MA1014 15/11/21

Applications of Differentiation

→ finding <u>local</u> extreme values

→ finding <u>global</u> extreme values

f, g are continuous on $[a, b]$ & differentiable on (a, b)

<u>Theorm</u> $f'(x) = g'(x) \ \forall x \in (a, b) \iff f(x) - g(x)$ is constant $= c \ \forall x \in [a, b]$

<u>Proof</u> First simplify: consider

$$h(x) = f(x) - g(x) \text{ still continuous}$$

$$h'(x) = f'(x) - g'(x) \text{ still differentiable}$$

So we have shown

$$h'(x) = 0 \ \forall x \iff h(x) \text{ is constant}$$

$\Leftarrow$) If $h(x) = c \ \forall x$, $h'(x) = 0$

$\Rightarrow$) Use the mean value theorm

for $h: [c, d] \longrightarrow \mathbb{R}$
$z \in$ for $a \leq c \leq d \leq b$

$$\exists z: \frac{h(d) - h(c)}{d - c} = h'(z) = 0 \Rightarrow h(c) = h(d)$$

So derivative always 0 $\Rightarrow \forall c<d,\ h(c)=h(d)$

So h is constant

<u>Theorm</u> $h'(x)>0$ for all $x \in (a,b) \Rightarrow h$ <u>is</u> strictly increasing

Same proof : if we chose any $c<d$ between a & b

the MVT says $\exists z$ between c & d

$$\frac{h(d)-h(c)}{d-c} = f'(z) > 0$$

$\Rightarrow h(d)-h(c)>0$ whenever.

$\Rightarrow h$ strictly increasing $c<d$.

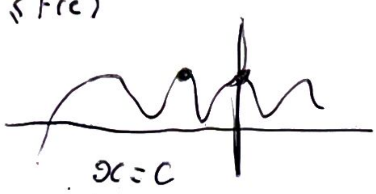
<u>Theorm</u> $h'(x)<0 \Rightarrow$ strictly decreasing

Finding max & min values

$f: [a,b] \to \mathbb{R}$ cts
$f': (a,b) \to \mathbb{R}$ exists

f has a <u>local</u> maximum at $x=c$

$\exists \delta>0$: $c-\delta < x \leq c \Rightarrow f(x) \leq f(c)$
& : $c \leq x \leq c+\delta \Rightarrow f(x) \leq f(c)$



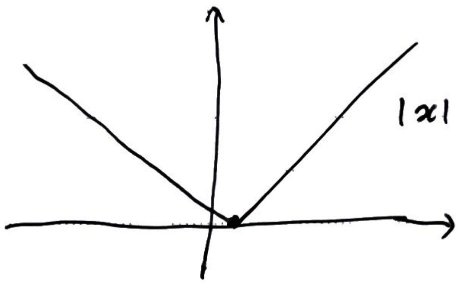
f has a local minimum at $x = c$

$\exists \delta > 0$ such that if $x \in (c-\delta, c+\delta)$ then $f(x) \geqslant f(c)$

$$\begin{cases} \text{local max } \nearrow \cdot \searrow \Rightarrow & f'(x) \geqslant 0 \quad \forall x \in (c-\delta, c) \\ & f'(x) \leqslant 0 \quad \forall x \in (c, c+\delta) \\ \text{local min } \searrow . \nearrow \Rightarrow & f'(x) \leqslant 0 \quad \forall x \in (c-\delta, c) \\ & f'(x) \geqslant 0 \quad \forall x \in (c, c+\delta) \end{cases}$$

$$\left. \begin{array}{c} \underline{\text{Local extreme value}} \\ \text{at } x = c \\ \text{\& if } f'(c) \text{ exists} \end{array} \right\} \Rightarrow f'(c) = 0$$

$f(x) = |x|$



local min but not differentiable

To distinguish local max from local min:

1) Look at $f(x)$ $x < c$, $x > c$
max: $f(x) \leqslant f(c)$ min: $f(x) \geqslant f(c)$

2) Look at $f'(x)$ $x < c$, $x > 0$
max: f' from +ve to −ve min: f' changes from −ve to +ve

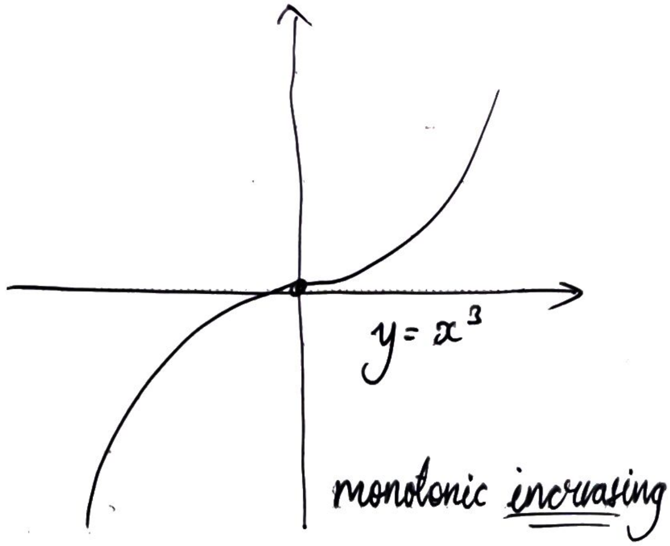
3) If $f''(c)$
max: $f''(c) < 0$ min: $f''(c) > 0$

careful: not every point $x = c$ with $f'(c) = 0$ is a local max or min

$f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$



$f(x) = x^4$ $\quad f'(0) = 0 \quad f''(0) = 0 \quad f'''(0) = 0$

Global Extreme Values

E.V.T. $f: [a,b] \to \mathbb{R}$ cts

$\Rightarrow \exists\, c, d$

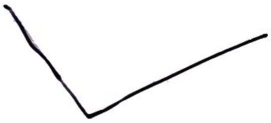
$$\underbrace{f(c)}_{\text{min}} \leq f(x) \leq \underbrace{f(d)}_{\text{max}}$$

Global extreme values could be

1) $x = a$
2) $x = b$
3) at $x = c$ with f not differentiable at $x = c$

× critical

4) at $x = c$ with $f'(c) = 0$



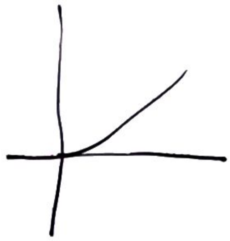
$f(x) = |x|$

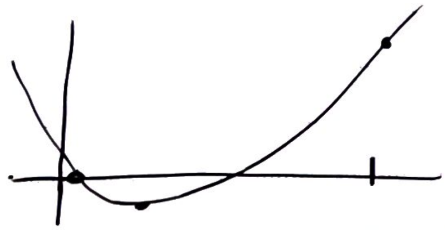
$x=0$ global minimum

(case 3)

$f:[0,1] \to \mathbb{R}$

$f(x) = x^2$





$x=0$ $\quad f(0)=0$ global min

$x=1$ $\quad f(1)=1$ global max

$f:[0,2] \to \mathbb{R}$ $\quad f(x) = x^2 - x$ $\quad f(0)=0$ $\quad f(2)=2$

$f'(x) = 2x-1$ $\quad f(\tfrac{1}{2}) = -\tfrac{1}{4}$