MA1114 17/11/21

$\det(A) \neq 0$ iff A invertible & $\det(AB) = \det(A)\det(B)$ & etc .

<u>Theorem</u>

$A \in M_{n,n}$ is invertible $\Leftrightarrow$ $\det(A) \neq 0$

<u>Proof</u>

$R = E_1 E_2 \dots E_r A$ be reduced echelon form of A

$\Rightarrow \det(r) = \det(E_1 \ \ . E_r A)$
$\quad = \det(E_1) \cdots \det(E_r) \det(A)$ $\quad (*)$

"$\Rightarrow$" suppose A is invertible
$\Rightarrow R = I_n$
$\Rightarrow 1 = \det(I_n) = \det(E_1) \cdots \det(E_r) \det(A)$

since $\det(E_i) \neq 0$
$\Rightarrow \det(A) = \det(E_1)^{-1} \cdots \det(E_r)^{-1} \neq 0$

"$\Leftarrow$" suppose $\det(A) \neq 0$

$(*) \Rightarrow \det(A) \neq 0$

$\Rightarrow R$ does not contain a row of zeros

$\Rightarrow A$ is invertible .

Theorm

If $A, B \in M_{n \times n}$ then $\det(AB) = \det(A)\det(B)$

Proof

case 1: A is singular (A is not invertible)

recall AB is invertible $\Leftrightarrow$ A and B is invertible

so AB is also singular $\Rightarrow$ $\det(AB) = \det(A)\det(B)$

$$= 0 \cdot \det(B)$$
$$= 0$$

case 2: A is invertible

Then $A = E_1 E_2 \cdots E_k$ for some elementary matrices $E_1 E_2 \cdots E_k$ (by proposition)

$\Rightarrow AB = E_1 E_2 \cdots E_k B$

$$\Rightarrow \det(AB) = \det(E_1 E_2 \cdots E_k B)$$
$$= \det(E_1)\det(E_2)\cdots\det(E_k)\det(B)$$

since $\det(A) = \det(A I_n)$

$$= \det(E_1 \cdots E_k I_n)$$
$$= \det(E_1)\cdots\det(E_k)$$

so $\det(AB) = \det(A)\det(B)$

Proposition 3.57

If $A \in M_{n,n}$ is invertible then $\det(A^{-1}) = \det(A)^{-1}$

Proof

$A^{-1}A = I_n$

so $\det(A^{-1}A) = \det(I_n)$

$\Rightarrow \det(AB) = 1$

$\Rightarrow$ implies $\det(A^{-1})\det(A) = 1$

$$\det(A^{-1}) = \frac{1}{\det(A)} \quad \text{since } \det(A) \neq 0 \quad \blacksquare$$