MA1114 18/1/22

Dimension Examples

Definition

The <u>dimension</u> of a vector space is the number of vectors in a basis

Example

Calculate the dimensions of the following vector spaces

① $\mathbb{R}^n$

② span(s) where $s \subset V$, a vector space and S is linearly independant.

③ $\{A \in M_{n,n}(\mathbb{R}) \mid A^T = A\}$

1) dimension$(\mathbb{R}^n)$ = $\dim(\mathbb{R}^n) = n$

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \middle| x_i \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$ is a basis for $\mathbb{R}^n$

2) $|s|$, s is a basis for span$(s) = \{\lambda_1 s_1 + \dots + \lambda_k v_k \mid \lambda_i \in \mathbb{R}\}$

where $s = \{s_1, s_2, s_2, \dots, s_k\}$

3) $\frac{1}{2}n(n+1)$

onsider the case $n=2$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

so $A = A^T \Rightarrow b = c$ so $A = \begin{bmatrix} a & c \\ c & d \end{bmatrix}$ or $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \{A \in M_{n,n}(\mathbb{R}) \mid A^T = A\}$

$\Rightarrow$ span $(\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \})$ $\Rightarrow$ dimensions $= 2$

consider the case $n=3$ $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

$A = A^T \Rightarrow b = d$ $\Rightarrow A \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & i \end{bmatrix}$
$c = g$
$f = h$

$$A = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dots$$

$\Rightarrow \dim(\{A \in M_{33}(\mathbb{R}) \mid A = A^T\}) = 6$

continuing this way, we can see that $\dim(\{A \in M_{n,n}(\mathbb{R}) \mid A^T = A\})$
$=$ # of upper triangular matrix entries
$= 1 + 2 + 3 + 4 + \dots + n$
$\Rightarrow \sum_i^n i = \frac{1}{2} n(n+1)$

<u>Basis of a null space.</u> Recall that if $A \in M_{n,n}(\mathbb{R})$, $\text{null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$

and recall $\text{null}(A) \subseteq \mathbb{R}^n$ (since $x \in$ null, $x \in \mathbb{R}$)

$$\Rightarrow A(\lambda x) = \lambda(Ax)$$
$$= \lambda(0)$$
$$= 0$$

$x, y \in Null(A)$

$$A(x+y) = Ax + Ay$$
$$= \underline{0} + \underline{0}$$
$$= \underline{0}$$

Calculate the dimension of null(A) when

① $A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 7 & 9 \\ 1 & 2 & 6 \end{bmatrix}$ ② $A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -4 & -3 \\ 2 & 8 & 6 \end{bmatrix}$

① $Ax = 0$ $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 7 & 9 \\ 1 & 2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 3 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 5 & 6 \\ 2 & 7 & 9 \\ 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ $x + 5y + 3z = 0$
$y + z = 0$

so general element null (A) is $\begin{pmatrix} -8y \\ y \\ y \end{pmatrix}$ null $(A) = \left\{ y \cdot \begin{pmatrix} -8 \\ 1 \\ 1 \end{pmatrix} \middle| y \in \mathbb{R} \right\}$
$= \text{span} \begin{pmatrix} -8 \\ 1 \\ 1 \end{pmatrix}$

② $\begin{bmatrix} 1 & 4 & 3 \\ -1 & -4 & -3 \\ 2 & 8 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{0}$ $\Leftarrow Ax = 0$

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} x+4y+3z \\ 0 \\ 0 \end{pmatrix} = 0 \Rightarrow x+4y+3z=0$$

so general element null (A) is $\begin{pmatrix} -4y-3z \\ y \\ z \end{pmatrix}$

$$\text{null}(A) = \left\{ y = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, z = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \middle| \; y, z \in \mathbb{R} \right\}$$

$$\lambda_1 \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = 0$$

$\Rightarrow \lambda_1 = \lambda_2 = 0$ so linearly independant

$$\Rightarrow \text{null}(A) = \left\langle \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\rangle \Rightarrow \dim(\text{null}(A)) = 2$$