MA1061 29/11/21

Joint Probability Distribution & Independant Random Variables

Multiple Random Variables

- In real life, we are often interested in several random variables that are related to each other
- e.g. a random variable representing heights and one representing weights Are they related? How?
- joint distribution ⇒ full probability table (not just the distribution of individuals r.v's)

Joint Probability Distributions of Random Variables

Let $X_1, \dots, X_r$ be random variables and $X = (X_1, \dots, X_r)$.

Definition (Joint Probability Mass Function)

The function $P_X : \mathbb{R}^r \mapsto [0,1]$

$$P_X(x_1, \dots, x_r) = P\{\omega : X_1(\omega) = x_1, \dots, X_r(\omega) = x_r\}.$$

is called the joint probability mass function (or joint probability dist.) of the random variables $(X_1, \dots, X_r)$.

For any $i = 1, \dots, r$, the marginal distribution of X_i is

$$P_{X_i}(x_i) = \sum_{x_1} \cdots \sum_{x_{i-1}} \sum_{x_{i+1}} \cdots \sum_{x_r} P_X(x_1, \dots, x_r)$$
$$= \sum_{x_1} \cdots \sum_{x_{i-1}} \sum_{x_{i+1}} \cdots \sum_{x_r} P\{\omega : X_1(\omega) = x_1 \cdots X_r(\omega) = x_r\}$$

Marginal Distribution

- In this situation x_i is also called marginal variable. The distribution of it. is obtained by marginalising - that is, focusing on the sums in the margin - over the variables being discarding, and the discarded variables are said to have been marginalized out.

- marginal distribution of random variable x_i = individual probability distribution of x_i
  it gives the probability of all possibles values of x_i without reference to the values of x_i without reference to the values of the other variables

- only x_i is retained, the probabilities of all the other variables are summed up.

- called "marginal" because they can be found by summing values in a table along rows or column, and writing the sum in, the margins of the table.

Two Random Variables

Specifically, Let X and Y be random variables take values

$$X = \{x_1, \dots, x_n\}, \; Y = \{y_1, \dots, y_n\}$$

The joint probability distribution of X and Y is the function $P(X_i, Y_j)$ defined by

$$P(x_i, y_j) = P\{w : X(w) = x_i, \; Y(w) = y_j\},$$

such that

$$P(x_i, y_j) \geq 0 \text{ and } \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) = 1$$

the joint probability distrubution table of X and Y is in the form of the table below.

$X \backslash Y$	y_1	$\cdots$	y_m	row sum
x_1	$P(x_1, y_1)$	$\cdots$	$P(x_1, y_m)$	$\sum_{j=1}^{m} P(x_1, y_j)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_n	$P(x_n, y_1)$	$\cdots$	$P(x_n, y_m)$	$\sum_{j=1}^{m} P(x_n, y_j)$
column sum	$\sum_{i=1}^{n} P(x_i, y_1)$	$\cdots$	$\sum_{i=1}^{n} P(x_i, y_m)$	1

Column sum and row sum are called marginal distrubutions, and are, in fact, the individual distribution of X and Y, respectively.

Joint Distribution Function

Definition

The function: $F_X(x_1, \dots, x_r) = P(\omega : X_1(\omega) \le x_1, \dots, X_r(\omega) \le x_r\}$,

where $x_i \in \mathbb{R}$ is called the <u>joint cumulative distribution function</u> of the random variables $X_1, \dots, X_r$.

Independant Random Variables

D: The random variables $X_1, \dots, X_r$ are said to be <u>mutually independant</u> if

$$P(X_1 = x_1, \dots, X_r = x_r) = P(X_1 = x_1) \cdots P(X_r = x_r),$$

for all $x_i \in \mathbb{R}$

$X_1, \ldots, X_r$ are pairwise independant when:

$$F_X(X_1, \ldots, X_r) = \prod_{i=1}^{r} F_{X_i}(X_i)$$

where $F_{X_i}(x_i)$ is the cumulative distribution function of X_i

Marginals

$X_2 \backslash X_1$	1	2	3	4	5	6	row sum
1	0	1/6	0	1/6	0	1/6	1/2
0	1/6	0	1/6	0	1/6	0	1/2
column sum	1/6	1/6	1/6	1/6	1/6	1/6	1

> Row sums: 1^{st} $P(X_2) = 1$
> 2^{nd} $P(X_2) = 0$

> Column sums: 1 $P(X_1) = 1$

Invert $X_2 \Rightarrow P(X_1 = x_1, X_2 = x_2)$ change