A vector field $\underline{F}$ is characterised by: curl $\underline{F}$ and div $\underline{F}$

A scalar field u is characterised by: grad u

$$\text{div}\,\underline{F} = \nabla \cdot \underline{F} \qquad \text{curl}\,\underline{F} = \nabla \times \underline{F}$$

$$\text{grad}\,u = \nabla u$$

we can apply ∇ to the characteristics of the fields:

$$\text{div}(\text{grad}\,u) = \nabla \cdot \nabla u$$

$$\text{curl}(\text{grad}\,u) = \nabla \times \nabla u$$

$$\text{curl}(\text{curl}\,\underline{F}) = \nabla \times \nabla \times \underline{F}$$

$$\text{grad}(\text{div}\,\underline{F}) = \nabla \cdot \nabla \cdot \underline{F}$$

$$\text{div}(\text{curl}\,\underline{F}) = \nabla \cdot \nabla \times \underline{F}$$

Laplacian:

Let $u: \mathbb{R}^n \to \mathbb{R}$ be twice differentiable scalar function. We define the Laplacian of u, as

$$\Delta u = \text{div}(\text{grad}\,u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \nabla^2 u$$

we can also use the "∇" notion for the Laplacian. We have

$$\Delta u = \nabla \cdot (\nabla u)$$

Thus, formally, the divergence is the scalar product of ∇ and ∇u!

Example: Find $\Delta f(r) = \nabla^2 f(r)$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

But $f = f(r)$ where $r = \sqrt{x^2+y^2+z^2}$

we apply the chain rule:

$$\frac{\partial f}{\partial x} = f'(r)\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Similarly $\frac{\partial f}{\partial y} = f'(r)\frac{\partial r}{\partial y}$, $\frac{\partial f}{\partial z} = f'(r)\frac{\partial r}{\partial z}$

$$\nabla f(r) = \left(f'(r)\frac{x}{r}, f'(r)\frac{y}{r}, f'(r)\frac{z}{r}\right)$$

$$= f'(r)\frac{1}{r}\underline{r}$$

$$\Delta f = \nabla \cdot \nabla f =$$

$$= \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] + \frac{\partial}{\partial y}\left[f'(r)\frac{y}{r}\right] + \frac{\partial}{\partial z}\left[f'(r)\frac{z}{r}\right]$$

consider

$$\frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] = f''(r)\frac{\partial r}{\partial x}\cdot\frac{x}{r} + \frac{f'(r)}{r} + f'(r) \times \frac{\partial}{\partial x}\left(\frac{1}{r}\right) =$$

$$= f''(r)\left(\frac{x}{r}\right)^2 + \frac{1}{r}f'(r) + f'(r) \times \left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} =$$

$$= f''(r)\left(\frac{x}{r}\right)^2 + \frac{1}{r}f'(r) - \frac{1}{r}f'(r)\left(\frac{x}{r}\right)^2$$

Similarly for $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$

Adding all terms together:

$$\Delta f(r) = f''(r)\frac{x^2+y^2+z^2}{r^2} + \frac{3}{r}f'(r) - \frac{1}{r}f'(r)\frac{x^2+y^2+z^2}{r^2} =$$

$$= f''(r) + \frac{2}{r}f'(r)$$

Laplace's Equation - Poisson equation

Let $u = \mathbb{R}^n \to \mathbb{R}$ be twice differentiable scalar function.

The partial differentiable equation (PDE in short)

$$\Delta u = 0$$

is called Laplaces equation.

Now consider another scalar function f. The PDE

$$\Delta u = f$$

is called Poisson's equations.

Laplace's and Poisson's equations are of fundamental importance in PDE theory and applications.

Usually u is not known in

$$\Delta u = 0 \quad (\text{or in } \Delta u = f$$

How do we find u?

Remark 1: (1) $u_1 = (x^2+y^2+z^2)^{-\frac{1}{2}} = 1/r$
(2) $u_2 = \text{const}$

But then every function of the form $\alpha u_1 + \beta u_2 \quad \forall \alpha, \beta \in \mathbb{R}$ is a solution. infinite # of sols.

Remark 2: it is not easy to solve this equation for u if we restrict ourselves to subsets of the space and assume specified values on the boundaries of these subsets. In certain cases there are methods for doing so.

$$\text{curl}(\text{grad}(U)) = \nabla \times \nabla U = 0$$

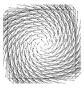
The curl of any potential ($\underline{E} = -\text{grad}(U)$) is zero. (see lecture 8)

or symbolically:

$$\nabla \times \nabla U = 0 \qquad (\nabla \text{ and } \nabla \text{ are colinear})$$

$$\text{div}(\text{curl } \underline{E}) = \nabla \cdot \nabla \times \underline{E} = 0$$

The field curl $\underline{E}$ is solenoid (incompressible). (see lecture 7-8)

div  = 0

$$\boxed{\text{curl}(\text{curl } \underline{E}) = \nabla \times \nabla \times \underline{E} = \text{grad}(\text{div } \underline{E}) - \Delta \underline{E}}$$

$$\boxed{\text{grad}(\text{div } \underline{E}) = \text{curl}(\text{curl } \underline{E}) + \Delta \underline{E}}$$

Example: Find curl(curl $\underline{E}$)
for $\underline{E}$ =

We shall use the above properties to prove the following fundamental theorem:

For any vector field $\underline{E}$ we have the following decomposition:

$$\underline{E} = \underline{a} + \underline{b}$$

where $\underline{a}$ is a potential field and $\underline{b}$ is a solenoidal field

sketch the proof:

$\underline{a} = -\text{grad}(u) \Rightarrow \underline{b} = \underline{E} + \text{grad}(u)$ — the field is solenoidal

$$\text{div } \underline{b} = \text{div}(\underline{E} + \text{grad}(u)) = \text{div } \underline{E} + \text{div}(\text{grad}(u)) = 0$$

we choose: $\text{div } \underline{E} + \text{div}(\text{grad}(u))\ \text{div } \underline{E} + \Delta u = 0$ (Poissons equation)

$$-\text{div } \underline{E} = \Delta u$$