MA1014 29/11/21

Vector Space Axioms

So far $\mathbb{R}^n$ → vectors
→ matrices
→ linear systems
→ determinants + inverse

Definition 4.1

A real vector space is a set of V of vectors equipped with two operations
- vector addition
- scalar multiplication

Satisfying for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$

VA 0 $u+v \in V$
VA 1 there exists $0 \in V$ such that $v+0 = v = 0+v$
VA 2 there exists $-v \in V$ with $v+(-v) = 0 = (-v)+v$
VA 3 $(u+v)+w = u+(v+w)$
VA 4 $u+v = v+u$

So far, this says $(V, +)$ is an abelian group

SM0 $\lambda v \in V$
SM1 $1 \cdot v \in V = v$
SM2 $\mu(\lambda v) = (\mu\lambda) \cdot v$
SM3 $(\mu+\lambda) \cdot v = \mu v + \lambda v$
SM4 $\lambda \cdot (v+w) = \lambda v + \lambda w$

Example

① $\mathbb{R}^n = \{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x \in \mathbb{R} \}$ with ordinary vector addition, scalar multiplication

② $M_{n,m}(\mathbb{R})$ (similar to example ①)

③ $P_n = \{ p(x) \mid p(x)$ is a real polynomial of degree less than $n \}$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0 x^0$$

where a_n is the coefficients $\in \mathbb{R}$
degree of $p(x)$ is largest i such that $a_i \neq 0$

check that this is a vector space
addition ordinary addition of polynomials
scalar multiplication .... scalar multiplication of polymial

VA0 $a_n x^n + a_{n-1} x^{n-1} + \ldots + b_n x^n + b_{n-1} x^{n-1} + \ldots$
$(a_n + b_n) x^n + \ldots$ (degree $\leq n$)

VA1 $0 \in p^n$ (zero polynomial)
$0 + p(x) = p(x) + 0 = p(x)$

VA2 $-p(x) \in p^n$
$p(x) + (-p(x)) = 0$ . . .

VA3 ✓ real number addition is associative

VA4 `` `` `` `` `` is commutative

SM0 – SM4 is similar for p_n

$\forall \varepsilon > 0 \ \exists \delta > 0$ such that if $|x - a| > \delta$ then $|f(x) - f(a)| > \varepsilon$

f is <u>continuous</u> if it is continuous at a for all $a \in [0,1]$

$V = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is continuous} \}$

claim: V is a vector space with vector addition $(f+g)(x) = f(x) + g(x)$
$f, g \in V$

scalar multiplication $(\lambda f)(x) = \lambda \cdot f(x)$
$\lambda \in \mathbb{R}$, $f \in V$

VA0: ✓ since sum of two continuous functions is continuous (e.g. limits respect addition)
VA1: $0 \in V$ $\quad \underline{0} + f = f + \underline{0} = f$
VA2: $(-f)(x) = -f(x) \qquad (-f(x)) + f(x) = \underline{0}$
VA3:
VA4: $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$

⑤ Recall a real sequence $(a_n)_{n \in \mathbb{N}}$ is a real sequence if $a_n \in \mathbb{R} \ \forall n \in \mathbb{N}$

Consider set $W = \{ (a_n)_{n \in \mathbb{N}} \}$ of all real sequences with vector addition

vector addition $(a_n)_{n \in \mathbb{N}} + (b_n)_{n \in \mathbb{N}} = (a_n + b_n) \in \mathbb{N}$
scalar multiplication $\lambda(a_n) = (\lambda a_n)$, $\lambda \in \mathbb{R}$

Exercise check W is a vector space

Example (weird)

$V = \mathbb{R}^{>0}$: positive real numbers (not zero) $u, v, w \in V$ $u+v = uv$

vector addition

$\lambda \in \mathbb{R}$, $v \in V$ $\qquad \lambda \cdot v = v^{\lambda}$ multiplication in $\mathbb{R}$

claim $(\underset{}{V}, \underset{VA}{+}, \underset{SM}{\cdot})$ is a vector space

VA0 ✓

VA1 $0 \in V \quad \leftrightarrow \quad 1 \in \mathbb{R}^{>0}$

$v + 0 = v \cdot 1 = v = 0 + v$

VA2 $\quad -v = \frac{1}{v}$

VA3 $(uv)w = u(vw)$

VA4

SM0 $\lambda v \in V$

SM1 $v^1 = v$

SM2 $\mu \cdot (\lambda \cdot v) = \mu(v^{\lambda}) = (v^{\lambda})^{\mu} = v^{\lambda\mu} = (\lambda\mu) \cdot v$

SM3 $(\lambda + \mu)\cancel{x} = v^{(\lambda+\mu)} = \lambda v + \mu v = v^{\lambda} + v^{\mu}$

SM4 $\lambda(v+w) = (vw)^{\lambda} = v^{\lambda}w^{\lambda} = \lambda v + \lambda w$