Let's say hydraulic head is $u(x)$

STRONG FORM

$$\frac{d}{dx}\left(-Ak\frac{du}{dx}\right) - s = 0$$

(S)

$$u(L) = 3 \quad \leftarrow \text{essential b.c.}$$

$$\frac{du}{dx}(0) = -0.2 \quad \leftarrow \text{natural b.c.}$$

Define :

$$S = \{ u \mid u \in H^1,\ u(L) = 3 \}$$
$$V = \{ w \mid w \in H^1,\ w(L) = 0 \}$$

- Multiply both sides by -1 to get strong form as:

$$\frac{d}{dx}\left(Ak\frac{du}{dx}\right) + s = 0$$

- Multiply both sides by the weight function and integrate:

$$\int_\Omega w\left[\frac{d}{dx}\left(Ak\frac{du}{dx}\right)\right]dx + \int_\Omega ws\,dx = 0$$

- Integration by parts on first term :

$$\left\{ \begin{array}{l} \int_a^b fg'\,dx = fg\Big|_a^b - \int_a^b f'g\,dx \\ \quad f = w \\ \quad g = Ak\,du/dx \end{array} \right.$$

$$wAk\frac{du}{dx}\Big|_0^L - \int_\Omega \frac{dw}{dx}Ak\frac{du}{dx}\,dx + \int_\Omega ws\,dx = 0$$

- Rearranging:

$$\int_\Omega \frac{dw}{dx} Ak \frac{du}{dx} dx = wAk\frac{du}{dx}\Big|_0^L + \int_\Omega ws\, dx$$

- Since $w(L) = 0$,

$$\int_\Omega \frac{dw}{dx} Ak \frac{du}{dx} dx = -wAk\frac{du}{dx}\Big|_{x=0} + \int_\Omega ws\, dx$$

WEAK FORM

(W) Find $u \in S$ such that $\forall\ w \in V$:

$$\int_\Omega \frac{dw}{dx} Ak \frac{du}{dx} dx = \int_\Omega ws\, dx - wAk\frac{du}{dx}\Big|_{x=0}$$

Define:

$S^h \subset S$, i.e. if $u^h \in S^h$, then $u^h \in S$

$V^h \subset V$, i.e. if $v^h \in V^h$, then $v^h \in V$

where the h superscript refers to a discretization of the domain Ω, which is parameterized by a characteristic length scale, h.

Thus we can infer that:

$u^h = g$ on Γ_g

$w^h = 0$ on Γ_g

Given a function $v^h \in V^h$ and a function g^h which satisfies the natural boundary conditions, we can define u^h to be:

$$u^h = v^h + g^h$$

- Note that u^h still satisfies the conditions of S that $u = g$ on Γ_g:

$$u^h\Big|_{\Gamma_g} = v^h\Big|_{\Gamma_g} + g^h\Big|_{\Gamma_g}$$

$$= 0 + g = g \checkmark$$

- Substitute the definitions of u^h and w^h into the weak form;

$$\int_\Omega \frac{dw^h}{dx} Ak \frac{d(v^h + g^h)}{dx}\, dx = \int_\Omega w^h s\, dx - w^h Ak \frac{du}{dx}\Big|_{x=0}$$

- The source term for this problem includes u, so substitute u in, and perform u^h substitution:

$$\int_\Omega \frac{dw^h}{dx} Ak \frac{d(v^h + g^h)}{dx}\, dx = \int_\Omega w^h \alpha_0 h\, dx - w^h Ak \frac{du}{dx}\Big|_{x=0}$$

$$\int_\Omega \frac{dw^h}{dx} Ak \frac{d(v^h + g^h)}{dx}\, dx = \int_\Omega w^h \alpha_0 (v^h + g^h)\, dx - w^h Ak \frac{du}{dx}\Big|_{x=0}$$

- Expanding ...

$$\int_\Omega \frac{dw^h}{dx} Ak \frac{dv^h}{dx} dx + \int_\Omega \frac{dw^h}{dx} Ak \frac{dg^h}{dx} dx = \int_\Omega w^h \alpha_0 v^h dx + \int_\Omega w^h \alpha_0 g^h dx - w^h Ak \frac{du}{dx}\Big|_{x=0}$$

- Rearranging ...

$$\int_\Omega \frac{dw^h}{dx} Ak \frac{dv^h}{dx} dx - \int_\Omega w^h \alpha_0 v^h dx = \int_\Omega w^h \alpha_0 g^h dx - \int_\Omega \frac{dw^h}{dx} Ak \frac{dg^h}{dx} dx - w^h Ak \frac{du}{dx}\Big|_{x=0}$$

