MA1114 28/2/22

Matrix representation & Similarity preserves trace/det

<u>Corollary</u>

If $T: V \to V$ is a linear map and $\mathcal{B}_1, \mathcal{B}_2$ are two bases for V then ${}_{B_1}[T]_{B_2} = P^{-1}_{B_1 \to B_2} \; {}_{B_2}[T]_{B_2} \; P_{B_1 \to B_2}$

$$
\begin{tikzcd}
V \arrow[r] \arrow[d] & V \arrow[d] \\
\mathbb{R}^n \arrow[r, "{[T]_B}"'] & \mathbb{R}^n
\end{tikzcd}
\qquad
v \mapsto [v]_B
\qquad
v \mapsto [v]_B
$$

<u>Example</u>

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix}$

$\mathcal{B}_1 = \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

$\mathcal{B}_2 = \mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$; $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$P_{E \to B} [v]_E \overset{?}{=} [v]_B$

let's check $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad [v]_E = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$P_{E \to B} [v]_E = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix}$

fine since $(2x_1 - x_2)\begin{pmatrix}1\\1\end{pmatrix} + (-x_1 + x_2)\begin{pmatrix}1\\2\end{pmatrix}$

$= \begin{pmatrix}x_1\\x_2\end{pmatrix}$

last line calculated

$${}_B[T]_B = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix} \quad {}_E[T]_E = \begin{pmatrix}1 & 1\\-2 & 4\end{pmatrix}$$

check $P_{E \to B}\, {}_E[T]_E\, P_{B \to E}$

$= P^{-1}_{B_1 \to B_2}\, {}_E[T]_E\, P_{B_1 \to B_2}$

$$= \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\-2 & 4\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix} = {}_B[T]_B$$

---

we say ${}_E[T]_E$ and ${}_B[T]_B$ are similar

<u>Definition</u>

$A, B \in M_{n,n}(\mathbb{C})$ are <u>similar</u> if there exists an invertible matrix P such that $P^{-1}BP = A$

<u>Proposition</u> If A and B are similar $\det(A) = \det(B)$

<u>Proof</u> [recall if $X, Y \in M_n(\mathbb{C})$ then $\det(XY) = \det(X)\det(Y)$]

$$\begin{aligned}\det(A) &= \det(P^{-1}BP)\\ &= \det(P^{-1})\det(B)\det(P)\\ &= \det(B)\det(P^{-1})\det(P)\end{aligned}$$

$$= \det(B)\det(P^{-1}P)$$
$$= \det(B)\det(Id)$$
$$= \det(B) \times 1$$
$$= \det(B) \qquad \square$$

<u>Definition</u>

$A \in M_n(\mathbb{C})$, then $tr(A) = \sum_{i=1}^{n} A_{ii}$

<u>Proposition</u>

If $A, B \in M_n(\mathbb{C})$ are similar then $tr(A) = tr(B)$

<u>Proof</u> * see notes *