class test
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A and G Multiplicities; Diagonalisation; Distinct e.values $\to$ LI e.vectors

Lemma $V_\lambda = \{v \in V \mid T(v) = \lambda v\}$, some λ

If $T: V \to V$ is linear and $\lambda \in \mathbb{C}$ is an eigenvalue of T then
$V_\lambda = \ker(\lambda\, id_V - T) = \{v \in V \mid T(v) = \lambda v\}$
(where $(\lambda\, id_V - T): V \to V$
$v \mapsto \lambda v - T(v)$)

Example $A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$ Find all e.values and e. vectors

$T = T_A : \mathbb{R}^2 \to \mathbb{R}^2$
$v \longmapsto Av$

$\ker(\lambda\, id_V - T) = \text{null}(\lambda\, id_2 - A)$
first need λ! $\det(\lambda I_2 - A) = \det \begin{bmatrix} \lambda & -3 \\ -4 & \lambda \end{bmatrix} = 0 \iff \lambda$ is e.value.

$\lambda = \pm 2\sqrt{3}$?
$\chi_A(t) = t^2 - 12$
$\lambda = 2\sqrt{3} \Rightarrow \pm\sqrt{12} = \pm 2\sqrt{3}$
$V_\lambda = \text{null}(2\sqrt{3}\, I_2 - A) = \text{null} \begin{bmatrix} 2\sqrt{3} & -3 \\ -4 & 2\sqrt{3} \end{bmatrix}$

a general $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in V_\lambda$
satisfy $x_1 - \frac{\sqrt{3}}{2} x_2 = 0$
so look like
(with $x_2 = \mu$) $\begin{bmatrix} \frac{\sqrt{3}}{2}\mu \\ \mu \end{bmatrix}$

$\begin{bmatrix} 2\sqrt{3} & -3 \\ -4 & 2\sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{0}$

$\begin{bmatrix} 2\sqrt{3} & -3 \\ -4 & 2\sqrt{3} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} \\ 0 & 0 \end{bmatrix}$

$\Rightarrow V_\lambda = \left\langle \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 1 \end{pmatrix} \right\rangle$

$= \left\langle \begin{bmatrix} \sqrt{3} \\ 2 \end{bmatrix} \right\rangle$

$V_{2\sqrt{3}} = \left\langle \begin{bmatrix} \sqrt{3} \\ 2 \end{bmatrix} \right\rangle$, $V_{-2\sqrt{3}} = \left\langle \begin{bmatrix} \sqrt{3} \\ -2 \end{bmatrix} \right\rangle$

$$\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ -2 \end{bmatrix} = \begin{bmatrix} -6 \\ 4\sqrt{3} \end{bmatrix} = -2\sqrt{3} \begin{bmatrix} \sqrt{3} \\ -2 \end{bmatrix}$$

Diagonalisation

Suppose $T : V \to V$ linear and suppose : $B = \{v_1, \ldots, v_n\}$ which is a basis of eigenvectors.

Then $[T]_B = \left[[T(v_1)]_B \; \cdots \; [T(v_n)]_B \right]$

$$= \left[[\lambda v_1]_B \cdots [\lambda v_n]_B \right]$$

where λ_i is an eigenvalue for v_i

But $[\lambda_1 v_1]_B = \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ similarly $[\lambda_i v_i]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

so $[T]_B = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ is a <u>diagonal</u> matrix

<u>Definition</u>

$T : V \to V$ is diagonalisable if it is a basis of V consisting of eigenvectors of T.

<u>Example</u> The map $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ with $A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$ is diagonalisable since $\left\{ \begin{pmatrix} \sqrt{3} \\ 2 \end{pmatrix}, \begin{pmatrix} \sqrt{3} \\ -2 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$

Example (not all linear maps are diagonaliseable)

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -1 & -5 & 6 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \chi_A(t) = \det(tI_3 - A) = t^2(t+4)$$

$\Rightarrow$ eigenvalues $t_1 = 0;\ t_2 = -4$

$$V_0 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \qquad V_{-4} = \left\langle \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right\rangle$$

check
$$\begin{bmatrix} 1 & -3 & 2 \\ -1 & -5 & 6 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 \\ -1 & -5 & 6 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ -16 \\ -4 \end{bmatrix} = -4 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

So not enough eigenvectors for a basis of V.

Let $T: V \to V$ be linear. Let λ be eigenvalues

- $\dim V_\lambda$ is the geo. multiplicity of λ
- the alg. multiplicity is the largest k with $\chi_T(t)$ divisible by $(\lambda - t)$

In previous examples

λ	alg	geo
0	2	1
-4	1	1

Remark: Neither of these numbers are 0. If λ is an eigenvalue

- There exists a non-zero eigenvectors $\Rightarrow V_\lambda \neq 0$
- $(\lambda - t)$ is always a factor of $\chi_A(t)$ since $\chi_A(t) = 0$