- DEFINE:

$$a(f,g) = \int_\Omega \frac{df}{dx} Ak \frac{dg}{dx} dx \qquad (f,g) = \int_\Omega f \alpha_0 g\, dx$$

Using these definitions, we can write the Galerkin form as:

$$(G) \qquad a(w^h, v^h) - (w^h, v^h) = (w^h, g^h) - a(w^h, g^h) - w^h A k \frac{du}{dx}\Big|_{x=0}$$

To get to the matrix form, let's first define some notations:

η = set of all nodes
η_g = nodes at boundary conditions (essential)
$\eta_f = \eta \backslash \eta_g$ = nodes at dofs

If $w^h \in V^h$, then there exists constants, C_A, for $A = 1, 2, \ldots, n$ such that

$$w^h = \sum_{A=1}^{n} C_A N_A$$

where N_A must satisfy

$$N_A \Big|_{\eta_g} = 0 \qquad (\text{ie } = 0 \text{ at essential b.c.s})$$

an C_A is an arbitrary constant. This can therefore be written as a sum of the non-boundary nodes:

$$w^h = \sum_{A \in \eta_f} C_A N_A$$

The same can be applied to v^h and g^h, keeping in mind that $v^h + g^h$ must equal u^h, giving us the following:

$$v^h = \sum_{B=\eta_f} d_B N_B \qquad g^h = \sum_{B=\eta_g} d_B N_B$$

Note this is at dofs

Note this is at essential b.c.

Now substitute into the Galerkin form:

$$a\left(\sum_{A\in\eta_f} c_A N_A, \sum_{B\in\eta_f} d_B N_B\right) - \left(\sum_{A\in\eta_f} c_A N_A, \sum_{B\in\eta_f} d_B N_B\right)$$

$$=$$

$$\left(\sum_{A\in\eta_f} c_A N_A, \sum_{B\in\eta_g} d_B N_B\right) - a\left(\sum_{A\in\eta_f} c_A N_A, \sum_{B\in\eta_g} d_B N_B\right) - Ak\frac{du}{dh}\left[\sum_{A\in\eta_f} c_A N_A\right]_{\Gamma_h}$$

Recall that this is a known value at Γ_h

Because c_A is arbitrary, we can choose $c_A = 1$ for a single value of A and $c_A = 0$ for all others, resulting in:

$$c_A\, a\left(N_A, \sum_{B\in\eta_f} d_B N_B\right) - c_A\left(N_A, \sum_{B\in\eta_f} d_B N_B\right)$$

$$=$$

$$c_A\left(N_A, \sum_{B\in\eta_g} d_B N_B\right) - c_A\, a\left(N_A, \sum_{B\in\eta_g} d_B N_B\right) - c_A N_A Ak\frac{du}{dh}\Big|_{\Gamma_h}$$

Divide the entire equation by c_A to get:

$$a\left(N_A, \sum_{B\in\eta_f} d_B N_B\right) - \left(N_A, \sum_{B\in\eta_f} d_B N_B\right)$$

$$=$$

$$\left(N_A, \sum_{B\in\eta_g} d_B N_B\right) - a\left(N_A, \sum_{B\in\eta_g} d_B N_B\right) - N_A Ak\frac{du}{dh}\Big|_{\Gamma_h}$$

Using the bilinearity of $(\cdot,\cdot)$ and $a(\circ,\circ)$, this becomes:

$$\sum_{B\in\eta_f} a(N_A,N_B)\,d_B - \sum_{B\in\eta_f}(N_A,N_B)\,d_B$$

$$=$$

$$\sum_{B\in\eta_g}(N_A,N_B)\,d_B - \sum_{B\in\eta_g} a(N_A,N_B)\,d_B - N_A A k \frac{du}{dh}\Big|_{\Gamma_h}$$

Now define the following:

$$K_{AB} = a(N_A,N_B)$$

$$M_{AB} = (N_A,N_B)$$

$$F = \sum_{B\in\eta_g}(N_A,N_B)d_B - \sum_{B\in\eta_g} a(N_A,N_B)d_B - N_A A k \frac{du}{dh}\Big|_{\Gamma_h}$$

And we're left with:

$$\sum_{B\in\eta_f} K_{AB}d_B - \sum_{B\in\eta_c} M_{AB}d_B = F$$

Which can be simplified and written in matrix form as:

(M) $$([K]-[M])\{d\} = [F]$$