MA1114 26/10/21

Number of solutions

<u>Example 2.27</u>

For all values of $a \in \mathbb{R}$

solve $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & -4 \\ 1 & 5 & a^2-7 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ a+14 \end{pmatrix}$

$R_3 \mapsto R_3 - R_1$ $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 7 \\ 0 & 2 & -4 & 4 \\ 0 & 2 & a^2-5 & a+7 \end{array}\right]$

$R_2 \mapsto \frac{1}{2} R_2$ $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 7 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & a^2-5 & a+7 \end{array}\right]$

$R_3 \mapsto R_3 - 2R_2$ $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 7 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & \underbrace{a^2-1}_{=(a+1)(a-1)} & a-1 \end{array}\right]$

Case 1 $a=-1$ $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 7 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & -2 \end{array}\right]$ - inconsistent system
- no solution

Case 2 $a=1$ $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 7 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$ - infinitely many solutions
(2 degrees of freedom)

Case 3 $a \neq -1, 1$ $a^2-1 \neq 0$

$R_3 \mapsto \dfrac{R_3}{a^2-1}$ $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & \frac{1}{a+1} \end{array}\right]$

$\Rightarrow$ reduced echelon form $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-5a-9}{a+1} \\ 0 & 1 & 0 & \frac{4a+6}{a+1} \\ 0 & 0 & 1 & \frac{1}{a+1} \end{array}\right]$

$\rightarrow$ unique solution (1 solution)

<u>Goal</u> Any linear system has 0, 1, or ∞ solution

<u>Proposition 2-28</u>

Any homogeneous system 1 or ∞ solutions

<u>Proof</u>

Suppose $A \in M_{n,n}$ and $Ax=0$

Put into reduced echelon form

case 1: every row of A (reduced echelon form) contains a leading 1

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ $x_1 = x_2 = x_3 = \cdots = x_n$ is the only solution

case 2: some column of echelonised and reduced does <u>not</u> contains a leading 1

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} =$

Corollary

Suppose $A \in M_{n,m}$ $m<n$ $\begin{array}{cc} & n \\ m & \begin{pmatrix} \quad & \quad & \quad \end{pmatrix} \end{array}$

Then $Ax=0$ has infinitely many solutions

Proof "A" echelonised must contain a column without a leading one

Proposition 2.29 (better than 2.28)

The solution to a homogeneous linear system in n variables form a subspace of $\mathbb{R}^n$

<u>reminder</u> A subset $S \subseteq \mathbb{R}^n$ is a subspace if
- $0 \in S$
- closed under vector addition
- closed under scalar multiplication

Proof

$$S = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

- $$A\,0 = \begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \Rightarrow 0 \in S$$

- Suppose $u, v \in S$

  $Au = 0\ Av$

  $A(u+v) = Au + Av = 0 + 0 = 0$ $(u+v \in S)$

$u \in S$, $\lambda \in \mathbb{R}$

$u \in S \Rightarrow Au = 0$

$A \times \lambda u = \lambda (Au) = \lambda \cdot 0 = 0$ $(\lambda u \in S)$