4 Invariant Subspaces

<u>Aim:</u> Relate block triangular (diag) matrices to linear operators

$A = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \longleftrightarrow T_A : \mathbb{R}^n \to \mathbb{R}^n$?

block triangular

$\begin{bmatrix} \square & * & * \\ 0 & \square & * \\ 0 & 0 & \square \end{bmatrix}$ block - Δ

$\begin{bmatrix} \square & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & \square \end{bmatrix}$ block - diag

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is also block - diagonal (size 1 blocks)

Recall $A = \begin{bmatrix} \boxed{A_1} & * & * \\ 0 & \boxed{A_2} & * \\ 0 & 0 & \boxed{A_3} \end{bmatrix}$ block-Δ

$\Rightarrow \det A = \det A_1 \cdot \det A_2 \cdot \det A_3$

so char. pol. $\Delta_A(t) = \Delta_{A_1}(t) \cdots \Delta_{A_2}(t)$

$A = \begin{bmatrix} \boxed{A_1} & 0 & 0 \\ 0 & \boxed{A_2} & 0 \\ 0 & 0 & \boxed{A_3} \end{bmatrix}$ block - diagonal

$\Rightarrow m_A = \mathrm{LCM}(m_1, m_2, m_3)$

(e.g for $A = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right] \Rightarrow m(t) = \mathrm{LCM}(t-1, t-1) = t-1$

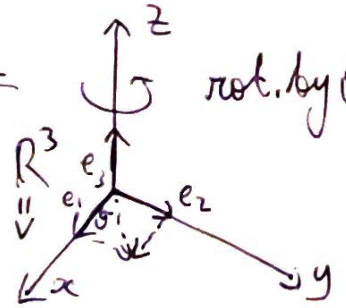
<u>Definition 4.1</u> Let $T : V \to V$ linear operators. A subspace W of V is <u>T-invariant</u> if $T(w) \in W \ \forall w \in W$ (i.e $T(W) \subseteq W$)

If W is T-invariant then we get new operator on W,

$\therefore T_W : W \to W$ (T_W is a restriction of T on W)

<u>Example 4.2</u> rot. by θ

$T : \mathbb{R}^3 \to \mathbb{R}^3$



$[T]_E = \Big[[T(e_1)]_E \cdots [T(e_3)]_E \Big]$

$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = [T]_{\mathcal{E}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}$$

T-invariant subspaces: $\{0\}, V$

$\rightarrow$ $\dim W = 2$: $W =$ xy-plane is T-invariant

the restriction of T to W: $T_W : W \rightarrow W$ (rot. by θ in 2D around the origin)

$$W = \left\{\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}\right\} \text{ so } T(W) \subseteq W$$

$$[T_W]_{\mathcal{E}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\rightarrow$ $\dim W = 1$: $W =$ z-axis is T-invariant

$$\begin{aligned} T_W : W &\rightarrow W \\ z &\mapsto z \end{aligned} \quad \text{so } T_W = Id_W \quad [T_W]_{\mathcal{E}} = [1]$$

$$W = \left\{\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : z \in \mathbb{R}\right\} \quad T\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$$

<u>Exercise 4.3</u> Show that $\ker T = \{v \in V \mid T(v) = 0\}$ and $\operatorname{Im} T = \{T(v) \mid v \in V\}$ are T-invariant

<u>Proposition 4.4</u> Let $T : V \longrightarrow V$ and let W be a 1-dimensional subspace of V

Then W is T-invariant $\Leftrightarrow$ W is spanned by an e.vector

Proof "$\Rightarrow$" Suppose $T(W) \subseteq W$. Take any $0 \neq u \in W$

Then $T(u) \in W = \text{span}\{u\} = \{\alpha u \mid \alpha \in \mathbb{R}\}$, so $T(u) = \lambda u$
for some $\lambda \in \mathbb{R}$
so u is e.vector

"$\Leftarrow$" Suppose $W = \text{span}\{u\}$ where $T(u) = \lambda u$ $(\lambda \in \mathbb{R})$

Let $w \in W$. Then $w = \alpha u$ $(\alpha \in \mathbb{R})$, so

$T(w) = T(\alpha u) = \alpha T(u) = \alpha \lambda u = (\alpha\lambda) u \in W$
so W is T-invariant

$\blacksquare$

More Examples Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $[T_A]_E = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

T-invariant: $\{0\}$, $V = \mathbb{R}$
dim $= 0$ dim $= 2$

All other T-invariant subspaces have dim $= 1$.

By 4 4 if W is T-invariant and 1-dim then

$W = \text{span}\{u\}$ where u is e.vector

Hence the basic vectors e_1, e_2 are eigenvectors

so $W_1 = x$-axis
$W_2 = y$-axis