2 $\langle A, B \rangle = \text{trace}(A^t B)$

Gram Schmidt

3. Let $B_S = \{v_1, v_2, \ldots, v_m\}$ be an orthonormal basis for $S$

It can be extended to a basis $B_V = \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$ for $V$

orthonormalise !

4. a orthonormalise $\{ v_1 , v_2 \}$ for an orthonormal basis, $\{ w_1 , w_2 \}$ of $U$

Solve $\langle w_1 , v \rangle = \langle w_2 , v \rangle = 0$

How does that help?

b $\pi : \mathbb{R}^3 \longrightarrow U$

$v \rightsquigarrow \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2$

$\pi(e_1), \pi(e_2), \pi(e_3) = (?) w_1 + (?) w_2$

what would the matrix look like?

## Digression :

$$L_A : U \longrightarrow V$$

$\nearrow$ basis $\qquad\qquad \nwarrow$ basis

$\{u_1, u_2, \ldots, u_n\} \qquad \{v_1, v_2, \ldots, v_m\}$

$$L_A(u_i) = \sum_{j=1}^{m} a_{ji} \, v_j \qquad \forall \; 1 \leqslant i \leqslant n$$

$$\Rightarrow A = (a_{ij})$$

5 Prove :

Given a basis $B = (v_1, v_2, \dots, v_n)$.

$T$ is upper triangular $\Leftrightarrow$

$\{v_1, v_2, \dots, v_j\}$ is a $T$- invariant subspace for all $j$.

use this along with another property from Gram Schmidt

6\. a) $V = \{(a_0, a_1, \ldots), \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$

$$\langle (a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty} \rangle = \sum_{n=0}^{\infty} a_n b_n$$

$U_1 = \{ a_n \mid \exists N \geq 0$ so that $a_m = 0 \ \forall m \geq N \}$

$\{ 1, 0, 3, 0, 0, 5, 0, 0, 0, 0$

$U \in U_1$

Let $v \in U^{\perp}$

$$\{ u_1, u_2, \dots \} \in U$$

$$\langle v, u_1 \rangle = \langle v, u_2 \rangle = \dots$$

$$u_1 = (1, 0, 0, \dots$$

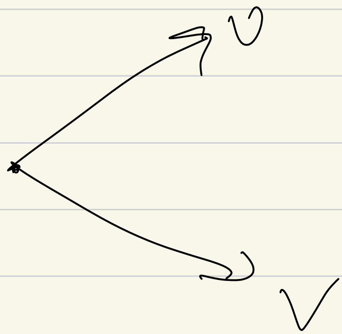
$$u_2 = (0, 1, 0, 0, \dots$$

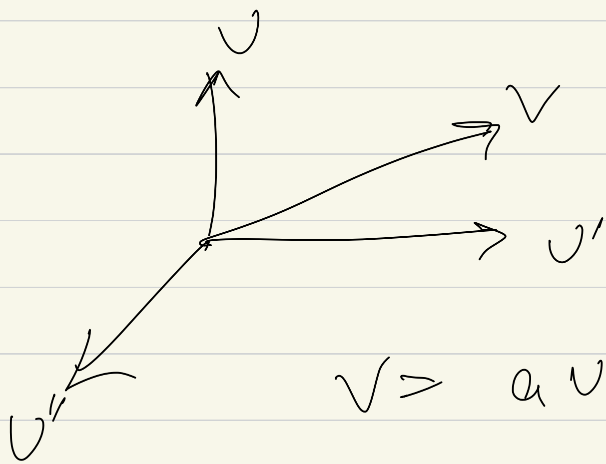
$$u_3 = (0, 0, 1, 0$$

$$\langle v, u_k \rangle = v_k = 0$$

$$v = (v_1, v_2, \dots)$$

$$\pi_u(v) = \frac{\langle u, v \rangle}{\langle u, u \rangle} u$$




$$v = au + bu' + cu''$$

$$\pi_u(v) = au$$

$\{v_1, v_2, \ldots, v_n\}$ for $V$

$u_1 = v_1$

$u_2 = v_2 - \pi_{u_1}(v_2)$

$u_3 = v_3 - \pi_{u_1}(v_3)$

$\qquad - \pi_{u_2}(v_3)$

$\{u_1, u_2, \ldots, u_n\}$

$\left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \ldots, \frac{u_n}{\|u_n\|} \right\}$

