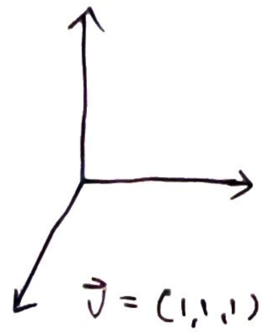
MA1014 29/3/22

Gradients and Local Extrema

$f(x, y, z)$ $\quad \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z}$

$\frac{\partial f}{\partial \vec{v}} =$



$\vec{v} = (1,1,1)$

$$\lim_{\substack{h_1 \to 0 \\ h_2 \to 0 \\ h_3 \to 0}} \frac{f(x_0 + h_1, y_0 + h_2, z_0 + h_3) - f(x_0, y_0, z_0)}{\sqrt{h_1^2 + h_2^2 + h_3^2}} \qquad \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\vec{v} = (a, b, c) \qquad \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{\quad}}, \frac{c}{\sqrt{\quad}} \right)$$

Theorem

$$\frac{\partial f}{\partial \vec{v}} = \frac{\partial f}{\partial x} \cdot \frac{a}{\sqrt{\quad}} + \frac{\partial f}{\partial y} \cdot \frac{b}{\sqrt{\quad}} + \frac{\partial f}{\partial z} \cdot \frac{c}{\sqrt{\quad}}$$
$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

Definition Let $f(x, y, z)$ be a 3-variable function with $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ exist then

(nabla) $\to$ $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$ is the gradient of f

$f(x, y) = \qquad \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$

Example 1 $u = x^2 e^y z$ at $(1,0,1)$, direction $(1,0,1)$

$$\downarrow$$

$$(2,1,1)$$

solution $\frac{\partial f}{\partial x} = 2x e^y z$ $\frac{\partial f}{\partial y} = u$

$$\frac{\partial f}{\partial z} = x^2 e^y$$

at $(1,0,1)$ $\nabla u \Big|_{(1,0,1)} = (2,1,1)$

$$\vec{v} = (2,1,1) - (1,0,1) = (1,1,0)$$

$$\frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\frac{\partial u}{\partial \vec{v}} = 2 \times \frac{1}{\sqrt{2}} + 1 \times \frac{1}{\sqrt{2}} + 1 \times 0 = \frac{3\sqrt{2}}{2}$$

$f(x,y)$

$f(x)$ $f'(x) = 0$ $f'(x_0)$ does not exists

$f''(x) > 0$ local minimum

$f''(x) < 0$ local maximum

$f''(x) = 0$ neither

Definition x_0 is said to be the local max point of $f(x,y)$ if $\exists \delta > 0$, $\forall (x,y)$:

$$\|(x,y) - (x_0, y_0)\| < \delta$$

$\sqrt{(x-x_0) + (y-y_0)}$

$$f(x_0, y_0) \geqslant f(x,y)$$

local minimum $\leqslant$

local max or local min $\rightarrow$ local extreme point

Theorem 1 If f attains local extreme at (x_0, y_0), and

$\frac{\partial f}{\partial x}\Big|_{(x_0, y_0)}$, $\frac{\partial f}{\partial y}\Big|_{(x_0, y_0}$ exist then

$$\nabla f|_{(x_0, y_0)} = 0$$

Theorem 2 If f has a 2nd continuous derivatives,

$$A = \frac{\partial^2 f}{\partial x^2} \qquad B = \frac{\partial^2 f}{\partial y^2} \qquad C = \frac{\partial^2 f}{\partial z^2}$$

suppose $\nabla f|_{(x_0, y_0)}$, then if $AC - B^2 > 0$
f attains local extreme at (x_0, y_0)

$$\begin{cases} A < 0 & \text{local max} \\ A > 0 & \text{local min} \end{cases}$$

if $AC - B^2 < 0$ (x_0, y_0) is not an extreme
$AC - B^2 = 0$ neither

example $f(x, y) = (6x - x^2)(4y - y^2)$

letting

$0 = \frac{\partial f}{\partial x} = (6 - 2x)(4y - y^2)$ $x = 3$, $y = 0, 4$

$0 = \frac{\partial f}{\partial y} = (6x - x^2)(4 - 2y)$ $x = 0, 6$ $y = 2$

$(3, 2)$ $(0, 0)$ $(0, 4)$ $(6, 0)$ $(6, 4)$

$\frac{\partial^2 f}{\partial x^2} = (-2)(4y - y^2)$ A $\qquad \frac{\partial^2 f}{\partial x \partial y} = (6 - 2x)(4 - 2y)$ B

$\frac{\partial^2 f}{\partial y^2} = (6x - x^2)(-2)$ C $\qquad AC - B^2$

$(3,2)$ $AC - B^2 = -8(-18) - 0^2 > 0$

$A = -8 < 0$ local max

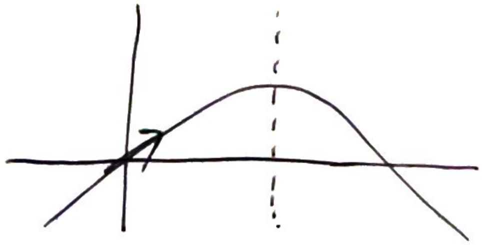
$(0,0)$ $AC - B^2 = 0$

$(0,4)$ $AC - B^2 = 0 \times 0 - (6(-4))^2 < 0$ not extreme

$(6,0)$ $AC - B^2 = 0 \times 0 - (-6(4))^2 < 0$ not extreme

$(6,4)$ $AC - B^2$ .... < 0 not extreme

fixe y, x near 0



so $(0,0)$ is not an extreme