MA1114 2/2/22

Complex Numbers, Matrix, Vectors and Vectorspaces

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$
hamiltonian octonians

Complex Numbers $\mathbb{C}$, $\mathbb{C}$ is a field

- there is commutative addition and multiplication
- every non-zero element has an inverse.

$\mathbb{C} = \{ x + yi \mid x, y \in \mathbb{R} \}$

real part imaginary part

$i = \sqrt{-1}$, satisfying $p(i) = 0$, $p(x) = x^2 + 1$

$(3+2i) + (2+8i) = 5+5i$

in general $z_1 = x_1 + y_1 i$, $z = x_2 + y_2 i$

$z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2) i$
(commutative)

$(3+i)(2-3i) = 3 \cdot 2 - 3 \cdot 3i + i \cdot 2 - i \cdot -3 \cdot i$
$= 6 - 9i + 2i - 3i = 9 - 7i$

z_1, z_2 as above.

$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i)$

$= x_1 x_2 + y_2 i x_1 + x_2 + y_1 i - (y_1 y_2)$

$= (x_1 x_2 - y_1 y_2) + i\,(y_2 x_1 + y_1 x_2) = z_1 z_2$

so commutitative.

Why does every $0 \neq z \in \mathbb{C}$ has an inverse? i.e an element $y \in \mathbb{C}$ such that $y = z = 1$

$z_j = (x_j + y_j i) \quad j = 1, 2$

$z_1 z_2 = 1 \Rightarrow x_1 x_2 - y_1 y_2 = 1$
$\quad y_1 x_2 + y_2 x_1 = 0$

$\Rightarrow y_2 = \dfrac{-y_1 x_2}{x_2} \qquad x_2 = \dfrac{1 + y_1 y_2}{x_1}$

$z_2 = \dfrac{x_1 - y_1 i}{\sqrt{x_1^2 + y_1^2}}$ check - $z_1 \cdot z_2 = 1$

$\dfrac{(x_1 + y_1 i)(x_1 - y_1 i)}{x_1^2 + y_1^2} = 1$

Definitions

Suppose $z = x + yi \in \mathbb{C}$

- $\mathrm{Re}(z) = x$
- $\mathrm{Im}(z) = y$

are the real imaginary parts of z

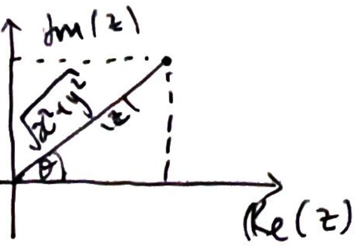
$|z| = \sqrt{x^2 + y^2}$ is the modulus of z

$\bar{z} = x - yi$ is conjugate of z

H commutivaty breakes down

observe $z\bar{z} = x^2 + y^2 = |z|^2$

θ argument of z.



argand diagram

we see that

$x = |z| \cos(\theta)$

$y = |z| \sin(\theta)$

we see that

$$
\begin{aligned}
z &= |z|\cos\theta + i|z|\sin\theta \\
&= |z|(\cos\theta + i(\sin\theta)) \\
&= e^{i\theta}|z| \qquad \text{(eulers formula)} \quad e^{i\pi}+1=0
\end{aligned}
$$

Complex vector space

slogan - "replace $\mathbb{R}$ by $\mathbb{C}$"

$$\mathbb{C}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \middle| \, x_i \in \mathbb{C}^n \right\}$$

$M_{mn}(\mathbb{C}) = \{ m \times n \text{ matrices with entries in } \mathbb{C} \}$

$P_n(\mathbb{C}) = \{ \text{polynomials of degree at most } n \text{ with coeff. in } \mathbb{C} \}$

Note $\mathbb{C}$ is a vector space — complex v.s $\neq$ $\mathbb{R}$ v.s.

- a $\mathbb{C}$ vectorspace of dim 1
- a $\mathbb{R}$ vector space of dim 2

$A \in M_{m,n}(\mathbb{C})$, define

$$
\begin{aligned}
Re(A)_{ij} &= Re(A_{ij}) \quad \text{for } 1 \le i, j \le m, n \\
Im(A)_{ij} &= Im(A_{ij}) \quad \text{for } 1 \le i, j \le m, n
\end{aligned}
$$

Example $Re\begin{pmatrix} 1+i & 3 \\ -i & \pi-6i \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & \pi \end{pmatrix}$

Also define (for A as above)

$$(\bar{A})_{ij} = (\bar{A}_{ij}) \quad \text{for } 1 \le i, j \le m, n$$

Proposition $u, v \in \mathbb{C}^n$

$A \in M_{m,n}(\mathbb{C})$, $B \in M_{m,n}(\mathbb{C})$

(a) $\overline{\overline{u}} = u$
(b) $\overline{\lambda u} = \overline{\lambda}\,\overline{u}$
(c) $\overline{u+v} = \overline{u} + \overline{v}$
(d) $\overline{A^T} = \overline{A}^T$
(e) $\overline{AB} = \overline{A} \cdot \overline{B}$

Example

$$\begin{pmatrix} i & 0 \\ 1 & 2+i \end{pmatrix}\begin{pmatrix} 7-3i & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 7i+3 & 0 \\ 7-3i & 8+4i \end{pmatrix}$$

$$\overline{\begin{pmatrix} i & 0 \\ 1 & 2+i \end{pmatrix}} = \begin{pmatrix} -i & 0 \\ -1 & 2-i \end{pmatrix}, \quad \overline{\begin{pmatrix} 7-3i & 0 \\ 0 & 4 \end{pmatrix}} = \begin{pmatrix} 7+3i & 0 \\ 0 & -4 \end{pmatrix}$$

$$\overline{\begin{pmatrix} 7i+3 & 0 \\ 7-3i & 8+4i \end{pmatrix}} = \begin{pmatrix} 7i-3 & 0 \\ 7+3i & 8-4i \end{pmatrix}$$

Lemma If $z_1, z_2 \in \mathbb{C}$

① $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, ② $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ (special case of (b) above).

Proof of ①

$$\overline{x_1 + y_1 i + x_2 + y_2 i} = x_1 + x_2 - (y_1 + y_2) i$$
$$= \overline{z_1} + \overline{z_2}$$

Proof of ②

$$\overline{z_1 z_2} = \overline{(x_1 + y_1 i)(x_2 + y_2 i)} = \overline{x_1 x_2 + x_1 y_2 i + x_2 y_2 i - y_2 y_1}$$
$$= (x_1 - y_1 i)(x_2 - y_2 i)$$
$$= \overline{z} \cdot \overline{z}$$

Proof of Proposition

(e) hardest one

$$\overline{(AB)}_{ij} = \sum_{r=1}^{k} A_{ij} B_{ij} = \sum_{r=1}^{k} \overline{A_{ij} B_{ij}} \quad \text{by lemma}$$

$$= \sum_{r=1}^{k} \bar{A}_{ij} \cdot \bar{B}_{ij} = (\bar{A} \cdot \bar{B})_{ij}$$

Theorem (Fundamental theory of algebra)

any complex polynomial can be factored uniquely as a product of linear polynomials. The roots come in complex conjugate pairs.

Example

$$p(x) = x^4 + x^3 + x^2 + x + 1$$

$$p(x) = (x - w)(x - w^2)(x - w^3)(x - w^4)$$

w is the 5^{th} root of unity

$$w = e^{\frac{2\pi i}{5}}$$