MA1061 15/11/21

Expectation

Expectation generalizes the basic idea of average, or mean value; e.g, what is the average age of the n students in a lecture theatre?

$$\text{average age} = \tfrac{1}{n}(a_1 + \cdots + a_n) \quad \text{where } a_i \text{ is the age of students } i$$

if we expect the average value from n endependant experiments to be

$$\tfrac{1}{n}\left[np_1 x_1 + \cdots + np_k x_k\right] = \sum_{i=1}^{k} p_i x_i .$$

<u>Definition</u> (Expectation / mean / expected value)

The <u>expectation</u> or <u>mean</u> or <u>expected value</u> of a random variable $X : \Omega \longrightarrow \{x_1, \ \ , x_k\}$, is denoted by $E[X]$ and defined a.

$$E[X] = \sum_{i=1}^{k} x_i P(A_i) = \sum_{i=1}^{k} x_i \, p(X = x_i),$$

where $A_i = \{\omega \in \Omega : X(\omega) = x_i\}$

<u>Example</u> (sum of two dice)

We roll two independant, unbiased dice and record their sum. Let X denote the sum of the numbers. Find the expectation of the random variable X

x	2	3	4	5	6	7	8	9	10	11	12
$P_X(x)$	$1/36$	$1/18$	$1/12$	$1/9$	$5/36$	$1/6$	$5/36$	$1/9$	$1/12$	$1/18$	$1/36$

$$E[X] = 2\left(\tfrac{1}{36}\right) + 3\left(\tfrac{1}{18}\right) + 4\left(\tfrac{1}{12}\right) + \cdots + 12\left(\tfrac{1}{36}\right) = \frac{252}{36} = 7$$

Let X denote the player's gain, then the distribution of X is as follows, where k denotes the unknown payoff to the player:

X	3	1	k
$P_X(x)$	$1/4$	$1/2$	$1/4$

$$\mu = E[X] = 3\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) + k\left(\frac{1}{4}\right) = \frac{5+k}{4}$$

for a fair game, $E[X]$ should be zero This yield $k = -5$ Thus, the player should lose £5 if no head occurs

Mean of Bernoulli Variable

Example

Let X be a Bernoulli Variable with $P(X=1) = p$ Then

$$E[X] = 1 \times P_X(1) + 0 \times P_X(0) = p.$$

Problem

Calculate the expectation of the binomial $Bi(n,p)$ random variable.

→ Let $X \sim Bi(n,p)$ then X takes values $0, 1, \ldots, n$. We use the basic formula.

$$E[X] = \sum_{k=0}^{n} k\, P_X(k) = \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} np \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k}$$

Then we change variables, let $s = k-1 \in \{0, \dots, n-1\}$

thus we get : $E[X] = np \sum_{s=0}^{n-1} C_s^{n-1} p^s q^{n-1-s}$ (A)

note (A) is just the binomial expansion of $(p+q)^{n-1}$ which is equal 1

thus we get $E[X] = np$

Basic Properties of the Expectation

In the following, let X and Y be random variables on the same sample space Ω and suppose that $X \geq Y$ means that $X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$

1. If $X \geq$, then $E[X] \geq 0$;
2. If $X = 1_A$, then $E[X] = P(A)$;
3. $|E[X]| \leq E[|X|]$;
4. Linearity: $E[aX + bY] = aE[X] + bE[Y]$ where a, b are constants;
5. If X and Y are independent then $E[XY] = E[X]E[Y]$;
6. If $X \leq Y$ then $E[X] \leq E[Y]$;
7. $(|E(XY)|)^2 \leq E[X^2]E[Y^2]$ (cauchy-schwarz inequality)

Proof

> (1) and (2) are evident from definition

> (3) notice that $|E[X]| = |\sum_{i=1}^{k} x_i P(A_i)| \leq \sum_{i=1}^{k} |x_i| P(A_i) = E[|X|]$
by triangle inequality