MA1114 15/2/22

Isomorphisms.

Recall, a function $f: X \to Y$ between sets is <u>bijective</u> if it is injective and surjective (1-1 map)

Definition

A linear map $T: V \to W$ is a liniar isomorphism if T is bijective

If there exists a linear isomorphism between V and W write $V = W$ and say isomorphic to W

Examples

a) $T_A: \mathbb{R}^n \to \mathbb{R}^n$, $A \in M_n(\mathbb{R})$

$v \mapsto Av$

is an isomorphic $\iff$ A is invertible

b) $id: \mathbb{C}^n \to \mathbb{C}^n$ is an isomorphism

$v \mapsto v$

More generally

$T: \mathbb{C}^n \to \mathbb{C}^n$

$v \mapsto rv \quad 0 \neq r \in \mathbb{C}$

is an isomorphism

$\pi: \mathbb{R}^m \to \mathbb{R}^n$

$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is an isomorphism $\iff n = m$

Proposition "Check one get one free for isomorphisms"

If $T: V \to V$ is linear then the following are equivalent.

(i) T is injective
(ii) T is surjective
(iii) T is isomorphic

Proof

notice (i) + (ii) = (iii)

so suffices to show (i) $\Leftrightarrow$ (ii)

T is injective $\Leftrightarrow \ker(T) = \{0\}$
$\Leftrightarrow \text{nullity}(T) = 0$
$\Leftrightarrow \text{rank}(T) = \dim(V)$ (by rank-nullity theorem $\text{rank}(T) + \text{nullity}(T) = \dim(V)$)
$\Leftrightarrow \dim(\text{im}(T)) = \dim(V)$
$\Leftrightarrow \text{im}(T) = V$
since $\text{im}(T)$ is spanned by $\dim(V)$ linearly independant vectors
$\Leftrightarrow$ T is surjective

Fact

Any linear isomorphisms $T: V \to W$ has unique inverse $T^{-1}: W \to V$, which is also linear isomorphism

Definition Suppose $T: U \to V$ is linear.
A map $S: V \to U$ is an inverse of T if $S \circ T = id_U$ and $T \circ S = id_V$

Exercise let $A \in M_n(\mathbb{C})$

$$\begin{aligned} T_A : \mathbb{C}^n &\to \mathbb{C}^n \text{ has an inverse} \\ v &\mapsto Av \end{aligned}$$

$\iff$ A is invertible

Suppose A is invertible, want $S : \mathbb{C}^n \to \mathbb{C}^n$
with $T_A \circ S$ id and $S \circ T_A = id$

set $S = T_{A^{-1}}$ $\quad (S \circ T_A)(v) = S(T_A(v)) = S(Av) = A^{-1}(Av)$

$= (A^{-1}A)v$

$= Iv \quad = v$

$\Rightarrow S \circ T_A = id$ similarly for $T_A \circ S = id$.

Exercise

Suppose $T : U \to V$
$S : V \to U$ are linear maps between v S_T U, V

then $S \circ T = id_U \Rightarrow$ T is injective
S is surjective

Example

$$\begin{aligned} T : \mathbb{C}^n &\to \mathbb{C}^n \\ v &\mapsto rv \end{aligned} \qquad 0 \neq r \in \mathbb{R}$$

is injective (yesterday)

Suppose $v \in \ker(T)$

$\Rightarrow rv = 0$

$\Rightarrow (1/r \, r)v = 1/r \, 0$

$\Rightarrow v = 0$