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Curl of a vector field (a.k.a rotation)

Let F_1, F_2, F_3 be a differentiable 3 dimensional vector field of 3 variables. We define the curl of the vector field $\underline{F}$, as

$$\text{curl}\ \underline{f} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

We can also use the "∇" notation for the curl. We have

$$\text{curl}\ \underline{f} = \nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

thus, formally, the curl is the vector product of ∇ and $\underline{F}$!

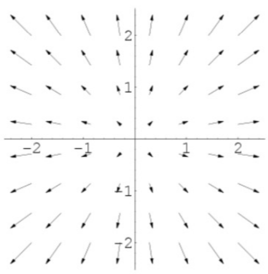
Components of a curl vector

$$\text{curl}\ \underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) - k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

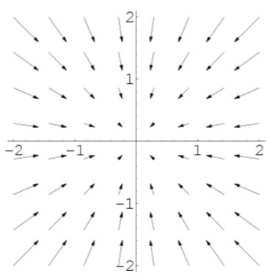
other notations

$\nabla = (\partial_x ; \partial_y ; \partial_z)$ $\quad$ $\text{curl}\ \underline{F} = \nabla \times \underline{F} = \text{rot}\ \underline{F}$

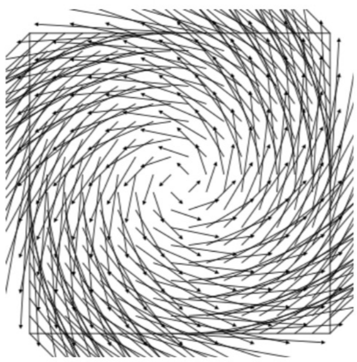
Positive divergence, zero curl



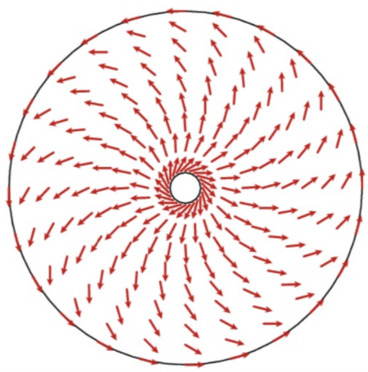
Negative divergence, zero curl



Positive curl, zero divergence



Positive curl, positive divergence



Examples from hydrodynamics:

Liquid rotates in a vessel around the axis Oz of with the angular velocity $\underline{\omega} = (0;0;\omega)$

The velocity fields $\underline{v}(\underline{r})$ can be represented as:

$$\underline{v} = \underline{\omega} \times \underline{r} = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = -\omega y \underline{i} + \omega x \underline{j}$$

$$\operatorname{curl} \underline{v} = \nabla \underline{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = \underline{i}\,0 - \underline{j}\,0 + 2\omega \underline{k} = 2\omega \underline{k}$$

The curl of any potential is zero:

$$\underline{F} = -\operatorname{grad}(U)$$

$$\operatorname{curl} \underline{F} = -\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ U'_x & U'_y & U'_z \end{vmatrix} = -i\left(\frac{\partial U'_z}{\partial y} - \frac{\partial U'_y}{\partial z}\right) + j\left(\frac{\partial U'_z}{\partial x} - \frac{\partial U'_x}{\partial z}\right) - k\left(\frac{\partial U'_y}{\partial x} - \frac{\partial U'_x}{\partial y}\right)$$

$$= -i(U''_{yz} - U''_{zy}) + j(U''_{xz} - U''_{zx}) - k(U''_{xy} - U''_{yx}) = 0$$

or symbolically:

$$\nabla \times \nabla \cdot U = 0 \qquad (\nabla \text{ and } \nabla \text{ are "colinear"})$$

Properties of the Curl:

Let $\underline{F}, \underline{G} : \mathbb{R}^n \to \mathbb{R}^n$, $n = 2,3$ vector fields, and $\varphi : \mathbb{R}^n \to \mathbb{R}$ a scalar function. Then

- $\nabla \times (\underline{F} + \underline{G}) = \nabla \times \underline{F} + \nabla \times \underline{G}$
- $\nabla \times (\lambda \underline{F}) = \lambda \nabla \times \underline{F}$ for $\lambda \in \mathbb{R}$
- $\nabla \times (\varphi \underline{F}) = (\nabla \varphi) \times \underline{F} + \varphi (\nabla \times \underline{F})$
- $\underline{F}$ is constant then $\nabla \times \underline{F} = 0$. The converse is NOT true
- A vector field such that $\nabla \times \underline{F} = \underline{0}$ is called irrotational

Vector Potential:

If a vector field $\underline{F}$ can be represented as:

$$\underline{f} = \text{curl}(\underline{A}) = \nabla \times \underline{A}$$

where $\underline{A}(x, y, z)$ is a vector field.

The vector field $\underline{a}$ called the vector potential of the field.

It is easy to show that:

$$\text{div}(\text{curl}(\underline{A}) = \nabla \cdot (\nabla \times \underline{A}) = 0$$

That is, $\underline{f} = \text{curl}(\underline{A})$ Is incompressible (solenoid) field.