MA1114 21/2/22

Matrix Representation of Linear Map.

Reminder

linear map $T: V \to V$

Theorem

Any real/complex vectorspace V of dimension n is isos to $\mathbb{R}^n / \mathbb{C}^n$

(isos $\Leftrightarrow$ $\exists$ a linear map $E: V \to \mathbb{R}^n / \mathbb{C}^n$ which is a bijection)

what is E?

E sends a vector to its coordinate vector with respect to a basis i.e $v \in V$ and V has bases

$$B = \{v_1, \dots, v_n\} \Rightarrow v = \sum_{i=1}^{n} \lambda_i v_i$$

$$\text{so } E(v) = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = [v]_B$$

$$\text{so } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \frac{x-y}{2} \end{pmatrix}$$

now suppose V, W real vectorspaces and $T: V \to W$ is a linear isos, suppose $V = \langle B \rangle$ and $W = \langle e \rangle$

we have two maps $E: V \to \mathbb{R}$
$v \mapsto [v]_B$

and $E': W \to \mathbb{R}^n$
$v \to [w]_e$

so we have a picture

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "E"'] & W \arrow[d, "E''"] \\ \mathbb{R}^n \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

<u>goal</u> find a matrix $A \in M_n(\mathbb{R})$

s.t. $[T(v)]_e = A[v]_B$ (for $v \in V$)

<u>lemma</u> A exists!

<u>Proof</u> suppose $B = \{v_1, \dots, v_n\}$

$A = ([T(v_1)]_e, \dots, [T(v_n)]_e)$

suppose $v \in V$

$v = \sum_{i=1}^{n} \lambda_i v_i$, want to show $[T(v)]_e = A[v]_B$

$E(v) = [v]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$

$A[v]_B = \lambda_1 [T(v_1)]_e + \lambda_2 [T(v_2)]_e + \dots + \lambda_n [T(v_n)]_e$

since $E: v \mapsto [v]_B$ is linear

this becomes $[\lambda_1 T(v_1) + \dots + \lambda_n T(v_n)]_e$ since T is linear

$[T(\lambda_1 v_1 + \dots + \lambda_n v_n)]_e = [T(v)]_e = [T(v)]_e$

Examples

(a) $id : V \longrightarrow V$ $\qquad V = [B]$

$\quad v \longmapsto v$

$$\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix} = \begin{tikzcd} v \arrow[d, mapsto] & V \arrow[r] \arrow[d] & V \arrow[d] & v \arrow[d, mapsto] \\ {[v]_B} & \mathbb{R}^n \arrow[r] & \mathbb{R}^n & {[v]_B} \end{tikzcd}$$

so $[T(v)]_{\mathcal{B}} = T_n$.

(b) $0 : V \longrightarrow W$

$\quad v \longmapsto \underline{0}$

is represented by $\dim(w) \times \dim(v)$ all 0 matrise

$$\begin{array}{cc} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} & \dim(w) \\ \dim(v) & \end{array}$$

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\quad v \longmapsto rv \qquad r \neq 0$

represented by ?

$\mathcal{B} = \{e_1, \ldots, e_n\}$ $\quad e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ ith position

$T(e_i) = re_i = \begin{pmatrix} 0 \\ \vdots \\ r \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ ith position so $A = \begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & r & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & r \end{bmatrix} = [T(e_1) \quad T(e_2) \qquad T(e_n)]$

(c) $P : P_{n+1} \longrightarrow P_n$

$p(x) \longmapsto \frac{d}{dx}(p(x))$

$P_n = \{$ polynomial of degree at most $n \}$

P is linear $\quad p(x) = \sum_{i=0}^{n+1} a_i x_i$

$q(x) = \sum_{i=0}^{n} b_i x_i$

$P(p(x) + q(x)) = P\left(\sum_{i=1}^{n+1} a_i x_i + \sum_{i=1}^{n} b_i x_i\right)$

$= \sum_{i=0}^{n+1} i(a_i + b_i)x^{i-1} \quad = \sum_{i=0}^{n+1} i a_i x^{i-1} + \cdots$

$= \sum_{i=0}^{n-1} i a_i x^{i-1} + \sum_{i=0}^{n+1} i b_i x^{i-1}$

$= P(p(x)) + P(q(x))$

$P(\lambda p(x)) = P\left(\sum_{i=1}^{n+1} \lambda_i a_i x^i\right)$

$= \sum_{i=0}^{n+1} \lambda_i a_i x^{i-1}$

$= \lambda P(p(x))$

with respect to the standard basis $\{1, x, x^2, x^3, \ldots, x^n\}$

P is represented by

$$
\begin{array}{c}
1 \\ x^2 \\ x^3 \\ \vdots \\ x^{n+1}
\end{array}
\begin{bmatrix}
1 & x & x^2 & x^3 & \cdots & x^n \\
0 & 1 & 0 & & & 0 \\
\vdots & & 2 & & & \vdots \\
\vdots & & & 3 & & \vdots \\
\vdots & & & & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & n+1
\end{bmatrix}
\quad n+2 \qquad = kx^{k-1}
$$

$n+1$

To work out n matrise, A, representing a linear map:

$T: V \to W$ where $V = \langle B \rangle$
$W = \langle e \rangle$

① calculating images of vectors $T(V_1), T(V_2), \cdots, T(V_n)$

where $B = \{V_1, \ldots, V_n\}$

② calculating coefficients $[T(V_i)]_e$ for each i

③ $A = \begin{bmatrix} [T(V_1)]_e & [T(V_2)]_e & \cdots\cdots & [T(V_n)]_e \end{bmatrix}$

(this is an $m \times n$ matrise where $\dim(V) = n$
$\dim(W) = m$)

good since $[V]_B$

so $A[V]_B$ make sense ☺

Calculate the matrix corresponding to $T: \mathbb{R}^2 \to \mathbb{R}^2$

(w.r.t the standard basis) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$

so $e = \{e_1, e_2\}$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad T(v_1) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad T(v_2) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$A = [T(v_1) \; T(v_2)]$

$A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$