Vector Algebra

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Components of a Vector in Space:

$$Q(x_2, y_2, z_2) \text{ and } P(x_1, y_1, z_1) \implies PQ(x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Vector Algebra

Let $\underline{a} = (a_1, a_2, a_3)$, $\underline{b} = (b_1, b_2, b_3)$, $\underline{c} = (c_1, c_2, c_3)$ vectors, and λ a scalar:

$$\underline{a} + \underline{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = \underline{b} + \underline{a}$$

$$(\underline{as} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$$

$$\lambda\underline{a} = (\lambda a_1, \lambda a_2, \lambda a_3)$$

$$\underline{a} + \underline{0} = \underline{a}, \text{ for } \underline{0} = (0, 0, 0)$$

$$-\underline{a} = (-1)\underline{a}$$

Length of a vector:

$$\boxed{|\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$$

Scalar Prodcut(dot product, inner product, etc)

Let $\underline{a} = (a_1, a_2, a_3)$, $\underline{b} = (b_1, b_2, b_3)$ vectors, and γ the angle between them

Scalar Products:

$$\begin{aligned} \underline{a} \cdot \underline{b} &= |\underline{a}||\underline{b}| \cos\gamma \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$$

Properties:

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$$
$$\underline{a} \cdot \underline{a} = a_1^2 + a_2^2 + a_3^2 = |\underline{a}|^2$$
$$(\underline{a} + \underline{b}) \cdot \underline{c} = \underline{a} \cdot \underline{c} + \underline{b} \cdot \underline{c}$$
$$\boxed{\text{If } \underline{a} \text{ perpendicular } \underline{b} \text{ then } \underline{a} \cdot \underline{b} = 0}$$

Applications:
Work done by a force: Consider a constant force $\underline{F}$ acting on a body that results to a $\underline{d}$ of the body. What is the work W done by $\underline{F}$?

$$\boxed{W = \underline{F} \cdot \underline{d}}$$

Vector Product (cross product, wedge product, etc)

Let $\underline{a} = (a_1, a_2, a_3)$, $\underline{b} = (b_1, b_2, b_3)$ vectors, and γ the angle between them

Vector Product:

$$\underline{a} \times \underline{b} = \underline{v} = (a_2 b_3 - b_2 a_3, a_3 b_1 - b_1 a_3)$$
$$\text{Length: } |v| = |\underline{a}||\underline{b}||\sin\gamma|$$

Applications:
Area of Parallelogram: Consider $\underline{a} = (a_1, a_2, a_3)$, $\underline{b} = (b_1, b_2, b_3)$ vectors. Then the area of the parallelogram having $\underline{a}$ and $\underline{b}$ as edges is given by

$$\boxed{\text{Area} = |\underline{a} \times \underline{b}|}$$

Conclusion: Area can be described using vector products

Moment of a force: Let $\underline{F}$ a force acting on a body. Body at distance $\underline{r}$ from a point O. The moment vector $\underline{m}$ of the force $\underline{F}$ about the point O is

$$\boxed{\underline{m} = \underline{r} \times \underline{F}}$$

Moment describes the rotation about the point O caused by $\underline{F}$

Conclusion: Rotation can be quantified using product!

Properties:
The formula

$$\boxed{\underline{a} \times \underline{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - b_3 a_1, a_1 b_2 - b_1 a_2) = (a_2 b_3 - a_3 b_2)i + (a_3 b_1 - b_3 a_1)j + (a_1 b_2 - b_1 a_2)k}$$

seems hard to remember... but let's see what we know already
Keyword: **Determinants!**

we get: $\boxed{\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$

Other Properties:

$$\underline{a} \times \underline{a} = 0$$

$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$$

$$(\underline{a} + \underline{b}) \times \underline{c} = \underline{a} \times \underline{c} + \underline{b} \times \underline{c} \text{ not the same as } \underline{c} \times (\underline{a} + \underline{b}) = \underline{c} \times \underline{a} + \underline{c} \times \underline{b}$$