MA1114 31/1/22

Intersections ($\dim(W) \leq \dim(V)$ with equality iff $W = V$)

<u>Proposition</u>

Suppose $W \subseteq V$, a vector space. Then $\dim(W) \leq \dim(V)$ with equality if and only if $W = V$.

<u>Proof</u> set $n = \dim(V)$

> If $W = \{0\}$ then it clear since $\dim(W) = 0$. if $W \neq \{0\}$ then choose $w_1 \in W$ with $w \neq 0$. if $\langle w \rangle = W \Rightarrow$ done.

> If $\langle w_1 \rangle \neq W$, then choose $w_2 \in W \setminus \{w_1\}$ so $\langle w_1, w_2 \rangle = \dim(2)$ by $\pm$ theorem.

> If $\langle w_1, w_2 \rangle = W$ done. Else continue as above until we have a basis $\{w_1, ..., w_k\}$ for W.

> If $k > n$ then $\{w_1, ..., w_k\}$ is not linearly independant (too many vectors) so $k \leq n \Rightarrow \dim(W) \leq \dim(V) = n$

> If $W = V$ then $\dim(W) = \dim(V)$. suppose $\dim(W) = \dim(V) = n$

> If $W \subset V$ then choose $v \in V \setminus W$

Then if $\{w_1, ..., w_n\}$ is a basis for W then $\{w_1, ..., w_n, v\}$ is LI

$\times$ too many vectors $W = V$

suppose $W \subseteq V = \langle S \rangle$ $S \subset V$ some spanning set
this does <u>NOT</u> imply $W = \langle \bar{S} \rangle$ for some subset of $\bar{S} \subset S$

Example $\mathbb{R} = \langle \{ \binom{1}{0} \binom{0}{1} \} \rangle$

$W = \langle \binom{1}{1} \rangle$ does not contain either $\binom{0}{1}$ or $\binom{1}{0}$

Intersection of Subspaces

Recall, if $U, W \leq V$ a vector space

that
- $U \cap W$ is a subspace of V
- $U + W$ is $\{ u + w \mid u \in U, w \in W \}$ is a subspace of V

Proposition 6.27?

For $U, W \leq V$ as above, $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

Example $V = \mathbb{R}^3$

$$U = \left\langle \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\} \right\rangle \quad (xy \text{ plane})$$

$$W = \left\langle \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} \right\rangle \quad (xz \text{ plane})$$

$$\mathbb{R}^3 \geq (U+W) \rightarrow \left\langle \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} \right\rangle = \mathbb{R}^3$$

$\Rightarrow U + W = \mathbb{R}^3$

$\Rightarrow \dim(U+W) = 3$

Example

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \, x_1 + x_2 + x_3 = 0 \right\} \leq V = \mathbb{R}^3$$

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \, x_1 + x_2 + x_3 = 0 \right\} \leq W = \mathbb{R}^3$$

$\dim(U) = 2$ how? why?

lets set $x_1 = \lambda$, $x_2 = \mu$

then $U = \left\{ \begin{pmatrix} \lambda \\ \mu \\ -\lambda-\mu \end{pmatrix} \middle| \, \lambda, \mu \in \mathbb{R} \right\}$

$\dim(U) = 2 = \dim(W)$.

$U \cap W = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ (x axis) $\Rightarrow \dim(U \cap W) = 1$

$$2 + 2 - 1 = 3$$

$$\Rightarrow U = \left\{ \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

$\Rightarrow \dim(U) = 2$ similarly $\dim(W) = 2$

$U \cap W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \, Ax = 0 \right\} = \text{null}(A)$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

$A \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ a general element of null(A) looks like.

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ where $x_1 = -x_3$, $x_2 = 0$ $\Rightarrow \text{null}(A) = \left\{ \begin{pmatrix} \lambda \\ 0 \\ \lambda \end{pmatrix} \middle| \, \lambda \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$\Rightarrow \dim(U \cap W) = 1$ by prop $U + W = \mathbb{R}^3$

Proof of Proposition 8.22

Set $m = \dim(U \cap W)$
$k = \dim(U)$
$l = \dim(W)$

let $B_{U \cap W} = \{v_1, \ldots, v_n\}$ be a basis for $U \cap W$
use + theorem to extend $B_{U \cap W}$ to bases U and W.

$$B_U = \{v_1, \ldots, v_n, u_{m+1}, \ldots, u_k\}$$

$$B_W = \{w_1, \ldots, w_m, w_{m+1}, \ldots, w_l\}$$

Claim

$B_U \cup B_W$ is a basis from $U+W$. once show we are done:

$$\dim(U+W) = |B_U \cup B_W| = |B_U| \cup |B_W| - |B_U \cap B_W|$$

$$= k + l - |B_U \cap B_W|$$
$$= k + l - m$$
$$= \dim(U) + \dim(W) - \dim(U \cap W)$$

Proof of Claim

① $B_U \cup B_W$ spans $U+W$
suppose $u+w \in U+W$, $u \in U$, $w \in W$

$$\Rightarrow u = \sum_{i=1} \lambda_i v_i + \sum_{i=1}^{k} \lambda u_i, \Rightarrow w = \sum_{i=1}^{m} \mu_i v_i + \sum_{i=1}^{l} \mu_i w_i$$

$$u+w = \sum_{i=1}^{m} v_i(\lambda_i \mu_i) + \sum_{i=m+1}^{k} \lambda_i u_i + \sum_{i=n+1}^{l} \mu_i w_i \in \operatorname{span}(B_U \cup B_W)$$

② $B_U \cup B_W$ is LI

suppose $(*)$ $\sum_{i=1}^{m} \lambda_i v_i + \sum_{i=m+1}^{k} \mu_i u_i + \sum_{i=m+1}^{l} \alpha_i w_i = 0$

Then $U \ni \sum_{i=1}^{m} \lambda_i v_i + \sum_{i=m+1}^{k} \mu_i u_i = -\sum_{i=m+1}^{l} \alpha_i w_i \in W$

$\Rightarrow \sum_{i=1}^{m} \lambda_i v_i + \sum_{i=m+1}^{k} \mu_i u_i \in U \cap W$

so equals $\sum_{i=1}^{m} \beta_i v_i$

so by $(*)$

$\sum_{i=1}^{m} \beta_i v_i + \sum_{i=m+1}^{l} \alpha_i w_i = 0$

$\Rightarrow \alpha_i = \beta_i = 0$ for all i (since $\mathcal{B}_W$ is a basis)

So by $(*)$ $\sum_{i=1}^{m} \lambda_i v_i + \sum_{i=m+1}^{k} \mu_i v_i = 0$

$\Rightarrow \lambda_i = \mu_i = 0$ for all i (since $\mathcal{B}_U$ is a basis)

$\Rightarrow \mathcal{B}_U \cup \mathcal{B}_W$ is LI as so a basis for $U+W$.