MA1014 27/10/21

Extreme Value Theorm

__Last time__ Bolzano, IVT

A continuous functions on a closed interval $\left\{ \begin{array}{l} \text{is bound} \\ \text{has bounded range} \end{array} \right.$ $\exists B$ $|f(x)| < B$

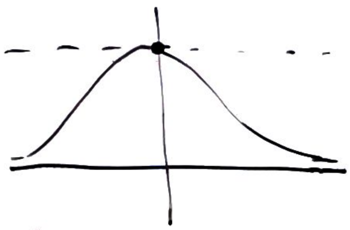
$f: [a,b] \to \mathbb{R}$ $\forall x \in$ domain $[a,b]$

range (f) bounded $\subseteq \mathbb{R}$

__Examples__

$f: \mathbb{R} \to \mathbb{R}$

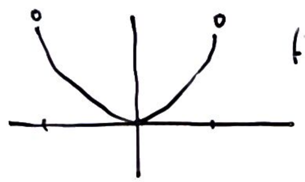
$f(x) = \frac{1}{1+x^2}$



$f(x) > 0$, $\lim_{x \to \pm\infty} f(x) = 0$ not attained

Bounded above by $1 = f(0)$

$f: (-1,1) \to \mathbb{R}$



$f(x) = x^2$

$f(0) = 0$

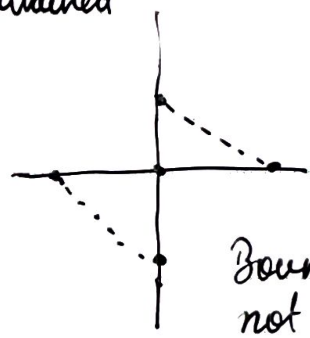
$\lim_{x \to 1^-} f(x) = 1$ not attained

$\lim_{x \to -1^+} f(x) = 1$

not cls

$f: [-1,1] \to \mathbb{R}$

$f(x) = \begin{cases} 1-x & x > 0 \\ 0 & 0 \\ -1-x & x < 0 \end{cases}$



Bounds ± 1 not attained

Extreme Value Theorm

If $f:[a,b] \to \mathbb{R}$ is continuos then it attains is bounds.

More explicitly: let $\underline{m} = GLB\ (\text{range}(f))$ & let $\bar{m} = (\text{range}(f))$
then $\exists c, d \in [a,b]$ $\qquad \underline{m} \leq f(x) \leq \bar{m}$
$\forall x \in [a,b]$

$$f(c) = \underline{m} \leq f(x) \leq \bar{m} = f(d)$$

Proof by Contradiction

Suppose $f(x) \neq \underline{m}\ \ \forall x \in [a,b]$

$d(x) = f(x) - \underline{m} > 0 \quad , \quad \underset{\text{cts}}{g(x)} = \frac{1}{d(x)} > 0$

$f, d, g : [a,b] \to \mathbb{R}$

g bounded as it's continuous on $[a,b]$

$0 < g(x) < B \quad \forall x \in [a,b]$

$d(x) > \frac{1}{B} \quad \forall x \in [a,b]$

$f(x) - \underline{m} > \frac{1}{B}$, $f(x) > \underline{m} + \frac{1}{B}$ contradiction greater lower bound than m

So if $\underline{m} = GLB\ (\text{range}(f))$

$\exists\, x = c \in [a,b]$ such that $f(c) = \underline{m}$

Leave the existence of d s.t. $f(d) = \underset{\bar{M}}{LUB}$
proof for the students