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Chapter 5 Recap & Dimensions

Definitions

Let V be the vector space and $X = \{v_1, v_2, \dots, v_n\} \subset V$

(a) X is linearly independant

if whenever $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \dots + \lambda_n v_n = \underline{0}$ $\forall \lambda_i$ $1 \leq i \leq n$

(b) X spans V, if every $v \in V$ is equal to some linear combination of elements of X.

(c) X is a basis if it spans V and is linearly independant.

(d) If X is a basis for V, $v \in V$ write $[v]_X = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbb{R}^n$, where

$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$ for the coordinate vector of v.

Example

$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$

(a) Suppose $\lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \underline{0}$

$$\begin{pmatrix} \lambda_1 + 3\lambda_2 \\ 2\lambda_1 + 4\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \underline{0}$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = 0$$

$\Rightarrow$ linearly independent

(b) we want λ_1, λ_2 such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ for any } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \left(\begin{array}{cc|c} 1 & 3 & x \\ 2 & 4 & y \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & x \\ 1 & 2 & y-2x \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 3 & x \\ 0 & 1 & -\frac{y-2x}{2} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & x + \frac{3}{2}(y-2x) \\ 0 & 1 & -\frac{y-2x}{2} \end{array}\right)$$

$$\lambda_1 = \tfrac{3}{2}y - 2x$$
$$\lambda_2 = x - \frac{y}{2}$$

$\underline{\text{check}} \to (\tfrac{3}{2}y - 2x) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (x + -y/2) \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

(c) X is a basis.

(d) $[V]_X = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ as above.

<u>Proposition 6.1</u>

Let V be a vector space and suppose $S = \{v_1, v_2, \ldots, v_n\} \subset V$ is a basis.
For any $\tilde{S} = \{w_1, w_2, w_3, \ldots, w_k\} \subset V$

(i) if $k > n$; linearly dependant.

(ii) if $k < n$; not span V

__Proof__

Suppose $k > n$ and suppose $\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_k w_k = 0$

there is a linear map

$$\begin{aligned} V &\to \mathbb{R}^n \\ w &\to [w]_S \end{aligned}$$

$\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_k w_k$ to $\lambda_1 [w_1]_S + \cdots + \lambda_k [w_k]_S$

so $\lambda_1 [w_1]_S + \cdots + \lambda_k [w_k]_S = 0 \in \mathbb{R}^k$

By proposition 5.1 ("too many vectors in $\mathbb{R}^n$")

$\Rightarrow \lambda_i \neq 0$, for some $1 \leq i \leq n$

$\Rightarrow \tilde{S}$ is linearly independent.

(ii) Suppose $k < n$ Suppose span$(\tilde{S}) = V$

Then $v_j = \sum_{i=1}^{k} v_{ij} w_i$ for all $1 \leq j \leq n$

since $\tilde{S}$ spans V

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{kn} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ a_{k1} & \cdots & \cdots & a_{kn} \end{pmatrix} \in M_{kn}$$

since $k < n$ more columns than rows. $\Rightarrow A = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = 0$ has a non trivial solution i.e. $\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix}$ write $\sum_{i=1}^{n} \lambda_i a_{ij} = 0$ with not all $\lambda_i = 0$ (since $\hat{S}$ spans V)

now $\underline{0} = \sum_{i=1}^{n} w_i \sum_{j=1}^{n} \lambda_j a_{ij}$

= "vice versa"

$= \sum_{j=1}^{n} \lambda_j v_j$ $\Rightarrow$ S not a basis (λ_j is not 0, some j)

a contradiction

Corollary.

Every basis of a vector space has the same size.

Proof Suppose S and $\hat{S}$ be a basis of V. if $|\hat{S}| > |S|$ by proposition.

$\hat{S}$ linearly independant (and not a basis) ※

suppose $|\hat{S}| < |S|$ then $\hat{S}$ not span V by prop ※

we conclude $|\hat{S}| = |S|$

Definition. (dimensions)

The dimension of a vector space V is the number of vectors in a basis of V.