2.2 Properties of Point Estimators - Unbiasedness pt2

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Recall that, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

Find $E(\hat{\sigma}^2)$:

Solution:
$E(\hat{\sigma}^2) = E(\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2) = E[\frac{1}{n}\sum_{i=1}^{n}((X_i - \mu) - (\bar{X} - \mu))^2]$
$= \frac{1}{n}E[\sum_{n}^{i=1}((X_i - \mu)^2 - 2(X_i - \mu)(\bar{X} - \mu) + (\bar{X} - \mu)^2)]$
$= \frac{1}{n}(E[\sum_{i=1}^{n}(X_i - \mu)^2] - 2E[\sum_{i=1}^{n}(X_i - \mu)(\bar{X} - \mu)] - E[\sum_{i=1}^{n}(\bar{X} - \mu)^2])$
$= \frac{1}{n}(\sum_{i=1}^{n} E[(X_i - \mu)^2] - 2E[(X_i - \mu)(\bar{X} - \mu)] - \sum_{i=1}^{n} E[(\bar{X} - \mu)^2])$
$= \frac{1}{n}(\sum_{i=i}^{n} var(X_i) - 2nvar(\bar{X}) + nvar(\bar{X}))$
$= \frac{1}{n}(\sum_{i=i}^{n} var(X_i) - nvar(\bar{X}))$
$= \frac{1}{n}(n\sigma^2 - \frac{n\sigma^2}{n})$

$$E(\hat{\sigma}^2) = E[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2] = \frac{n-1}{n}\sigma^2 \neq \sigma^2$$

Theorem:

Let $X_1, \ldots, X_n$ be random sample drawn from an infinite population with variance $\sigma^2 < \infty$. If $S^2 \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is the variance of the random sample, then S^2 is an unbiased estimator for σ^2.

Proof:

$E(S^2) = \frac{1}{n-1}E[\sum_{i=1}^{n}(X_i - \bar{X})^2] = \frac{1}{n-1}E[\sum_{i=1}^{n}((X_i - \mu) - (\bar{X} - \mu))^2]$
$= \frac{1}{n-1}\sum_{i=1}^{n} E[(X_i - \mu)^2] - n(E[(\bar{X} - \mu)])^2$
$= \frac{1}{n-1}(\sum_{i=1}^{n} \sigma^2 - n\frac{\sigma^2}{n}) = \sigma^2$

Hence, S^2 is an unbiased estiamtor for σ^2.