MA1114 22/3/22

Inner Products

Definition

The function $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

defined by $\langle v, w \rangle = v^T w$

$$v \in M_{n,1}(\mathbb{R}) = v^T \in M_{1,n}(\mathbb{R})$$
$$w \in M_{n,1}(\mathbb{R}) \Rightarrow v^T w \in M_{1,n}(\mathbb{R}) = \mathbb{R}$$

Alternatively if $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$

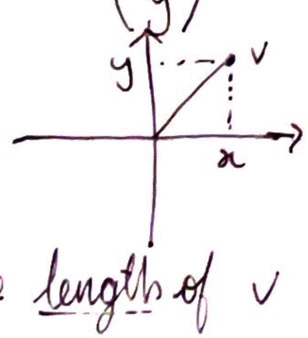
then $v^T w = (v_1, v_2, \ldots, v_n) \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \sum_{i=1}^{n} v_i w_i$

$\langle v, w \rangle$ is called standard (Euclidean) inner product.

$|v| = \sqrt{\langle v, v \rangle}$ is the norm of v.

Examples $n = 2$, $v = \begin{pmatrix} x \\ y \end{pmatrix}$

$\sqrt{x^2 + y^2}$
(pythagoras)



$$\langle v, v \rangle = (x, y) \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= x^2 + y^2$$
$$\Rightarrow \sqrt{\langle v, v \rangle} = \sqrt{x^2 + y^2}$$

$|v|$ is the length of v

$n = 4$

$$\underset{v}{\overset{\shortparallel}{\begin{pmatrix} 2 \\ -1 \\ 3 \\ -5 \end{pmatrix}}} \underset{w}{\overset{\shortparallel}{\begin{pmatrix} -3 \\ -4 \\ 1 \\ 0 \end{pmatrix}}} \in \mathbb{R}^4 \qquad \langle v, w \rangle = 1$$

don't confuse

$\langle v, w \rangle$ with $\underset{\text{span}}{\langle v, w \rangle}$

__Theorem__ $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

satisfies for $\lambda, \mu \in \mathbb{R}$ $\quad u, v, w \in \mathbb{R}^n$

(i) $\langle \lambda u + \mu v, w \rangle = \lambda \langle u, w \rangle + \mu \langle v, w \rangle$ and satisfies for $\langle u, \lambda v + \mu w \rangle$

(ii) $\langle v, w \rangle = \langle w, v \rangle$

(iii) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \Leftrightarrow v = 0$

__Proof__

(ii) $\langle v, w \rangle = v^T w \in M_{1,1}(\mathbb{R})$

$$= (v^T w)^T$$
$$= w^T (v^T)^T = \underset{\langle w, v \rangle}{w^T v}$$

(i) $\langle v, \lambda u + \mu w \rangle = v^T(\lambda u + \mu w)$

$$= v^T \lambda u + v^T \mu w$$
$$= \lambda v^T u + \mu v^T w$$
$$= \lambda \langle v, u \rangle + \mu \langle v, w \rangle$$

using (ii) we can deduce the "other one" by symmetry.

(iii) let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ Then $\langle v, v \rangle = v^T v$

$$= \sum_{i=1}^{n} v_i v_i$$
$$= \sum_{i=1}^{n} (v_i)^2 \geq 0$$

since $v_i^2 \geq 0$

also $\sum_{i=1}^{n} v_i^2 = 0 \implies v_i = 0$ for all i

<u>Corollary</u> for $v \in \mathbb{R}^n$

$|v| \geq 0$, $|v| = 0 \iff v = 0$

<u>Proof</u>

$|v| = \sqrt{\langle v, v \rangle} \geq 0$

and $|v| = 0 \iff \langle v, v \rangle = 0$
$\iff v = 0$

Notice $|\lambda v| = |\lambda||v|$, for

$\lambda \in \mathbb{R}, v \in \mathbb{R}^n$ since