$$u_1 = \frac{v_1}{\|v_1\|}$$

$$u_2 = \frac{v_2 - \langle v_2, u_1 \rangle u_1}{\|v_2 - \langle v_2, u_1 \rangle u_1\|}$$

2. $V$ of polynomials of degree $\leq n$

$$\langle p, q \rangle = \int_0^\infty p(t) q(t) e^{-t} dt$$

$$p = p_0 + p_1 x + p_2 x^2 + \dots + p_n x^n$$

$$p_c = [p_0 \quad p_1 \quad p_2 \quad \dots \quad p_n]$$

$$\langle p, q \rangle = p_c^t A q_c$$

$$A = (a_{ij})$$

$$a_{ij} = \langle x^{i-1}, x^{j-1} \rangle$$

$\{ v_1, v_2, \ldots, v_n \}$ for $V$

$$A = (a_{ij})$$

$$a_{ij} = \langle v_i, v_j \rangle$$

$v_1 =$ $\quad v_2 = x, \ldots v_{n+1} = x^n$

$$a_{ij} = \langle x^{i-1}, x^{j-1} \rangle$$

$$= \int_0^\infty x^{i+j-2} e^{-t} \, dt$$

$$= (i+j-2)!$$

3. $T : \mathrm{Hom}(V)$

$$\forall v \quad \|Tv\| \leq \|v\|$$

show : $T - \sqrt{2} I$ is invertible

$\exists\ v \neq 0,$

$$(T - \sqrt{2} I) v = 0$$

$$Tv = \sqrt{2} v$$

$$\|Tv\| = \sqrt{2} \|v\|$$

$$\|Tv\| \leq \|v\|$$

$$\langle\ ,\ \rangle : V \times V \to \mathbb{R}$$

- $\langle v_1 + v_2, u \rangle = \langle v_1, u \rangle + \langle v_2, u \rangle$

  $\langle v, u_1 + u_2 \rangle = \langle v, u_1 \rangle + \langle v, u_2 \rangle$

- $\langle \alpha v, u \rangle = \alpha \langle v, u \rangle$

  $\langle v, \alpha u \rangle = \alpha \langle v, u \rangle$

- $\langle v, u \rangle = \langle u, v \rangle$

- $\langle v, v \rangle > 0$

$$\langle\ ,\ \rangle : V \times V \to \mathbb{C}$$

- $\langle v_1 + v_2, u \rangle = \langle v_1, u \rangle + \langle v_2, u \rangle$

$$\langle v, u_1 + u_2 \rangle = \langle v, u_1 \rangle + \langle v, u_2 \rangle$$

- $\langle \alpha v, u \rangle = \alpha \langle v, u \rangle$

$$\langle v, \alpha u \rangle = \overline{\alpha} \langle v, u \rangle$$

- $\langle v, u \rangle = \overline{\langle u, v \rangle}$
- $\langle v, v \rangle > 0$

$$\langle v, u_1 + u_2 \rangle = \langle v, u_1 \rangle + \langle v, u_2 \rangle$$

$$\langle v, u_1 + u_2 \rangle$$

$$= \overline{\langle u_1 + u_2, v \rangle}$$

$$= \overline{\langle u_1, v \rangle + \langle u_2, v \rangle}$$

$$= \overline{\langle u_1, v \rangle} + \overline{\langle u_2, v \rangle}$$

$$= \langle v, u_1 \rangle + \langle v, u_2 \rangle$$

$$\langle v, \alpha u \rangle = \bar{\alpha} \langle v, u \rangle$$

follows