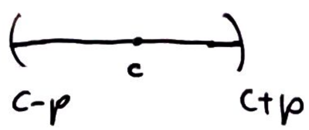
MA1014 13/10/21

Definition of a Limit

__Definition__ Let $c \in \mathbb{R}$, $f : D \to \mathbb{R}$
where
$(c-p, c+p) \setminus \{c\} \in D \subseteq \mathbb{R}$



punctured open interval

So c does not itself need to be in D

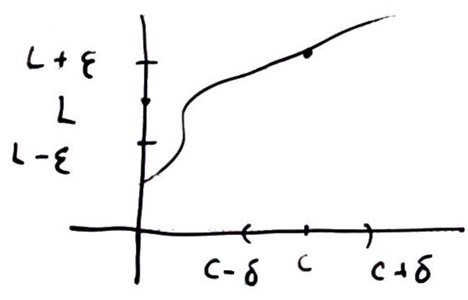
We say f(x) has __limit__ L as x approaces c

$$\boxed{\lim_{x \to c} f(x) = L}$$ if

$\forall \varepsilon > 0 \quad \exists \delta > 0 : 0 < |x-c| < \delta$

$\Rightarrow |f(x) - L| < \varepsilon$

__Def__: We say f is continuous at $c \in \mathbb{R}$ if $f(c)$ exists & $\lim_{x \to c} f(x)$ exists & these are EQUAL



we could <u>combine</u> these two definitions

f continuous at c if

$$\forall \epsilon > 0 \ \exists \ \delta > 0 : |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

in other words $\lim_{x \to c} f(x) = f(c)$

<u>Examples</u> i) $f(x) = 2x - 3$

Proof of "f continuous at -1"

$c = -1$, we want prove

$$\lim_{x \to -1} f(x) = f(-1) = 2(-1) - 3 = -5$$

Given any $\varepsilon > 0$, we need to find some $\delta > 0$ such that

If $|x - (-1)| < \delta$ then $|f(x) - (-5)| < \varepsilon$

$|x + 1| < \delta \implies |2x - 3 + 5| < \varepsilon$

$|x + 1| < \delta \implies |2x + 2| < \varepsilon$

let $\delta = \varepsilon/2$ then

$$|x - c| = |x + 1| < \delta \implies |f(x) - f(c)| = |2x + 2| = 2|x + 1| < 2\delta = \varepsilon$$

where ⊙ means equal each other

ii) $f(x) = x$ continuous everywhere easy to prove

at all $c \in \mathbb{R}$

$\forall \varepsilon > 0 \ \exists \delta > 0 : |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

Given any ε, let $\delta = \varepsilon$

iii) Hard example:

$$f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

we will not prove $f(x)$ continuous now, but a useful theorm

<u>Pinching theorm</u> $f, g, h : D \to \mathbb{R}$

where D contains $(c-p, c+p) - \{c\}$ & suppose

$f(x) \leq g(x) \leq h(x) \ \forall x \in D$

Then if $\lim_{x \to c} f(x) = L = \lim_{x \to} h(x)$

then $\lim_{x \to c} g(x) = L$ also.