MA1061 27/10/21

Conditional Probability: Definition, Examples

Conditional probability allow us to answer the questions like: as you obtain additional information, how should you update the probability of an event?

Why conditional probability?

solution:

$\Omega = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5\}$, $A = \{1, 2, 3\}$

- $P(B) = \frac{|B|}{|\Omega|} = \frac{3}{6} = \frac{1}{2}$

- if A has occurred, the outcome must be $\{1, 2, 3\}$
  - for B to occur too the outcome must be among $\{1, 3\}$

$$\frac{2}{3} = \frac{|A \cap B|}{|A|} = P(B|A)$$

in fact: $$P(B|A) = \frac{|A \cap B|}{|A|} = \frac{\frac{|A \cap B|}{|\Omega|}}{\frac{|A|}{|\Omega|}} = \frac{P(A \cap B)}{P(A)}$$

<u>Definition of Conditional Probability</u>

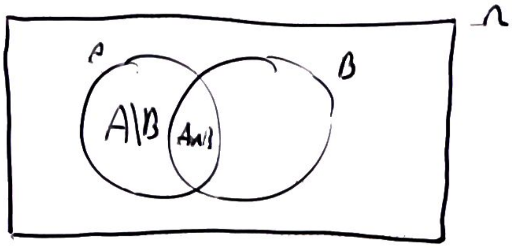
Let $A, B \in \mathcal{A}$ be two events. The <u>conditional probability</u> $P(B|A)$ of event A with $P(A) > 0$ is defined as

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

This gives us a new formula: $P(A \cap B) = P(B|A)\,P(A)$

called the multiplication formula for probs

If A occurred, the outcome must have been located in the red circle.
In the situation, for B to occur too, the outcome cannot be out of the green area



For equiprobable space we have $P(A) = \frac{|A|}{|\Omega|}$, and $P(A \cap B) = \frac{|A \cap B|}{|\Omega|}$;

Therefor

$$P(B|A) = \frac{|A \cap B|}{|A|}$$

Example

A bag contains n balls, n_1 white and n_2 black. Pick two balls from the bag without replacement. What is the probability that the 2nd ball is white (event B) given that the first ball is also white (event A)?

By intuition $\quad P(B|A) = \frac{n_1 - 1}{n - 1}$

By definition of conditional probability

$$\Omega = \{\omega : \omega = (a_{i_1}, a_{i_2}), a_{i_j} \in \{a_1, .. a_n\}, a_{i_1} \neq a_{i_2}\}$$

$$A = \{\omega : \omega = (a_{i_1}, a_{i_2}), a_{i_1} \in \{a_1, \dots, a_2\}, a_{i_2} \in \{a_1, \dots, a_2\}, a_{i_1} \notin a_{i_2}\}$$

$$|A| = n_1 \times (n-1)$$

$$B = \{\omega \in \Omega : \omega = (a_{i_1}, a_{i_2}), a_{i_1} \in \{a_1, \dots, a_n\}, a_{i_2} \in \{a_1, \dots, a_{n_1}\}\}$$

- $A \cap B = \{\omega \in \Omega : \omega = (a_{i_1}, a_{i_2})\ a_{i_j} \in \{a_1, \dots, a_{n_1}\},\ a_{i_1} \neq a_{i_2}\}$

$|A \cap B| = n_1 \times (n_1 - 1)$

thus as desired

$$P(B|A) = \frac{n_1(n_1-1)}{n_1(n-1)} = \frac{n_1 - 1}{n-1}$$ which agrees with our intuition.

Properties of Conditional Probability

1. $P(A|A) = 1$;
2. $P(\emptyset | A) = 0$;
3. if $A \subseteq B$ then $P(B|A) = 1$
4. if B_1, B_2 are disjoint then

$$P((B_1 \cup B_2)|A) = P(B_1|A) + P(B_2|A)$$

5. $P(B|A) + P(\bar{B}|A) = 1$

$P(\cdot|A)$ as a measure on $\mathcal{A}$

$P(\cdot|A)$ is a probability because it satisfies

- For any $B \in \mathcal{A}$, $P(B|A) \geq 0$
- $P(\Omega|A) = 1$
- If $B_1, B_2, \dots$ are disjoint events, then

$$P\left(\bigcup_i B_i | A\right) = \sum_i P(B_i|A)$$

Problem

- Consider a family with two children.

(a) What is the probability that both are boys given that the first child is a boy?

(b) You know at least one of them is a boy. Find the probability that both children are boys. Find the probability that one child is boy and the other is a girl.

$\Omega = \{BB, BG, GB, GG\}$, where BG Boy first, Girl second.

$$P(BB) = P(BG) = P(GB) = P(GG) = 1/4$$

$$P(\{BB\} \mid \{BB, BG\}) = \frac{1/4}{1/4 + 1/4} = 1/2$$

$$P(\{BG, GB\} \mid \{BG, BB, GB\}) = \frac{P(BG) + P(GB)}{P(BB) + P(BG) + P(GB)} = 2/3$$

$$P(\{BB\} \mid \{BG, BB, GB\}) = \frac{P(BB)}{P(BB) + P(BG) + P(GB)} = 1/3$$

finally recall that $P(\{BG, BB, GB\} \mid \{BG, BB, GB\}) = 1$