MA1114 1/12/21

Vector Space Axioms ctd...

Recall

A real vector space is a triple $(V, +, \cdot)$ satisfying $\forall\, u, v, w \in V$, $\lambda, \mu \in \mathbb{R}$

VA0 $u+v \in V$	SM0
VA1	SM1
VA2	SM2
VA3	SM3
VA4	SM4

non-example

$V = \mathbb{R}^2$, usual addition of vectors
scalar multiplication defined by

$$\lambda \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ 0 \end{pmatrix} \quad \lambda \in \mathbb{R}$$

VA0 – VA4

SM0 ✓, SM1 ✗: $1 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Properties

Let's V be a vector space

(i) $\underline{0}$ is unique

(ii) $0 \cdot v = \underline{0} \quad \forall v \in V$

(iii) $\lambda \cdot \underline{0} = \underline{0} \quad \forall \lambda \in \mathbb{R}$

(iv) $\lambda \cdot v = \underline{0} \Rightarrow \lambda = 0$ or $v = \underline{0}$

(v) $-v$ is unique for all $v \in V$

(vi) $(-1) \cdot v = -v \quad \forall v \in V$

Proof (only use axioms)

(VA0 − VA4) and (SM0 − SM4)

(i) Suppose $0'$, 0 both zeros in V

Then $0 = 0' + 0 = 0'$ ✓

(ii) for any $v \in V$

$v = 1 \cdot v$ by SM1

$= (0+1) \cdot v$

$= 0 \cdot v + 1 \cdot v$ by SM3

$= 0 \cdot v + v$ by SM1

So $\underline{0} = (-v) + v = (-v) + (v + 0 \cdot v)$

$= (-v) + 0 \cdot v$ by VA3

$= \underline{0} + 0v$ by VA2

$= 0 \cdot v$ by VA1 ✓

(iii) exercise

(iv) Suppose $\lambda \cdot v = \underline{0}$

If $\lambda = 0$, then refer back to (ii)

suppose $\lambda \neq 0 \Rightarrow \frac{1}{\lambda} \in \mathbb{R}$

so $v = 1 \cdot v = (\frac{1}{\lambda}\lambda) \cdot v = \frac{1}{\lambda} \cdot (\lambda \cdot v)$ by SM2

$= \frac{1}{\lambda} \cdot \underline{0} = \underline{0}$ ✓

(v) Suppose v has two negatives

u, w

$$\Rightarrow \quad u+v = v+u = \underline{0}$$
$$w+v = v+w = \underline{0}$$

$u = u + \underline{0}$ by VA1

$= u + (v+w)$

$= (u+v) + w$ by VA3

$= \underline{0} + w$

$u = w$ by VA1 ✓

(vi) $(-1)\cdot v = -v$

$v + (-1)\cdot v = 1\cdot v + (-1)v$ by SM1

$= (1+(-1))\cdot v$ by SM3

$= 0 \cdot v$

$= \underline{0}$ by part (ii)

similarly (or by VA4)

$(-1)\cdot v + v = \underline{0}$

So $(-1)\cdot v$ satisfys VA2

i.e behaves like negative but by (v) negatives are unique

$$\Rightarrow (-1)\cdot v = -v$$