MA1114 3/11/21

Inverse Algorithm

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \qquad \left(\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right)$$

$$R_1 \mapsto \tfrac{1}{3} R_1 \quad \left(\begin{array}{cc|cc} 1 & 4/3 & 1/3 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right)$$

$$R_2 \mapsto R_2 - 2R_1 \quad \left(\begin{array}{cc|cc} 1 & 4/3 & 1/3 & 0 \\ 0 & 1/3 & -2/3 & 1 \end{array}\right)$$

$$R_2 \mapsto 3R_2 \quad \left(\begin{array}{cc|cc} 1 & 4/3 & 3 & -4 \\ 0 & 1 & -2 & 3 \end{array}\right)$$

$$R_1 \mapsto R_1 - \tfrac{4}{3} R_2 \quad \left(\begin{array}{cc|cc} 1 & 0 & 3 & -4 \\ 0 & 1 & -2 & 3 \end{array}\right)$$

$$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Inverse Algorithm

Let $A \in M_{n,n}$ be a matrix

Find the inverse of A (if it exists)

① form augmented matrix $(A \mid I_n)$

② apply the Gauss-Jordan algorithm

③ if reduced echelonised augmented matrix is $(I_n \mid B)$, B is inverse of A

Definition 3.18 ($n \times n$ elementary matrices)

$$X_n(i,\lambda) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow \text{row } i. \ 0 \neq \lambda \in \mathbb{R} \\ \\ \\ \end{array}$$

$$Y_n(i,j) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & \cdots & 1 & \\ & & \vdots & \ddots & \vdots & \\ & & 1 & \cdots & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow \text{row } i \quad I_n \text{ with rows } i, j \text{ swapped} \\ \\ \leftarrow \text{row } j \\ \\ \\ \end{array}$$

$$Z_n(i,j,\lambda) = \begin{bmatrix} 1 & & \lambda \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{array}{l} i \neq j \\ \\ \lambda \text{ in position } (i,j) \end{array}$$

Proposition 3.19 let $A \in M_{n,n}$

$X_n(i,\lambda)A \quad R_i \mapsto \lambda R_i$ on A

$Y_n(i,j)A \quad R_i \leftrightarrow R_j$ on A

$Z_n(i,j,\lambda) \quad R_i \mapsto R_i + \lambda R_j$

Proof (inverse algorithm)

[suppose the algorithm terminates for $A \in M_{n,n}$]

$(A \mid I_n) \quad (E_1 A \mid E_1 I_n) \quad E_1$ some elementary matrix

$(E_2 E_1 A \mid E_2 E_1 I_n) \to \cdots \to (E_k E_{k-1} \cdots E_1 A \mid I_n)$

where E_i is an elementary matrix for each $1 \leq i \leq k$ where $E_k E_{k-1} \cdots E_1 A$ is the I_n.

Set $B = E_k E_{k-1} \cdots E_1$

so algorithm has produced a matrix

$(BA \mid B)$ where $BA = I_n$

Suppose B is invertible

$$
\begin{aligned}
\text{then } A = I_n A &= (B B^{-1}) A \\
&= B^{-1}(BA) \\
&= B^{-1}(I_n) \\
&= B^{-1}
\end{aligned}
$$

$\Rightarrow A^{-1} = B$ as needed. $\blacksquare$