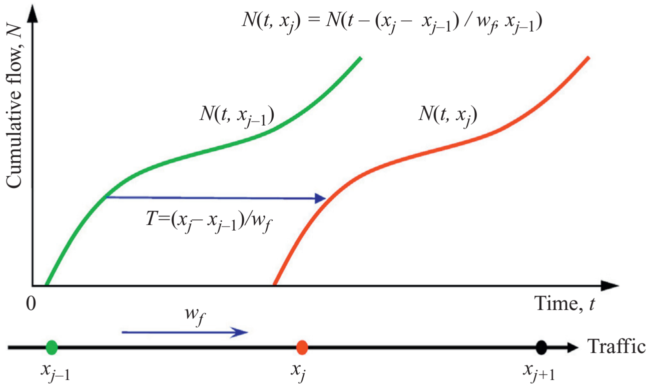


Figure 10.2 Forward wave propagation.

Suppose the cumulative flow recorded at location x_{j-1} over time t is $N(t, x_{j-1})$ and there is no congestion between x_{j-1} and x_{j+1}. The traffic will be dictated by (uncongested) upstream arrival from x_{j-1}, and these vehicles will arrive at downstream location x_j after a duration of $T = \frac{x_j - x_{j-1}}{v_\mathrm{f}}$ if the vehicles preserve their order (i.e., first in, first-out). The traffic also carries a kinematic wave whose speed w_f happens to be v_f, as noted above, so it is equivalent to saying that the kinematic wave will propagate forward and arrive at x_j after $T = \frac{x_j - x_{j-1}}{w_\mathrm{f}}$. Graphically, this forward wave propagation can be constructed as in Figure 10.2, where the profile $N(t, x_j)$ is simply a horizontal translation of profile $N(t, x_{j-1})$ to the right by T:

$$N(t, x_j) = N(t - T, x_{j-1}) = N\left(t - \frac{x_j - x_{j-1}}{w_\mathrm{f}}, x_{j-1}\right).$$

10.3 BACKWARD WAVE PROPAGATION

Suppose the cumulative flow recorded at location x_{j+1} over time t is $N(t, x_{j+1})$ and there is congestion between x_{j-1} and x_{j+1} (see Figure 10.3). Then the kinematic wave carried by the traffic will propagate backward at speed w_b. Hence, the traffic condition at location x_j ($x_{j-1} < x_j < x_{j+1}$) will be dictated by downstream congestion. Consequently, cumulative flow at x_j, $N(t, x_j)$, will be a horizontal translation of $N(t, x_{j+1})$ to the right by $T = \frac{x_{j+1} - x_j}{w_\mathrm{b}}$ shifted upward by a jam storage $n = K_j(x_{j+1} - x_j)$:

$$N(t, x_j) = N(t - T, x_{j+1}) + n = N(t - \frac{x_{j+1} - x_j}{w_\mathrm{b}}, x_{j+1}) + K_j(x_{j+1} - x_j).$$

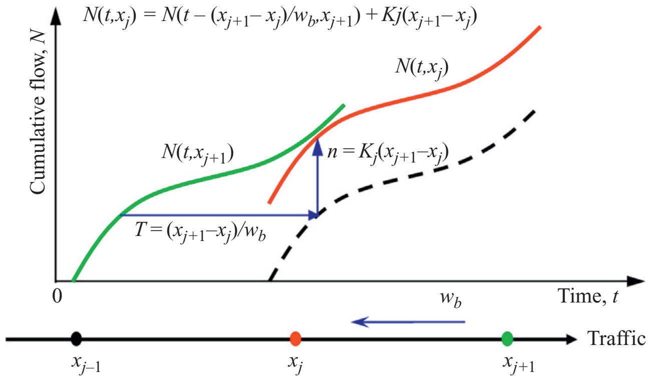


Figure 10.3 Backward wave propagation.

