Vector Fields, Directional Derivatives and Gradient

Steven Cheung

12 October 2022

Vector Fields:

Until now we have considered scalar functions of $1, 2, 3$ variables, and vector functions of 1 variable.

We now consider vector functions of 2 or 3 variables

Let $\underline{F} : \mathbb{R}^n \to \mathbb{R}^n$, $n = 2, 3$ with

$$\underline{F}(x, y) = (F_1(x, y), F_2(x, y)) \text{ for } n = 2$$
$$\underline{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) \text{ for } n = 3$$

with F_1, F_2, F_3 scalar functions. Using the notation $\underline{x} = (x, y)$ or $\underline{x} = (x, y, z)$ then the above can be shortened to:

$$\underline{F}(\underline{x}) = (F_1(\underline{x}), F_2(\underline{x})) \text{ or } \underline{F}(\underline{x}) = (F_1(\underline{x}), F_2(\underline{x}), F_3(\underline{x}))$$

Vector functions of many functions are customarily called **Vector Fields**

Directional Derivative:

Problem: Find the slope of the surface $z = f(x, y)$ at the point (x, y) in the direction $\underline{a} = (a_1, a_2)$, where $|\underline{a}| = 1$.

Solution:

$$D_{\underline{a}} f(x, y) = \underline{a} \cdot \underset{\overrightarrow{a}}{\nabla f(x, y)} = a_1 \tfrac{\partial f}{\partial x} + a_2 \tfrac{\partial f}{\partial y}$$

We call $D_{\underline{a}} f(x, y)$ the **directional derivative of f in the direction $\underline{a}$**.

Gradient of a Scalar Function:

Let f be a scalar differentiable function of 3 variables. We define:

$$grad(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

We can also use the following notion for the gradient. Denote (formally) by:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

Then We can write:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

Direction of Maximum Increase:

Problem: Find the direction of the maximum increase and the slope of the surface $z = f(x, y)$ at the point (x, y).
Solution: We seek $\underline{a} = (a_1, a_2)$ with $|\underline{a}| = 1$ such that

$$D_{\underline{a}} f(x, y) = \underline{a} \cdot \nabla f(x, y)$$

is maximum!

Recall

$$\underline{a} \cdot \nabla f(x, y) = |\underline{a}| \cdot |\nabla f(x, y)| cos\gamma$$

Hence the **direction of maximum increase** is when $\gamma = 0$, i.e. the direction of the gradient!

The **slope** at the direction of maximu increase is $|\nabla f(x, y)|$, i.e. the length of the gradient!

Example - The Fly:

Suppose the temperature distribution in a room in given by the function

$$f(x, y, z) = sin(x)cosh(y - 1)z, \quad 0 \leq x \leq 5 \quad 0 \leq y \leq 3 \quad 0 \leq z \leq 3$$

A fly is at the point $(\pi, 1, 2)$ but it feels cold. Which direction it should fly towards in order to get warmer most rapidly?
Solution: The direction of maximum increase is given by

$$\nabla f(x, y, z) = (cos(x)cosh(y - 1)z, sin(x)sinh(y - 1)z, sin(x)cosh(y - 1))$$

Hence, the fly should fly towards the direction given by the vector

$$\nabla f(\pi, 1, 2) = (cos(\pi)cosh(1 - 1)2, sin(\pi)sinh(1 - 1)2, sin(\pi)cosh(1 - 1)) = (-2, 0, 0)$$

Properties of Gradient:

Let $f, g : \mathbb{R}^2 \to \mathbb{R}$, $n = 2, 3$ scalar functions. Then

- $\nabla(f + g) = \nabla f + \nabla g$
- $\nabla(\lambda f) = \lambda \nabla f$ for $\lambda \in \mathbb{R}$
- $\nabla(fg) = g\nabla f + f\nabla g$
- $\nabla f = 0$ **if and only if** f is constant

Stationary Points of a Function $f(x, y, z)$

These are the points where: $grad(f) = 0$
Hence, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$,$\frac{\partial f}{\partial z} = 0$