MA1014 17/11/21

Subsequences & Cauchy Sequences.

<u>Proof</u> If $(a_n)_{n \in \mathbb{Z}}$ <u>bounded above</u>

then let $L = L.U.B \{ a_n : n \in \mathbb{N} \}$

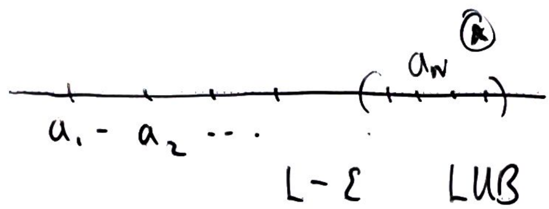
Need to prove $a_n \to L$ as $n \to \infty$

Suppose we are given any $\varepsilon > 0$

L is the <u>least</u> upper bound

$\Rightarrow L - \varepsilon$ is not an upper bound

$\Rightarrow \exists N \in \mathbb{N}$ such that $a_N > L - \varepsilon$ ⊛



If we know (a_N) is monotonic <u>increasing</u>
$a_{n+1} \geq a_n \ \forall n$, so if

$n > N$ we have $a_n \geq a_N$

⊛ so $a_n > L - \varepsilon \quad \forall n > N$

$|a_n - L| < \varepsilon \quad \forall n > N$

& a_n converges to L

Basic Law for Limits of Sequences

Suppose $(a_n)_{n\in\mathbb{N}}$ $(b_n)_{n\in\mathbb{N}}$

are both convergent, with limits L M

then

$(a_n+b_n)_{n\in\mathbb{N}}$ converges to $L+M$

$(a_n \cdot k)_{n\in\mathbb{N}}$ converges to kL

$(a_n b_n)_{n\in\mathbb{N}}$ converges to LM

$(a_n/b_n)_{n\in\mathbb{N}}$ (if b_n is never 0) converges to L/M (if $M\neq 0$)

converges $\begin{cases} a_n = \frac{1+n}{n} = 1+\frac{1}{n} \longrightarrow 1 = L \text{ as } n\to\infty \\ b_n = \frac{1}{n} \longrightarrow 0 = M \text{ as } n\to\infty \end{cases}$

diverges $\frac{a_n}{b_n} = 1+n$ unbounded
So not convergent

Proofs of limit laws are similar to those in

topic 2 "δ" "N"

Pinching Theorms

Suppose that $(a_n)_{n\in N}$, $(b_n)_{n\in N}$, $(C_n)_{n\in N}$ are 3 sequences such that

① $a_n \leq b_n \leq c_n \quad \forall n \geq k$

② $(a_n)_{n\in N}$, $(C_n)_{n\in N}$ convert

③ their limits are equal

Then $(b_n)_{n\in N}$ also converges to this limit

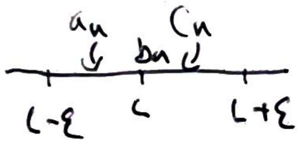
Proof Suppose $a_n \to L$, $c_n \to L$ as $n \to \infty$

Given any $\varepsilon > 0$, so we know

$\exists N_a \in N: \; n > N_a \Rightarrow |a_n - L| < \varepsilon$

$\exists N_c \in N \quad n > N_c \Rightarrow |C_n - L| < \varepsilon$

If $N = \max(N_a, N_k, k)$
$n > N \Rightarrow L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$



$\Rightarrow |b_n - L| < \varepsilon$

i.e $b_n \to L$ as $n \to \infty$

We know convergent sequences are bounded, but the converse is
general the converse is false

Example $a_n = (-1)^n$

$1, -1, 1, -1, 1, -1, 1, \dots$ bounded
not convergent

But it does have convergent

subsequences

$1, 1, 1, 1, 1, \dots$ (a_{2n})
Still bounded but now also convergent

Similarly $a_{2n+1} = -1 \quad \forall n$

Definition A subsequence of $(a_n)_{n \in \mathbb{N}}$ is a subsequence $(b_m)_{m \in \mathbb{N}}$

defined by $b_m = a_{n_m} \quad \forall m$

where $n_0 < n_1 < n_2 < n_3 < \dots$

is a strictly increasing sequence of natural numbers.