Single Choice

(1) Let $\dim V = 4$. Then, there is a $\varphi \in V^*$ with $\dim \ker \varphi = 2$

(a) True

(b) False

Solution. (b) False.

$\varphi \in V^* = \hom_K(V, K)$[1], where V is a vector space over a field F. Let us assume such a function φ exists. We use rank-nullity[2] to realise that

$$\begin{aligned}\dim \operatorname{Im} \varphi &= \dim V - \dim \ker \varphi \\ &= 4 - 2 = 2\end{aligned}$$

The dimension of the image of φ has to be less than the dimensionality of the codomain i.e $\dim \operatorname{Im} \varphi \leq \dim K = 1$, which is a contradiction. $\square$

(2) Every finite-dimensional vector space is the dual space of another finite-dimensional vector space.

(a) True

(b) False

Solution. (a) True

We can define an isomorphism from a vector space V to its dual V^* in the following manner.

Since V is finite dimensional, we can pick a basis $\{v_1, v_2, \dots, v_n\}$ for V. Then we can define the dual basis $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ of V^* as $\varphi_i(v_j) = \delta_{ij}$ **extended linearly**.

That means that for a vector $v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n \in V$,

$$\begin{aligned}\varphi_i(v) &= a_1 \varphi_i(v_1) + a_2 \varphi_i(v_2) + \ldots + a_n \varphi_i(v_n) \\ &= a_i\end{aligned}$$

Remark. Now, just because I call this set a basis for V^* doesn't make it one. Do convince yourself that this indeed a basis. You can go about doing this by showing linear independence and that V^* is spanned by this set

This defines an isomorphism between V and V^*. It is not a canonical isomorphism as the construction of our isomorphism depends on the choice of our basis. $\square$

(3) The set of invertible real $n \times n$ matrices is

(a) not a real subspace of $\mathbb{M}_n(\mathbb{R})$

(b) a real subspace of $\mathbb{M}_n(\mathbb{R})$

Solution. (a) not a real subspace of $\mathbb{M}_n(\mathbb{R})$

The set of invertible matrices is not closed under addition as we can see from the below example

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

as the null matrix is not invertible □

(4) Let $f : V \to W$ be an arbitrary homomorphism between two K-vectorspaces. Which of the following five statements is not equivalent to the others?

(a) f is injective

(b) The dual mapping $f^* : W^* \to V^*$ is surjective

(c) The zero element of V is the only element that is mapped to the zero element of W

(d) There is a homomorphism $g : W \to V$ with $f \circ g = \mathrm{id}_W$

(e) For every $v \in V \backslash \{0\}$ there exists an $l \in W^*$ with $l(f(v)) \neq 0$

(f) All five statements are equivalent.

Solution. Statement (d) is not equivalent to the others

Firstly, notice that (a) and (c) are equivalent. An element $w \in W$ is non-zero only when an $l \in W^*$ exists with $l(w) \neq 0$. (e) is equivalent to saying that $\forall v \in V \backslash \{0\} : f(v) \neq 0$ which makes (e) equivalent to (a) and (c). You should have seen the equivalence of (b) and (a) in an earlier exercise (d) is equivalent to the surjectivity of f but not its injectivity. □

Multiple Choice

(1) For which values of x is the matrix $A = \begin{bmatrix} 1 & x & 1 \\ 3 & 3 & x \\ 0 & 3 & 1 \end{bmatrix}$ not invertible?

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

Solution. (c) 2

A square matrix is invertible if and only if its determinant is non-zero Therefore, we can check by setting the determinant of A to be 0 i.e

$$\begin{aligned} \det A &= 0 \\ \implies 1(3-3x) - 3(x-3) &= 0 \\ \implies x &= 2 \end{aligned}$$

□

Write it out

(1) Calculate the determinant of the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \end{bmatrix}$$

over $\mathbb{R}$ and $\mathbb{F}_5$. Is it invertible?

Solution.

Remark. For large square matrices, you can compute the determinant in a smart manner by the choice of the row/column you expand along. More often than not, it is the row/column with most 0s.

In this case, you can start by expanding along C_1. Figure out which rows/columns you should be expanding along for the two 4×4 sub-matrices to get to the answer (relatively quickly).

You should obtain $\det B = 55$. Notice, that in $\mathbb{R}$, B is invertible as the determinant is non zero. In $\mathbb{F}_5$[3], $\det B = \overline{55} = \overline{0}$. Hence, B is not invertible.

