MA1114 19/1/22

# Cartesian Product of vector spaces and their dimensions.

## Definition

Let $V$ and $W$ be vector spaces. The cartesian product of $V$ and $W$ is the set $V \times W = \{(v, w) \mid v \in V, w \in W\}$
of all formal pairs with VA and SM given by:

VA: $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ for all $v_1, v_2 \in V$ $w_1, w_2 \in W$
SM: $\lambda(v_1, w_1) = (\lambda v_1, \lambda w_1)$ for all $v \in V$, $w \in W$ and $\lambda \in \mathbb{R}$

## Exercise

check $(V \times W, +, \cdot)$ is a vector space.

## example

$V = \mathbb{R}^n$, $W = \mathbb{R}^m$ $n, m \in \mathbb{N}$

$V \times W = \mathbb{R}^{n+m}$ (note if $n = m = 1$)
$V \times W = \{(x, y), x, y \in \mathbb{R}\}$ is the $xy$-plane.

## Theorm

Let $V, W$ be a finite dimensional vector spaces then $\dim(V \times W) = \dim(V) + \dim(W)$

## Proof

Let $\{v_1, v_2, v_3, \ldots, v_n\}$ and $\{w_1, w_2, w_3, \ldots, w_n\}$ be bases for $V$ and $W$ respectively.

$\left[ \text{note } \mathbb{R} = \langle 1 \rangle \text{ and } \mathbb{R}^2 = \langle \binom{0}{1}, \binom{1}{0} \rangle \right]$ it suffices to show $\mathcal{B} = \{(v_1, 0), (v_2, 0), \ldots, (v_n, 0), (0, w_1), (0, w_2), \ldots, (0, w_m)\}$ is a basis for $V \times W$.

first check $\mathcal{B}$ spans $V \times W$

suppose $(v, w) \in V \times W$

so now $v = \sum_{i=1}^{n} \lambda_i v_i$, some $\lambda_i \in \mathbb{R}$

$w = \sum_{j=1}^{m} \mu_j w_j$, some $\mu_i \in \mathbb{R}$

$$(v, w) = \left(\sum_{i=1}^{n} \lambda_i v_i, \sum_{j=1}^{m} \mu_j w_j\right)$$

$$= \sum_{i=1}^{n} \lambda_i (v_i, 0) + \sum_{j=1}^{m} \mu_j (0, w_j)$$

we have written $(v, w)$ as a linear combination of elements of $\mathcal{B}$ so $\mathcal{B}$ spans

suppose $\sum_{i=1}^{n} \lambda_i (v_i, 0) + \sum_{j=1}^{m} \mu_j (0, w_j)$

then $\underline{0} = \left(\sum_{i=1}^{n} \lambda_i v_i, \sum_{j=1}^{m} \mu_j w_j\right)$

$\Rightarrow \sum_{i=1}^{n} \lambda_i v_i = 0$ & $\sum_{j=1}^{m} \mu_j w_j = 0$

$\Rightarrow \lambda_i = 0 = \lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_m$

$\mu_j = 0 = \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_m$