MA1114 13/10/21

Matrix Multiplication

Examples

$$(a, b, c)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = ax_1 + bx_2 + cx_3$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \quad \text{e.g} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 50 \\ 122 \end{pmatrix}$$

Definition 1.54 "matrices as functions"

If $A \in M_{mn}$, there is a function

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \longmapsto Av$$

Example

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 1 & 9 \end{pmatrix} \quad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

$$T_A(v) = Av = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 1 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + 2y + z \\ 5x + y + 9z \end{pmatrix}$$

Definition 1.53 (matrix multiplication)

Suppose $A \in M_{m,r}$, $B \in M_{r,n}$

$$A \cdot B \in M_{mn} \text{ with } (AB)_{ij} = \sum_{k=1}^{r} A_{ik} B_{kj} \qquad \begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix}$$

Equivalently,

$$B = \begin{pmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ b_1 & b_2 & b_3 & \cdots & b_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

$r \times 1$ vectors

$$AB = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ Ab_1 & Ab_2 & \cdots & Ab_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \in M_{mn}$$

<u>Example</u>

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Exercise

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 32 \end{pmatrix}$$

<u>Proposition</u>

$A \in M_{mn}$ Then

$$I_m A = A = A I_n$$

Proof first $I_m A$, $I_n A$ at $m \times n$ matrix

$$(AI_n)_{ij} = \sum_{k=1}^{n} A_{ik} (I_n)_{kj} = A_{ij} \text{ since } (I_n)_{kj} = \underset{1 \; k=j}{\overset{0 \; k \neq j}{}}$$ similar for AI_m

Proposition 1.59 (Associativity of matrix multiplication)

$$A \in M_{mr} \quad B \in M_{rs} \quad C \in M_{sn} \rightarrow (AB)C = A(BC)$$

Example

① $\left((1,2)\begin{pmatrix} 3 \\ 4 \end{pmatrix} \right)(4,5)$ ② $(1,2)\left(\begin{pmatrix} 3 \\ 4 \end{pmatrix}(5,6) \right)$

1×1 matrix 2×2 matrix

① $= (11)(5,6)$ "

$= (55, 66)$ 1×2 matrix

② $= (1,2)\begin{pmatrix} 15 & 18 \\ 20 & 24 \end{pmatrix}$ "

Proof sizes $(m \times n)$

$$((AB)C)_{ij} = \sum_{k=1}^{s} (AB)_{ik} C_{kj} = \sum_{k=1}^{s}\left(\sum_{l=1}^{r} A_{il} B_{lk}\right) C_{kj}$$

$$= \sum_{l=1}^{r} \sum_{k=1}^{s} A_{il} B_{lk} C_{kj}$$

$$= \sum_{l=1}^{r} A_{il} (BC)_{lj} = (A(BC))_{ij}$$

Definition 1.61

If $A \in M_{nn}$ square matrix $k \in N = \{1, 2, 3\}$

$$A^k = \underbrace{A \cdot A \cdot A \cdot A \ldots A}_{k \text{ times}}$$

$$A^0 = I_n$$