MA1114 29/3/22

Gram-Schmidt O-normalisation Process; its proofs; Complex Inner Prod.

<u>Definition</u> (reminder) $(V, \langle -, - \rangle)$

$\{v_1, \ldots, v_k\} \subset V$ is <u>orthogonal</u> if $\langle v_i, v_j \rangle = 0$ for all $i \neq j$.

It is also orthogonal if $||v_i|| = 1$ for $1 \leq i \leq k$

If $\mathcal{B}$ is an orthonormal basis of V and $v \in V$ then
$[v]_{\mathcal{B}} = \begin{pmatrix} \langle v_1, v \rangle \\ \langle v_2, v \rangle \\ \vdots \\ \langle v_n, v \rangle \end{pmatrix}$ (where $\mathcal{B} = \{v_1, \ldots, v_n\}$)

want to make orthonormal bases from ordinary bases

<u>Gram-Schmidt Orthonormalisation</u>

Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V. Then $\text{span}(u_1, \ldots, u_k) = \text{span}(v_1, \ldots, v_k)$ for all $1 \leq i \leq k$ where

$$u_k = \frac{1}{||\hat{u}_k||} \tilde{u}_k \text{ and}$$

$\tilde{u}_k$ is defined <u>inductively</u> as follows

$\tilde{u}_1 = v_1$ and for $k \geq 2$

$$\hat{u}_k = v_k - \sum_{i=1}^{k-1} \frac{\langle v_k, \tilde{v}_i \rangle \hat{u}_i}{||v_i||^2}$$

moreover

$\{u_1, u_2, \ldots, u_k\}$ is orthonormal

Proof by induction

we prove orthonormality first. Suppose $\langle \tilde{u}_i, \tilde{u}_j \rangle = 0 \ \forall\, 1 \le i \ne j \le k$
consider $\langle \tilde{u}_i, \tilde{u}_{k+1} \rangle$ where $1 \le l \le k$

$$\langle \tilde{u}_l, \tilde{u}_k \rangle = \langle \tilde{u}_l, v_{k+1} - \sum \frac{\langle v_{k+1}, \tilde{u}_i \rangle \tilde{u}_i}{\|\tilde{u}_i\|^2} \rangle$$

$$= \langle \tilde{u}_i, v_{k+1} \rangle - \sum_{i=1}^{k} \frac{\langle v_{k+1}, \tilde{u}_i \rangle}{\|\tilde{u}_i\|^2} \langle \tilde{u}_l, \tilde{u}_i \rangle$$

all i

$$= \langle \tilde{u}_l, v_{k+1} \rangle - \frac{\langle v_{k+1}, \tilde{u}_i \rangle}{\|\tilde{u}_i\|^2} \langle \tilde{u}_l, \tilde{u}_l \rangle$$ except when $l=i$ by induction

$$= \langle \tilde{u}_l, v_{k+1} \rangle - \langle v_{k+1}, \tilde{u}_l \rangle = 0$$

By construction

$\mathrm{span}(u_1, \dots, u_k) \subset \mathrm{span}(v_1, \dots, v_k)$ (each v_i is a linear combination of the v_i)

But LHS is orthogonal $\Rightarrow$ linearly independant.
so $\mathrm{span}(u_1, \dots, u_k) = \mathrm{span}(v_1, \dots, v_k)$

Example $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$B = \{v_1, v_2, v_3\}$ span $\mathbb{R}^3$ not orthogonal $\langle v_1, v_2 \rangle = (1\ 1\ 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2 \ne 0$

using Gram-Schmidt:

$$\tilde{u}_1 = v_1, \quad \tilde{u}_2 = v_2 - \frac{\langle v_2, \tilde{u}_1 \rangle \tilde{u}_i}{\|\tilde{u}_1\|^2}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{(1\ 1\ 0)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{(1^2+1^2+1^2)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\tilde{u}_3 = v_3 - \frac{\langle v_3, \tilde{u}_1 \rangle}{\|\tilde{u}_1\|} \tilde{u}_1 - \frac{\langle v_3, \tilde{u}_2 \rangle \tilde{u}_2}{\|\hat{u}_2\|^2}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{(1\,0\,0)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{\frac{1}{3}(1\,0\,0)\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{(6/9)} \tfrac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{8}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$$

