MA1114 8/2/22

Definition & Basic Examples of Linear Maps.

$V = \mathbb{R}^n \qquad W = \mathbb{R}^m$

$$T_A \, V \longrightarrow W$$
$$v \longmapsto Av$$

$$T_A(\lambda v + \mu w) = \lambda T_A(v) + \mu T_A(w)$$

<u>Definition</u> Let V, W be vector spaces

A map $T: V \to W$ is linear if for any $v, w \in V$ $\lambda, \mu \in \mathbb{C}$

$$T_A(\lambda v + \mu w) = \lambda T_A(v) + \mu T_A(w)$$

$0: V \longrightarrow 0$ zero map
$1: V \longrightarrow V$ Identity map.

$\pi : \mathbb{C}^m \to \mathbb{C}^n \qquad m \geq n$

$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is linear

(example yesterday).

<u>non-examples</u>

$$P: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$P\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix}\right) = P\begin{pmatrix} 5 \\ 5 \end{pmatrix} \neq \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

However. $P\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) + P\left(\begin{pmatrix} 3 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 7 \\ 5 \end{pmatrix} + \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

$Q: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 x_2$$

$$\lambda = \mu = 1 \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$Q\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix}\right) = P\left(\begin{pmatrix} 4 \\ 6 \end{pmatrix}\right) = 24$$

$$Q\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) + Q\left(\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right) = 2 + 12 = 14$$

Another example

- $P = \{$ polynomial with complex coefficient $\}$

$T: P \to P$

$p(x) \mapsto x p(x)$

to show T is linear

let $p(x), q(x) \in P \quad \lambda, \mu \in \mathbb{C}$

then $T(\lambda p(x) + \mu q(x)) = x(\lambda p(x) + \mu q(x))$

$= x \lambda p(x) + x \mu q(x)$

$= \lambda T(p(x)) + \mu T(q(x))$

Proposition U, V, W are vector spaces

and $T: U \to V$

$S: V \to W$

linear maps

(a) $T(0) = 0$ and $T(u - v) = T(u) - T(v)$

Example

$\mathcal{U} = \mathbb{C}^n \qquad \mathcal{V} = \mathbb{C}^k \qquad \mathcal{W} = \mathbb{C}^m$

$T = T_B \qquad B \in M_{kn}(\mathbb{C})$

$S = T_A \qquad A \in M_{mk}(\mathbb{C})$

$v \in \mathcal{U}$

$$S \circ T(v) = S(T(v)) = S(Bv)$$
$$= ABv$$

$\Rightarrow S \circ T = T_{AB}$ is linear

Proof of Proposition

(a) $\underline{0} + \underline{0} = \underline{0}$

$\Rightarrow T(\underline{0} + \underline{0}) = T(\underline{0})$

$\Rightarrow T(\underline{0}) + T(\underline{0}) = T(\underline{0})$ since T is linear

$\Rightarrow T(\underline{0}) = \underline{0}$

$T(u-v) = T(u + (-1)v)$

$= T(u) + (-1)T(v)$

$= T(u) - T(v)$

(b) Suppose $u_1, u_2 \in \mathcal{U}$ $\lambda_1, \lambda_2 \in \mathbb{C}$

$S \circ T(\lambda_1 u_1 + \lambda_2 u_2)$

$= S(T(\lambda_1 u_1 + \lambda_2 u_2))$ linear

$= S(\lambda_1 T(u_1) + \lambda_2 T(u_2))$ linear

$= \lambda_1 (S \circ T(u_1)) + \lambda_2 (S \circ T(u_2))$

$\Rightarrow S \circ T$ is linear

Proposition "enough to understand T on a basis"

Suppose V, W are vectorspaces and $T: V \to W$ a map. Let $\{v_1, \dots, v_n\}$ be a basis for V

T is linear $\iff$ $T(v) = \sum_{i=1}^{n} \lambda_i T(v_i)$ where $v = \sum_{i=1}^{n} \lambda_i v_i \in V$

Proof

"$\Rightarrow$" by induction on n

$n = 2$, $v = \lambda_1 v_1 + \lambda_2 v_2$

$\Rightarrow T(v) = \lambda_1 T(v_1) + \lambda_2 T(v_2)$ (since linear)

so results hold.

In general, suppose true for $\dim(V) = n-1$

$$v = \sum_{i=1}^{n} \lambda_i v_i = \sum_{i=1}^{n-1} \lambda_i v_i + \lambda_n v_n$$

$$\Rightarrow T(v) = T\left(\sum_{i=1}^{n-1} \lambda_i v_i + \lambda_n v_n\right)$$

$$= T\left(\sum_{i=1}^{n-1} \lambda_i v_i\right) + \lambda_n T(v_n)$$

$$= \sum_{i=1}^{n-1} \lambda_i T(v_i) + \lambda_i T(v_n)$$

(by induction)

$$= \sum_{i=1}^{n} \lambda_i T(v_i)$$

"$\Leftarrow$" Suppose $w = T(v_i)$ is specified for all $1 \leq i \leq n$ (i.e. we understand T on a basis)

and $(T(v)) = \sum_{i=1}^{n} \lambda_i w_i$

for any $v = \sum_{i=1}^{n} \lambda_i v_i$

Why is the linear map T linear?

$\alpha, \beta \in \mathbb{C} \quad u, v \in V$

where $u = \sum_{i=1}^{n} x_i v_i \quad , \quad v = \sum_{i=1}^{n} \mu_i v_i$

$$T(\alpha u + \beta v) = T(\alpha \sum_{i=1}^{n} \lambda_i v_i) + T(\beta \sum_{i=1}^{n} \mu_i v_i)$$

$$= T(\sum_{i=1}^{n} \alpha \lambda_i v_i + \sum_{i=1}^{n} \beta \mu_i v_i)$$

$$= T(\sum_{i=1}^{n} (\alpha \lambda_i + \beta \mu_i) v_i)$$

$$= \sum_{i=1}^{n} (\alpha \lambda_i + \beta \mu_i) v_i$$

$$= \sum_{i=1}^{n} (\alpha \lambda_i + \beta \mu_i) T(v_i)$$

$$= \alpha \sum_{i=1}^{n} \lambda_i T(v_i) + \beta \sum_{i=1}^{n} \mu_i T(v_i)$$

$$= \alpha T(V) + \; + \beta T(v)$$