MA1114 22/3/22

Fundamental Theorem of Algebra; e.values & e-vectors (finish)

Proposition

If $p(t) \in \mathbb{R}[t]$ then for $z \in \mathbb{C}$, $p(z) = 0 \Rightarrow p(\bar{z}) = 0$
(so roots come in conjugate pairs)

eg. $t^2 + t + 1$

Corollary

If $A \in M_n(\mathbb{R})$ then $\chi_A(t) \in \mathbb{R}[t]$ $(\chi_A(t) = \det(tI - A))$
If λ is an eigenvalue then so is λ

Proof

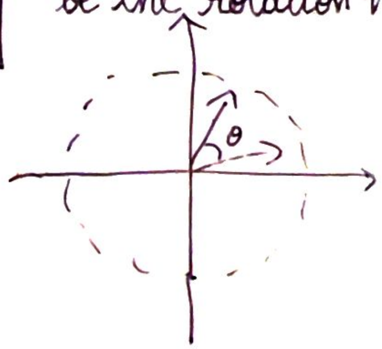
$\chi_A(t) \in \mathbb{R}[t]$ so $\chi_A(\lambda) = 0 \Rightarrow \chi_A(\bar{\lambda})$

so $\bar{\lambda}$ is an eigenvalue

Examples

Let $A = R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be the rotation matrix

> Calculate $\chi_A(t)$
> Solve $\chi_A(t) = 0$



$$\chi_A(t) = \det(It - A)$$

$$= \det \begin{bmatrix} t-\cos\theta & \sin\theta \\ -\sin\theta & t-\cos\theta \end{bmatrix}$$

$$= (t-\cos\theta)^2 + (\sin\theta)^2$$

$$= t^2 - 2t\cos\theta + \cos^2\theta + \sin^2\theta$$

$$= t^2 - 2t\cos\theta + 1 \implies t = \cos\theta \pm i\sin\theta$$

<u>Theorem</u>

If V is complex vector space and $T: V \to V$ is linear, then T is diagonalisable $\iff$ geometric multiplicity of λ = alg multiplicity of λ for all eigenvalues λ.

<u>Proof</u>

By 9.36 $\displaystyle\sum_{\substack{\lambda \text{ distinct} \\ e \text{ values}}} \overbrace{\dim(V_\lambda)}^{g_\lambda} = n$

But geometric multiplicity of $\lambda = g_\lambda$

$\leqslant$ algebraic ------ $\lambda = a_\lambda$

$\iff n = \displaystyle\sum_{\lambda \text{ distinct}} g_\lambda \leqslant \sum_{\lambda \text{ distinct}} a_\lambda = n \leftarrow$ by fundamental theorem of algebra

$$\begin{array}{ccc} g_{\lambda_1} & g_{\lambda_2} & \Sigma = n \\ \wedge & \wedge\!\!| & \\ a_{\lambda_1} & a_{\lambda_2} & \Sigma = n \end{array}$$

$\iff g_\lambda = a_\lambda \ \forall \lambda$