MA1114 14/12/22

Surjective and Injective Functions

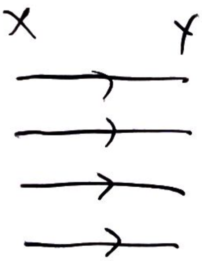
Definition 8.21

Let $f: X \to Y$ be a function between sets X and Y

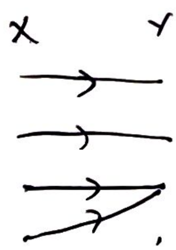
f is injective if $a, b \in X$ are subtracted then $a = b$

f is surjective if $\operatorname{im}(f) = \{f(a) \mid a \in X\} = Y$

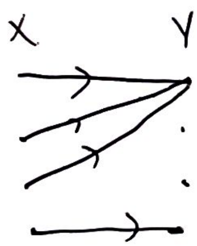
Picture



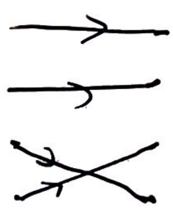
injective not surjective



not injective
surjective

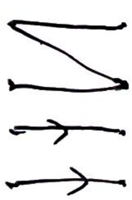


not both



both

Caution



not a function $f(a)$ not well defined

Example

$n \in \mathbb{N}$

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad , \; 0 \neq r \in \mathbb{R}$$
$$v \longmapsto rv$$

injective?

suppose $T(v) = T(w)$ for $v, w \in \mathbb{R}^n$

$\Rightarrow rv = rw$
$\Rightarrow r(\frac{1}{r})v = r(\frac{1}{r})w$
$\Rightarrow (r(\frac{1}{r}))v = (r(\frac{1}{r}))w$
$\Rightarrow \quad v = w$
$\Rightarrow$ T is injective

surjective?

let $w \in \mathbb{R}^n$. Can we find $v \in \mathbb{R}^n$ with $T(v) = w$

guess $v = w$ then $T(v) = T(w) = rw \neq w$

aha! so $v = \frac{1}{r}w$ then $T(v) = T(\frac{1}{r}w)$
$= \frac{1}{r}rw = w$
$\Rightarrow$ T is surjective

Suppose $0 : V \longrightarrow$ is injective then $0(v) = 0(w)$
$\Rightarrow v = w$
$0 = 0 \Rightarrow v = w$

only works if $|V| = 1 \quad V = \{0\}$

Actually 0 is injective $V = \{0\}$

Claim 0 is surjective $W = \{0\}$

suppose $W \neq \{0\} \Rightarrow$ can choose $0 \neq w \in W$

Question $w = 0(v)$ some $v \in U$

no! $0v = 0$, not surjective

"$\Rightarrow$" holds

"$\Leftarrow$" excercise

If V is a vectorspace then

let $V \to V$ is surjective
$v \mapsto v$ injective

Proof $id(v) = id(u)$ some $v, w \in V$

$\Rightarrow v = w$
$\Rightarrow id$ is injective
if $v \in V$ then $id(v) = v$
$\Rightarrow im(id) = v$
$\Rightarrow id$ is surjective

Definition A function that is both, is bijective

e.g. $\pi : \mathbb{R}^m \to \mathbb{R}^n$ $(m \geq n)$

$$① \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \to ② \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$ ① surjective ② not injective

e.g. $T : P \to P$
$p(x) \mapsto xp(x)$

P is a set of all polys in x with $\mathbb{C}$ coeff.