MA1114 9/2/22

Kernals and Images

<u>Definition</u>

U, V vector space

$T: U \to V$ linear

- $\ker(T) = \{u \in U \mid T(u) = 0\}$ is the kernal of T.
- $\text{im}(T) = \{v \in V \mid \text{exists } u \in U \text{ with } T(u) = v\}$ is the image of T.

<u>Examples</u>

Suppose $U = \mathbb{R}^n$, $V = \mathbb{R}^m$

$A \in M_{m,n}(\mathbb{R})$, $T = T_A$ linear

$$T_A : U \to V$$
$$u \mapsto Au$$

<u>claim</u> $\ker(T) = \text{null}(A)$
$\text{im}(T) = \text{col}(A)$

<u>Proof</u>

$$\begin{aligned}\ker(T) &= \{u \in U \mid T(u) = 0\} \\ &= \{u \in U \mid T_A(u) = 0\} \\ &= \{u \in U \mid Au = 0\} \\ &= \text{null}(A)\end{aligned}$$

write $A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ a_1 & a_2 & \cdots & a_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$

e_i, the i-th basis vector (1 in ith position, 0 otherwise)

$$Ae_i = a_i$$

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = T\left(\sum_{i=1}^{n} x_i e_i\right)$$

$$= \sum_{i=1}^{n} x_i T(e_i)$$

$$= \sum_{i=1}^{n} x_i a_i$$

$$= \mathrm{col}(A)$$

$$= \langle a_1, \dots, a_n \rangle$$

$$\Rightarrow \mathrm{im}(T) = \mathrm{col}(A)$$

Suppose $\pi : \mathbb{R}^m \to \mathbb{R}^n$ $(0 \leq n)$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$\ker(\pi) = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ x_{n+1} \\ x_m \end{pmatrix} \middle| \, x_i \in \mathbb{R} \right\}$$

$\mathrm{im}(\pi) = \mathbb{R}^n$ $\left(\text{e.g. } n=3,\ m=5 \quad \pi = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \to \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right)$

any junk

Exercise

Let V be a vectorspace, calculate $\ker(T)$ and $\operatorname{im}(T)$ for.

$$T=0: V \to V, \quad T=1: V \to V$$
$$\quad v \mapsto 0 \qquad\qquad v \mapsto v$$

$$\ker(T) = \{v \in V \mid T(v) = 0\}$$
$$= V$$
$$\operatorname{im}(T) = \{0\}$$

$T = 1$ $\ker(T) = \{0\}$
$\operatorname{im}(T) = V$

Proposition

Let U, V vector space and $T: U \to V$, a linear map. then

a - $\ker(T) \leq U$
b - $\operatorname{im}(T) \leq V$
subspace

Proof

(a) $0 \in \ker(T)$?

$$T(0) = 0 \quad (\text{since } T(0+0) = T(0) + T(0))$$
$$\ \overset{T(0)}{\|}$$

$\lambda \in \mathbb{R}$, $v \in \ker(T)$

$$T(\lambda v) = \lambda T(v)$$
$$= \lambda \cdot 0$$
$$= 0 \quad \Rightarrow \lambda v \in \ker(T)$$

$u, v \in \ker(T)$

$$T(u+v) = T(u) + T(v)$$
$$= 0 + 0$$
$$= 0$$
$$\Rightarrow u+v \in \ker(T)$$

So $\ker(T)$ is a subspace

$\operatorname{im}(T) \leq V$ is exercise

<u>Definition</u>

Let $T: U \to W$ be a linear map (V, W vector space)

$\operatorname{rank}(T) = \dim(\operatorname{im}(T))$ is rank of T
$\operatorname{null}(T) = \dim(\ker(T))$ is nullity of T

<u>Theorem</u>

If $T: V \to W$ is a linear map as above and suppose $\dim(V) < \infty$ ← strictly inf

Then $\operatorname{rank}(T) + \operatorname{null}(T) = \dim(V)$

<u>Example.</u>

consider $1: V \to V$, a vectorspace
$v \mapsto v$

$$\ker(1) = \{0\}$$
$$\operatorname{im}(1) = V$$

$\operatorname{rank}(1_V) = \dim V$
$\operatorname{nullity}(1_V) = 0$ $\quad\quad \dim(V) + 0 = \dim(V)$

Similar for $0 : V \to V$
$v \mapsto 0$

Also for

$T : \mathbb{R}^m \to \mathbb{R}^n$, $m \geq n$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \to \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$Im(\pi) = \mathbb{R}^n$
$$\ker(\pi) = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{n+1} \\ \vdots \\ x_m \end{pmatrix} \middle| \, x_i \in \mathbb{R} \right\}$$

$\dim(V) = m$

$\text{rank}(\pi) = \dim(\mathbb{R}^n) = n$
$$\text{nullity}(\pi) = \dim\left(\left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{n+1} \\ \vdots \\ x_m \end{pmatrix} \middle| \, x_i \in \mathbb{R} \right\} \right)$$
$$= \dim(\langle e_{n+1}, \dots, e_m \rangle)$$
$$= m - n$$

$\dim(V) = m = n + (m-n)$