MA1114 9/11/21

Determinants

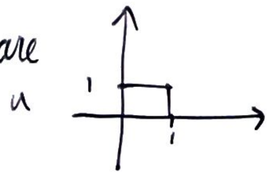
Determinants of 2×2 matrix

<u>Definition</u> $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A is $\det(A) = ad - bc$

Recall $\det(A) \neq 0 \iff A$ is invertible for 2×2 matrix

Geometric interpretation

Consider a unit square



let $A \in M_{2,2}$

$$T_A : \mathbb{R}^2 \to \mathbb{R}^2$$
$$v \mapsto Av$$

consider $T_A(u)$ with vertices

$$T_A\begin{pmatrix}0\\0\end{pmatrix}, T_A\begin{pmatrix}1\\0\end{pmatrix}, T_A\begin{pmatrix}0\\1\end{pmatrix}, T_A\begin{pmatrix}1\\1\end{pmatrix}$$

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

this is $\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}a\\c\end{pmatrix} \begin{pmatrix}b\\d\end{pmatrix}, \begin{pmatrix}a+b\\c+d\end{pmatrix}\right\}$

<u>Example</u> $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\det(A)\ \cos^2\theta + \sin^2\theta = 1$

<u>Proposition</u> $A \in M_{2,2}$ is invertible $\iff \det(A) \neq 0$

<u>Proof</u> Assume $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

"$\Leftarrow$" if $ad - bc \neq 0$

$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and we checked

$A^{-1}A = AA^{-1} = I_n$

"$\Rightarrow$" prove counter positive

$x \Rightarrow y \iff y \Rightarrow x$

start by assuming $\det(A) = 0$ and want to show A is not invertible

$ad - bc = 0 \Rightarrow ad = bc$

case 1 $\quad a = 0 \Rightarrow b = 0$ or $c = 0$

$\Rightarrow$ zero row or column

$\Rightarrow$ A is not invertible

case 2 $\quad a \neq 0 \Rightarrow E_2(2,1,-c/a)A$

$$\begin{bmatrix} 1 & 0 \\ -c/a & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ -c/a & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ -c+c & -\frac{bc}{a} + d \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ 0 & \frac{-bc + ad}{a} \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$ is not invertible.