MA1061 9/11/21

Probability Distribution & Cumulative Distribution Function

In an experiment described by random variables, we may maybe mainly interested in:

> probabilities with which the random variable takes particular values

we may not care about the actual probability of each elementary event:

> we are more interested in the distribution of probability over the possible values of the random variable

e.g consider an experiment where we roll three dica and X is the random variable giving the sum of the three dice;

we only care about the probability with which each possible sum will arise.

Probability Distribution

The set of numbers $\{P_x(x_1), \ldots P_x(X_n)\}$, is called the probability distribution of the random variable x. Note that this is also called the probability mass function (pmf) of the random variable x if x is the discrete random variable.

obviously $P_x(X_i) \geq 0$ $\sum_{i=0}^{n} P_x(X_i) = 1$

Example

Three fair coins are tossed together. If X is a random variable denoting the number of heads obtained, find the probability distribution of X

$\Omega = \{HHH, HHT, HTH, \ldots, TTT\}$ 8 elements.
$|\Omega| = 8$

let A_i be the event that i heads are obtained.

$P(A_0) = 1/8 \quad P(A_1) = 3/8 \quad P(A_2) = 3/8 \quad P(A_3) = 1/8$

the random variable X takes values $\{0, 1, 2, 3\}$

$$P_X(i) = P(X = i) = P(\{\omega \in \Omega : X(\omega) = i\}) = P(A_i)$$

i	0	1	2	3
$P_X(i)$	$1/8$	$3/8$	$3/8$	$1/8$

Example (sum of two dice)

We roll two independant unbiased dice and record there sum. Find the distribution of the random variable X giving the sum.

X takes in values $\{2, 3, 4, \ldots, 12\}$

x	2	3	4	5	6	7	8	9	10	11	12
$\{\omega \mid X(\omega) = x\}$	1	2	3	4	5	6	5	4	3	2	1
$P_X(x)$	$1/36$	$1/18$	$1/12$	$1/9$	$5/36$	$1/6$	$5/36$	$1/9$	$1/12$	$1/18$	$1/36$

Cumulative Distribution Function

Let $X \in \mathbb{R}$. The function F_X defined by

$$F_X(x) = P(X \leq x) = P\{w : X(w) \leq x\},$$

is called the (cumulative) distrubution function of the random variable X

$$F_X(x) = \sum_{\{i : x_i \leq x\}} P_X(x_i)$$

if we suppose that $x_1 < x_2 < \cdots < x_n$, then

$$P_X(X_1) = F_X(X_1)$$

$$P_X(X_i) = F_X(X_i) - F_X(X_{i-1}), \text{ for } i = 2, \ldots, n$$

Hence, cumulative distribdution functions can be determined given probability distribution, and vice versa

when 3 coins are tossed, we already saw the probability distribution for the number of heads to be:

i	0	1	2	3
$P_X(i)$	$1/8$	$3/8$	$3/8$	$1/8$

the cumulative distribution function is then

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1/8 & 0 \leq x < 1 \\ 1/2 & 1 \leq x < 2 \\ 7/8 & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

e.g if $x=-1$, $F_X(-1) = P(X \leq -1) = 0$

If $x=0$, $F_X(0) = P(X \leq 0) = P(X<0) + P(X=0) = 0 + \frac{1}{8} = \frac{1}{8}$

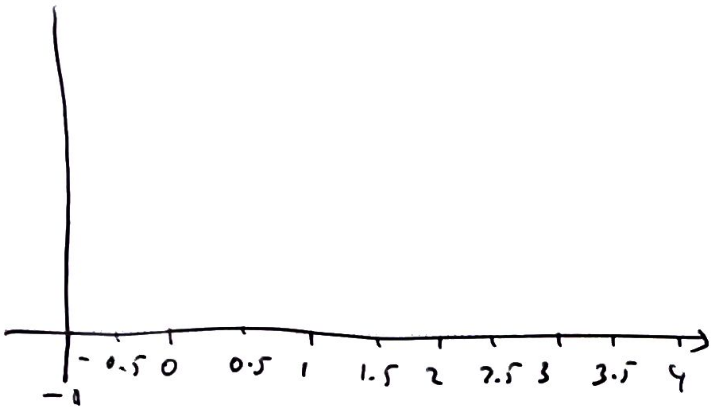
If $x=0.5$, $F_X(0.5) = P(X \leq 0.5) = P(X<0) + P(X=0) + P(0<X<0.5)$

$= 0 + \frac{1}{8} + 0$

$= \frac{1}{8}$

If $x=1$, $F_X(1) = P(X \leq 1) = P(X<0) + P(X=0) + P(0<X<1) + P(X=1)$

$= 0 + \frac{1}{8} + 0 + \frac{3}{8} = \frac{1}{2}$



It follows immediately from the definition that the distribution function $F = F_X$ has the following properties:

1. $F_X(-\infty) = 0$;
2. $F_X(+\infty) = 1$;
3. $F_X(x)$ is right-continuous, that is, $F_X(x) = \lim_{h \to 0^+} F_X(x+h)$ at every point x
4. F_X is not decreasing.
5. F_X is piecewise constant (for discrete r.v's)