MA1014 2/11/21

Chain Rule

Derivative of composition of differentiable functions

$$\mathbb{R} \xrightarrow{\sin} \mathbb{R} \xrightarrow{\exp} \mathbb{R} \qquad e^{\sin(x)}$$

$$\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R} \qquad f \circ g$$

$$x \qquad \underset{y}{g(x)} \qquad \underset{z}{f(g(x))} = (f \circ g)(x)$$

__Theorm__ If f and g are differentiable functions

g is differentiable at x $\quad g'(x)$

f —— " —— $g(x)$ $\quad f'(g(x))$ exists

$y = g(x) \quad z = f(y) = f(g(x))$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \quad \boxed{\text{i.e } (f \circ g)'(x) \cdot g'(x)} \; f \circ g \text{ is differentiable}$$

__Proof__ Define $H(y) = \begin{cases} f(g(x)) & y = g(x) \\ \dfrac{f(y) - f(g(x))}{y - g(x)} & y \neq g(x) \end{cases}$

Calculate $f'(g(x)$ ...

$$(f \circ g)'(x) = \lim_{t \to x} \frac{f(g(t)) - f(g(x))}{t - x} = \lim_{t \to x} H(g(t)) \cdot \frac{g(t) - g(x)}{t - x}$$

$$= f'(g(x) \cdot g'(x)) \quad \blacksquare$$

Examples

$f(x) = \sin(x) \qquad f'(x)$

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$\lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$\lim_{h\to 0} \left(\sin x \, \frac{\cos h - 1}{h} + \frac{\sin h}{h} \cos x \right)$$

$$\frac{\cos h - 1}{h} = \underbrace{h}_{\text{limit } 0} \cdot \underbrace{\frac{\cos h - 1}{h^2}}_{\text{limit } -\frac{1}{2}} \rightarrow 0$$

$$\frac{\sin h}{h} \rightarrow 1 \quad \text{as } h \rightarrow 0$$

So $\sin'(x) = \sin x \cdot 0 + 1 \cdot \cos x$

$\underline{\sin' = \cos}$

Similarly $\cos'(x) = \sin(x)$

Derivative of $\sin^2(x)$ $\qquad \frac{d}{dx}(y^2) = 2y$

$y = \sin x$

$z = y^2$

is $\underbrace{2\sin(x)}_{\frac{dz}{dy}} \cdot \underbrace{\cos(x)}_{\frac{dy}{dx}}$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{\sin(x)}\right) = (f \circ g)'(x)$$

$$g(x) = \sin(x)$$
$$f(y) = 1/y$$

$$-\frac{1}{\sin^2(x)} \cdot \cos(x)$$

$$\frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = -\frac{1}{\cos^2(x)} \cdot (-\sin(x)$$

by chain rule.

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{d}{dx}\left(\sin x \cdot \frac{1}{\cos x}\right.$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x) \cdot \frac{1}{g(x)}\right.$$

$$= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{-1}{g^2(x)} \cdot g'(x)$$

$$= \frac{f'(x)}{g(x)} - f(x)\frac{g'(x)}{g^2(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Proved quotient rule from product & chain rules

$$\tan'(x) = \left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$((\sin x)^2)'$$
$$= (\sin^2(x))' = 2 \sin x \cos x$$
$$= (\sin(x^2))' = \cos(x^2) \cdot 2x$$