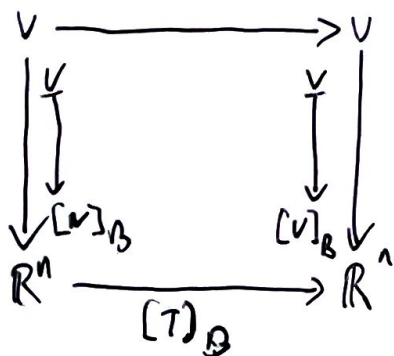


MA1114 28/2/22

## Matrix representations & similarity preserves trace/det

### Corollary

If  $T: V \rightarrow V$  is a linear map and  $B_1, B_2$  are two bases for  $V$   
then  ${}_{B_1} [T]_{B_2} = P_{B_1 \rightarrow B_2}^{-1} {}_{B_2} [T]_{B_2} P_{B_1 \rightarrow B_2}$



### Example

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix}$$

$$B_1 = B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

$$B_2 = E = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$P_{E \rightarrow B} [v]_E = [v]_B$$

$$\text{let's check } v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad [v]_E = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$P_{E \rightarrow B} [v]_E = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix}$$

fine since  $(2x_1, -x_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-x_1, x_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

last line calculated

$${}_B[T]_B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \quad {}_E[T]_E = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

check  $P_{E \rightarrow B} {}_E[T]_E P_{B \rightarrow E}$

$$= P_{B_1 \rightarrow B_2}^{-1} {}_E[T]_E P_{B_1 \rightarrow B_2}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = {}_B[T]_B$$


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we say  ${}_E[T]_E$  and  ${}_B[T]_B$  are similar

Definition

$A, B \in M_{n,n}(\mathbb{C})$  are similar if there exists an invertible matrix  $P$  such that  $P^{-1}BP = A$

Proposition If  $A$  and  $B$  are similar  $\det(A) = \det(B)$

Proof [recall if  $X, Y \in M_n(\mathbb{C})$  then  $\det(XY) = \det(X)\det(Y)$ ]

$$\begin{aligned} \det(A) &= \det(P^{-1}BP) \\ &= \det(P^{-1}) \det(B) \det(P) \\ &= \det(B) \det(P^{-1}) \det(P) \end{aligned}$$

$$\begin{aligned}
&= \det(B) \det(P^{-1}P) \\
&= \det(B) \det(I_d) \\
&= \det(B) \times 1 \\
&= \det(B) \quad \square
\end{aligned}$$

### Definition

$$A \in M_n(\mathbb{C}), \text{ then } \text{tr}(A) = \sum_{i=1}^n A_{ii}$$

### Proposition

If  $A, B \in M_n(\mathbb{C})$  are similar then  $\text{tr}(A) = \text{tr}(B)$

Proof \* see notes \*