MA1114 24/1/22

Plus / Minus Theorm

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Suppose $\emptyset = S \subset V$, a vector space

(i) if S is linearly independant and, $v \notin \text{span}(S)$ then $S \cup \{u\}$ is "plus" linear independant

(ii) if $\text{span}(S) = V$ and $v \in S$ is such that $v \in \text{span}(S \setminus \{u\})$ then $\text{span}(S \setminus \{u\}) = V$

Proof

let $S = \{v_1, \dots, v_k\}$

(i) $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k + \lambda v = 0$

$\lambda \neq 0 \Rightarrow v \in \text{span}(S)$ $\times$

$\Rightarrow \lambda = 0 \Rightarrow \lambda_i = 0$ (since S is linearly independance)

(ii) $\text{span}(S) = V$, $S = \{v_1, \dots, v_k\}$

assume $v_k \in \text{span}(S \setminus \{v_k\})$; up to the reordering the v_i

$\exists \lambda_j \in \mathbb{R}$ such that $v_k = \sum_{j=1}^{k-1} \lambda_j v_j$

since S spans V for any $w \in V$ there are $\mu_i \subset \mathbb{R}$ such that

$$w = \sum_{i=1}^{k} \mu_i v_i = \sum_{i=1}^{k-1} \mu_i v_i + \mu_k v_k = \sum_{i=1}^{k-1} \mu_i v_i + \mu_k \sum_{j=1}^{k-1} \lambda_j v_j = \sum_{i=1}^{k-1} v_i (\mu_i + \mu_k \lambda_i)$$

$\Rightarrow \{v_1, \dots, v_{k-1}\}$ spans V.

__Proposition 6.1__

V, a vector space, $S = \{v_1, \dots, v_n\}$ is a basis of V.
For __any__ subset $\tilde{S}$ of V of size $k \in \mathbb{N}$

(i) $k > n \Rightarrow \tilde{S}$ linear dependant
(ii) $k < n \Rightarrow S$ does not span V

__Proposition 6.13__ ("check one get one free")

Let V be a vector space of dimension n and $S = \{v_1, v_2, \dots, v_n\} \subset V$

Then $\text{span}(S) = V \Leftrightarrow S$ is linearly independance.

__Proof__

"$\Rightarrow$" suppose $\text{span}(S) = V$ and S is linearly dependant

up to reordering the v_i, there are $\lambda_i \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} \lambda_n v_n = 0 \quad \text{with } \lambda_n \neq 0$$

$$\Rightarrow v_n = \frac{-1}{\lambda_n} \sum_{i=1}^{n} \lambda_n v_n = \sum_{i=1}^{n} \frac{-\lambda_i}{\lambda_n} v_i$$

$$= \text{span}(\{v_1, \dots, v_n\})$$

plus / minus theorm

$$\Rightarrow \text{span}(\{v_1, \dots, v_{n-1}\}) = V \;\text{※}$$

by prop 6.1 with $\tilde{S} = \{v_1, \dots, v_{n-1}\}$, $k = n-1$

"$\Leftarrow$" assuming S is linearly independant

suppose (for a contradiction) that

$\mathrm{span}(S) \neq V$. Can choose $v \in V \setminus \mathrm{span}(S)$ $(v \notin \mathrm{span}(S))$

by $\pm$ theorm

$S \cup \{v\}$ is linearly independant ※

by prop 6.1 with $\tilde{S} = \{S \cup \{v\}\}$ $(k = n+1 > n)$

<u>Example</u>

$\mathrm{span}\left(\left\{\begin{pmatrix}-3\\7\end{pmatrix}, \begin{pmatrix}5\\5\end{pmatrix}\right\}\right)$ is a basis by prop6.13 $n=2$

(clearly $\left\{\begin{pmatrix}-3\\7\end{pmatrix}, \begin{pmatrix}5\\5\end{pmatrix}\right\}$ is linearly independant)

$\mathrm{span}\left(\left\{\underbrace{\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}4\\0\\7\end{pmatrix}}_{\text{clearly linearly independant}}, \begin{pmatrix}-1\\1\\4\end{pmatrix}\right\}\right)$ is a basis for $\mathbb{R}^3$

$\begin{pmatrix}-1\\1\\4\end{pmatrix} \notin \mathrm{span}\left(\left\{\begin{pmatrix}2\\0\\-1\end{pmatrix}, \begin{pmatrix}4\\0\\7\end{pmatrix}\right\}\right)$

$\Rightarrow \left\{\begin{pmatrix}2\\0\\-1\end{pmatrix}, \begin{pmatrix}4\\0\\7\end{pmatrix}, \begin{pmatrix}-1\\1\\4\end{pmatrix}\right\}$ is linearly independant.

by $\pm$ theorm.

Similarly

$$\operatorname{span}(\{1, 1+x, 1+x+x^2+ \dots + (1+x+x^2+\dots+x^n)\})$$

$$= \operatorname{span}\left(\left\{\frac{x-1}{x-1}, \frac{x^2-1}{x-1}, \frac{x^3-1}{x-1}, \dots, \frac{x^n-1}{x-1}\right\}\right)$$

$= P_n = \{p_n \mid \text{polynomials of degree} \leq n\}$ by I theorem