MA1014 14/3/22

Infinite Series & Vanishing Condition & The Harmonic Series.

$\{a_n\}$

$a_1 + a_2 + \cdots\cdots + a_n + \cdots\cdots$ $\sum_{n=1}^{\infty} a_n$

$\lim_{n\to\infty} (a_1 + \cdots + a_n)$ $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$

Definition 7.1

Let $\{a_n\}$ be a real sequence, $S_n = a_1 + a_2 + a_3 + \cdots + a_n$
If $\lim_{n\to\infty} S_n$ exists, then we can say $\sum_{n=1}^{\infty} a_n$ is convergent, $\updownarrow$ divergent

define the sum $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

a_n : general term S_n : partial sum

Example 1 $\sum_{n=1}^{\infty} n$ is divergent.

$$S_n = 1 + 2 + \cdots + n = \frac{(1+n)n}{2} \to \infty$$

Example 2 $\sum_{n=1}^{\infty} q^n$ $q \in \mathbb{R}$ $q \neq 1$

$$S_n = 1 + q + q^2 + \cdots + q^n = \frac{1(1-q^{n+1})}{1-q}$$

when $|q| < 1$ $S_n \to \frac{1}{1-q}$

$|q| > 1$ S_n divergent

$q = 1$ $1 + 1 + \cdots + 1 + \cdots$ $S_n = n \to \infty$ divergent

$q = -1$ $S_n = -1 + 1 - 1 + 1 - 1 \cdots = \begin{cases} 0, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$ divergent

$\sum_{n=1}^{\infty} q^n$ is $\begin{cases} \text{con}, & \text{when } |q| < 1 \\ \text{div}, & \text{when } |q| \geq 1 \end{cases}$

Example 3 $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$S_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)}$$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} \to 1 \text{ as } n \; n \to \infty$$

Then $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.

Example 4 $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$1 + \frac{1}{2} + \cdots + \frac{1}{n}$ $\qquad$ $1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$.

Theorem 7 4 (vanishing condition)

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Proof

Since $\sum_{n=1}^{\infty} a_n$ is convergent, then set $S_n = a_1 + \cdots + a_n$

then $\lim_{n\to\infty} S_n = S$

then $a_n = S_n - S_{n-1} \to S - S = 0$ (as $n \to \infty$)

$\sum_{n=1}^{\infty} (-1)^n$ $\qquad$ $\sum_{n=1}^{\infty} \sin(n)$ $\qquad$ $\sum_{n=1}^{\infty} n^2$ $\qquad$ $\sum_{n=1}^{\infty} 2$

Example 5 $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, but $\lim_{n\to\infty} \frac{1}{n} = 0$

S_n. Cauchy's theorem: $\{a_n\}$ is convergent $\Leftrightarrow$ $\{a_n\}$ is a cauchy sequence

that is, $\forall \varepsilon>0, \exists N, \forall m \geq n > N$

$|a_m - a_n| < \varepsilon$

set $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Letting $\varepsilon = \frac{1}{2}$ $\forall N \in \mathbb{N}$.

setting $m = 2n$, $n = N+1$

$$|S_m - S_n| = |1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m} - (1 + \frac{1}{2} + \cdots + \frac{1}{n})|$$

$$= \frac{1}{n+1} + \cdots + \frac{1}{m} \geq n \cdot \frac{1}{m} = \frac{1}{2} = \varepsilon$$

By Cauchy's theorem, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

harmonic series

$\sum_{n=1}^{\infty} a_n$ $(a_n \geq 0)$ non negative series

$a_1 + a_2 + \cdots + a_n$ increasing

$S_n \geq S_{n-1}$ unbounded : $+\infty$; bounded : convergent

Theorem

Let $\sum_{n=1}^{\infty} a_n$ be an non-negative series, S_n its partial sum. Then $\sum a_n$ is convergent if and only if S_n has upper bound

Theorem 7.14 (comparison tests)

Let $\sum a_n$ and $\sum b_n$ be non-negative with $a_n \le b_n \ \forall n$

Then (1) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent;

(2) If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Proof

(1) Set $S_n^{(a)} = a_1 + \cdots + a_n$
$\le b_1 + \cdots + b_n \le S^{(b)}$

Then $\sum a_n$ is convergent.

(2) By contradiction. Assume $\sum b_n$ is conv

by (1) $\sum a_n$ is conv. ※ (a contradiction)

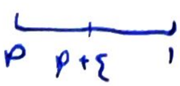
Theorem 7.16.1 (ratio test)

Let $\sum_{n=1}^{\infty} a_n$ be non-negative and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$ exists then

$$\sum a_n = \begin{cases} \text{conv} & \text{when } \rho < 1 \\ \text{div.} & \text{when } \rho > 1 \\ ? & \text{when } \rho = 1 \end{cases}$$

Proof

Since $\rho < 1$, then $\exists\ \varepsilon > 0$ s.t. $\rho + \varepsilon < 1$



since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, then $\exists N \quad \forall n > N$

$$\left| \frac{a_{n+1}}{a_n} - \rho \right| < \varepsilon$$

then $\frac{a_{n+1}}{a_n} < \rho + \varepsilon$ i.e

$$a_{n+1} < (\rho+\varepsilon) a_n < (\rho+\varepsilon)^2 a_{n-1} < \cdots < (\rho+\varepsilon)^{n-N} a_N$$

$$= \underbrace{(\rho+\varepsilon)^n}_{\text{con}} \underbrace{\frac{a_N}{(\rho+\varepsilon)^N}}_{\text{constant}}$$

By comparison test $\sum a_n$ is con.

<u>Remark</u> geometric series $\Rightarrow$ Ratio test

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \qquad a_n = \frac{1}{2^n} \qquad a_{n+1} = \frac{1}{2^{n+1}}$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{2^{n+1}} \cdot 2^n = \frac{1}{2} < 1$$