MA1061 16/11/21

Mean of Function of R.V. & Variance & Markov's Inequality.

Another Property of the expectation

the expectation of a constant is the constant itself, such that $E[c] = c$ where c is the constant.

Expectation of Function of Random Variable

Let X be a discrete r.v.

suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is a function

Then $Y := \varphi(X(\omega))$ is a function of x and also a discrete random variable, taking values $\varphi(x_1), \ldots, \varphi(x_k)$

Denote $y_j = \varphi(x_j)$. by definition $E[Y] = \sum_j y_j \, P_Y(y_j)$.

Proposition

Let X be random variable and let $\varphi : \mathbb{R} \to \mathbb{R}$. Then

$$E[\varphi(X)] = \sum_i \varphi(x_i) \, P(X = x_i).$$

this formula allows us to find the mean of $\varphi(X)$ without having to actually find the distribution of $\varphi(X)$.

instead we use the distrubution of X.

the proof is omitted.

Problem

An unbiased die has 6 faces labelled -2, -1, 0, 1, 2, 3. If the die is tossed once and X is the score obtained, find the expectations of X, X^2 and e^X

- each face has probability $1/6$
- range is $X = \{-2, -1, 0, 1, 2, 3\}$

$$E[X] = \sum_{i=1}^{6} x_i P(X = x_i) = \frac{1}{6}(-2 - 1 + 0 + 1 + 2 + 3) = \frac{1}{2}$$

now let $g(x) = x^2$ then

$$E[X^2] = E[g(x)] = \sum_{i=1}^{6} g(x_i) P(X = x_i)$$

$$= \frac{1}{6}(4 + 1 + 0 + 1 + 4 + 9) = \frac{19}{6}$$

now let $h(x) = e^x$ then

$$E[e^x] = E[h(x)] = \sum_{i=1}^{6} h(x_i) P(X = x_i)$$

$$= \frac{1}{6}(e^{-2} + e^{-1} + e^{0} + e^{1} + e^{2} + e^{3}) \approx 5.2827.$$

Variance

Variance is an equally important concept which measures how scattered the values of X are around $E[X]$.

If $E[X]$ gives a first estimate of the random variables of X, the variance tells us how good this estimate is.

Definition (Variance)

The variance of the random variable X, denoted by var(x), is defined as

$$var(X) = E[(X - E[X])^2]$$

- not the same scale as X.

- a more meaningful measure, because it is on the same scale, is the standard deviation of X denoted by s.d(X) which defined as

$$s.d.(X) = \sqrt{var(X)}$$

Properties of Variation

- the standard deviation gives us an idea of how far the random variable wanders on average from its expected value.

Proposition

For any random variable X and Y we have

① $var(X) \geq 0$;
② $var(X) = E[X^2] - (E[X])^2$;
③ $var(a+bX) = b^2 var(X)$.
④ if X and Y are independant, then

$$var(X+Y) = var(X) + var(Y)$$

① Is obvious;
② $var(X) = E[(X-E[X])^2] = E[X^2 - 2X E[X] + (E[X])^2]$
$= E[X]^2 - 2[E[X]]^2 + (E[X])^2$

③ $\text{var}(a+bX) = E[(a+bX - E[a+bX])^2] = E[(bX - bE[X])^2] =$
$= b^2 E[(X-E[X])^2] = b^2 \text{var}(X)$

④ will be proved later. (when multiple r.v.'s are introduced).

<u>Example</u> (sum of two dice)

We roll two independant unbiased dice and record their sum. Let X denote the sum of their numbers. Find the expectation of the random variable X.

X	2	3	4	5	6	7	8	9	10	11	12
$P_X(x)$	$1/36$	$1/18$	$1/12$	$1/9$	$5/36$	$1/6$	$5/36$	$1/9$	$1/12$	$1/18$	$1/36$

$$E[X] = 2\left(\tfrac{1}{36}\right) + 3\left(\tfrac{1}{18}\right) + \left(\tfrac{4}{12}\right) + \dots + 12\left(\tfrac{1}{36}\right) = \frac{252}{36} = 7$$

$$\text{var}(X) = (2-7)^2\left(\tfrac{1}{36}\right) + (3-7)^2\left(\tfrac{1}{18}\right) + \dots + (12-7)^2\left(\tfrac{1}{36}\right) = \tfrac{35}{6}$$

<u>Markov's and Chebyshev's inequality</u>

<u>Theorm</u> (Markov's inequality)

Let X be a non-negative random variable. Then for all $c > 0$ we have

$$P\{X \geq c\} \leq \frac{E[X]}{c}$$

<u>Proof</u> Let A be the event $X \geq c$ and 1_A be the indicator of event A.

when $X < c$ we have $1_A = 0$ and so $c1_A = 0 \leq X$

$X \geq c$, we have $1_A = 1$ and so $c1_A = c \leq X$

thus we have $c1_A \leq X$

then by properties of expectation

$$E[X] \geq E[c1_A] = cE[1_A] = cP(A)$$