MA1114 23/3/22

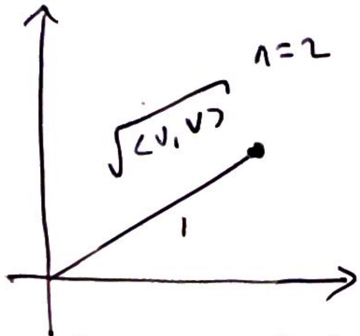
Angles, Inequalities, General Inner Product

Definition $v, w \in \mathbb{R}^n$

$\langle v, w \rangle = v^T w$ is called the standard inner product of v, w

$|v| = \sqrt{\langle v, v \rangle}$ if $|v| = 1$ v is a unit vector

fact $\frac{1}{|v|} v$ is a unit vector.



Example $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$

$|v| = \sqrt{\langle v, v \rangle} = \sqrt{5}$

$w = \frac{v}{|v|} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \Rightarrow |w| = \sqrt{(1/\sqrt{5})^2 + (2/\sqrt{5})^2}$
$= \sqrt{1/5 + 4/5} = \sqrt{1} = 1$

Angles Between Vectors

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

x, y norm 1

$|x| = |y| = 1$

$x_1 = \cos\theta_1$
$x_2 = \sin\theta_1$
$y_1 = \cos\theta_2$
$y_2 = \sin\theta_2$

$\langle x, y \rangle = x_1 y_1 + x_2 y_2$
$= \cos\theta_1 \sin\theta_1 + \cos\theta_2 \sin\theta_2$
$= \cos(\theta_1 - \theta_2)$
$= \cos(\theta)$

In general, we have for $\underline{x}, \underline{y} \in \mathbb{R}^n$

$$\langle \underline{x}, \underline{y} \rangle = |\underline{x}|\,|\underline{y}| \cos\theta$$

where θ is the angle between $\underline{x}$ and $\underline{y}$

In particular, if v and w are orthogonal (i.e angle θ between v and w is $90°$) then $\langle v, w \rangle = 0$

or perpendicular

Example $n=2$

- $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \in \mathbb{R}^2$

$$\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\rangle = 0$$

- $\left\langle \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\rangle = 0$
- $\left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\rangle = 0$

write $v^\perp = \{ w \in \mathbb{R}^n \mid \langle v, w \rangle = 0 \}$
"v perp"

example if $v = \begin{pmatrix}1\\1\\1\end{pmatrix}$ then

$$v^\perp = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \middle| \, x_1 + x_2 + x_3 = 0 \right\}$$

$x_1 = \lambda \quad x_2 = \mu, \quad v^\perp = \left\{ \begin{pmatrix}\lambda\\\mu\\-\lambda-\mu\end{pmatrix} \middle| \, \lambda, \mu \in \mathbb{R} \right\}$

$$= \left\{ \lambda \begin{pmatrix}1\\0\\-1\end{pmatrix} + \mu \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\} = \left\langle \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rangle$$

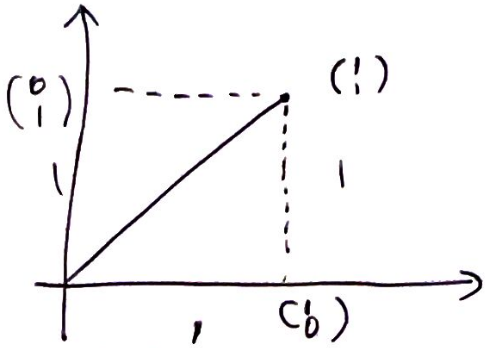
Triangle Inequality

If $v, w \in \mathbb{R}^n$ then $|v+w| \leq |v| + |w|$

Example if $n=2$ and $v = \begin{pmatrix}1\\0\end{pmatrix}$ $w = \begin{pmatrix}0\\1\end{pmatrix}$

$|v| = \sqrt{1^2 + 0^2} = 1$
$|w| = 1$

$|v+w| = \left|\begin{pmatrix}1\\1\end{pmatrix}\right|$
$= \sqrt{1^2+1^2} = \sqrt{2}$



so $|v+w| = \sqrt{2} < 2 = 1+1 = |v| + |w|$

Proof of Triangle Inequality : requires the Cauchy-Schwarts inequality.