Remark. If you have to compute the determinant of a matrix in $\mathbb{F}_n$, you can reduce the elements of the matrix modulo n before computing the determinant to make calculations (and life) easier. You will get the same value of the determinant modulo n. Reason to yourself why that works.

□

(2) (a) Let K be a field, $\lambda \in K$ and let $A \in M_{n\times n}(K)$. Show that:

i. Let B be so that $A \xrightarrow{\lambda L_i \to L_i} B$. Then, $\det B = \lambda \det A$

ii. Let B be so that $A \xrightarrow{L_i \leftrightarrow L_i} B$. Then, $\det B = -\det A$

iii. Let B be so that $A \xrightarrow{\lambda L_i + L_j \to L_j} B$ with $i \neq j$. Then, $\det B = \det A$

Solution. We follow a general template for all three proofs. **Without loss of generality**, we assume that the operations are being performed on R_1 and R_2 Consider a matrix E such $EA = B$. If E is invertible, then $\det B = \det E \times \det A$ as A is already known to be invertible.

i.

$$E = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, $\det E = \lambda$

ii.

$$E = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, $\det E = -1$

iii.

$$E = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, $\det E = 1$

Remark. Why can you do this without loss of generality? You can make this kind of augmentation to other entries of E to affect the corresponding rows. To do the same thing for columns, consider the same matrices but with $B = AE$ instead of $B = EA$

□

(b) The numbers 2014, 1484, 3710 and 6996 are all divisible by 106. Show without calculating that

$$\det \begin{bmatrix} 2 & 1 & 3 & 6 \\ 0 & 4 & 7 & 9 \\ 1 & 8 & 1 & 9 \\ 4 & 4 & 0 & 6 \end{bmatrix}$$

is also divisible by 106

Hint: Read the numbers in each column from top to bottom.

Solution. Assume $2014 = 106a_1$, $1484 = 106a_2$, $3710 = 106a_3$ and $6996 = 106a_4$

$$det \begin{bmatrix} 2 & 1 & 3 & 6 \\ 0 & 4 & 7 & 9 \\ 1 & 8 & 1 & 9 \\ 4 & 4 & 0 & 6 \end{bmatrix} \overset{i}{=} \det \begin{bmatrix} 2 & 1 & 3 & 6 \\ 0 & 4 & 7 & 9 \\ 1 & 8 & 1 & 9 \\ 2014 & 1484 & 3710 & 6996 \end{bmatrix}$$
$$\overset{ii}{=} 106 \det \begin{bmatrix} 2 & 1 & 3 & 6 \\ 0 & 4 & 7 & 9 \\ 1 & 8 & 1 & 9 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix}$$

Step i: $1000R_1 + 100R_2 + 10R_3 + R_4 \to R_4$

Step ii: $106R_4 \to R_4$

The determinant is divisible by 106 □

(3) Compute the determinants for the matrices,

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & 1 \\ 1 & 2 & -3 & 1 \\ 0 & -4 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & -3 & 5 & 1 & 4 \\ 2 & -3 & 1 & -6 & 18 \\ 4 & -3 & 9 & 6 & 10 \\ -2 & 4 & -6 & -1 & -1 \\ -6 & 11 & -23 & -14 & 9 \end{bmatrix}$$

Solution. You can observe that for B, $R_1 = R_3 + R_5$ which leads you to conclude that $\det B = 0$. The rest we compute by Gaussian elimination[4].

$$\det A = -4$$
$$\det B = 0$$
$$\det C = 24$$

□

References

[1] Wikipedia contributors, "Dual space — Wikipedia, the free encyclopedia." https://en.wikipedia.org/w/index.php?title=Dual_space&oldid=1206157626, 2024. [Online; accessed 17-February-2024].

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[3] Wikipedia contributors, "Cyclic group — Wikipedia, the free encyclopedia." https://en.wikipedia.org/w/index.php?title=Cyclic_group&oldid=1194831654, 2024. [Online; accessed 17-February-2024].

[4] Wikipedia contributors, "Gaussian elimination — Wikipedia, the free encyclopedia." https://en.wikipedia.org/w/index.php?title=Gaussian_elimination&oldid=1200010964, 2024. [Online; accessed 17-February-2024].