10.4 LOCAL CAPACITY

Suppose the cumulative flow to pass location x_j is $N(t, x_j)$ and the local capacity is Q_j. Since vehicles cannot be discharged beyond the capacity, this is equivalent to saying that the tangent to the profile $N(t, x_j)$ at any point should not exceed Q_j. Hence, the cumulative flow constrained by local capacity Q_j, $N^Q(t, x_j)$ is constructed as follows. Draw a line with slope Q_j from the right toward the profile $N(t, x_j)$ till the line is tangent to the profile. Any portion of the profile above the line is replaced by the latter. Continue the above process until no portion of the profile has a tangent greater than Q_j (see Figure 10.4).

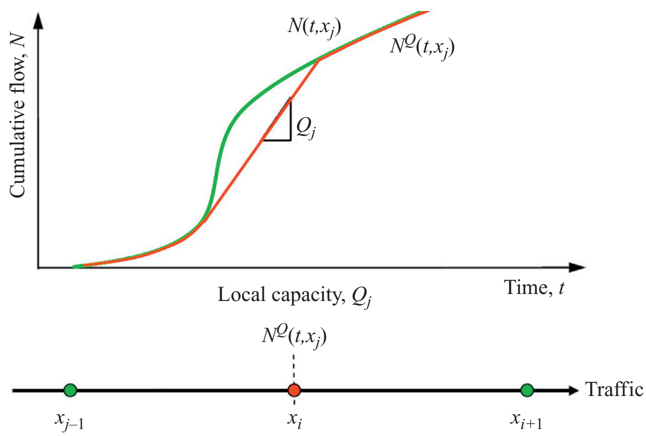


Figure 10.4 Flow constrained by local capacity.

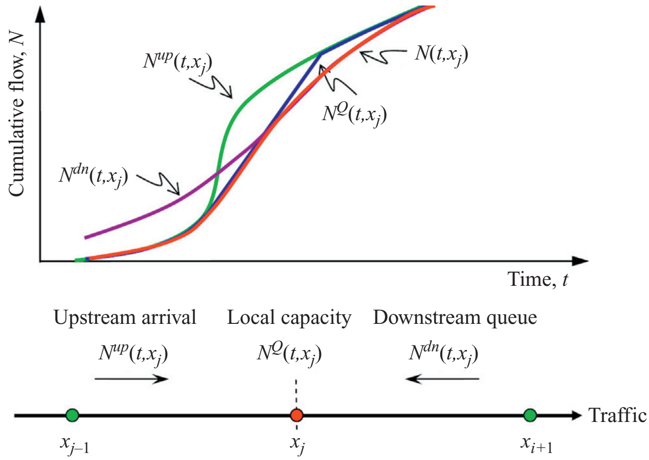


Figure 10.5 The minimum principle.

10.5 MINIMUM PRINCIPLE

Intuitively, the minimum principle means that any point on a roadway x_j cannot admit more vehicles than arrive from an upstream location $N^{\mathrm{up}}(t, x_j)$, which is allowed by local capacity $N^{Q}(t, x_j)$, and which the downstream location is able to receive $N^{\mathrm{dn}}(t, x_j)$. Graphically, this involves superimposing the above three curves on a single graph, and the cumulative flow that actually passes x_j, $N(t, x_j)$ is the lower envelope of the three (see Figure 10.5):

$$N(t, x_j) = \min\{N^{\mathrm{up}}(t, x_j), N^{Q}(t, x_j), N^{\mathrm{dn}}(t, x_j)\}.$$

10.6 SINGLE BOTTLENECK

In Figure 10.5, if there is an on-ramp at x_j, the location slightly downstream (to the right of x_j), x_j^+, may be a bottleneck since both traffic streams from the upstream mainline and the on-ramp meet here. To keep track of arrival and departure flows, cumulative flow $N(t, x)$ will be replaced by two notations:

- cumulative arrival flow $A(t, x)$, which denotes cumulative flow having arrived at location x by time t waiting to pass x, and
- cumulative departure flow $D(t, x)$, which denotes cumulative flow having departed location x by time t.

