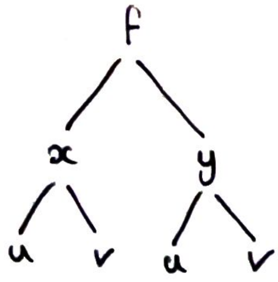
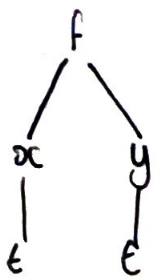
MA1014 28/3/22

Multivariable Tangents



$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$



$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Implicit Function Theorem

Let $z = f(x, y)$ be such that

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are continuous

and $\frac{\partial z}{\partial y} \neq 0 \quad \forall (x, y) \in D$

If we consider $f(x, y) = 0$

$$\boxed{\frac{dy}{dx} = -\frac{z_x}{z_y}}$$

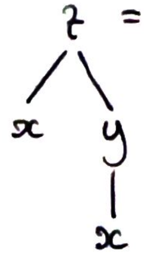
Example 1 $x^2+y^2=1, \quad \underline{y>0}$

$$z = x^2+y^2-1$$

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y \neq 0$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

$$f(x, \overset{y(x)}{y}) = 0$$



$= f(x,y)$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{z_x}{z_y}$$

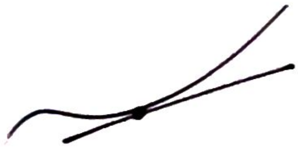
$$f_1' + f_2'\cdot\frac{dy}{dx} = 0$$

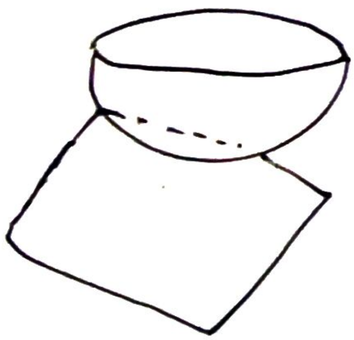
$$\frac{dy}{dx} = -\frac{f_1'}{f_2'} \qquad \frac{z_x}{z_y}$$

Example 2 $\sin^2 x + \tan(e^y) = 1 \qquad y = y(x)$

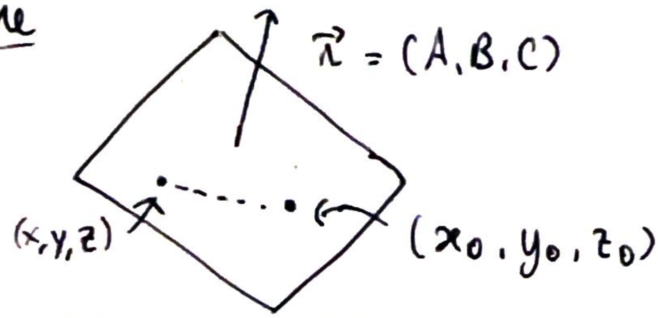
$$2\cos x + \sec^2 e^y \cdot e^y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\cos x}{(\sec^2 e^y)\, e^y}$$





Plane

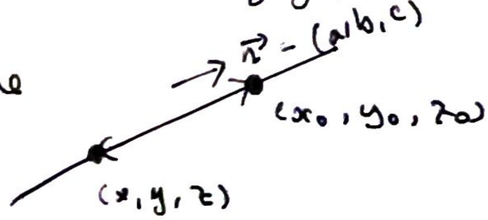


$\forall \ (x, y, z) \in \Sigma$

$$\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Straight line



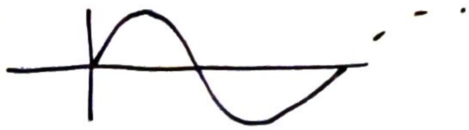
$$\vec{n} \parallel (x - x_0, y - y_0, z - z_0)$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

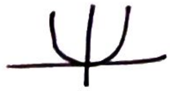
when $a = 0$ $\begin{cases} \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c} \\ x = x_0 \end{cases}$

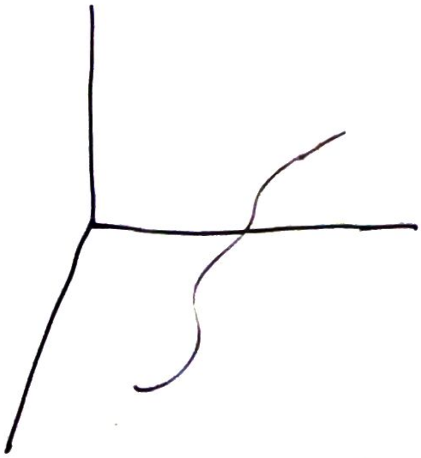
$a = b = 0$ $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$