If $u, v \in \mathbb{R}^n$ then $\langle u, v\rangle^2 \leq |u|^2 |v|^2$
with equality if u, v are parallel

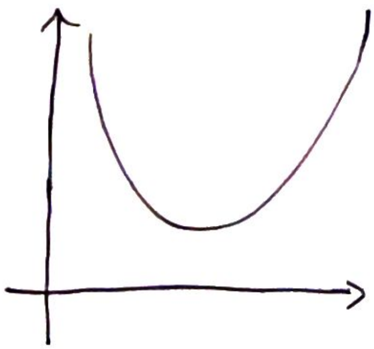
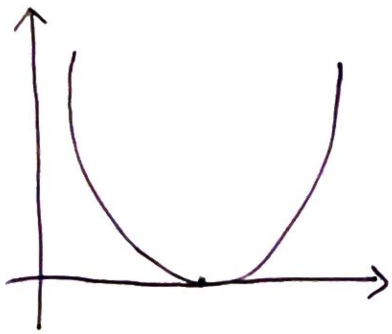
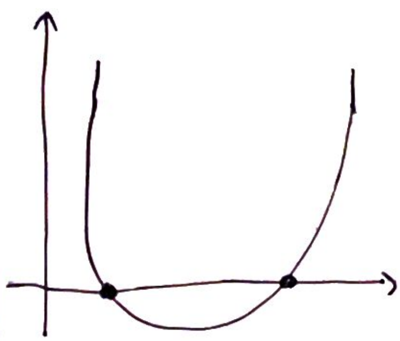
Proof $\forall t \in \mathbb{R}$

$0 \leq \langle u+tv, u+tv\rangle$
$= \langle u,v\rangle + \langle u, tv\rangle + \langle tv, u\rangle + \langle tv, tv\rangle$

$= \langle u,v\rangle + t\langle u,v\rangle + t\langle v,u\rangle + t^2\langle u,v\rangle$

$= t^2\langle v,v\rangle + 2t\langle u,v\rangle + \langle u,u\rangle$

3 types of quadratic $at^2 + bt + c$

type	picture	$0 = at^2 + bt + c$
① no real roots		$D < 0$
② exactly one real roots		$D = 0$
③ two real roots		$D > 0$

we are of type 1 or 2

$$0 \geq D = b^2 - 4ac$$

$$= (2\langle u,v\rangle)^2 - 4(\langle u,u\rangle)(\langle v,v\rangle)$$

$$= 4\langle u,v\rangle^2 - 4|u|^2|v|^2 \quad = p(t)$$

$$\Rightarrow 4|u|^2|v|^2 \geq 4\langle u,v\rangle^2$$

$$\Rightarrow \langle u,v\rangle^2 \leq \langle u,v\rangle^2$$

equality when cxactly one root of $p(t) = 0$ (i.e $D=0$)

this happens when $\langle u+tv, u+tv\rangle = 0$

(some t) $\Rightarrow u+tv = 0$

$\Rightarrow u = -tv$

$\Rightarrow u, v$ are parallel.

<u>Corollary</u> (triangle enequality)

$|v+w| \leq |v| + |w|$ for all $v, w \in \mathbb{R}^n$

<u>Proo/</u> $|v+w|^2 = \langle v+w, v+w\rangle$

$$= \langle v,v\rangle + \langle w,v\rangle + \langle v,w\rangle + \langle w,w\rangle$$
$$= \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$$

<u>cauchy-schwarts enequality</u>

$\Rightarrow \langle v,w\rangle^2 \leq |v|^2 |w|^2$

$\Rightarrow |\langle v,w\rangle| \leq |v||w|$ so $|v+w|^2 \leq |v|^2 + 2|v||w| + |w|^2$

$= (|v|+|w|)^2$

square rooting both sides we get $|v+w| \leq |v|+|w|$

General Inner Products (for vector spaces)

<u>Definition</u> Let V be a vector space. A real inner product is a function

$$\langle -, - \rangle : V \times V \to \mathbb{R}$$

satisfying $\forall \lambda, \mu \in \mathbb{R}$, $u, w \in V$

(i) $\langle \lambda v + \mu u, w \rangle = \lambda \langle v, w \rangle + \mu \langle u, w \rangle$

(ii) $\langle v, w \rangle = \langle w, v \rangle$

(iii) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \Rightarrow v = \underline{0}$

<u>Examples</u>

- Euclidean product with $V = \mathbb{R}^n$

$$\langle v, w \rangle = v^T w$$

- $V = M_{m,n}(\mathbb{R})$, $\langle A, B \rangle = \mathrm{trace}(A^T B)$ where $A, B \in V$
- $V = P_n = \{ p(x) \mid \deg(p(x)) \leq n \}$

$$\langle p(x), q(x) \rangle = \int p(x) q(x) \, dx$$

- V probability space

$$\langle X, Y \rangle = E(XY)$$