Note that their difference $D(t,x)-A(t,x)$ gives the length of the queue at time t.

Central to the single bottleneck idea is to determine its cumulative arrival and departure flows, $A(t,x_j)$ and $D(t,x_j)$, given by

- upstream departure at earlier times $D(t,x_{j-1})$;
- downstream departure at earlier times $D(t,x_{j+1})$;
- on-ramp cumulative inflow $A_j(t)$;

From wave forward propagation (Figure 10.2), the cumulative flow arriving at location slightly upstream of the bottleneck (to the left of x_j), x_j^- is

$$A(t,x_j^-) = A(t - \frac{x_j - x_{j-1}}{w_{\mathrm{f}}}, x_{j-1}).$$

Given on-ramp traffic $A_j(t)$, the cumulative flow arriving to the right of x_j is

$$N^{\mathrm{up}}(t,x_j^+) = A(t,x_j^+) = A(t,x_j^-) + A_j(t).$$

From wave backward propagation (Figure 10.3), the cumulative flow allowed to depart is

$$N^{\mathrm{dn}}(t,x_j^+) = N(t - \frac{x_{j+1} - x_j}{w_{\mathrm{b}}}, x_{j+1}^-) + K_j(x_{j+1} - x_j).$$

Considering local capacity (Figure 10.4), the cumulative flow departing x_j^+ should not exceed $N_Q(t,x_j^+)$.

Therefore, on the basis of the minimum principle (Figure 10.5), the cumulative flow that actually departed at x_j^+ is

$$D(t,x_j^+) = \min\{N^{\mathrm{up}}(t,x_j^+), N_Q(t,x_j^+), N^{\mathrm{dn}}(t,x_j^+)\}.$$

If on-ramp traffic, $A_j(t)$, has priority over mainline traffic and can always bypass any queue at the bottleneck (this assumption is a limitation of the theory of kinematic waves since it eliminates queuing on ramps), then the cumulative departure flow to the left of x_j can be determined as

$$D(t,x_j^-) = D(t,x_j^+) - A_j(t).$$

The above procedure is illustrated in Figure 10.6.

10.7 COMPUTATIONAL ALGORITHM

With the above knowledge, traffic flow on a freeway involving multiple segments and bottlenecks can be numerically modeled as follows. First, the time and space are partitioned using the discretization scheme in

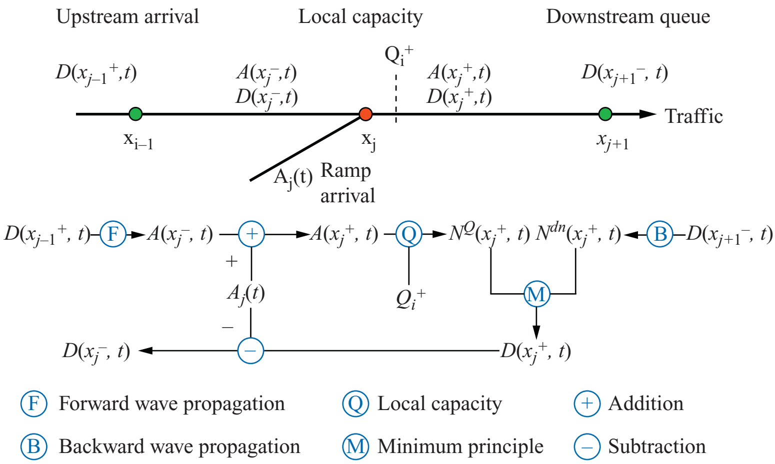


Figure 10.6 Single bottleneck.

