MA1014 19/10/21

Examples and applications

$$\lim_{x \to c} (f_1(x) \cdot f_2(x)) = \lim_{x \to c} f_1(x) \cdot \lim_{x \to c} f_2(x)$$

Before that that let's do an easy case

Suppose $k(x) \to M$ and $h(x) \to 0$ as $x \to c$

Then $h(x)k(x) \to 0$ as $x - c$

<u>Proof</u> Take $\varepsilon = 1$ in the definition of $\lim_{x \to c} k(x) = M$

$\exists \delta < 0$ such that if $0 < |x - c| < \delta$

then $M - 1 < k(x) < M + 1$

So $\underbrace{(M-1)|h(x)|} < |h(x)|k(x) < \underbrace{(M+1)|h(x)|}$

As $x \to c$ has limit zero has limit zero

So by the Pinching Theorem

$|h(x)|k(x) \to 0 \Rightarrow \underline{h(x)k(x) \to 0}$

Used: $\lim_{x \to c} |f(x)| = |\lim_{x \to c} f(x)|$

Used: M constant $\lim_{x \to c} M \cdot f(x) = M \cdot \lim_{x \to c} f(x)$

Theorm

$$\lim_{x \to c} (f_1(x) \cdot f_2(x)) = \underbrace{\lim_{x \to c} f_1(x)}_{L_1} \cdot \underbrace{\lim_{x \to c} f_2(x)}_{L_2}$$

Proof Algebra trick

$$|f_1(x) f_2(x) - L_1 L_2|$$

$$= |f_1(x)(f_2(x) - L_2) + (f_1(x) - L_2) L_2|$$

$$\leq |f_1(x)||f_2(x) - L_2| + |f_1(x) - L_1||L_2|$$

limit L_1 limit 0 limit 0 limit limit $|L_2|$

As $x \to c$, $|f_1(x) f_2(x) - L_1 L_2| \leq h(x) k(x) + r(x)|L_2|$

limit 0 : limit 0 limit $0 + 0$

by pinching theorm

$f_1(x) \cdot f_2(x)$ has limit $L_1 L_2$

Examples a) $\sin(x)$ is continuous at $x=0$

i.e. $\lim_{x\to 0} \sin(x) = \sin(0) = 0$

b) $\frac{1}{x}$ is continuous at all $x \neq 0$

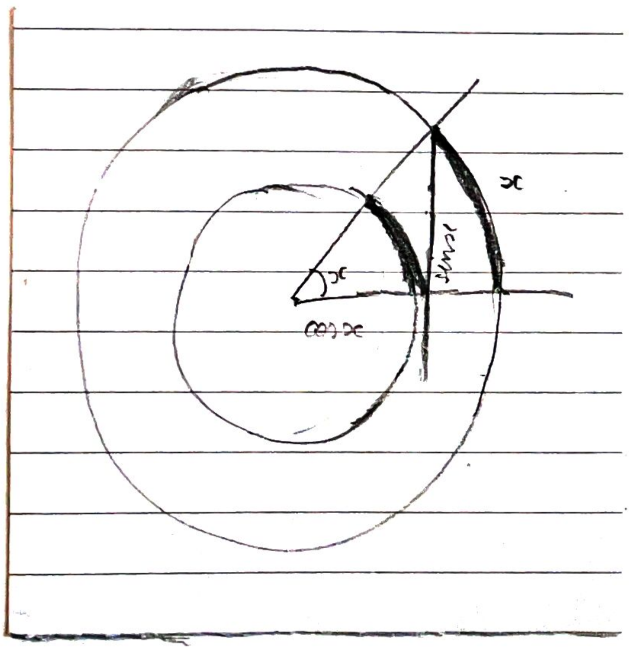
c) $f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

continuous for all x

d) $\cos(x)$

e) $f(x) = \begin{cases} \frac{1-\cos^2(x)}{x^2} & x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases}$

For a) I define the sine function



x radians

compare lengths

$$x\cos x \leq \sin x \leq x$$
$$\downarrow \qquad\qquad \downarrow$$
$$0 \qquad\qquad 0$$

Pinching Theorm: $\sin x \to 0$

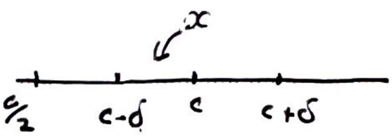
sin continuous at $x=0$

b) $\frac{1}{x} \to \frac{1}{c}$ as $x \to c$ $\quad (c \neq 0)$

$\to \left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c-x}{xc}\right| = \frac{|x-c|}{|x||c|}$

($|x-c|$ small $\Rightarrow \left|\frac{1}{x} - \frac{1}{c}\right|$ small)

Given any $\varepsilon > 0$



choose $\delta < \frac{|c|}{2}$

$|x| > \frac{|c|}{2}$, $\frac{1}{|x|} < \frac{2}{|c|}$

& and $\delta < \frac{c^2}{2}\varepsilon$

$0 < |x-c| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{c}\right| < \frac{c^2}{2}\varepsilon \cdot \frac{2}{|c|} \frac{1}{|c|}$

$= \varepsilon$

So $\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$

$1/x$ continuous.

Idea $\cos(x) \to \underset{=1}{\cos(0)}$ as $x \to 0$

Eventually. $\sqrt{1-\sin^2(x)} = \cos(x)$

$\to 1(1-0^2) = 1$ as $x \to 0$