

CE 594 - Final Project

This project and all associated code are available on Tristan's website:

http://atdyer.github.io/ncsu/courses/ce594/project/index.html

For this project, we are going to analyze a seepage problem. Imagine there is a sand filled pipe buried underground below the water table and oriented from left to right. The pipe has a cross-sectional area of 2m^2 and is 10m long. Water is flowing into the left end of the pipe at a rate of 100m^3/day. At the right end of the pipe we measure the hydraulic head, h, (look it up on wiki if you need to) to be 3m. The wall of the pipe is made of a permeable material so that water can pass through, which effectively creates a source or sink depending on the hydraulic head inside the pipe. We will represent this as a source term of the form $\alpha^0 h$.

The governing differential equation can be derived from Darcy's Law, which relates the volume rate of flow per unit area to the rate of change of the hydraulic head, given by

$$q = -k\frac{dh}{dx}$$

where k is the hydraulic conductivity. For a steady state flow and assuming that water is incompressible, conservation of mass tells us that the amount of water flowing into any part of the pipe must be equal to the amount of water flowing out. In other words, any change in the volume rate of flow must balance any water gained or lost through a source or sink, respectively. Mathematically, this is stated as

$$\frac{d}{dx}(Aq) - s = 0$$

where s is the source term. For this problem we can replace$s$ with $\alpha^0 h$.

Using the following parameters

1. State the governing strong form problem, identifying essential and natural boundary conditions
2. Derive the weak form
3. Define appropriate approximation spaces and the Galerkin problem
4. Develop the matrix equations showing how all coefficients are calculated
5. Modify your finite element code to solve this problem using at least 8 elements
6. Plot your solution and state the hydraulic head at the left end of the pipe and the amount of water flowing out of the right end

$$\begin{aligned} A &= 2\text{m}^2 \\ k &= 250\text{m/day} \\ \alpha^0 &= -5\text{m/day} \end{aligned}$$

Solution

The complete hand-written solution can be found in the following pdf document:

Solution

The hand-written solution contains the complete derivation of the matrix form starting from the strong form. Below, the strong form, weak form, Galerkin form, and matrix form are shown for reference.

Strong Form

$$\frac{d}{dx}(-Ak\frac{du}{dx}) - S = 0$$

$$u(L) = 3$$

$$\frac{du}{dx}(0) = -0.2$$

Weak Form

$$\int_{\Omega} \frac{dw}{dx} Ak \frac{du}{dx} dx = \int_{\Omega} wSdx - wAk\frac{du}{dx}\bigg|_{\Gamma^h}$$

Galerkin Form

$$a(w^h, v^h) - (w^h, v^h) = (w^h, g^h) - a(w^h, g^h) - w^h Ak\frac{du}{dx}\bigg|_{\Gamma^h}$$

Matrix Form

$$([K] - [M])\{d\} = [F]$$

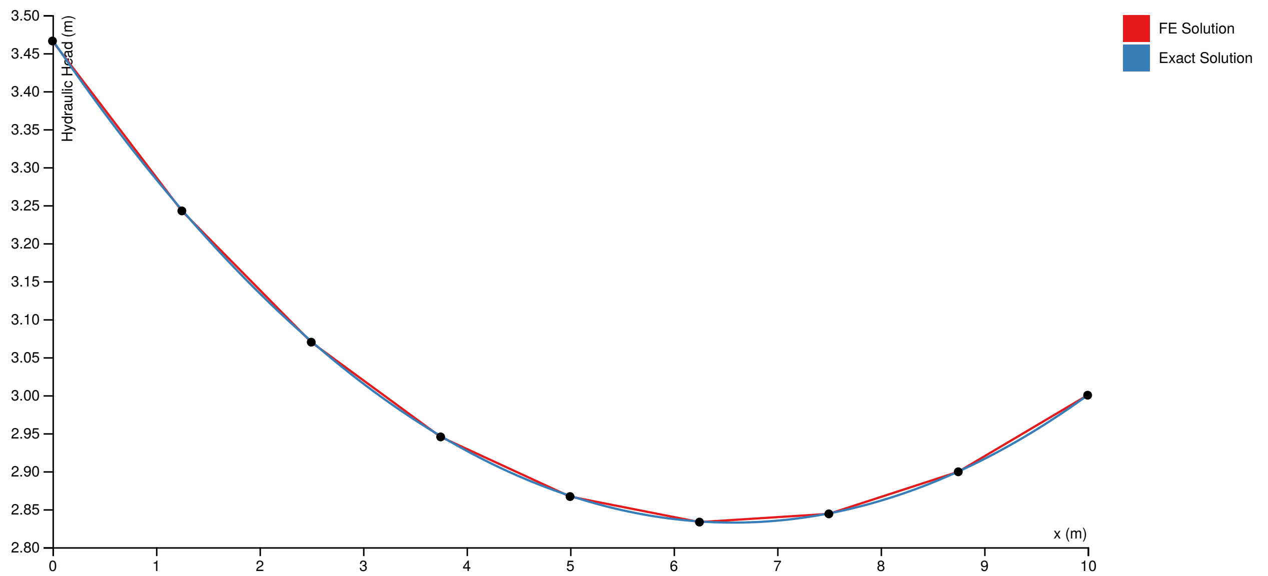
Code and Implementation

The code used to solve the matrix form of the equation is written in Python and composed of 3 scripts. A fourth script, containing the problem definition is used to execute the code and plot the results.

- **project.py** - This script contains the problem definition. Run this script to solve the problem and generate a plot.
- **fe.py** - This is the finite element implementation used to solve this problem. It builds the matrices defined in the above matrix form of the governing equation and solves for the profile of the hydraulic head along the length of the pipe.
- **jacobian.py** - This script contains functions used in coordinate transformations.
- **shape_functions.py** - This script contains a function that can be used to generate shape functions and their derivatives for any number of element nodes. Further implementation details can be found on the page.

Results

The following interactive plot shows the results of the finite element code compared to the exact solution. Use the number pickers below the plot to change the number of element nodes and number of elements.



Number of Element Nodes: 2 **Number of Elements:** 8

We can see that as the number of element nodes and number of elements increase, the finite element solution converges on the exact solution, and we're able to calculate the following values:

- Hydraulic head at the left end of the pipe:**3.47m**
- Flow rate at the right end of the pipe:**-49.43m^3/day**