MA1114 30/3/22

Real Symmetric Matrices & Properties.

Theorem

If $A \in M_n(\mathbb{R})$ and $A = A^T$ then all eigenvalues of A are real.

(yesterday did complex inner product)

Proof $(AB)^T = B^T A^T$

View $A \in M_n(\mathbb{C})$

Suppose $Av = \lambda v$ for some $v \neq 0$ $\lambda \in \mathbb{C}$ so λ is an eigenvalue with eigenvectors v (possibly $v \in \mathbb{C}^n$)

$$\lambda \langle v, v \rangle = \lambda v^T \bar{v} = (\lambda v)^T \bar{v} = (Av)^T \bar{v}$$

$$= v^T A^T \bar{v} = v^T A \bar{v} = v^T \overline{Av} = v^T \overline{\lambda v}$$

$$= v^T \bar{\lambda} \bar{v} = \bar{\lambda} v^T \bar{v} = \bar{\lambda} \langle v, v \rangle$$

since $v \neq 0$ $\langle v, v \rangle \neq 0$

so $\lambda = \bar{\lambda}$ $\Rightarrow \lambda \in \mathbb{R}$

Proposition Suppose $A \in M_n(\mathbb{R})$ with $A^T = A$. If $\lambda_1 \neq \lambda_2$ are eigenvalues then $\langle v_1, v_2 \rangle = 0$ where

$$A v_1 = \lambda_1 v_1$$

and $A v_2 = \lambda_2 v_2$

Example $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} \in M_2(\mathbb{R})$

2 eigenvalues $\lambda_1 = 4$, $\lambda_1 = 0$

with eigenvectors $v_1 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, $v_2 = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$

(check $Av_1 = \lambda v_1$ holds true)

$$\left\langle \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}, \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \right\rangle = \begin{pmatrix} 1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$$

$$= 0$$

Proof : notice $V_i \in \mathbb{R}^n$

since $\underbrace{(A - \lambda_i I)}_{M_n(\mathbb{R})} v_i = 0 \in \mathbb{R}$

$$(A - \lambda_i I)\bar{v}_i = \overline{(A - \lambda_i I)\bar{v}_i} = 0 \Rightarrow \bar{V}_i = V_i$$

$V_i \in \mathbb{R}^n$ so use real inner product

$$\begin{aligned}
\lambda_i \langle v_1, v_2 \rangle &= \langle \lambda_i v_1 , v_2 \rangle \\
&= \langle Av_1 , v_2 \rangle \\
&= (Av_1)^T v_2 \\
&= v_1^T A^T v_2 \\
&= v_1^T A v_2 = v_1^T \lambda_2 v_2 \\
&= \lambda_2 v_1^T v_2 = \lambda_2 \langle v_1 , v_2 \rangle
\end{aligned}$$

$$(\lambda_1 - \lambda_2)\langle v_1, v_2 \rangle = 0$$

$$\lambda_1 \neq \lambda_2 \Rightarrow (\lambda_1 - \lambda_2) = 0$$

$$\Rightarrow \langle v_1 , v_2 \rangle = 0 \quad \blacksquare$$

<u>lemma</u> $V = \mathbb{R}$, $0 \neq v \in V$

then $V = \langle v \rangle \oplus v^{\perp}$

where $v^{\perp} = \{ w \in V \mid \langle w, v \rangle = 0 \}$

<u>Proof</u> $v \neq 0 \Rightarrow v \notin v^{\perp}$
(else $v \in V^{\perp} \Rightarrow \langle v, v \rangle = 0$
$\Rightarrow v = 0$ ※)

$\Rightarrow V = \langle v \rangle \oplus v^{\perp}$
enough to show direct sum (ie intersection is $\{0\}$)

if $w \in \langle v \rangle \wedge v^{\perp}$

$\Rightarrow w = \lambda v$ some $\lambda \in \mathbb{R}$ <u>and</u> $\langle w, v \rangle = 0$
$\Rightarrow 0 = \langle w, v \rangle = \langle \lambda v, v \rangle$
$\Rightarrow \lambda = 0 \qquad (v \neq 0)$
$\Rightarrow w = \lambda v = 0$
$\Rightarrow v^{\perp} \wedge \langle v \rangle = \{0\}$

since $v^{\perp} = \ker (T : w \longmapsto \langle w, v \rangle)$

by rank-nullity $\Rightarrow V = v^{\perp} \oplus \langle v \rangle$ ☒

<u>Theorem</u>

If $A \in M_n(\mathbb{R})$ and $A^T = A$ then there is an orthonormal bases of eigenvectors for $T_A : \mathbb{R}^n \to \mathbb{R}^n$
$v \longmapsto Av$

Example $A = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}$ as seen previously

$\left\{ \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^3$

Proof by induction on n

Plainly true when $n=1$ (only one eigenvalue/space)

assume true for $n-1$

let $0 \neq v \in \mathbb{R}^n$ be an eigenvector for T_A

by lemma $V = \langle v \rangle \oplus v^\perp$

idea restrict T_A to $v^\perp$ (dimension $n-1$)

i.e consider the map

$$T_A|_{v^\perp} : v^\perp \to v^\perp$$
$$w \to Aw$$

check well defined i.e $Aw \in v^\perp$

$$\langle v, Aw \rangle = v^T Aw$$
$$= v^T A^T w$$
$$= (Av)^T w$$
$$= (\lambda v)^T w$$

some $\lambda \in \mathbb{R}$

$$= \lambda v^T w$$
$$= \lambda \langle v, w \rangle = 0$$

since w is in $v^\perp$

so by induction, the restricted linear map

$T_A|_{v^\perp}$ has an orthonormal basis of eigenvectors

now $\mathcal{B} = \mathcal{B}' \cup \{v\}$ is orthogonal

since $\langle v, w \rangle = 0$ for all $w \in \mathcal{B}$

so $\mathcal{B}$ is an orthonormal bases of eigenvectors

can orthonormalise (i.e divide every vector by it own norms)

to make an orthonormalise basis