MA1114 22/2/22

Multiplication of Matrices vs Composition of Linear Maps ; linear isos $\Leftrightarrow$ invertible

$T: V \longrightarrow W$, linear isos

$V = \langle \mathcal{B} \rangle \quad W = \langle e \rangle$

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\varepsilon"', "v \mapsto {[v]_{\mathcal{B}}}"] & W \arrow[d, "\varepsilon'", "w \mapsto {[w]_e}"'] \\ \mathbb{R}^n \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

$\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$

$A = {}_e[T]_{\mathcal{B}} = \big[[T(v_1)]_e \; [T(v_2)]_e \; \cdots \; [T(v_n)]_e\big]$

$$A[v]_{\mathcal{B}} = [T(v)]_e$$

Example

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix}$

$V = \mathbb{R}^2 \quad W = \mathbb{R}^2 \; ; \; \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad e = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \; T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$\left.\begin{aligned} \left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_e &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \left[T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_e &= \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}\right\} \rightarrow A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

Does it work?

$$\left[\begin{pmatrix}x_1\\x_2\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$A = \left[\begin{pmatrix}x_1\\x_2\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}1 & 1\\-2 & 4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1+x_2\\-2x_1+4x_2\end{pmatrix} = [T(v)]_e$$

Difference Bases, $e = \mathcal{B} = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$
$= \{v_1, v_2\}$

$[T(v_1)]_e \quad [T(v_2)]_e$?

$$T(v_1) = \begin{pmatrix}2\\2\end{pmatrix} = 2v_1 + 0v_2 \Rightarrow [T(v_1)]_e = \begin{pmatrix}2\\0\end{pmatrix}$$

$$T(v_2) = \begin{pmatrix}3\\6\end{pmatrix} = 0v_1 + 3v_2 \Rightarrow [T(v_2)]_e = \begin{pmatrix}0\\3\end{pmatrix}$$

diff matrise $A = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}$

Example

$\mathbb{R}^3 = \langle \mathcal{B} \rangle$

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$
$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \mapsto \begin{pmatrix}x_1+x_2\\x_1-x_3\end{pmatrix}$$

$\mathbb{R}^2 = \langle e \rangle$

$$\mathcal{B} = \left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\} \quad e = \left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}\right\}$$

calculator $A = [T]_{\mathcal{B}}$

$$T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} = -2/5\begin{pmatrix}1\\2\end{pmatrix} + 4/5\begin{pmatrix}3\\1\end{pmatrix} \Rightarrow \left[T\begin{pmatrix}1\\1\\1\end{pmatrix}\right]_e = \begin{pmatrix}-2/5\\4/5\end{pmatrix}$$

$$T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} = 1/5\begin{pmatrix}1\\2\end{pmatrix} + 3/5\begin{pmatrix}3\\1\end{pmatrix} \Rightarrow \left[T\begin{pmatrix}1\\1\\0\end{pmatrix}\right]_e = \begin{pmatrix}1/5\\3/5\end{pmatrix}$$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = 2/5\begin{pmatrix}1\\2\end{pmatrix} + 1/5\begin{pmatrix}3\\1\end{pmatrix} = \left[T\begin{pmatrix}1\\0\\0\end{pmatrix}\right]_e = \begin{pmatrix}2/5\\1/5\end{pmatrix} \quad {}_e[T]_{\mathcal{B}} = \begin{pmatrix}-2/5 & 1/5 & 2/5\\4/5 & 3/5 & 1/5\end{pmatrix}$$

Proposition 8.47

if $T : U \to V$ and $S : V \to W$

and $U = \langle B_1 \rangle$, $V = \langle B_2 \rangle$, $W = \langle B_3 \rangle$

$$_{B_3}[S \circ T]_{B_1} = {}_{B_3}[S]_{B_2} \; {}_{B_2}[T]_{B_1}$$

Proof (see notes)

Proposition 8.48

let $V = \langle \mathcal{B} \rangle$, $W = \langle \mathcal{C} \rangle$

and $T : V \to W$ is a linear map

T is an isos $\Leftrightarrow$ $_{\mathcal{C}}[T]_{\mathcal{B}}$ is invertible

Proof

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "E_1"'] & W \arrow[d, "E_2"] \\ \mathbb{R}^n \arrow[r, "T_A"'] & \mathbb{R}^n \end{tikzcd}$$

$A = {}_{\mathcal{C}}[T]_{\mathcal{B}}$ $\quad T_A = E_2 \circ T \circ E_1^{-1}$

$T_A(v) = Av$

If T is an isos then
$T^{-1} : W \to \mathcal{C}$ is an isos and is represented by matrix C

where $T_C = E_1 T E_2^{-1}$ $\quad T_C T_A = E_1 \cancel{T^{-1}} \cancel{E_2^{-1}} \cancel{E_2} \cancel{T} \cancel{E_1^{-1}}$
similarly $T_{A_C} = \mathrm{id}_{\mathbb{R}^n}$ $\quad = \mathrm{id}_n$

$AeV = v = eAV$ for all $v \in \mathbb{R}^n$

$\Rightarrow Ae = eA = \mathcal{I}n$

$\Rightarrow$ invertible

Conversely if A is invertible

then $s = E_1^{-1} \cdot T_A \cdot E_2$ is the inverse of T since

$T = E_2^{-1} T_A E_1$

$$\Rightarrow s \circ T = E_1^{-1} T_A^{-1} E_2 E_2^{-1} T_A E_1$$
$$= id_V$$

similarly $T \circ s = id_W$ $\Rightarrow$ s is inverse of T $\Rightarrow$ T is isomorphic

Example

$\mathcal{B} = e = \{e_1, e_2\}$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix}$$

$$_{C}[A]_{B} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

$\therefore$ invertible since $\det \neq 0$

$\ker(T) = \{v \in \mathbb{R}^2 \mid T(v) = \underline{0}\}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \ker(T) \iff \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow x_1 = -x_2$$
$$\Rightarrow 2x_1 = 4x_2$$
$$\Rightarrow -2x_1 = 4x_2$$
$$\Rightarrow x_2 = 0 \Rightarrow x_1 = 0$$

so $\ker(T) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \Rightarrow T$ is injective $\Rightarrow T$ is isos by COGOF for isos