MA1114 21/3/22

Pf of diagonalisable $\Leftrightarrow$ n ; Change of variables with e. vectors

Proposition

If $T: V \to V$ is linear ($\dim(V) = n$) $\leftarrow$ speacial case of theorem with n distinct eigenvalues.
and $v_1, \dots, v_k$ are eigenvectors with k <u>distinct</u> eigenvalues then $\{v_1, \dots, v_k\}$ is LI

Theorem (better)

T is diagonalisable $\Leftrightarrow$ $n = \sum_{\substack{\lambda \text{ distinct} \\ \text{e. values}}} \overset{\text{geom mult.}}{\text{of } \lambda}$ $(= \dim(V_\lambda))$ exists a basis of e-values.

example $T: V \to V$
$\mathbb{R}^3 \mapsto \mathbb{R}^3$

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

two eigenvalues $\lambda_1 = 1, \lambda_2 = 2$

$$V_1 = \left\langle \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rangle \quad V_2 = \left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

LI

Proof of Theorem

step 1: Let $B_1, B_2, \dots, B_n$ be a basis for the distinct eigenspaces then claim $B_1 \cup B_2 \cup B_3 \cup \dots \cup B_k$ is LI (similar to prop 2.28)

step 2: "$\Leftarrow$" suppose $\sum_{i=1}^{k} |B_i| = n$ then $|B_1 \cup B_2 \cup \dots \cup B_k| = n = \dim V$

$\Rightarrow B_1 \cup B_2 \cup \cdots \cup B_k$ is a basis (by COGOF for basis)

$\Rightarrow T$ is diagonalisable

conversely "$\Rightarrow$" suppose T is diagonalisable

$\Rightarrow \exists$ a basis of eigenvectors

$\left|\bigcup_{\lambda \text{ distinct}} B_i\right| \geq n$ where $\langle B_\lambda \rangle$ V_λ

$\Rightarrow \sum_{\lambda \text{ distinct}} \dim(V_\lambda) \geq n$

Recall algebraic multiplicity of λ_k with $(t-\lambda)^k$ $\geq$
geometric " .... factor of $\chi_T(t)$

$\Rightarrow n \leq \sum_{\lambda \text{ dist}}$ geom mult. of $\lambda \leq \sum_{\lambda \text{ dist}}$ alg mult. $\leq n$

$\Rightarrow n = \sum_{\lambda \text{ distinct}} (\dim(V_\lambda))$

$\boxed{\text{W}}$

<u>Theorem</u>

If T is diagonalisable and $B = \{v_1, \ldots, v_n\}$ is a basis of eigenvectors.
Then $[T]_B = D$ is a diagonal matrix moreover

if $V = \mathbb{C}^n$ and $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$

is the standard basis,

then $P^{-1} [T]_{\mathcal{E}} P = D$

where $P = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$

Example $T: \mathbb{C}^3 \to \mathbb{C}^3$ where $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

$v \mapsto Av$

$([T]_{\mathcal{E}} = A)$

$$P = \begin{pmatrix} V_1 & V_2 & \cdots & V_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

Then check $P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -2 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$