MA1014 25/1/22

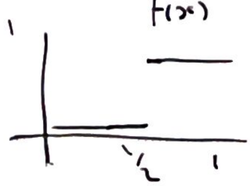
Substitution and Change of Variables

a) if $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(t)$

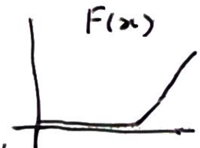
b) if $f(x) = F'(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$

In (a) <u>f must be continuous</u>

Otherwise: $f(x) = \begin{cases} 0 & x < 1/2 \\ 1 & x \geq 1/2 \end{cases}$



$F(x) = \int_0^x f(t)\,dt = \begin{cases} 0 & x < 1/2 \\ x - 1/2 & x \geq 1/2 \end{cases}$



is not differentiable at $x = 1/2$

or/ $f(x) = \begin{cases} 0 & x \neq 1/2 \\ 1 & x = 1/2 \end{cases}$

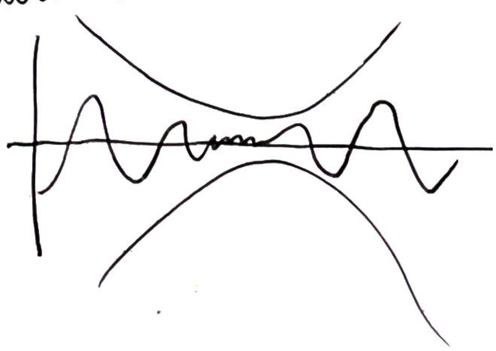
$\Rightarrow F(x) = 0 \quad \forall x$

is differentiable $F'(x) = f(x)$ when $x = 1/2$

(b) f must be integrable

not uniformly continuous

$F(x) = \begin{cases} (x - 1/2)^2 \sin\left(\frac{1}{(x-1/2)^2}\right) & (x \neq 1/2) \\ 0 & (x = 1/2) \end{cases}$



chapter 3 : $F'(x) = \begin{cases} 0 & \text{if } x = 1/2 \\ -2/_{x-1/2} \sin(\ldots) & x \neq 1/2 \end{cases}$

unbounded

not integrable

$\int_a^b f(x)\,dx$ numbers, area

<u>definite</u> integral.

$\int_c^x f(t)\,dt = F(x) + C$

antiderivative or <u>indefinite integrable.</u>

C can be any (arbitary) constant

$F'(x) = f(x) \qquad (F(x)+C)' = f(x)$

$\int_c^x f(t)\,dt = \underline{F(x)} - \underbrace{F(c)}_{\text{constant}}$

$F(x)$ $\xrightarrow{d/dx}$	$f(x)$ $\xrightarrow{\int dx}$
C	0
x	1
$ax+b$	a
x^2	$2x$
x^n	nx^{n-1}
$\frac{1}{n+1} x^{n+1}$ $(n \neq -1)$	x^n
$\ln(x)$ $(x>0)$	$1/x$
$u(v(x))$	$u'(v(x)) \cdot v'(x) = \frac{du}{dv} \cdot \frac{dv}{dx}$
$\sin(x^2)$	$2x\cos(x^2$
?	$\cos(x^2)$

Integration by Substitution

$$\int u'(v(x)) \cdot v'(x)\, dx = u(v(x)) + C$$

$$\int f(v(x)) \frac{dv}{dx}\, dx = F(v(x)) + C$$

Example $I = \int \sqrt{(1+4\cos^2 x)}\sin 2x \, dx$

Let $v(x) = 1 + 4\cos^2 x$

so $v'(x) = -4 \times 2\cos x \sin x = -4 \sin 2x$

$$I = \int \sqrt{u} \cdot \frac{1}{-4} \frac{dv}{dx}\, dx = \frac{1}{-4} \int \sqrt{u}\, dv$$

$$= -\tfrac{1}{4} \cdot \tfrac{2}{3} v^{3/2} + C$$

$$= \underline{-\tfrac{1}{6}(1+4\cos^2 x)^{3/2} + C}$$

$\int \sqrt{(1+4\cos^2 x)}\, dx = ?$

! **Definite integral**

$$I = \int_0^{\pi/4} \sqrt{(1+4\cos^2 x)} \sin 2x \, dx$$

Make same substitution, don't forget the range of integration

$x = 0 \Rightarrow v = 1 + 4\cos^2 0 = 5$
$x = \pi/4 \Rightarrow v = 1 + 4\cos^2 \pi/4 = 3$

$$I = \left[-\tfrac{1}{6} v^{3/2}\right]_5^3 = -\tfrac{1}{6}\left(3^{3/2} - 5^{3/2}\right)$$

$\int_0^{2\pi}$ would go wrong as $v(x)$ not $1 \to$

$x = 0 \Rightarrow v = 5$
$x = 2\pi \Rightarrow v = 5$ $\qquad \int_5^5 \quad = 0$

More examples:

a) $\int (x^3 - 1)^3 \, dx = ?$

b) $\int 12 (x^3 - 1)^3 \, x^2 \, dx$ ✓

a) is much "harder."

$$\int (x^9 - 3x^6 + 3x^3 - 1)$$

$$= \frac{1}{10} x^{10} - \frac{3}{7} x^7 + \frac{3}{4} x^4 - x + C$$

b) using substitution

$\frac{dv}{dx} = 3x^2$

$v(x) = x^3 - 1 \qquad v' = 3x^2$

$\frac{1}{3} dv = x^2 dx$

$\frac{1}{3} \int 12 v^3 \, dv$

$\int 4v^3 dv = v^4 + c$
$= (3^3 - 1)^4 + c$

$\int \ln(x) \, dx = ?$
[$x \ln x - x + c$
parts]

$\int (\ln x)^3 / x \, dx$

$v = \ln x \qquad \frac{1}{x} dx = dv$

$\int v^3 dv \qquad \frac{dv}{dx} = \frac{1}{x}$

$\frac{1}{4} v^4 + c = \frac{1}{4} (\ln x)^4 + c$