MA1114 1/11/21

Matrix Inverse

Definition 3.4

$A \in M_{n,n}$ is invertible if it has an inverse

"A" has an inverse if there exists $B \in M_{n,n}$ such that $AB = BA = I_n$

If an inverse exists its is unique and we write A^{-1} for the inverse of A

Proposition 3.7 (properties of inverses

(i) $(A^{-1})^{-1} = A$

(ii) A^T is invertible $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

Proof 3.7

(i) Since A is invertible $A^{-1}A = AA^{-1} = I_n$

A^{-1} is invertible since there exists a matrix $(B = A)$ such that $A^{-1}B = BA^{-1} = I_n$

$(A^{-1})^{-1} = A$ since inverses are unique.

(ii) $AA^{-1} = I_n = A^{-1}A$ [recall: for any $n \times n$ matrix x, y, $(xy)^T = x^T y^T$]

$(A^{-1})^T A^T = (A^{-1}A)^T = (I_n)^T = I_n$ similarly $A^T (A^{-1})^T = (AA^{-1})^T = (I_n)^T = I_n$

so $(A^T)^{-1} = (A^{-1})^T$ by uniqueness of inverses

(iii) $(AB)(B^{-1}A^{-1})$

$$= A(BB^{-1})A^{-1} = A(I_n)A^{-1}$$
$$= AA^{-1}$$
$$= I_n$$

similarly $(B^{-1}A^{-1})(AB)$

$$= B^{-1}(A^{-1}A)B = B^{-1}(I_n)B$$
$$= B^{-1}B$$
$$= I_n$$ by uniqueness of inverses

<u>Corollary</u>

The product of two invertible matrices is invertible.

<u>Exercise</u>

① Prove that A is invertible $\Rightarrow A^k$ is invertible for any $k \in N$

$A^{-1} \Rightarrow (A^k)^{-1}$

$A^{-1} \Rightarrow (A^{-1})^k$

$(A^{-1})^k A = A^{-1} \cdots \cancel{A^{-1}A} \cdots A$

$= A^{-1}A$

$= I_n$

$A^k (A^{-1})^k = I_n$ so $(A^{-1})^k = (A^k)^{-1}$

② Prove that A is invertible then λA is invertible for any $0 \neq \lambda \in \mathbb{R}$

$(\lambda A)(\lambda^{-1}A^{-1}) = \lambda\lambda^{-1} AA^{-1} = I_n \times 1$
$= I_n$

similarly $(\lambda^{-1}A^{-1})(\lambda A)$