MA1114 7/12/21

Column Space and Null Space

(yesterday) made a remark $\rightarrow$ if $W \leq V$ is a vector space

then $\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_r w_r \in W$ $\leftarrow$ linear combination

for $\lambda_i \in \mathbb{R}$, $w_i \in W$ for $1 \leq i \leq r$

$\Rightarrow \{\sum \lambda_i w_i \mid \lambda_i \in \mathbb{R}\} \subset V$, is this a subspace ?

<u>Definition 4.23</u>

Let V be a vector space and suppose $S = \{w_1, w_2, \ldots, w_r\} \subset V$

Then
$$\begin{aligned} \operatorname{span}(S) &= \operatorname{span}(w_1, w_2, \ldots, w_r) \\ &= \langle w_1, w_2, \ldots, w_r \rangle \\ &= \langle S \rangle \\ &= \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_r w_r \mid \lambda_i \in \mathbb{R}\} \subseteq V \end{aligned}$$

<u>Proposition</u>

For any $S = \{w_1, w_2, \ldots, w_r\} \subset V$ as above, span (S) is a subspace; it is the smallest subspace containing S.

<u>Proof</u> we check $\underline{0} \subseteq \operatorname{span}(S)$ and closure under VA & SM

> $0 = 0 \cdot w_1 + 0 \cdot w_2 + \cdots + 0 \cdot w_r \in \operatorname{span}(S)$

> $u, v \in \operatorname{span}(S)$, $u = \sum_{i=1}^{r} \lambda_i w_i$, $v = \sum_{i=1} \mu_i w_i$

$$u + v = \sum_{i=1} (\lambda + \mu)_i w_i \in \operatorname{span}(S)$$

> u as above $\mu \in \mathbb{R}$ $\mu \cdot u = \mu \cdot \sum_{i=1}^{r} \lambda_i w_i$ by SM $\in \operatorname{span}(S)$

Now let $W \leq V$ be a subspace containing S then since W is closed under VA & SM

$$\lambda_1 w_1 + \lambda_2 w_2 + \dots + \lambda_r w_r \in W, \quad \lambda_i \in \mathbb{R}$$

$$\Rightarrow \text{span}(S) \leq W \quad \blacksquare$$

Definition 4.25

If $S = \{w_1, w_2, \dots, w_r\} \in V$ a vector space and $\text{span}(S) = V$ then is called a <u>spanning set</u>

Examples

① $S = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\} \subset \mathbb{R}^2$

$$\text{span}(S) = \mathbb{R}^2$$

$$\mathbb{R}^2 \ni \begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}$$

② $S = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\}$

$\text{span}(S) = \mathbb{R}^2$

since $\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix} + 0\begin{pmatrix}1\\1\end{pmatrix}$

$= 0\begin{pmatrix}1\\0\end{pmatrix} + (y-x)\begin{pmatrix}0\\1\end{pmatrix} + y\begin{pmatrix}1\\1\end{pmatrix}$

So linear combination need not be unique

③ $S = \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \subset \mathbb{R}^3 \quad \text{span}(S) = \mathbb{R}^3$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \left(\frac{x+y}{2} - z\right)\begin{pmatrix}1\\1\\0\end{pmatrix} + \left(\frac{y-x}{2}\right)\begin{pmatrix}-1\\1\\0\end{pmatrix} + z\begin{pmatrix}1\\1\\1\end{pmatrix}$$

e.g. first row $\frac{x+y}{2} - z + \frac{x-y}{2} + z = 1$

True or False?

$A = B$ where

$$A = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle \subset \mathbb{R}^3$$

$$B = \left\langle \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix} \right\rangle \subset \mathbb{R}^3$$

$w_1 = \begin{pmatrix}2\\1\\0\end{pmatrix}$ $\quad w_1 = 2v_1 + v_2 \in A$ where $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$w_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\quad w_2 = v_1 - v_2 \in A$ $\quad \text{span}(v_1, v_2)$

$$\Rightarrow \text{span}(w_1, w_2) \subset \text{span}(v_1, v_2)$$

$$v_1 = \tfrac{1}{3} w_1 + \tfrac{1}{3} w_2 \in \text{span}(w_1, w_2)$$

$$v_2 = \tfrac{1}{3} w_1 - \tfrac{2}{3} w_2 \in \text{span}(w_1, w_2)$$

$$\Rightarrow \text{span}(v_1, v_2) \subset \text{span}(w_1, w_2)$$

Column Space

If $A = \begin{pmatrix} \uparrow & & \uparrow \\ a_1 & \cdots & a_n \\ \downarrow & & \downarrow \end{pmatrix} \in M_{m,n}$

the column space of A is $\text{col}(A) = \text{span}(a_1, a_2, \ldots, a_n)$

Proposition 4.29

Let $A \in M_{m,n}$. The linear system $Ax = b$ is consistent

$\iff b \in col(A)$

Proof

$$b = Ax = \begin{pmatrix} \uparrow & & \uparrow \\ a_1 & \cdots & a_n \\ \downarrow & & \downarrow \end{pmatrix} x = \begin{pmatrix} a_{11}x_1 + a_{11}x_2 & \cdots & a_{1m} \\ a_{12}x_1 + a_{12}x_2 & \cdots & \vdots \\ \vdots & \cdots & \end{pmatrix}$$

$$= x_1 \begin{pmatrix} \uparrow \\ a_1 \\ \downarrow \end{pmatrix} + x_2 \begin{pmatrix} \uparrow \\ a_2 \\ \downarrow \end{pmatrix} + \cdots + x_n \begin{pmatrix} \uparrow \\ a_n \\ \downarrow \end{pmatrix} \in col(A)$$

Corollary Under the hypothesis of proposition 4.29

$Ax = b$ has a solution for all $b \in \mathbb{R}^m$

$\iff col(A) = \mathbb{R}^m$