Minimal Polynomial of Block Matrices

<u>Theorem 2.6</u> Let $f(t) \in K[t]$. Suppose $f(A) = 0$. Then minimal polynomial $m(t)$ of A divides $f(t)$. In particular, $m(t)$ divides characteristic polynomial $\Delta(t)$ of A.

<u>Proof(of 2.6)</u> By division algorithm, $\exists$ polynomials $q(t)$ and $r(t)$ for which

$$f(t) = m(t)q(t) + r(t)$$

and $r(t) = 0$ or $\deg r(t) < \deg m(t)$

substituting $t = A$ we get $f(A) = \underbrace{m(A)}_{0}\, q(A) + r(A)$ with $f(A) = 0$

so $r(A) = 0$, another polynomial (which kills A)

but $\deg r(t) < \deg m(t)$.

by def. of $m(t)$, $r(t) \equiv 0 \Rightarrow f(t) = m(t)q(t)$

so $m(t) \mid f(t)$

<u>Theorem 2.7</u> The char. pol. $\Delta(t)$ and min. pol. $m(t)$ of a matrix A have the same irreducible factors. In particular, they have the same roots

Corollary 2.8 Let $\lambda \in K$. TFAE (the following are equiv.)

(1) λ is an eigenvalue of A
(2) λ is a root of the characteristic polynomial of A.
(3) λ is a root of the minimal polynomial of A.

Corollary 2.9 Suppose

$$\Delta_A(t) = (t-\lambda_1)^{n_1} \cdots (t-\lambda_q)^{n_q}$$

where $\lambda_1, \dots, \lambda_q$ are distinct roots of $\Delta_A(t)$ and $n_1, \dots, n_q$ are their alg. multiplicity then

$$m_A(t) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_q)^{m_q}$$

where $m_1, \dots, m_q$ are the smallest integers s.t. $1 \leq m_i \leq n_i \ \forall i$ and $m_A(t) = 0$

Example minimal polynomial $m(t)$ of $A = \begin{bmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{bmatrix}$

sol: $\Delta(t) = \det(tI_n - A) = t^3 - 5t + 7t - 3$
$= (t-1)^2(t-3)$

$m_1(t) = (t-1)(t-3)$ or $m_2(t) = (t-1)^2(t-3)$

start with the smallest one and check whether $m_i(A) = 0$

$$m_1(A) = (A - I_3)(A - 3I_3) = \begin{bmatrix} 1 & 2 & -5 \\ 3 & 6 & -15 \\ 1 & 2 & -5 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 & -5 \\ 3 & 4 & -15 \\ 1 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $m(t) = (t-1)(t-3)$

Example 2.13

$$J_r(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$ (jordan block)

$\lambda \in K$

can show that $M(t) = \Delta(t) = (t-\lambda)^r$ for $J_r(\lambda)$

Remark 2.14 The characteristic polynomial of a linear operator $T: V \to V$ are defined as those of $[T]_S$ (the matrix of T in any basis S of V). Check it don't depend on S.

$A = P^{-1} B P$

$\det A \neq \det P' \det B \det P = \det B$

$$[T]_{S'} = P^{-1} [T]_S P$$

similar

3. Characteristic and Minimal Polynomial of Block Matrices.

block triangular matrix $M = \left[\begin{array}{c|c} A_1 & B \\ \hline 0 & A_2 \end{array}\right]$ A_1 & A_2 are square matrices

$$\Delta_M(t) = \det(tI - M) = \det \begin{bmatrix} tI - A_1 & -B \\ 0 & tI - A_2 \end{bmatrix}$$
$$= \det(tI - A_1) \times \det(tI - A_2)$$
$$= \Delta_{A_1}(t) \Delta_{A_2}(t)$$

Theorem 3.1 If M is on the diag is a block triangular matrix with blocks $A_1, \ldots, A_r$

then $\Delta_M(t) = \Delta_{A_1}(t) \Delta_{A_2}(t) \Delta_{A_3}(t) \ldots \Delta_{A_r}(t)$

Theorem 3.2 Suppose M is a block diagonal matrix with diagonal blocks $A_1, A_2, \ldots, A_r$ · minimal polynomial of M is

$$m_M(t) = LCM(m_{A_1}(t), m_{A_2}(t), \ldots, m_{A_r}(t))$$

Example 3.4 $\Delta(t)$ & $m(t)$? 7×7 matrix

$$M = \mathrm{diag}(A_1, A_2, A_3) \text{ with } A_1 = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 7 & 2 & 0 \\ 0 & 7 & \end{bmatrix}$$

$\Delta_1(t)$ of A_1 $\quad (t-2)^2$
$\Delta_2(t)$ of A_2 $\quad (t-7)(t-2)$
$\Delta_3(t)$ of A_3 $\quad (t-7)^3$

$$\Delta(t) = (t-2)^3(t-7)^4 \qquad (\deg(\Delta(t)) = 7)$$

$m_1(t) = (t-2)^2$
$m_2(t) = (t-2)(t-7)$
$m_2(t) = (t-7)$

$$m(t) = LCM((t-2)^2, (t-2)(t-7), (t-7))$$

$$= (t-7)(t-2)^2$$