MA1114 20/10/21

Elementary Row Operations

$Ax = b$ $\qquad A \in M_{mn}$ $\qquad x$ "variable" vector $\quad b \in \mathbb{R}^n$

$A = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

upper triangular

$$\begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$$\begin{aligned} 4x_1 + 5x_2 + 6x_3 &= 7 \\ 2x_2 + 3x_3 &= 8 \\ x_3 &= 9 \end{aligned}$$

sol $x_3 = 9$

$2x_2 + 27 = 8 \Rightarrow x_2 = -\frac{19}{2}$

$4x_1 + -\frac{95}{2} + 54 = 7 \quad \Rightarrow x_1 = \frac{1}{8}$

Elementary row operations

$A \in M_{mn}, \ b \in \mathbb{R}^n$

idea: represent the linear system $Ax = b$ via an augmented matrix $(A|b)$ "an $m \times (n+1)$ matrix"

now transform $(A|b) \to (A'|b') \to (A''|b'') \to (A'''|b''') \to \ldots \to (A^k|b^k)$

where $A^{(k)}x = b^{(k)}$ is easy to solve has the same solution as $Ax = b$

Example $A = \begin{pmatrix} 7 & 3 \\ 1 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$Ax = b \Rightarrow \begin{aligned} 7x_1 + 3x_2 &= 3 \\ x_1 + x_2 &= 1 \end{aligned}$$

Augmented matrix $= \left[\begin{array}{cc|c} 7 & 3 & 3 \\ 1 & 1 & 1 \end{array}\right]$ we now transform to work out x

Operation	System	Augmented Matrix
switch 1st & 2nd row	$x_1 + x_2 = 1$ $7x_1 + 3x_2 = 3$	$\left[\begin{array}{cc\|c} 1 & 1 & 1 \\ 7 & 3 & 3 \end{array}\right]$
add -7 times 1st row to 2nd row	$x_1 + x_2 = 1$ $0 - 4x_2 = -4$	$\left[\begin{array}{cc\|c} 1 & 1 & 1 \\ 0 & -4 & -4 \end{array}\right]$
multiply 2nd row by $-1/4$	$x_1 + x_2 = 1$ $0 + x_2 = 1$	$\left[\begin{array}{cc\|c} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right]$
add -1 times 2nd row to 1st row	$x_1 = 0$ $x_2 = 1$	$\left[\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$

Definition 2.16

Three types of row operations are:

① multiply a row by a scalar

② swap two rows

③ add a multiplication of one row to another