Derivatives

Steven Cheung

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Recall the notion of differentiation in one variable ...

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{r\to x} \frac{f(r)-f(x)}{r-x}$$

Basic Rules of Differentiation:

$$(f(x)g(x))') = f'(x)g(x) + f(x)g'(x)$$
$$(\frac{f(x)}{g(x)})') = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$$
$$(f(g(x)))' = f'(g(x))g'(x)$$

Partial Derivatives:

For real functions of 2 or 3 variables, recall the notion of partial differentiation ...

$$(x,y,z) = \lim_{h\to 0} \frac{f(x+h,y,z)}{h} = f_x(x,y,z)$$
$$\frac{\partial f}{\partial y}(x,y,z) = \lim_{h\to 0} \frac{f(x,y+h,z)}{h} = f_y(x,y,z)$$
$$\frac{\partial f}{\partial z}(x,y,z) = \lim_{h\to 0} \frac{f(x,y,z+h)}{h} = f_y(x,y,z)$$

Theorem:

Let $f : \mathbb{R}^2 \to \mathbb{R}$, and assume that

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$$

exist and that $\frac{\partial^2 f}{\partial x \partial y}$ is continuous. **Then the mixed derivatives are equal!** $\square$

Chain Rule in many Dimension:

Let $f(x,y,z)$, with $x(u,v,w)$, $y(u,v,w)$ and $z(u,v,w)$, and let
$g(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w))$

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$
$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$
$$\frac{\partial g}{\partial w} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$$

If f is a function of x and y only, then the last terms on the right-hand sides disappear