MA2132 11/10/22 L-6

Polynomials ; Minimal Polynomial

k a field ($= \mathbb{R}, \mathbb{C}$)

Definition 1.1 A polynomial over k is a formal expression

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 + a_0 \qquad (a_i \in k)$$

a_n ($\neq 0$) is the leading coefficient
$n = \deg(f)$ degree of f.
$f(t)$ is monic if $a_n = 1$

Examples 1.2 (1) By convention, $\deg(0) = -\infty$
(2) If $a_0 \neq 0$ then $\deg(a_0) = 0$ (scalar polynomial)
(3) $\deg(3t-2) = 1$ (linear polynomial)
(4) $\deg(5t^2+3t-1) = 2$ (quadratic)

Fundamental Property of Degree:

$$\deg(fg) = \deg f + \deg g \qquad \forall f, g \in k[t]$$

all pol-s over k in var. t

(e.g $(x^2+2)(x^3-3) = x^{2+3} + \cdots$)

Theorem 1.3 (Division Algorithm) Let $f, g \in k[t]$ with $g \neq 0$

Then $f = qg + r$ for some $q \in k[t]$ and $r \in k[t]$ (remainder)

with either $r = 0$ or $\deg r < \deg g$

Definition 1.4 A polynomial $p \in k[t]$ is irreducible (over k) if

$p = fg$ implies f or g is a scalar.

Examples 1.5 (1) all linear polynomials are irreducible.
(2) t^2+1 over $\mathbb{R}$ is irreducible
(3) $t^2+1 = (t-i)(t+i)$ not irreducible over $\mathbb{C}$

Theorem 1.6 (Unique Factorisation) Let $f \in k[t]$. Then

$$f = k p_1, p_2, \dots, p_n \quad \text{(unique factorisation.)}$$

where $k \in k$ and $p_1, \dots, p_n$ are irreducible polynomials over k.

One can prove that
all irreducible polynomials over $\mathbb{C}$ are linear (deg = 1)
— // — over $\mathbb{R}$ are of deg 1 or 2.

Theorem 1.7 (FTA: Fundamental Theorem of Algebra)

Let $f \in \mathbb{C}[t]$. Then f is the product of linear polynomials:

$$f(t) = k(t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$$

where $k \in \mathbb{C}$ and λ_i (roots of $f(t)$)

Theorem 1.9 Let $f \in \mathbb{R}[t]$. Then

$$f = k p_1(t) \cdots p_m(t)$$

where $k \in \mathbb{R}$ and $p_i(t) \in \mathbb{R}[t]$ of degree 1 or 2. and irreducible and monic

Everywhere A is a square matrix or linear operator.

Definition 2.1 The characteristic polynomial of A is

$$\Delta(t) = \Delta_A(t) = \det(tI_n - A) \quad (= \det(A - tI_n))$$

note: $\Delta(t)$ is always monic ($\Delta(t) = t^n + \dots$)

Example 2.2 $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ then $\Delta_A(t) = t^2 - 3t + 2 = (t-1)(t-2)$

note: that $\Delta_A(t) = A^2 - 3 \cdot A + 2 \cdot I_2 \overset{or}{=} (A - I_2)(A - 2I_2) = \underline{0}_{4\times 4}$

Theorem 2.3 (Cayley-Hamilton) $\Delta_A(A) = 0 \quad \forall A$

Definition 2.4 The minimal polynomial of A is the monic polynomial

$$m(t) = m_A(t) \text{ of lowest degree s.t. } m(A) = 0$$

Exercise 2.5 $m_A(t) = (t - \lambda)^1$ ($\deg = 1$) $\Leftarrow$ $A = \lambda \cdot I_n = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$(\lambda \in k)$

note: $m(t)$ exists (as $\Delta_A(A) = 0$).

and $\deg m_A(t) \leq \deg \Delta_A(t) = n =$ size of A.