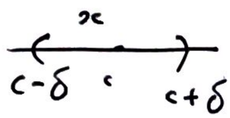
MA1014 18/10/21

Definition of Limit ctd...

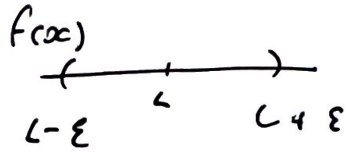
$\lim_{x \to c} f(x) = L$ means $\forall \varepsilon > 0 \ \exists \delta > 0 \ \ 0 < |x - c| < \delta \Rightarrow$

$\Rightarrow |f(x) - L| < \varepsilon$

given margin of error



$x = c$ guarantees



limit from $\begin{cases} \text{above / right} \\ \text{below / left} \end{cases}$

$\lim_{x \to c^-} f(x) = L$

$\forall \varepsilon > 0 \ \exists \delta > 0 \quad 0 < c - x < \delta$
$\Rightarrow |f(x) - L| < \varepsilon$

$\lim_{x + c^+} f(x) = L$

$\forall \varepsilon > 0 \ \exists \delta > \quad 0 < c - x < \delta$
$\Rightarrow |f(x) - L| < \varepsilon$

<u>Continous</u> $f(x)$ continuous at $x = c$

$\lim_{x \to c} f(x) = f(c)$

$f(c) =$ exists, limits exists. & they are equally. same

Example a) $f(x) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases}$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 1 = 1$ $\qquad \lim_{x \to 0^+} f(x) = 0$

$\lim_{x \to 0} f(x)$ does not exists

$1 \neq 0$ so not continuous

When $\lim_{x \to c}$ exists it is same as $\lim_{x \to c^-} = \lim_{x \to c^+}$

b) $f(x) = \frac{1}{x}$

$c \neq 0$
then $f(x)$ is continuous at $x = c$

$\lim_{x \to c} \frac{1}{x} = \frac{1}{c}$

Neither $\lim_{x \to 0^-} \frac{1}{x}$ nor $\lim_{x \to 0^+} \frac{1}{x}$ exists

c) $f(x) = \begin{cases} 2x+1 & x \neq 0 \\ 17 & x = 0 \end{cases}$

$\lim_{x \to 0^-} 2x+1 = 1 \qquad \lim_{x \to 0^+} 2x+1 = 1$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} 2x+1 = 1 \neq f(0)$

not continuous

exists

d) $\boxed{f(x) = \dfrac{x^2 - 1}{x - 1}}$ $\overset{x \neq 1}{=} x + 1$

$$f(x) = \begin{cases} x+1 & x \neq 1 \\ \text{undefined} & x = 1 \end{cases}$$

domain $(f) = \mathbb{R} \setminus \{1\}$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 2$$

Basic "Limit Theorms"

Theorm 1 If a limit exists then it is unique:

If $\lim_{x \to c} f(x) = L_1$, $\lim_{x \to c} f(x) = L_2$

then $L_1 = L_2$

$\forall \varepsilon > 0 \; \exists \; \delta_1 > 0 \quad 0 < |x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$

$\forall \varepsilon > 0 \; \exists \; \delta_2 > 0 \quad 0 < |x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$

Need to prove $L_1 = L_2$

Given any $\varepsilon > 0$, we know there exists δ_1, δ_2

If $0 < |x - c| < \min(\delta_1, \delta_2) \leq \delta_1, \delta_2$

$|L_1 - L_2| \leq |f(x) - L_1| + |-f(x) + L_2)$

triangle inequality $< \varepsilon + \varepsilon$

$|x - c| < \delta_1$ $\qquad |x - c| < \delta_2$

So $|L_2 - L_1| < 2\varepsilon \quad \forall \varepsilon > 0$

So $L_2 = L_1$

Theorem 2 $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$

then i) $\lim_{x \to c} f(x) + g(x) = L + M$

ii) $\lim_{x \to c} f(x) \cdot g(x) = L \cdot M$

iii) if $M \neq 0$ then $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}$

Proof of (i) Given any $\varepsilon > 0$

need to find some $\underline{\delta > 0}$ such that

if $0 < |x - c| < \delta$ then

$$|f(x) + g(x) - (L + M)| < \varepsilon$$

Consider $\varepsilon/2 > 0$ so $\exists \, \delta_F > 0$ $0 < |x - c| < \delta_F$
$\Rightarrow |f(x) - L| < \varepsilon/2$

& $\exists \, \delta_g > 0$ $0 < |x - c| \, \delta_g$
$\Rightarrow |g(x) - L| < \varepsilon/2$

Choose $\delta = \min(\delta_f, \delta_g) \leq \delta_f, \delta_g$
$0 < |x - c| < \delta \Rightarrow$

$$\left| \begin{matrix} f(x) - L \\ + g(x) - M \end{matrix} \right| \leq |f(x) - L| + |g(x) - M|$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon$$

Theorem 3 <u>Pinching Theorem</u>

If $f(x) \leq g(x) \leq h(x)$

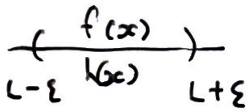
& $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$

then $\lim_{x \to c} g(x) = L$

$\forall \varepsilon > 0 \ \exists \ \delta_f, \delta_h :$

$0 < |x - c| < \delta_f \implies |f(x) - L| < \varepsilon$

$0 < |x - c| < \delta_h \implies |h(x) - L| < \varepsilon$



If $\delta = \min(\delta_f, \delta_h) \leq \delta_f, \delta_h$

$0 < |x - c| < \delta \implies L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$

$\implies L - \varepsilon < g(x) < L + \varepsilon$

So $|g(x) - L| < \varepsilon$