MA1061 25/10/21

Ordered Sampling with(out) Replacement

Sampling

Many problems in probability and statistics are concerned with choosing an element from a set repeatedly, i.e. sampling

Four Cases:
- ordered with replacement
- ordered without replacement
- unordered with
- unordered without

Ordered Sampling with Replacement

The element is replaced in the set before the next element is chosen, and the order matters.

Proposition (2)

Consider an ordered sample of size r taken from the set: $\{a_1, .. a_n\}$ as follows:
1. select an element a_i, and record it;
2. return the element back into the set (replacement);
3. repeat r times,

So that a sample $(a_{i_1}, .., a_{i_r}$ is registered, where a_{i_k} is the element selected in the k-th step

Then the number of all possible such samples is $N = n^r$

Proof

- After an element is chosen we replace it back in the set

- Therefore the experiment is identical to choosing one item from r distinct sets $A_1, A_2, \ldots, A_r$ each containing n elements

- Proposition (1) then applies with $n_1 = n_2 = \ldots = n_r = n$, showing that there are

$$N = n^r, \quad \text{distinct ordered samples } (a_{i_1}, \ldots, a_{i_r}$$

- Obviously, this is a special case of the multiplication principle where there are r groups and each group has n elements

Ordered Sampling without Replacement

Here the element is not replaced in the set and is taken away.

Proposition (3)

Given a set of n element $\{a_1, \ldots, a_n\}$, form all possible combinations of r $(r \le n)$ distinct elements taking into account their order. Then then number of all the samples is

$$N = \frac{n!}{(n-r)!} = n \times (n-1) \times \ldots \times (n-(r-1)).$$

Proof

- no replacement, $n-1$
- k elements chosen, $n-k$
- corresponds to proposition (1) $n_1 = n, n_2 = n-1, \ldots, n_r = n-r+1$
- proposition (1) gives

$$N = n \times (n-1) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}$$

Example : Birthday Problem

What is the probability that N ($N \leq 365$) students have distinct birthdays? Assume a year has 365, and that a student's birthday can fall on any day with the same probability

$$\Omega = \{ w = (w_1, w_2, \ldots, w_N), w_i \in \{1, 2, \ldots, 365\}\}$$

$$n_1 = n_2 = \ldots = n_{365} \qquad 365^N \text{ samples}$$

Let $A = \{$ at least two have the same birthday $\}$
$\bar{A} = \{$ all bdays are distinct $\}$

$N(\bar{A}) = 365 \times \ldots \times (365 - N + 1)$ hence

$$P(\bar{A}) = \frac{365 \times \cdots \times (365 - N + 1)}{365^N}$$

$$\therefore \quad P(A) = 1 - \frac{365 \times \ldots \times (365 - N + 1)}{365^N}$$

N	10	20	23	30	40	50	60
P(A)	.117	.411	.507	.706	.891	.970	.994

Class of 23 : $P(A) = 0.507 = 50.7\%$

Class of 60 : $P(A) = 0.994 = 99.4\%$

Another Example

A train with n coaches is boarded by r passengers ($r \leq n$) each entering a coach at random. What is the probability of all the passengers endup in different coaches?

$$\Omega = \{(i_1, \dots i_r) \mid i_j \in \{1, 2, \dots, n\} \text{ for } 1 \leq j \leq r\}$$

$|\Omega| = n^r$ $\qquad A = \{\text{all pasengers choose a different coach}\}$

A occurs $\iff i_j \neq i_k$ for $j \neq k$

1st passenger $= n$
2nd .. $= n-1$

$|A| = n \times (n-1) \times \dots \times (n-r+1)$ elementary events for which A occurs

thus $P(A) = \dfrac{n(n-1)\cdots(n-r+1)}{n^r}$