MA1014 25/10/21

Continuity

cts = continuous

- $\lim_{x\to c} (f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
- $\lim_{x\to c} (f(x)g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
- $\lim_{x\to c} \frac{1}{x} = \frac{1}{c} \quad (c \neq 0)$
- $f(x)$ cts at $x = c$ & $g(y)$ cts at $y = f(c)$

  $\Rightarrow g(f(x))$ cts at $x = c$
- Pinching Theorm $f(x), h(x) \to L$ as $x \to c$

  and $f(x) \leq g(x) \leq h(x)$ then $g(x) \to L$ as $x \to c$

<u>Examples</u>

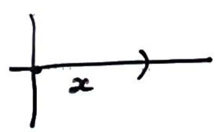
0) $x\cos x \leq \sin x \leq x$

$$\begin{array}{ccc} x\cos x & \sin x & x \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array} \quad \text{as } x \to 0$$

$0 = \sin(0)$ $\quad \sin(x)$ cts at $x = 0$

1) $\sqrt{x}$ cts (from the right) at $x = 0$

$\forall \varepsilon > 0 \ \exists \delta > 0 \quad 0 \leq x < \delta \Rightarrow \sqrt{x} < \varepsilon$



Given $\varepsilon > 0$, choose $\delta = \varepsilon^2$, so that

if $0 \leq x < \delta = \varepsilon^2 \Rightarrow \sqrt{x} < \varepsilon$

2) If $\lim_{x \to c} f_1(x) = L_1 = f_1(c)$, $\lim_{x \to c} f_2(x) = L_2 = f_2(c) \neq 0$

f_1, f_2 cts at $x = c$

Then $\dfrac{f_1(x)}{f_2(x)}$ is cts at $x = c$

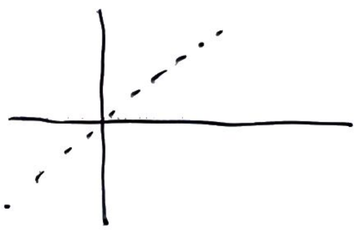
<u>Proof</u> $g(x) = \frac{1}{x}$ cts at $f_2(c) \neq 0$

$\Rightarrow g(f_2(x)) = \dfrac{1}{f_2(x)}$ cts $x = c$

$\Rightarrow f_1(x) \cdot \dfrac{1}{f_2(x)} = \dfrac{f_1(x)}{f_2(x)}$

3) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \quad \text{rational} \\ 0 & \text{if } x \notin \mathbb{Q} \quad \text{irrational} \end{cases}$

is cts at $x = 0$ ✓



$$\underset{\substack{\downarrow \\ 0}}{-|x|} \leq f(x) \leq \underset{\substack{\downarrow \\ 0}}{|x|}$$

by pinching this $0 = f(0)$ as $x \to 0$

4) $\cos(x)$ cts at $x=0$

$$\lim_{x \to 0} \cos(x) = \cos(0) = 1$$

$$\cos(x) = +\sqrt{(1-\sin^2(x))} \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

$\sin(x) \times \sin(x)$

As $x \to 0$ $\quad \sin^2(x) \to 0$

$$1 - \sin^2(x) \to 1$$

$\sqrt{x}$ continuous at $x = c$, $c > 0$

$$\cos(x) = \sqrt{1-\sin^2(x)} \to \sqrt{1} = 1$$
$$= \cos(0)$$

as $x \to 0$

5) ① $f(x) \to f(c)$ as $x \to c$

② $\lim_{x \to c} f(x) = f(c)$

③ $\lim_{h \to 0} f(c+h) = f(c)$

④ $f(c+h) \to f(c)$ as $h \to 0$

$\sin(x)$, $\cos(x)$ are continuous everywhere, not just at $x=0$

$\sin(c+h) = \sin(c)\cos(h) + \cos(c)\sin(h)$ as $h \to 0$

$$\to \sin(c) \cdot 1 + \cos(c) \cdot 0$$
$$= \sin(c)$$

$\lim_{h \to 0} \cos(c+h) = \cos(c)$ $\qquad$ $\cos(x)$ cts at $x = c$

$$\cos(c+h) = \cos c \cosh - \sin c \sinh$$

$$\to \cos c \cdot 1 - \sin c \cdot 0 = \cos(c)$$

6) $f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

is continuous everywhere

if $c \neq 0$, continuous at $x = c$

as $\sin(x)$ and x are and quotients are (by 7)

At $x = 0$, we need to prove $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

From <u>Example (0)</u> above

$$x \cos(x) \leq \sin x \leq x$$

$0 < x < \frac{\pi}{2}$ $\quad \cos(x) \leq \frac{\sin x}{x} \leq 1$

$x \to 0^+$ $\quad \downarrow 1 \quad \downarrow 1 \quad \downarrow 1$

$-\frac{\pi}{2} < x < 0$ $\quad$ * same thing *

$\Rightarrow$ limit is 1 $\qquad\qquad x \cos x \leq \sin x \leq x$

7) $\lim_{x \to 0} \frac{\sin(2x)}{4x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin(2x)}{2x}$

$[A = 2x]$ $\quad = \frac{1}{2} \lim_{A \to 0} \frac{\sin A}{A}$

$= \frac{1}{2} \cdot 1 = \frac{1}{2}$

8) $\lim_{x \to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$

$[x = 2A]$

$1 - \cos(x) = 1 - \cos 2A$
$= 2\sin^2 A$

If $x \neq 0$ $\quad \frac{1-\cos(x)}{x^2} = \frac{2\sin^2 A}{4A^2} = \frac{1}{2}\left(\frac{\sin A}{A}\right)^2$

As $x \to 0$, $A \to 0$, $\left(\frac{\sin A}{A}\right)^2 \to 1^2 = 1$

9) $\frac{1-\cos(x)}{x} = x \cdot \frac{1-\cos(x)}{x}$

as $x \to 0$ $\quad \frac{1-\cos(x)}{x} \to 0 \cdot 1/2$
$= 0$

$\frac{1-\cos(x)}{x} \to 0$ as $x \to 0$