MA1114 1/3/22

More fun with e.values & e.vectors; Calculating e-values of matrix / linear map.

Definition

If $T: V \to V$ is linear and $T(v) = \lambda v$ for some non-zero $v \in V$, $\lambda \in \mathbb{C}$ then v is an eigenvector of T with eigenvalue λ.

If λ is an eigenvalue of T then $V_\lambda = \{ v \in V \mid T(v) = \lambda v \} \leq V$ is the eigenspace of λ

Example

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_1 + 2x_2 + x_3 \\ x_1 + 3x_3 \end{pmatrix}$$

eigenvectors

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{array}{ccc} 2 & 2 & 1 \\ v_1 & v_2 & v_3 \end{array}$$

$$T(\lambda v_1 + \mu v_2) = 2(\lambda v_1 + \mu v_2) \quad \forall \lambda, \mu \in \mathbb{C}$$

$$\lambda T(v_1) + \mu T(v_2)$$

$$= \lambda 2 v_1 + \mu 2 v_2$$

$$= 2(\lambda v_1 + \mu v_2)$$

Examples

a) $T: V \to V$

$v \mapsto \lambda v \qquad \lambda \in \mathbb{C}$

$V_\lambda = \{v \in V \mid T(v) = \lambda v\} = V$

b) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix}$

two eigenvalues 2, 3 (yesterday)

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$V_2 = \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \}$

$\Leftrightarrow \begin{matrix} x_1 + x_2 = 2x_1 \\ -2x_1 + 4x_2 = 2x_2 \end{matrix}$

$\Leftrightarrow \begin{matrix} -x_1 + x_2 = 0 \\ -2x_1 + 2x_2 = 0 \end{matrix}$

$\Leftrightarrow x_1 = x_2$

$\Rightarrow V_2 = \{ \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid \lambda \in \mathbb{R}$

$V_3 = \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle$ (check)

c) $T: V \to V$, suppose 0 is an eigenvalue $\Leftrightarrow \ker(T) \neq \{0\}$

$V_0 = \{ v \in V \mid T(v) = 0v = 0 \}$

$= \ker(T)$

Corollary T is an isomorphism $\Leftrightarrow$ 0 not an eigenvalue

Finding eigenvalues

If $T: V \to V$ is linear and $V = \langle B \rangle$ some basis B

Have matrix $A = [T]_B$

$$= \left([T(v_1)]_B \; [T(v_2)]_B \cdots [T(v_n)]_B \right)$$

where $B = \{v_1, v_2, \ldots, v_n\}$

we showed that T is invertible (i.e isos)

$\Leftrightarrow$ A invertible

$\Leftrightarrow$ $\det(A) \neq 0$

so $\ker(T) = 0 \Leftrightarrow \det(T) = \det(A) \neq 0$

(Also by COGOF for isomorphism)

T is invertible $\Leftrightarrow \ker(T) = 0$

Theorem For T as above

λ is an eigenvalue of $T: V \to V$

$\Leftrightarrow \det(\lambda \, id - T) = 0 \Leftrightarrow \det(\lambda I - A) = 0$

Proof The map $\lambda id - T : V \to V$

$v \mapsto \lambda v - T(v)$

λ is an eigenvalue of T there exists $0 \neq v \in V$ with $T(v) = \lambda v$

$\Leftrightarrow \lambda v - T(v) = 0$

$= (\lambda id_V - T)(v) = 0$

$\Leftrightarrow v \in \ker(\lambda id - T)$

$\Leftrightarrow \ker(\lambda id - T) \neq 0$

$\Leftrightarrow$ T is not invertible $\Rightarrow \det(T) = 0$ by ⊛

Also $\det(\lambda id - T) = \det(\lambda I - A)$ where $A = [T]_B$

So result proved, provided it didn't matter which basis we chose for A.

"this is on theorem 9.9"

check $\det(\lambda id - [T]_B) = \det(\lambda id - [T]_C)$

Definition

Let $T: V \to V$ be a linear map the characteristic polynomial of T

$$\chi_T(t) = \det(t\, id_V - T) \Rightarrow \chi_A(t) = \det(\lambda I_n - A)$$

where A is any possible matrix representation of T

Fact λ is an eigenvalue $T \Leftrightarrow \chi_T(\lambda) = 0$ (by previous result)

Example

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix}$ $B = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$

then $[T]_B = \left([T\begin{pmatrix} 1 \\ 0 \end{pmatrix}]_B \; [T\begin{pmatrix} 0 \\ 1 \end{pmatrix}]_B \right) = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$

so $\chi_T(t) = \chi_A(t) = \det(tI_2 - A)$

$$= \det\left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} t-1 & 1 \\ -2 & t-4 \end{pmatrix}$$

$$= (t-1)(t-4) + 2$$

$$= t^2 - 5t + 4 + 2$$

$$= t^2 - 5t + 6$$

$$(t-3)(t-2)$$

so λ is the eigenvalue of

$\iff \chi_A(t) = 0$

$\iff t = 3$ or $t = 2$ $\implies$ so 2 and 3 are eigenvalues

<u>Proposition</u> $\chi_A(t)$ is a polynomial

$\dim(A) = 2$ ✓ since $\det\begin{pmatrix} t-a & -b \\ -c & t-d \end{pmatrix}$

$$= (t-a)(t-d) - bc$$

$$= t^2 + (a+d)t + ad \quad = t^2 - \text{trace}(A) + \det(A)$$

generally

$$\det(tI-A) = \det\begin{pmatrix} t-a_{11} & a_{12} & \cdots & a_{1n} \\ -a_{21} & t-a_{22} & & \vdots \\ \vdots & & \ddots & \\ -a_{n1} & \cdots & & t-a_{nn} \end{pmatrix}$$

is a polynomial by eg expanding along the first row using induction

<u>Proposition</u> If $A \in M_n(\mathbb{C})$

is diagonal or upper / lower triangular then the eigenvalues are

$\{A_{ii} \mid 1 \leq i \leq n\}$

<u>Proof</u>

<u>upper</u> $\begin{pmatrix} a_{11} & 0 & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & a_{nn} \end{pmatrix}$

$$\det(tI-A) = \begin{pmatrix} t-a_{11} & -a_{12} & \cdots & -a_{1n} \\ 0 & t-a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & t-a_{nn} \end{pmatrix}$$

visible $\chi_A(t) = \det(tI-A) = (t-a_{11})(t-a_{22})\cdots(t-a_{nn})$

so $\chi_A(t) = 0 \iff t \in \{a_{ii} \mid 1 \leq i \leq n\}$

similarly for lower triangular