MA1114 19/10/21

Linear Equations

Definition 2.1

A linear equation, in n unknowns (variables) $x_1, x_2, x_3, \dots, x_n$ is of the form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n = b$$
for $a_j \in \mathbb{R}$ for all $1 \leq j \leq n$, $b \in \mathbb{R}$

If $b = 0$ we say the equation is homogeneous

$3x - y = 2$

- $x_1 + x_2 - 3x_3 = 0$ is homogenous in 3 unknowns

non-examples

$x^2 + 2y = 0$ X no powers of x

$xy = 0$ X no products of unknowns

$2x - \sqrt{y}$ X not an equation

$\left.\begin{array}{l} \sin(x) - \cos(y) = 1 \\ e^x = 2 \end{array}\right]$ - no functions of variables

Question: What are all solutions to?

$$3x - 4y = 0 \quad \Rightarrow \quad y = \tfrac{3}{4}x$$

$$\left\{ \begin{pmatrix} x \\ \frac{3x}{4} \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} = \left\{ \lambda \begin{pmatrix} 1 \\ 3/4 \end{pmatrix} \middle| \, \lambda \in \mathbb{R} \right\}$$ line

$$= \left\{ \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix} \middle| \, \lambda \in \mathbb{R} \right\}$$

and what about?

$$3x - 4y = 2 \quad \Rightarrow \quad \tfrac{3}{4}x - y = \tfrac{1}{2} \quad \Rightarrow \quad y = \tfrac{3}{4}x - \tfrac{1}{2}$$

$$\left\{ \begin{pmatrix} x \\ \frac{3}{4}x - \frac{1}{2} \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix} \middle| \, \lambda \in \mathbb{R} \right\}$$

Example

$$2x_1 - 3x_2 + 6x_3 = 0$$

$$\text{set } \left. \begin{matrix} x_2 = \lambda \\ x_3 = \mu \end{matrix} \right\} \Rightarrow 2x_1 = \frac{3\lambda - 6\mu}{2}$$

$$\left\{ \begin{pmatrix} \frac{3\lambda - 6\mu}{2} \\ \lambda \\ \mu \end{pmatrix} \middle| \, \lambda, \mu \in \mathbb{R} \right\} = \left\{ \lambda \begin{pmatrix} 3/2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \middle| \, \lambda, \mu \in \mathbb{R} \right\}$$

(a plane through origin) so subspace

Definition 2.6

A system of lines equations in n unknowns is a collection of linear equations

It is homogeneous if every equation in the system is homogeneous $(b=0)$

A solution is an n-tuple $(y_1, \ldots y_n)$ $y \in \mathbb{R}$ where it solves all equations simultaneous.

Examples

Ⓐ ① $3x + 4y = 2$

② $2x - y = 0$ (these represent lines)

① $\{ \binom{2}{1} + \lambda \binom{4}{3} \mid \lambda \in \mathbb{R} \}$

② $\{ \lambda \binom{1}{2} \mid \lambda \in \mathbb{R} \}$

A solution is the intersection of these lines

$y = 2x$

$y = \frac{1}{2} - \frac{3}{4} x$ $x = -\frac{2}{5}$, $y = -\frac{4}{5}$

$(-\frac{2}{5}, -\frac{4}{5})$ this system has a unique solution

Ⓑ ① $3x - 4y = 2$ $\{ \lambda \binom{4}{3} \mid \lambda \in \mathbb{R} \}$

② $6x - 8y = 4$ $\{ \lambda \binom{4}{3} (?) \mid \lambda \in \mathbb{R} \}$

① and ② do not intersect since they are parallel no solution

ⓒ ① $3x - 4y = 2$ $\left\{ \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{R} \right.$

② $6x - 8y = 4$

these are the same line so we have infinitely many solutions

Definition 2.8

A linear system is consistent if it has a solution it is consistent if it has no solutions.

Notation 2.10

let $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + a_{1n} x_n = b_1$

$$a_{21} x_1 + a_{22} x_2 + a_{2n} x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + a_{mn} x_n = b_m$$

be a system of m lines equations

let $A_{ij} = a_{ij}$, so $A \in M_{mn}(\mathbb{R})$

then $Ax = b$ is the linear system

where $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Some easy to solve systems

<u>Definition 2.14, 2.15</u>

$A \in M_{nm}$ is - diagonal if $A_{ij} = 0$ for all $1 \le i, j \le n$ $i \neq j$

- upper triangular if $A_{i,j} = 0$ $1 \le i, j \le n$, $j < i$

- lower triangular if $A_{i,j} = 0$ $1 \le i, j \le n$, $i < j$

<u>Example</u>

A diagonal matrix is both upper and lower triangular

I_n is a diagonal matrix