MA1114 25/10/21

Gauss Algorithm

We want to transform the system $(A|b)$ to a system $(B|c)$

where B is in row echelon form using row operations (so the solutions can be easily read off)

<u>Definition</u> (row echelon form)

A matrix is in (row) echelon form if:

- first non-zero (leading) entry in every row is 1
- in any two successive non-zero rows, the leading 1 in the lower row occurs further to the right then that of the highest row
- zero rows occur at the bottom

in reduced echelon if every column containing a leading 1 has zeros in every other entry

<u>definition of row echelon form</u>

$$\begin{pmatrix} 1 & * & \cdots & \cdots & * \\ 0 & 1 & * & \cdots & * \\ \vdots & & \ddots & & \vdots \\ \vdots & \cdots & 0 & \ddots & \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Gauss Algorithm

① find the left most non-zero column.

② swap rows (if necessary) to bring a non-zero entry to the top of this column

③ multiply the first row by a scalar to make the top left entry 1

④ add multiples of the first row to other rows until the entries below the top left entry are all 0

⑤ cover first row and repeat the process ① to ④ for n rows

Example 2.25

① $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$ $b = \begin{pmatrix} 0 \\ 8 \\ -9 \end{pmatrix}$ we want to solve $Ax=b$ $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$

augmented matrix

$R_3 \mapsto R_3 + 4(R_1)$ $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$

$R_2 \mapsto \frac{1}{2} R_2$ $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$

$R_3 \mapsto R_3 + 3R_2$ $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

② $\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right] = (A|b)$

$R_1 \leftrightarrow R_2$ $\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right]$

$R_1 \mapsto \frac{1}{2}R_1$ $\left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right]$

$R_3 \mapsto R_3 - 5R_1$ & $R_3 + \frac{1}{2}R_2$ $\left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & -5/2 \end{array}\right]$ inconsistent system $\Rightarrow Ax = b$ has no solution

<u>Gauss - Jordan Algorithm</u> (slightly better than Gauss)

Transform into reduced echelon form

① transform echelon form using the Gauss Algorithm

② working upwards from the lowest non-zero row; clear all entries above each leading 1

Example 2.25

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_2 \mapsto R_2 + 4R_3 \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_1 \mapsto R_1 - R_3 \quad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_1 \mapsto R_1 - 2R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$