Chapter 9, resulting the lattice shown in Figure 10.7. Next, starting from the initial conditions, one applies the numerical solution procedure outlined in Chapter 9. At each lattice point (t_i, x_j), the cumulative arrival and departure flows are determined as follows:

1. Determine upstream arrival to x_j^-:

$$A(t_i, x_j^-) = D(t_i - \frac{x_j - x_{j-1}}{w_f}, x_{j-1}^+).$$

2. Determine upstream arrival to x_j^+:

$$N^{\text{up}}(t_i, x_j^+) = A(t_i, x_j^+) = A(t_i, x_j^-) + A_j(t_i).$$

3. Apply capacity constraint at x_j^+:

$$N^Q(t_i, x_j^+) = D(t_{i-1}, x_j^+) + Q_j^+ \times \Delta t,$$

   where Q_j^+ is the capacity at x_j^+ and Δt is $t_i - t_{i-1}$.

4. Determine departure allowed by x_{j+1}^-:

$$N^{\text{dn}}(t_i, x_j^+) = N(t_i - \frac{x_{j+1} - x_j}{w_b}, x_{j+1}^-) + K_j(x_{j+1} - x_j).$$

5. Determine actual departure at x_j^+:

$$D(t_i, x_j^+) = \min\{N^{\text{up}}(t_i, x_j^+), N^Q(t_i, x_j^+), N^{\text{dn}}(t_i, x_j^+)\}.$$

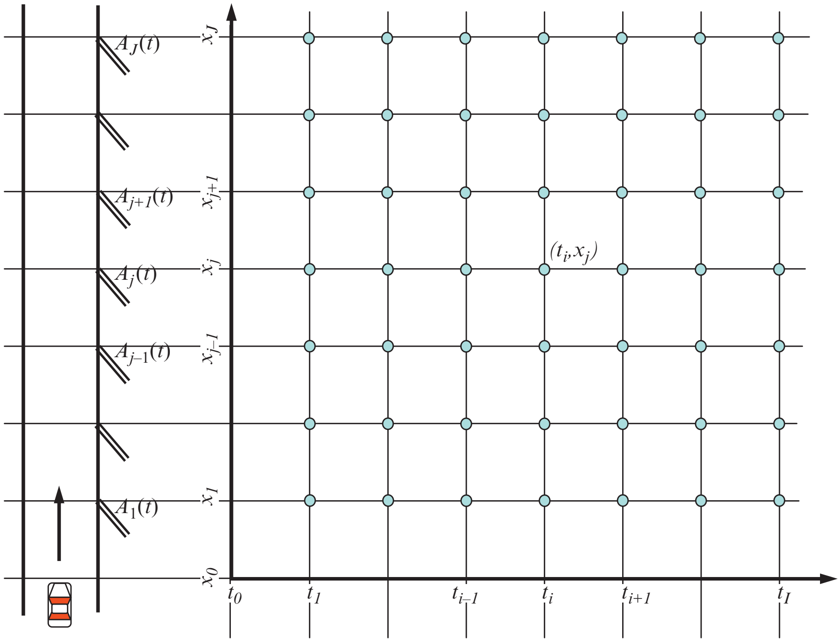


Figure 10.7 Lattice of kinematic waves.

6. Determine actual departure at x_j^-:

$$D(t_i, x_j^-) = D(t_i, x_j^+) - A_j(t_i)$$

7. Proceed to the next lattice point (t_i, x_{j+1}).

Repeat the above steps at lattice point (t_i, x_{j+1}) till the end (t_i, x_J). Then advance time to t_{i+1} and start over again from (t_{i+1}, x_1) to (t_{i+1}, x_J). Repeat the above steps till all the lattice points have been traversed.

