MA1114 16/2/22

Isos Preserve Basis | Every $\mathbb{R}$ or $\mathbb{C}$ VS is isos to $\mathbb{R}^n$ or $\mathbb{C}^n$

last time

- A linear map $T: V \to W$ is an isomorphisms if it is a bijection (injective & surjective)
- S is an <u>inverse</u> of T if $S \circ T = id_V$, $T \circ S = id_W$

$$S: W \to W$$

- A linear isomorphisms has an unique inverse (which is also an isomorphisms).

lemma

If $T: U \to V$, $S: V \to W$ are invertible (i.e there exists unique inverses to the linear map)
then $(T \circ S)^{-1} = T^{-1} \circ S^{-1}$

Proof

enough to show $(S \circ T) \circ (T^{-1} \circ S^{-1}) = id_W$
$(T^{-1} \circ S^{-1}) \circ (S \circ T) = id_U$

if $u \in U$

$$\begin{aligned}(T^{-1} \circ S^{-1} \circ S \circ T) &= (T^{-1} \circ (S^{-1} \circ S) \circ T)(u) \\ &= (T \circ id_V \circ T)(u) \\ &= (T^{-1} \circ T)(u) = id_U(u) = u.\end{aligned}$$

so $(T^{-1} \circ S^{-1}) \circ (S \circ T) = id_u$
similarly $(S \circ T) \circ (T^{-1} \circ S^{-1}) = id_v$

so $(T^{-1} \circ S^{-1})$ is an inverse of $(S \circ T)$ and therefore the inverse so $(S \circ T)^{-1} - T^{-1} \circ S^{-1}$ $\blacksquare$

Proposition 8.37

If $T: V \longrightarrow W$ is a linear isomorphisms and $\{v_1, \dots, v_n\}$ is a basis for V. Then $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is a bases for W

Proof

By rank-nullity theorem $\dim(V) = \text{nullity}(T) + \text{rank}(T)$
$= \dim(\ker(T)) + \dim(\text{im}(T))$
$= \dim(\{0\}) + \dim(W)$

since T is an isomorphisms

injective $\Leftrightarrow \ker(T) = \{0\}$
surjective $\Leftrightarrow \text{im}(T) = \text{codomain}(T)$

so $\dim(V) = 0 + \dim(W)$
$= \dim(W)$

so by "check one get one free for basis" is enough to show linear independance

$\{T(v_1), T(v_2), \dots, T(v_n)\}$ is linearly independant
suppose $\lambda_1 T(v_1) + \lambda_2 T(v_2) + \dots + \lambda_n T(v_n) = 0$

$\Rightarrow T\left(\sum_{i=1}^{n} \lambda_i v_i\right) = 0$

$\Rightarrow \sum_{i=1}^{n} \lambda_i v_i \in \ker(T) = \{0\}$ since T is injective

$\Rightarrow \sum_{i=1}^{n} \lambda_i v_i = 0$ since $(v_1, \dots, v_n)$ is basis and linear

so $\{T(v_1), \dots, T(v_n)\}$ is a basis of W

Theorem 8.38

- Any real vector space of dimension n is isomorphic to $\mathbb{R}^n$
- Any complex vector space of dimension n is isomorphic to $\mathbb{C}^n$

Proof

let F be $\mathbb{R}$ or $\mathbb{C}$
let V be a vector space of F with bases $\{v_1, v_2, \dots, v_n\}$

The map. $V \to F^n$
$E : v_i \mapsto e_i$
extends to an unique linear map $E : V \to F^n$

$$E\left(\sum_{i=1}^{n} \lambda_i v_i\right) = \sum_{i=1}^{n} \lambda_i E(v_i) = \sum_{i=1}^{n} \lambda_i e_i = \begin{pmatrix} \lambda_i \\ \vdots \\ \lambda_n \end{pmatrix}$$

E has an inverse $E^{-1} : F \to V$
$e_1 \to v_1$

so E is an isomorphism (since E has an inverse must be bijection)

$\boxed{}$

Remark

the map E sends $v = \sum_{i=1}^{n} \lambda_i v_i$ to $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$

so $E(v)$ is the coordinate vectors of v,

Example

Let P_1 be polynomials in x of degree at most 1, $P_1 = \{ax+b \mid a,b \in \mathbb{C}\}$

$P_1 = \langle 1+x, 1-x \rangle$

by theorem there is an isomorphism $P_1 \to \mathbb{C}^2$

$$ax+b = \left(\frac{a+b}{2}\right)(1+x) + \left(\frac{a-b}{2}\right)(1-x)$$

so $E(v) = \left(\frac{a+b}{2}\right)e_1 + \left(\frac{a-b}{2}\right)e_2 = \begin{pmatrix} \frac{a+b}{2} \\ \frac{a-b}{2} \end{pmatrix}$

E is an isomorphism

$\ker(E) = \{ax+b \mid \frac{a+b}{2} = \frac{a-b}{2} = 0\} \Rightarrow \frac{a+b}{2} = 0, \ \frac{a-b}{2} = 0$

$\Rightarrow \left(\frac{a+b}{2}\right) + \left(\frac{a-b}{2}\right) = 0 + 0 \quad \Rightarrow a = b = 0$

$\Rightarrow \ker(E) = \{0\} \Rightarrow$ injective

<u>E surjective?</u>

$\begin{pmatrix} \frac{a+b}{2} \\ \frac{a-b}{2} \end{pmatrix} = \frac{a}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{b}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ already know $\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle = \mathbb{C}^2$

so any elt in $\mathbb{C}^2$ is of form $a/2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b/2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ surjective

Corollary 8.10

Let V and W be vector spaces then $V \cong W \Rightarrow \dim(V) = \dim(W)$

Proof "$\Rightarrow$" suppose $V \cong W$

$\Rightarrow$ there exists an isomorphism $f: V \to W$

$\Rightarrow$ $\operatorname{im}(f) = W$ $\ker(f) = \{0\}$

$$\Rightarrow \dim(V) = \dim(\ker(f)) + \dim(W)$$
$$= \dim(W)$$

"$\Leftarrow$" suppose $\dim(V) = \dim(W)$

$\Rightarrow$ by theorem 8.38

there exists maps

$$T: V \to F^n$$
$$S: W \to F^n$$

which are isomorphism

by earlier fact there exists

$S^{-1}: F^n \to W$ linear isomorphism

then $S^{-1} \circ T: V \to W$ is an isomorphism (since composition of inj / surj map)

$\Rightarrow V = W$