MA1014 7/2/22

# $2^{nd}$ Order O.D.E.s

First order $\frac{dy}{dx} + p(x)y = q(x)$

integration — seperation of variables — integrating factor

Aim: (easy) second order ODEs

(*) $y'' + p(x)y' + q(x)y = f(x)$

Linear $2^{nd}$ order differential equation

<u>Homogenous</u> if $f(x)$ is always 0

[we should always try to solve a homogenous version <u>first</u>]

<u>Theorem 6.21</u> Let $k_0, k_1 \in \mathbb{R}$ $x_0 \in (a,b)$, & $f, p, q : (a,b) \to \mathbb{R}$ continuous

Then the IVP

$$\begin{cases} (*)\ y'' + p(x)y' + q(x)y = f(x) \\ y(x_0) = k_0 \\ y'(x_0) = k_1 \end{cases}$$ initial conditions

has an <u>unique</u> solution

without specifying the initial there will be infinite number of solutions parameterised by <u>two</u> arbitary constants

(which are fixed by imposing the initial conditions).

## Two Basic Types of Examples $k \in \mathbb{R}$

① $y'' - k^2 y = 0$

② $y'' + k^2 y = 0$

both homogeneous.

① has general solution

$$y = c_1 e^{kx} + c_2 e^{-kx} \quad (\text{check } y'' = ky)$$

② $y = c_1 \cos kx + c_2 \sin kx$ (check $y'' = -ky$)

so for the IVP : $\begin{cases} y'' + 4y = 0 \\ y(0) = 1 \end{cases}$

we have the unique solution $y'(0) = 2$ $k=2$

$y(0) = 1 \iff 1 = c_1 \cos 0 + c_2 \sin 0$
i.e. $\underline{c_1 = 1}$.

$y'(0) = 2 \iff 2 = -2c_1 \sin 2\times 0 + 2c_2 \cos 2\times 0$
$2 = 2c_2$ i.e $\underline{c_2 = 1}$

& $y = \cos 2x + \sin 2x$ satisfies ODE & initial value conditions

<u>Theorem 6.24</u> If the ODE is <u>homogenous</u> then the general solutions form a vector space

$y_{(1)}, y_{(2)}$ solutions $\Rightarrow c_1 y_{(1)} + c_2 y_{(2)}$ is a solution

Another special case : assume $p(x)$, $q(x)$ are constant functions

Let $a, b, c \in \mathbb{R}$ and consider

$$(*) \quad a y'' + b y' + c y = f(x)$$

Linear $2^{nd}$ order differential equation with constant coefficients $a, b, c$

Consider the homogenous version

$$(\circledast) \quad a y'' + b y' + c = 0$$

<u>Method</u> step ① solve $(\circledast)$ to get general solution $y_H$ of homogenous ODE

step ② find any particular $y_p$ of $(*)$

step ③ $y = y_H + y_p$ is general solution of $(*)$

step ④ find the arbitary constants if two initial conditions are given

More details:

<u>Step 1A</u>: Consider the auxilary equation $p(\lambda) = 0$
where $p(\lambda)$ is the "characteristic polynomial"
$$a\lambda^2 + b\lambda + c = 0$$
& find its solutions $\lambda = \lambda_1, \lambda_2$

i) $\Delta = b^2 - 4ac > 0$ :

$\exists$ two distinct real solutions $\lambda = \lambda_1, \lambda_2$

& $y_H = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

ii) $\Delta = 0$ :

$\exists$ solution $\lambda = \lambda_1$ (twice) . repeated root

& $y_H = (C_1 + C_2 x) e^{\lambda_1 x}$

iii) $\Delta = b^2 - 4ac < 0$ :

$\exists$ two complex conjugate solutions to the auxilary equation

& $y_H = (C_1 \cos(\omega x) + C_2 \sin(\omega x)) e^{\mu x}$

since $\lambda = \mu \pm \omega i$

$\mu, \omega \in \mathbb{R}$

Step 2: How do you guess a particular solution of

(*) $ay'' + by' + cy = f(x)$

Idea : guess something of the same form as $f(x)$

For now one example

$$y'' + 4y' + 4y = 3x^2 + 1$$

Guess

$y_p = Ax^2 + Bx + C = \frac{3}{4}x^2 - \frac{3}{2}x + \frac{11}{8}$

$y_p' = 2Ax + B$

$y_p'' = 2A$

So $y_p'' + 4y_p' + 4y_p =$

$$\begin{aligned} & \quad\quad\quad\quad\quad\quad\quad 2A \\ & \quad + 8Ax \quad + 4B \\ & + 4Ax^2 \quad + 4Bx \quad + 4C \\ & = 3x^2 \end{aligned}$$

$\Rightarrow y'' + 4y' + 4y = 3x^2 + 1$

has general solution

$$y = (C_1 + C_2 x)e^{-2x} + \tfrac{3}{4}x^2 - \tfrac{3}{2}x + \tfrac{11}{8}$$

IVP $\rightarrow$ find $C_1$, $C_2$