MA1014 29/11/21

Bolzano - Weierstrass Theorem.

<u>sequences</u> monotonic, bounded $\Rightarrow$ convergent ← has limit L

(a_n) $a_{n+1} \geq a_n$ $\forall n$ $|a_n| < B$

$\forall \varepsilon > 0 \; \exists N \quad n > N \Rightarrow |a_n - L| < \varepsilon$

<u>cauchy</u> sequence:

$\forall \varepsilon > 0 \; \exists N \quad m, n > N \Rightarrow |a_n - a_m| < \varepsilon$

If $(a_n)_{n \in \mathbb{N}}$ is Cauchy then it is <u>bounded</u>

1 If $(a_n)_{n \in \mathbb{N}}$ is a <u>convergent</u> <u>sequence</u> then it is <u>cauchy</u>

Proving cauchy $\Rightarrow$ convergent

Either ① Any sequence has a monotonic sequence

or ② Any <u>bounded</u> sequence has a convergent subsequence

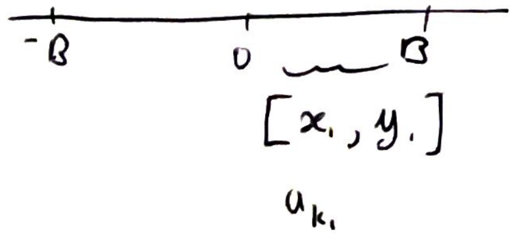
<u>Proof of ②</u> Bolzano - Weierstrass Theorem

Given $(a_n)_{n \geq 0}$ with $|a_n| < B$

i.e $-B < a_n < B$ for all n $\forall n$

$a_0 \in [-B, B]$ let $k_0 = 0$ looking for $(a_{k_m}) \to L$

Bisection Method



Let $x_0 = -B$, $y_0 = B$

now let $[x_1, y_1]$ be either

$[-B, 0]$ or $[0, B]$

so that $[x_1, y_1]$ contains infinitely many terms of the sequence

choose $k_1 > k_0$ such that $a_{k_1} \in [x_1, y_1]$

Now inductively: suppose

$[x_n, y_n]$ contains ∞'ly many terms of the sequence.

Let $[x_{n+1}, y_{n+1}]$ be $[x_n, \frac{x_n + y_n}{2}]$ or $[\frac{x_n + y_n}{2}, y_n]$

so that $[x_{n+1}, y_{n+1}]$ still contains infinitely many terms

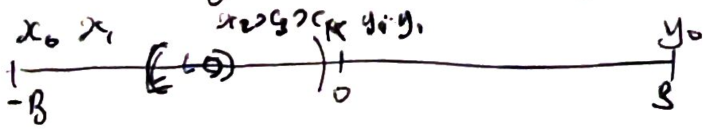
suppose we have chosen k_n with $a_{k_n} \in [x_n, y_n]$ then choose

$k_{n+1} > k_n$ with $a_{k_{n+1}} \in [x_{n+1}, y_{n+1}]$

so finally:

we have a subsequence $(a_{k_m})_{m \geq 0}$ $a_{k_m} \in [x_m, y_m]$ $\forall m$

each $[x_m, y_m] \supseteq [x_{m+1}, y_{m+1}] \subseteq [-B, B]$



All the x_m form increasing seq

All the y_m form decreasing seq

→ By Pinching theorem

$$x_m \le a_{k_m} \le y_m \quad \forall m$$

$x_m \to$ some limit L_x as increasing and bounded above

$y_m \to$ some limit L_y as decreasing and bounded below.

& $y_m - x_m$ = length of interval $[x_m, y_m]$

$$= \frac{\text{length of } [-B, B]}{2^m}$$

$$\to 0 = L_y - L_x$$

L_x & L_y are same limit

$L_x = L_y = L$

Pinching theorem says that we have constructed a subsequence $(a_{k_m})_{m \geq 0}$ with limit L

Alternative Proof of Bolzano - Weierstrass

Prove ①: every sequence has a monotonic subsequence.

then ②: every bounded sequence has a monotonic bounded subsequence

i.e. every bounded sequence has a convergent subsequence

Proof that every Cauchy sequence has a limit.

a) If $(a_n)_{n \in \mathbb{N}}$ satisfies the cauchy property then it is bounded.

b) so it has a subsequence $a_{k_m} \to$ as $m \to \infty$

c) we have to prove the original Cauchy sequence had limit L.

Proof Given $\varepsilon > 0$ we need to prove there $\exists N$ such that $|a_n - L| < \varepsilon$ for all $n > L$

we know - there is a subsequence $(a_{k_m}) \to L$

- sequence (a_n) Cauchy

Consider $\varepsilon/2$

$\exists M : |a_{k_m} - L| < \varepsilon/2$ if $m > M$

$\exists N : |a_{N_1} - a_{n_1}| < \varepsilon/2$ if $n_1, n_2 > N$

so choose $N' > M, N, k_{M+1}$

Then $|a_n - L| \leq |a_n - a_{k_{M+1}}| + |a_{k_{M+1}} - L|$

$< \varepsilon/2 + \varepsilon/2$ if $n > N'$

not $< \varepsilon$