Potential Fields, Divergence

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Application of the gradient: Potential Vector Field:

A vector field is called **potential** if this field can be represented in the form:

$$\underline{f} = -grad(U) = -\nabla U$$

where $U(x, y, z)$ is a scalar functiuon. This function is called the **scalar potential** of the field.

Applications of the gradient: Level Curves

Let $f(x, y) = C$ be the equation for a level curve that passes through $\overrightarrow{r_0} = (x_0, y_0)$
∇f exists and the $\nabla f \neq 0$

parametrize the level curve as $\underline{r}(t) = (x(t), y(t))$, with $\overrightarrow{r}(t_0) = (x_0, y_0)$

$$\frac{d}{dt} f(\overrightarrow{r}(t)) = \nabla f(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t) = \frac{d}{dt} C = 0$$

$\boxed{\nabla f(\overrightarrow{r_0}(t)) \cdot \overrightarrow{a}(\overrightarrow{a_0}) = 0}$ Gradient is perpendicular to the level curve!

Applications of the gradient: Tangent Plane

Problem: a surface is given by the equation $f(x, y, z) = c$, where c is a constant, Assume that ∇f exists and that $\nabla f(x_0, y_0, z_0) \neq \underline{0}$ at a point (x_0, y_0, z_0). Find the tangent plane of the surface at the point (x_0, y_0, z_0).

Solution:

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Important Idea: $\nabla f(x_0, y_0, z_0)$ is perpendicular to the tangent plane of the surface $f(x, y, z) = C$

The normal line to a surface $f(x, y, z) = C$ passing through the point (x_0, y_0, z_0) is determined by:

$$\boxed{\frac{x-x_0}{f'_x} = \frac{y-y_0}{f'_y} = \frac{z-z_0}{f'_z}}$$

Divergence of a Vector Field:

Let $\underline{F} = (F_1, F_2, F_3)$ be a differentiable 3-dimensional vector field of 3 variables. We define the divergence of the vector field $\underline{\mathrm{F}}$, as

$$div f = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

We can also use the "∇" **notation for the divergence**. We have

$$\nabla \cdot \underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Thus, formally, the divergence is the scalar product of ∇ and $\underline{\mathrm{F}}$!

The divergence is a measure of the rate of change of a vector field in the radial direction!

Properties of the Divergence:

Let $\underline{F}, \underline{G} : \mathbb{R}^n \to \mathbb{R}^n$, $n = 2, 3$ vector fields, and $\phi : \mathbb{R}^n \to \mathbb{R}$ a scalar function. Then

- $\nabla \cdot (\underline{F} + \underline{G}) = \nabla \cdot \underline{F} + \nabla \cdot \underline{G}$
- $\nabla \cdot (\lambda \underline{F}) = \lambda \nabla \cdot \underline{F}$ for $\lambda \in \mathbb{R}$
- $\nabla \cdot (\phi \underline{F}) = (\nabla \phi) \cdot \underline{F} + \phi (\nabla \cdot \underline{F})$
- $\underline{\mathrm{F}}$ is a constant **then** $\nabla \cdot \underline{F} = 0$. The converse is **NOT true**
- A vector field $\underline{\mathrm{F}}$ such taht $\nabla \cdot \underline{F} = 0$ is called **incompressible**