The result of this computational algorithm is a set of cumulative arrival and departure flows:

$$\begin{gathered}
A(t_1, x_1^-), D(t_1, x_1^-), A(t_1, x_1^+), D(t_1, x_1^+) \\
\ldots \\
A(t_1, x_J^-), D(t_1, x_J^-), A(t_1, x_J^+), D(t_1, x_J^+) \\
A(t_2, x_1^-), D(t_2, x_1^-), A(t_2, x_1^+), D(t_2, x_1^+) \\
\ldots \\
A(t_I, x_J^-), D(t_I, x_J^-), A(t_I, x_J^+), D(t_I, x_J^+).
\end{gathered}$$

10.8 FURTHER NOTE ON THE THEORY OF KINEMATIC WAVES

The above discussion summarizes the first two papers of Newell's simplified theory of kinematic waves [30, 31] involving bottlenecks with on-ramps only. The third paper [32] takes off-ramps into consideration, and hence multiple destination flows. Discussion of this subject is quite involved, and readers are encouraged to read the original paper for full information. In addition, supplementary information on the simplified theory of kinematic waves can be found in Son [33] and Hurdle and Son [34] for model validation and extraction of information of traffic dynamics and in Ni [35] and Ni et al. [36] for extension of the theory and associated computational algorithms.

Though Newell's theory involves partitioning a highway into a series of segments, the lengths of these segments do not necessarily have to be equal and small. Such a partitioning is necessary only at locations where capacity changes (e.g., lane drop), there is an on-ramp, and there is an off-ramp. Therefore, the resulting number of segments can be much less than in cell-based models such as FREFLOW, KRONOS, and the cell transmission model, whose accuracy relies on cell size (length of segment). Consequently, the computation and storage requirements can be significantly reduced.

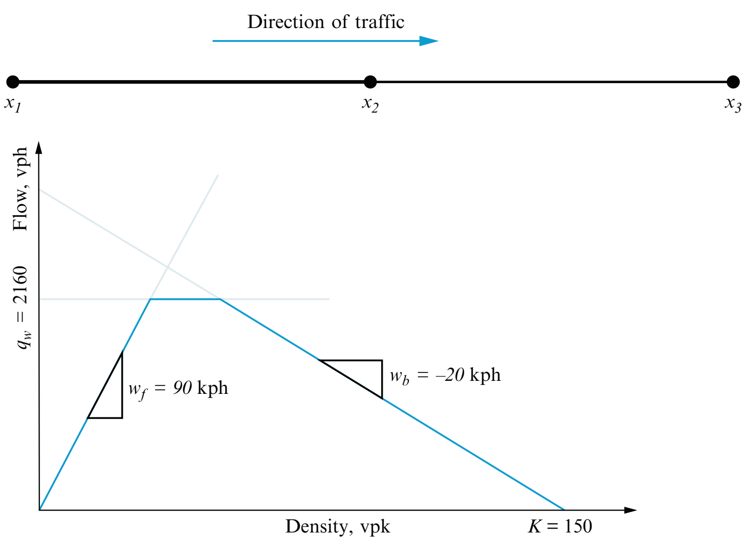
In addition to the assumption of a triangular flow-density relationship, another limitation of the theory of kinematic waves is its assumption that on-ramp traffic has priority over mainline traffic and can always bypass any queue at a bottleneck. Consequently, the theory of kinematic waves is unable to model network traffic where queuing at ramps has to be accounted for. A further attempt to address this issue can be found in Ni [35] and Ni et al. [36], where on-ramp and off-ramp queuing models were proposed, on the basis of which the theory of kinematic waves was extended to network flows.

PROBLEMS

1. A 1.8-km link AB connects nodes A and B. The free-flow speed is 30 m/s and a triangular flow-density relationship is assumed for this link. How long does it take for traffic passing node A to arrive at node B if there is no congestion in this link?
2. Link AB above is followed by link BC of length 1 km. The same flow-density relationship applies to link BC with backward wave propagation speed −5 m/s. If an accident occurred at node C at 5:00 p.m., when will drivers at node B notice the impact of the accident?