$$u_1 = \frac{\tilde{u}_1}{\|u_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{\sqrt{6}}{3}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

check orthogality

$$\langle u_1, u_2 \rangle = \frac{1}{\sqrt{3}} \frac{\sqrt{6}}{3} (1\ 1\ 1) \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 0$$

$$u_3 = \frac{1}{4}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Complese Inner Product

Definition

$\langle -, - \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$
defined by $\langle v, w \rangle = v^T \bar{w}$ is the complese inner on $\mathbb{C}^n$

$\|v\| = \sqrt{\langle v, v \rangle} \in \mathbb{R}$ is the <u>norm</u> of v

if $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ then $v^T v = (v_1, \ldots, v_n)\begin{pmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{pmatrix}$

$$= v_1\bar{v}_1 + v_2\bar{v}_2 + \cdots + v_n\bar{v}_n$$

<u>Recall</u> $x + iy \in \mathbb{C}$

$$\Rightarrow x + iy = x - iy$$

$$\Rightarrow (x + iy) + (x - iy) = x^2 + y^2 \in \mathbb{R}$$

<u>Example</u>

$$v = \begin{pmatrix} i \\ 1-i \end{pmatrix} \in \mathbb{C}^2 \qquad w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2 \subseteq \mathbb{C}^2$$

$$\begin{aligned}
\langle v, w \rangle &= v^T \overline{w} \\
&= (i \; (1-i)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= i \\
\langle w, v \rangle &= w^T \overline{v} \\
&= (1 \quad 0) \overline{\begin{pmatrix} i \\ 1-i \end{pmatrix}} \\
&= (1 \quad 0) \begin{pmatrix} -i \\ 1+i \end{pmatrix} = -i = \overline{i}
\end{aligned}$$

In general, we have $\langle v, w \rangle = \overline{\langle w, v \rangle}$

$\langle \overset{T}{\_} , v \rangle : V \to V$ is linear for $v \in V$

$$\ker(T) = \{ w \mid \langle w, v \rangle = 0 \} = v^{\perp}$$

Theorem

$\langle -, - \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ satisfies for $\lambda, \mu \in \mathbb{C}$, $u, v, w \in \mathbb{C}^n$

(i) $\langle \lambda u + \mu v, w \rangle = \lambda \langle u, w \rangle + \mu \langle u, w \rangle$

(ii) $\langle v, \lambda u + \mu w \rangle = \bar{\lambda} \langle v, u \rangle + \bar{\mu} \langle v, w \rangle$

(iii) $\langle v, w \rangle = \overline{\langle w, v \rangle}$

(iv) $\langle v, v \rangle \geqslant 0$ and $\langle v, v \rangle = 0 \iff v = 0$

Application

Theorem

If $A \in M_n(\mathbb{R})$ and $A^T = A$ then eigenvalues are real
note: not always true $A \in M_n(\mathbb{R})$

e.g. $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ evalues $e^{\pm i\theta} \notin \mathbb{R}$

For $A \in M_n(\mathbb{R})$

$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is symmetric $\chi_A(t) = \det(tI - A)$

$$= \det \begin{pmatrix} t-a & (-b) \\ (-b) & t-c \end{pmatrix}$$

$$= (t-a)(t-c) - b^2$$

$$= t^2 - (a+c)t + ac - b^2$$

notice $D =$ " $b^2 - 4ac$ "

$$D = (a+c)^2 - 4(ac - b^2)$$

$$= (a+c)^2 + (2b)^2 \geqslant 0$$

$\Rightarrow$ both eigenvalues are real $(n=2)$