MA1114 25/1/22

Extending/Reducing to a Basis

Proposition 6.13 ("check one get one free")

Let V be a vector space of dimension n. If $S \subset V$ has a size n. then S is linearly independant $\Leftrightarrow$ span$(S) = V$

Examples

Are the following basis for $\mathbb{R}^n$?

1) $\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ when $n=3$

2) $\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \right\}$ when $n=4$

3) $S_n = \{ e_1+e_2, e_2+e_3, e_3+e_4, \dots, e_{n-1}+e_n, e_n+e_1 \}$ where $e_i = \begin{pmatrix}0\\ \vdots\\0\\ \vdots\\0\\ \vdots \\0\end{pmatrix}$ ith $\rightarrow$

1) $\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$

$$\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix} \to \begin{bmatrix}1&0&1\\0&1&-1\\0&1&1\end{bmatrix} \to \begin{bmatrix}1&0&1\\0&1&-1\\0&0&2\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

L.I ✓ $\lambda_1 = \lambda_2 = \lambda_3 = 0$ $\Rightarrow$ spans

$S = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\} \rightarrow$ linearly independant

$\begin{pmatrix}1\\0\\1\end{pmatrix} \notin \text{span}(S) \Rightarrow S \cup \left\{ \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ is linearly independant

therefore basis.

2) $\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \right\}$

$S = \left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix} \right\} \Rightarrow$ linearly independant

in fact $\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\}$ linearly independant

$\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$ $\therefore$ <u>not</u> a basis

3) S_n is a basis when n is odd and is not a basis when n is even.
(basis when n is odd.)

Proof / sketch first $n-1$ vectors are linearly independant by $\pm$ theorem.

n even, notice $= (e_1 + e_2) - (e_2 + e_3) + (e_3 + e_4) - \cdots + (e_{n-1} + e_n)$
$- (e_n + e_1)$

$= \underline{0}$

so n even S_n is not a basis

n odd <u>exercise</u>

<u>Theorem Extend to a basis</u>

(i) Let V finite dimensional vector space and $S \subset V$ finite.
If S is linearly independant there exists a basis $\hat{S}$ of V containing S.

(ii) If $\text{span}(S) = V$ then there is a basis $\hat{S}$ contained in S

<u>Proof</u> (i) Suppose S is not a basis $|S| < \dim(V)$. Choose $v_1 \notin \text{span}(S)$
(since $\text{span}(S) \subsetneq V$

$\Rightarrow S \cup \{v_1\}$ is linearly independant

if $\dim(\text{span}(S \cup \{v_1\})) = |S| + 1$

if $|S| + 1 < \dim V$ then $\exists\, v_2 \notin \text{span}(S \cup \{v_1 \cup v_2\})$

and $\Rightarrow \dim(\text{span}(S \cup \{v_1, v_2\}))$

eventually we would obtain $v_1, \ldots, v_k$ with

$k = \dim(V) - |S|$

and $\hat{S} = S \cup \{v_1, \ldots, v_k\}$ is linearly independant

$\Rightarrow \hat{S}$ is a basis containing S

(ii) is $|S| > \dim(V)$ and too many vectors

then by plus/minus theorm

$\exists v_1 \in S$ with $\operatorname{span}(S \setminus \{v_1\}) = V$

if $|S| - 1 > \dim(V)$ $\exists v_2 \in S$ with $\operatorname{span}(S \setminus \{v_1, v_2\}) = V$

$\vdots$

eventually we would obtain $v_1, \dots, v_k \in S$ with where

$k = |S| - \dim(V)$ $\qquad$ $\operatorname{span}(S \setminus \{v_1 \dots v_k\}) = V$

set $\hat{S} = S \setminus \{v_1, \dots, v_l\}$

$$|\hat{S}| = |S| - l = |S| - (|S| - \dim(V)) = \dim(V)$$

$\Rightarrow \hat{S}$ is a basis by GOGOF

Extend to a basis for the following:

$$\left\{ \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} \right\} \quad \cup \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

and then $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ -2 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$