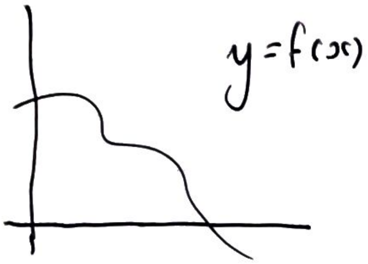
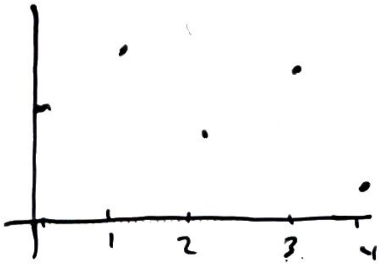
M1014 16/11/21

Sequences

Functions ✓ sequences $(a_n)_{n \in \mathbb{N}}$





$f: \mathbb{R} \to \mathbb{R}$ $\qquad$ $a: \mathbb{N} \to \mathbb{R}$

$f: D \to \mathbb{R}$

domain $\qquad$ $a_0, a_1, a_2, a_3, \ldots \in \mathbb{R}$

<u>examples</u> $f_0 = 0,\ f_1 = 1,$

inductively $f_{n+1} = f_n + f_{n-1}$ $\qquad$ $0, 1, 1, 2, 3, 5, 8, \ldots$

Behaviours: 1) unbounded
2) bounded

$a_n = (-1)^n$ $\quad 1, -1, 1, -1, 1, \ldots,$

$-1 \leq a_n \leq 1 \ \forall n$

3) monotonic increasing

$n < m \quad a_n \leq a_m$

$[\ x_1 < x_2 \quad f(x_1) < f(x_2)\ ]$

or monotonic decreasing

Limits For $f: \mathbb{R} \to \mathbb{R}$

we have defined $\lim_{x \to c} f(x)$

we can define

$\lim_{x \to \infty} f(x) = L$ if and only if

$\forall \varepsilon > 0 \ \exists N$ such that if $x > N$ then $|f(x) - L| < \varepsilon$

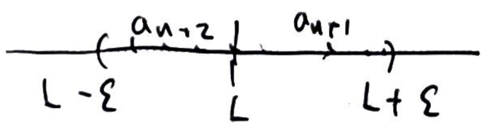
For sequences, we say

$(a_n)_{n \in \mathbb{N}}$ converges to L

$a_n \to L$ as $n \to \infty$, or

$\lim_{n \to \infty} a_n = L$ means

$$\forall \varepsilon > 0 \ \exists N : n > N \Rightarrow |a_n - L| < \varepsilon$$



sequences can be $\begin{cases} \text{bounded} \\ \text{unbounded} \end{cases}$ $\begin{cases} \text{monotonic} \\ \text{not} \end{cases}$ $\begin{cases} \text{convergent} \\ \text{divergent} \end{cases}$

Theorem If a limit of a sequence exists then it is unique

$a_n \to L$ & $a_n \to M \Rightarrow L = M$

Proof By Contradiction, If $L \neq M$

let $\varepsilon = \frac{|L-M|}{2} > 0$

$\exists N : |a_n - L| < \varepsilon$ if $n > N$
$\exists N' : |a_n - M| < \varepsilon$ if $n > N'$

triangle inequality

if $n > N, N'$ $|L-M| \leq |a_n - L| + |a_n - M| < 2\varepsilon$ $= \frac{|L-M|}{2}$

contradiction

Examples $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$a_n = \frac{1}{n+1}$. $n \in \mathbb{N}$

Does this sequence converge?
yes $a_n \to 0$

Proof Given any $\varepsilon > 0$, choose $N \in \mathbb{N}$ larger than $1/\varepsilon$. That $\frac{1}{N} < \varepsilon$
& $n > N \Rightarrow |a_n - 0| < \frac{1}{N} < \varepsilon$

Example $\frac{3}{2}, 1, \frac{5}{6}, \frac{3}{4}, \frac{7}{10}, \frac{8}{12}, \frac{9}{14}, \ldots$

$$a_n = \frac{n+3}{2n+2}$$

monotone decreasing: $\forall n \in \mathbb{N}$

$$a_{n+1} < a_n \iff \frac{n+4}{2n+4} < \frac{n+3}{2n+2}$$

$$\iff (n+4)(2n+2) < (n+3)(2n+4)$$

$$\iff 2n^2+10n+8 < 2n^2+10n+12$$

$$8 < 12$$

convergent ?

$$a_{1000000} = \frac{1000003}{2000002} \sim \frac{1}{2}$$

Proof $\lim_{n\to\infty} \frac{n+3}{2n+2} = \frac{1}{2}$

Control $|a_n - \frac{1}{2}| = \left| \frac{n+3}{2n+2} - \frac{1}{2} \right| = \left| \frac{(n+3)-(n+1)}{2n+2} \right| = \frac{2}{2n+2}$

$$= \frac{1}{n+1}$$

Given $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon}$

$$n > N \Rightarrow |a_n - \frac{1}{2}| = \frac{1}{n+1} < \frac{1}{N} < \varepsilon$$

so a_n converges to a $\frac{1}{2}$

We just saw two examples of monotonic bounded sequences.

<u>Theorem</u> If a sequence $(a_n)_{n\in\mathbb{N}}$ is

monotonic increasing and bounded above or

monotonic decreasing and bounded below then a_n is convergant to

$LUB\{a_n : n\in\mathbb{N}\}$ or $GLB\{a_n : n\in\mathbb{N}\}$

<u>Theorem</u> If a sequence

$(a_n)_{n\in\mathbb{N}}$ is convergent then it is bounded

<u>Proof</u> Given $\varepsilon=1$ in $\lim_{n\to\infty} a_n = L$

$\exists N$: if $n>N$ then $|a_n-L|<1$

$a_{N+1}, a_{N+2}, a_{N+3}, \ldots \in (L-1, L+1)$

So $|a_n| <$ maximum$(|L+1|, |L-1|)$ if $n>N$

& $|a_n| \leq \max\{|a_0|, |a_1|, \ldots, |a_N|, |L+1|, |L-1|\}$

$\forall n\in\mathbb{N}$

& a_n is bounded.

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bounded monotonic $\Rightarrow$ convergent

convergent $\Rightarrow$ bounded.