MA1114 22/3/22

Inner Products

Definition

The function $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

defined by $\langle v, w \rangle = v^T w$

$v \in M_{n,1}(\mathbb{R}) = v^T \in M_{1,n}(\mathbb{R})$
$w \in M_{n,1}(\mathbb{R}) \Rightarrow v^T w \in M_{1,1}(\mathbb{R}) = \mathbb{R}$

Alternatively if $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$

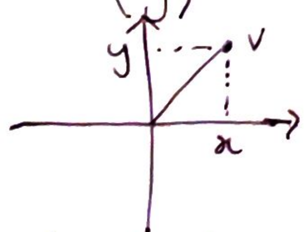
then $v^T w = (v_1, v_2, \dots, v_n) \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \sum_{i=1}^{n} v_i w_i$

$\langle v, w \rangle$ is called standard (Euclidean) inner product.

$|v| = \sqrt{\langle v, v \rangle}$ is the <u>norm</u> of v.

<u>Examples</u> $n = 2$, $v = \begin{pmatrix} x \\ y \end{pmatrix}$

$\sqrt{x^2 + y^2}$
(pythagoras)



$\langle v, v \rangle = (x, y) \begin{pmatrix} x \\ y \end{pmatrix}$
$= x^2 + y^2$
$\Rightarrow \sqrt{\langle v, v \rangle} = \sqrt{x^2 + y^2}$

$|v|$ is the <u>length</u> of v

$n=4$

$$\underset{v}{\begin{pmatrix}2\\-1\\3\\-5\end{pmatrix}} \underset{w}{\begin{pmatrix}-3\\-4\\1\\0\end{pmatrix}} \in \mathbb{R}^4 \qquad \langle v, w\rangle = 1$$

don't confuse

$\langle v, w\rangle$ with $\underset{\text{span}}{\langle v, w \rangle}$

<u>Theorem</u> $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

satisfies for $\lambda, \mu \in \mathbb{R}$ $\quad u, v, w \in \mathbb{R}^n$

(i) $\langle \lambda u + \mu v, w\rangle = \lambda \langle u, w\rangle + \mu \langle v, w\rangle$ and satisfies for $\langle u, \lambda v + \mu w\rangle$

(ii) $\langle v, w\rangle = \langle w, v\rangle$

(iii) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0 \Leftrightarrow v = 0$

<u>Proof</u>

(ii) $\langle v, w\rangle = v^T w \in M_{1,1}(\mathbb{R})$

$$= (v^T w)^T$$
$$= w^T (v^T)^T = \underset{\langle w, v\rangle}{w^T v}$$

(i) $\langle v, \lambda u + \mu w \rangle = v^T(\lambda u + \mu w)$

$$= v^T \lambda u + v^T \mu w$$
$$= \lambda v^T u + \mu v^T w$$
$$= \lambda \langle v, u \rangle + \mu \langle v, w \rangle$$

using (ii) we can deduce the "other one" by symmetry.

(iii) let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ Then $\langle v, v \rangle = v^T v$

$$= \sum_{i=1}^{n} v_i v_i$$
$$= \sum_{i=1}^{n} (v_i)^2 \geq 0$$

since $v_i^2 \geq 0$

also $\sum_{i=1}^{n} v_i^2 = 0 \Rightarrow v_i = 0$ for all i

<u>Corollary</u> for $v \in \mathbb{R}^n$

$|v| \geq 0$, $|v| = 0 \Leftrightarrow v = 0$

<u>Proof</u>

$|v| = \sqrt{\langle v, v \rangle} \geq 0$

and $|v| = 0 \Leftrightarrow \langle v, v \rangle = 0$
$\Leftrightarrow v = 0$

Notice $|\lambda v| = |\lambda||v|$, for

$\lambda \in \mathbb{R}$, $v \in \mathbb{R}^n$ since

$$|\lambda v| = \sqrt{\langle \lambda v, \lambda v \rangle}$$

$$= \sqrt{\lambda \langle v, \lambda v \rangle}$$

$$= \sqrt{\lambda^2 \langle v, v \rangle}$$

$$= |\lambda|\,|v|$$

For any $0 \neq v \in \mathbb{R}^n$, $\frac{1}{|v|} v$ is a unit vector, since with $\lambda = 1/|v|$

$$\left|\frac{1}{|v|}\right| = |\lambda v| = |\lambda|\,|v|$$

$$= \left|\frac{1}{|v|}\right| |v|$$

$$= 1/|v|\,|v| \quad (\text{since } |v| > 0 \Rightarrow 1/|v| > 0)$$

$$= 1$$