MA1061 10/11/21

Discrete Uniform, Bernoulli, Binomial Distribution.

Different situations may have the same random variable

- very different experiments can lead naturally to essentially the same random variable.

<u>Example</u>

- In human populations, the probability of male and female are approximately equal:

$$P(\text{male}) = P(\text{female}) = \tfrac{1}{2}$$

- consider now a family with 3 children;
- let X count the number of male children;
- what is the distribution of X?

X	0	1	2	3
$P_X(x)$	$1/8$	$3/8$	$3/8$	$1/8$

Discrete Uniform Distribution - equally likely

<u>Definition</u> (Discrete uniform distribution)

Let X take each of the values $\{x_1, \dots, x_n\}$ with the same probability $1/n$. Then we say that X is uniformly distributed on the set $\{x_1, \dots, x_n\}$. The distribution of X is called discrete uniform distribution.

e.g

X	1	2	3	4	5	6
$P_X(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

Bernoulli Random Variable / Bernoulli Distribution

<u>Definition</u> (Bernoulli Random Variable)

A random variable x that takes the value 0 and 1 with probabilities p, and $q = 1-p$ resp. is called a Bernoulli Random Variable.

- any experiment with two possible outcomes can be modelled by a Bernoulli variable. eg the sex of the child. the outcome of a toss of a coin.

- How is a Bernoulli random variable related to an indicator?

Bernoulli variables as indicators

> let $A \subset \Omega$. in some probability space. such that $P(A) = p$.
> consider $X = 1_A$
> then $P(X=1) = p$. and $P(X=0) = 1-p$

<u>Definition</u> (Sequence of trials)

A sequence of trials for experiments

i) the outcomes of the trials are independant;
(ii) outcomes are of two types (success, failure); and
(iii) the probabilities for the two types of outcomes remain the same for all trials,

is called a sequence of <u>Bernoulli trials</u>

> A sequence of trials is essentially a sequence of 'independant' random variables $X_1, X_2, \ , X_n$.
> Each random variable must have the same distribution
> In each case of Bernoulli trials:

$$P(X_1=1) = \dots = P(X_n=1) = p,$$
$$P(X_1=0) = \dots = P(X_n=0) = q = 1-p,$$

when we have n Bernoulli trials the sample space is

$$\Omega = \{\omega : \omega = (a_1, \dots, a_n).\ a_i = 0 \text{ or } 1\}$$

by independance of trials, the probability of an elementary event $\omega \in \Omega$ in which there are precisely k successes (i.e. 1) and $n-k$ failures (i.e 0) is just

$$P(\omega) = P((a_1, \dots, a_n)) = p^k q^{n-k}, \quad k = \sum_{i=1}^{n} a_i.$$

Example

Four students buy lottery tickets. The probability of each ticket to win a prize is 10%
> let k denote the number of students that win
> What is the probability of k student win?

$X : \Omega \to \mathbb{R}$ by $\quad X(\omega) = X((a_1, \dots, a_n)) = \sum_{i=1}^{n} a_i$ = number of successes

- thus if in the elementary event ω there are exactly k successes, then $X(\omega) = k$
- we want to find the (probability) distribution of X Range? $\{0, 1, \dots, n\}$
- let $k = 0, \dots, n$. What is $P_X(k)$?

by definition $P_X(k) = P(\{\omega : X(\omega) = k\})$. let $B_k = \{\omega : X(\omega) = k\}$

if $\omega \in \Omega$ then $P(\omega) = P((a_1, \ldots, a_n)) = p^k q^{n-k}$.

therefore $P(\{X = k\}) = P_X(k) = {}^nC_r \, p^k q^{n-k}$ $k = 0, \ldots, n$

$$P_X(k) = C^n_r \, p^k q^{n-k}, \; k = 0, \ldots, n$$

the above distribution is also known as the <u>binomial distribution</u>.

we write $X \sim Bi(n, p)$

<u>Definition</u> (Binomial random variable)

A random variable X such that

$$P(X = k) = C^n_k \, p^k (1-p)^{n-k},$$

for some $p \in [0,1]$, and $k = 0, 1, \ldots, n$, is called a <u>Binomial random variable</u> with parameters n, p.

The distribution of X defined as above is called the <u>Binomial Distribution</u> with parameters n, p.

If $X \sim Bi(n, p)$ then X counts the number of successes in n independant experiments each with probability of success p.

It's distribution $P_X(k)$ gives the prob of having k successes in n independant trials

<u>Theorm</u> Let $X_1, X_2, \ldots, X_n$ be n 'independant' Bernoulli random variables of parameter p. Then

$$X = X_1 + X_2 + \ldots + X_n$$

has a binomial distribution. In fact

$$X \sim Bi(n, p)$$

Example

Suppose a family has three children and suppose that it is equally likely to have a boy or a girl.
If X is the number of male children What is the probability distribution of X?
Using this distribution, calculate the probability that the family has 0, 1, 2 or 3 boys

X	b 1	g 0
P	1/2	1/2

$P(1) = P(0) = \frac{1}{2}$

$p = \frac{1}{2} \quad q = 1 - \frac{1}{2} = 1 - p$
$= \frac{1}{2}$

$n = 3 \qquad k = 0, 1, 2, 3$

$P(X=0) = C^3_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^3 = \frac{1}{8}$
$P(X=1) = C^3_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2 = \frac{3}{8}$
$P(X=2) = C^3_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = \frac{3}{8}$
$P(X=3) = C^3_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^0 = \frac{1}{8}$

X	0	1	2	3
P	1/8	3/8	3/8	1/8

Example

Suppose 10% of items produced in a factory are defective Choose 3 items at random. What is the probability that

(i) all are defective
(ii) 2 are defective
(iii) none are defective.

$n = 3$
$k = \{0, 1, 2, 3\}$

X	1	0
P	0.1	0.9

use binomial distribution $P(X=2) = C^3_2 (0.1)^2 0.9 = 3 \times 0.01 \times 0.9$
$= 0.027$