MA1114 12/10/21

Matrices

Definition 1.32

A $m \times n$ matrix A is an $m \times n$ rectangular array

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \text{ rows}$$

columns

$a_{ij} \in \mathbb{R}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$

If $m = n$ say A is square

Examples

- $(1, 0)$ is a 1×2 matrix $a_{11} = 1$ $a_{12} = 0$
- $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ is a 2×3 matrix

a column vectors on $\mathbb{R}^n$ is a $n \times 1$ matrix

Definition 1.35

Two matrix A, B are equal if they have the same size and entries

Definition 1.37

If $A = (A_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ is an $m \times n$ matrix

A_{11}, A_{22}, A_{33} A_{nn} are the diagonal entries

$tr(A)$ "the trace of A"

$$= \sum_{i=1}^{n} A_{ii}$$

$$tr\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3 \qquad tr\begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 1 \\ 3 & 1 & 2 & 5 \\ 1 & 2 & 4 & 6 \end{pmatrix} = 16$$

Definition 1.41

A, B, $m \times n$ matrix

$$(A+B)_{ij} = A_{ij} + B_{ij} \qquad \begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix}$$

$$(\lambda A)_{ij} = \lambda \times A_{ij} \qquad \begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix} \qquad \lambda \in \mathbb{R}$$

Example

$$\begin{pmatrix} 3 & 1 \\ 2 & 9 \end{pmatrix} + \begin{pmatrix} 7 & 2 \\ 8 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 3 \\ 10 & 12 \end{pmatrix}$$

$$2 \cdot \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 8 & 4 \end{pmatrix}$$

Proposition 1.44

The set of all $m \times n$ matrices $M_{m,n} = \{(A) \mid A \text{ is a } m \times n \text{ matrix}\}$ forms a vector space.

Proof

closed under addition ✓   0 m×n matrices $\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

closed under scalar multiple ✓

Definition 1.45

$A \in M_{m,n}$ (A is a $m \times n$ matrix)

$(A^T)_{ij} = A_{ji}$

$$\begin{pmatrix} 1 & 3 & 2 \\ 9 & 2 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 9 \\ 3 & 2 \\ 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 8 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ 3 & 8 \end{pmatrix}$$

Definition 1.39

The $m \times n$ matrix I_n is the matrix whose diagonal entries are 1 and the rest are 0.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proposition 1.47 transpose properties

$A, B \in M_{mn}$, $\lambda \in \mathbb{R}$

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(\lambda A)^T = \lambda \cdot A^T$

Proof

First notice that the matrices on either side of the equations have the same size

(i) $(A^T)^T_{ij} = (A_{ji})^T = A_{ij}$

(ii) $(A+B)^T_{ij} = A_{ji} + B_{ji}$

(iii) $((\lambda A)^T)_{ij} = (\lambda A)_{ji} = \lambda \cdot A_{ji} = \lambda\, (A^T)_{ij}$