3. A uniform freeway link x_1x_3 is 6 km long as illustrated in the figure below. The triangular flow-density relationship given below applies to this link. Node x_2 is the midpoint of this link. A segment of data is given below where D1, D2, and D3 are cumulative traffic counts at x_1, x_2, and x_3, respectively. On the basis of the simplified theory of kinematic waves, complete a one-step simulation by answering the following questions:

Time	D1	D2	D3
8:50	1965	1950	1551
8:51	1970	1957	1555
8:52	1978	1961	1560
8:53	1982	1968	1566
8:54	1987	1973	1571
8:55	1991	1980	1578
8:56	1996	1984	1583
8:57	2000	1990	1589
8:58	2008	1996	1594
8:59	2012	2000	1600
9:00	2018	2005	1605
9:01	2024		1610



a. How many vehicles are expected to arrive x_2 from 9:00 to 9:01?

b. Dictated by capacity only, what is the cumulative number of vehicles that are allowed to pass x_2 by 9:01?

c. What is the jam storage in x_2x_3—that is, the number of vehicles that can be stored in x_2x_3 at jam density?

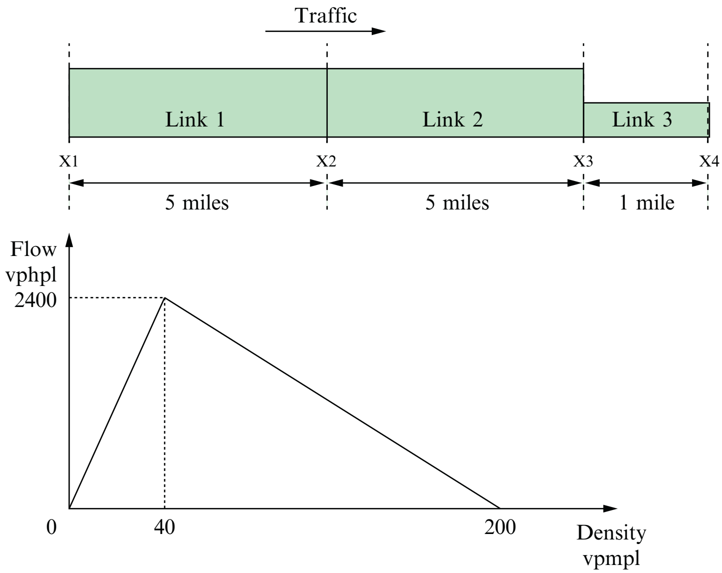
d. On the basis of the condition in x_2x_3 only, how many vehicles are allowed to enter x_2x_3 by 9:01 at most?

e. What is the cumulative number of vehicles that actually pass x_2 by 9:01?

4. A freeway corridor consists of three links whose physical properties are tabulated below. Also provided in the figure below is the underlying flow-density relationship of the freeway corridor. Assume the freeway corridor was initially empty and subsequent traffic arrival from the upstream end is as given in the second table below. Use the simplified theory of kinematic waves to simulate traffic evolution on this freeway corridor. You may use an Excel spreadsheet or a computer program such as MATLAB if necessary.

	Link 1	Link 2	Link 3
Lanes	2	2	1
Free-flow speed	60	60	60
Capacity	4800	4800	2400
Jam	400	400	200

Time	Flow (vehicles/h)
0:00:00	0
1:00:00	120
2:00:00	240
3:00:00	480
4:00:00	600
5:00:00	1200
6:00:00	1500
7:00:00	1800
8:00:00	3000
9:00:00	3600
10:00:00	1800
11:00:00	1200
12:00:00	1500
13:00:00	900
14:00:00	1200
15:00:00	1500
16:00:00	2400
17:00:00	3600
18:00:00	2100
19:00:00	1500
20:00:00	1200
21:00:00	600
22:00:00	240
23:00:00	0


Traffic
Link 1
Link 2
Link 3
X1
X2
X3
X4
5 miles
5 miles
1 mile
Flow
vphpl
2400
0
40
200
Density
vpmpl