MA1014 18/1/22

Integrable Functions

① Which of the following function $f(x)$ are integrable on $[0,1]$? what is the value of $\int_0^1 f(x)\,dx$?

a) $f(x) = \begin{cases} 0 & x = \frac{m}{n} \text{ rational} \\ 1 & x \text{ irrational} \end{cases}$

b) $f(x) = \begin{cases} 1 & x = \frac{m}{n} \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$

c) $f(x) = \begin{cases} 1 & x = \frac{1}{n},\ n \text{ positive integer} \\ 0 & \text{otherwise} \end{cases}$

(a) and (b) Any interval $[x_{i-1}, x_i]$ of any P will contain rational & irrational number

$$m_i = 0 \qquad M_i = 1$$

$L_f(P) = \sum m_i \Delta x_i = \underline{0}$ $\qquad U_f(P) = \sum M_i \Delta x_i$

$\qquad\qquad\qquad\qquad\qquad = \sum \Delta x_i \quad = \underline{1}$

$\nexists P : U_f(P) - L_f(P) < \frac{1}{2}$, so f not integrable. (lesbesgue integrable)

(c) show $f(x)$ integrable we need to find P making $U_f(P)$ as close to zero as we like

"$\forall \varepsilon > 0\ \exists P : U_f(P) < \varepsilon$"

For each positive integer m define partitions P_m $U_f(P_m) \to 0$ as $m \to 0$
$\Rightarrow f$ integrable

$P_1 = \{0, 1\}$ $\quad$ $P_2 = \{0, \frac{3}{4}, 1\}$ $\quad$ $P_m = \{0, \frac{1}{m} \pm \frac{1}{2m^2}, \ldots, \frac{1}{3} \pm \frac{1}{2m^2}, \frac{1}{2} \mp \frac{1}{2m^2},$
partition into $2m$ subintervals $1 - \frac{1}{m^2}$,
$1\}$

$[0, x_1]$ $\quad M_1 = 1 \quad \Delta x_1 = \frac{1}{m} - \frac{1}{2m^2}$

$[x_{i-1}, x_i]$ where i is <u>even</u> $\quad M_i = 1 \quad \Delta x_i = \frac{1}{m^2}$

Others : $M_i = 0 \quad \Delta x_i \ldots$

$U_f(P) = 1 \times (\frac{1}{m} - \frac{1}{m^2}) + m \times 1 \times \frac{1}{m^2} + 0 \quad = \frac{2}{m} - \frac{1}{m^2} \to 0$

interval $(0,1)$, $[0,1]$, $\mathbb{R}$, $(0, \infty)$

② $f: D \to \mathbb{R}$ is continuous everywhere

$\forall x \in D$ $\exists \delta > 0 : \forall c, |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$
$\forall \varepsilon > 0$

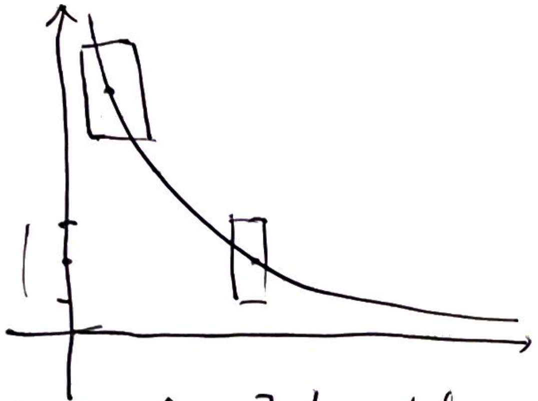
<u>Definition</u> $f: D \to D$ is <u>uniformly</u> continuous

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, c \quad |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

this δ must work for <u>all</u> x

__Examples__ $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$ __not__ uniformly continuous

$f: (0,1) \to \mathbb{R}$ $f(x) = 1/x$ not uniformly continuous



$f: (0,1) \to [0,1]$ bounded, continuous
$f(x) = \sin(1/x)$ __not__ uniformly continuous

__Theorem__ Any continuous function on a closed bounded interval is __uniformly continuous__ $f: [a,b] \to \mathbb{R}$

Proof using Bolzano - Weierstrass

__Theorem__ Any __continuous__ function $f: [0,1] \to \mathbb{R}$ is __integrable.__

__Proof__ f is uniformly continuous by previous theorem.

Given any $\varepsilon > 0$, $\exists \delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

choose any partition P with all $\Delta x_i < \delta$

then m_i, M_i are the bounds of f on $[x_{i-1}, x_i]$, $m_i = f(x)$, $M_i = f(y)$
$x, y \in [x_{i-1}, x_i]$

$|x-y| < \delta \Rightarrow M_i - m_i < \varepsilon$

$$U_f(P) - L_f(P) = \sum M_i \Delta x_i - \sum m_i \Delta x_i$$

$$= \sum (M_i - m_i) \Delta x_i$$

$$< \sum_{i=1}^{n} \varepsilon \Delta x_i = \varepsilon \left(\sum_{i=1}^{n} \Delta x_i \right)$$

$$= \varepsilon$$