Curve $y = \sin x$



$y = x^2$



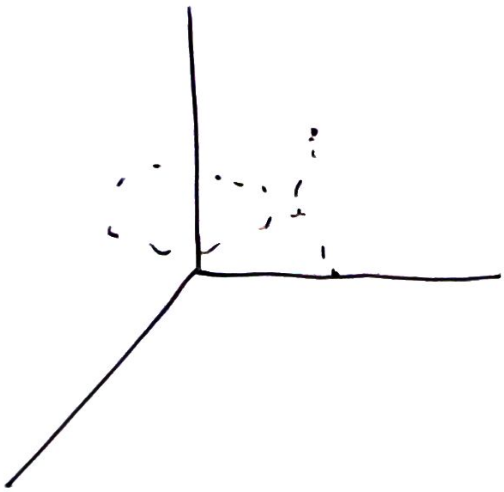


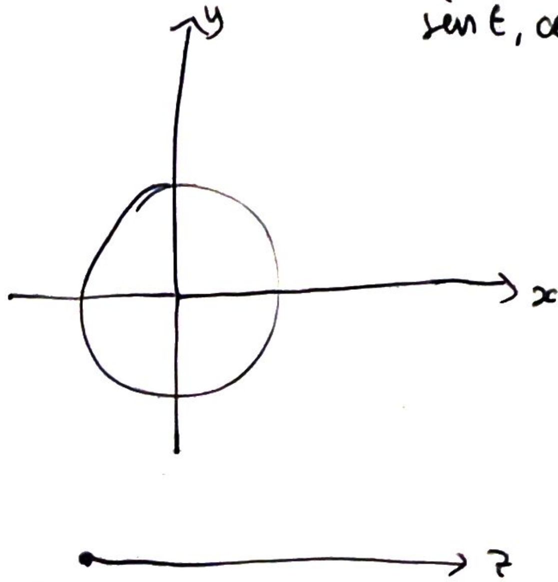
$t: x(t), y(t), z(t)$

$$\begin{cases} x(t) = \sin(t) \\ y(t) = \cos(t) \\ z(t) = kt \end{cases} \qquad t \in [0, 2\pi]$$

parameterised formula.

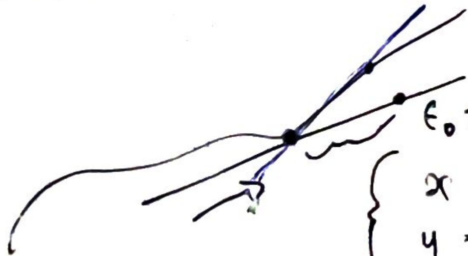
$\sin t, \cos t$





$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

$$\begin{cases} x = x_0 + at \\ y = y + bt \\ z = z + ct \end{cases}$$



$t_0 + \Delta t$

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

$t = t_0 \quad x(t_0), y(t_0), z(t_0)$

$(x(t_0 + \Delta t) - x(t_0),\ y(t_0 - \Delta t) - y(t_0),\ z(t_0 - \Delta t) - z(t_0))$

$$\frac{x - x(t_0)}{x(t_0-\Delta t) - x(t_0)} = \frac{y - y(t_0)}{y(t_0-\Delta t) - y(t_0)} = \frac{z - z(t_0)}{z(t_0-\Delta t) - y(t_0)}$$

letting $\Delta t \to 0$

$$\frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y(t_0)} = \frac{z - z(t_0)}{z(t_0)}$$

Example $\begin{cases} x(t) = \sin t \\ y(t) = \cos t \\ z = kt \end{cases}$ $\quad t \in [0, 2\pi]$

$t = \pi$

tangent point $(0, -1, k\pi)$

$x'(t) = \cos t \qquad y'(t) = -\sin t \qquad z'(t) = k$

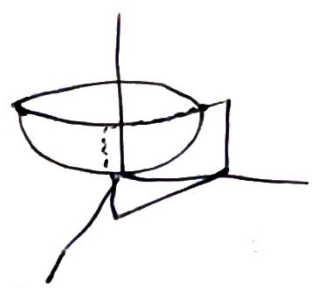
directional vector $(-1, 0, k)$

tangent line

$$\begin{cases} \dfrac{x}{-1} = \dfrac{z - k\pi}{k} \\ y = -1 \end{cases}$$

$x^2 + y^2 + z^2 = R^3 \qquad x^2 + y^2 + z^2 - R^3 \Rightarrow$

$F(x, y, z) = 0$



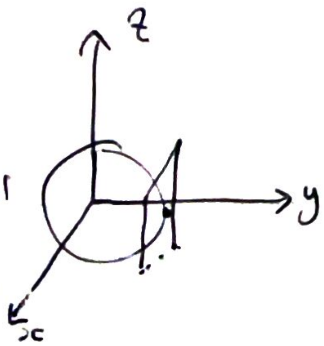
Theorem The tangent vectors of all curves on Σ and passing through M are in a joint plane

$\vec{a} + \vec{b}$

$$(F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0))$$

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0)}$$

Example $x^2 + y^2 + z^2 = 1$



set $F(x, y, z) = x^2 + y^2 + z^2 - 1$

$F_x = 2x \quad F_y = 2y \quad F_z = 2z$

normal vector $(0, 2, 0)$

tangent plane

$2(y - 1) = 0 \qquad \underline{y = 1}$