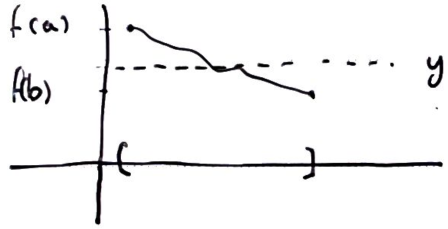
MA1014 26/10/21

Intermediate Value Theorem

Applications of continuity

① Bolzano's Theorem & the Intermediate Value Theorem (IVT)

Intermediate Value Theorem If $f:[a,b]\to\mathbb{R}$ is continuous & $k\in\mathbb{R}$ strictly between $f(a)$ and $f(b)$. Then $\exists c\in(a,b)$ such that $f(c)=k$



* can cross more than once. *

Bolzano's Theorem (special case for $k=0$)

If $f:[a,b]\to\mathbb{R}$ continuous
& $f(a)$ & $f(b)$ have different signs
(i.e $f(a)f(b)<0$)
Then $\exists c\in(a,b)$ such that $f(c)=0$

$$\text{IVT} \iff \text{Bolzano}$$

Let $g(x)=f(x)-k$ $\quad f,g:[a,b]\to\mathbb{R}$

f cts $\iff$ g cts

$f(a)<k<f(b) \iff g(a)<0<g(b)$

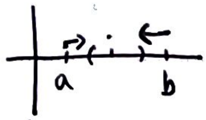
or $f(b)<k<f(a) \iff g(b)<0<g(a)$

$\exists c\in(a,b)\ f(c)=k \iff \exists c\in(a,b)\ \ g(c)=0$

Before we prove Bolzano's Theorem...

lemma-small theorem

lemma $f:[a,b] \to \mathbb{R}$



a) f continuous from above at a & $f(a)>0$ then $\exists\, \delta>0$ such that f is positive on all of $[a, a+\delta)$

b) f continuous from below at b & $f(b)>0$ then $\exists\, \delta>0$ such that $f(x)>0 \;\; \forall x \in (b-\delta, b]$

c) f continuous at $x \in (a,b)$ & $f(x)>0$ then $\exists\, \delta>0$ such that f is strictly positive for all values in $(x-\delta, x+\delta)$

a') b') c') $\Big\}$ same as (a), (b), (c) but with $f(x)<0$

Proof (of (c) for example)

Take $\varepsilon = f(x) > 0$ continuous $\Rightarrow \exists\, \delta>0$

such that if $x' \in (x-\delta, x+\delta)$

we have $\underbrace{f(x)-\varepsilon} < f(x') < f(x)+\varepsilon$

$0 < f(x')$

Proof of Bolzano's Theorem

Suppose $f : [a, b] \to \mathbb{R}$ continuous & $f(a) < 0$, $f(b) > 0$
($f(a) > 0$, $f(b) < 0$ similar)

Let $c = L.U.B\ (S)$ where

$S = \{ x \in (a, b) \mid f \text{ is negative on all of } [a, x) \}$

(aim: prove $f(c) = 0$!)

① $f(c) < 0$ impossible, as the lemma would say f is negative on all of some interval $(c - \delta, c + \delta)$
so negative on $[a, c - \delta/2] \cup [c - \delta/2, c + \delta/2]$
as c is least upper bound.

$= [a, c + \delta/2]$

$c + \delta/2 \in S$!

② $f(c) > 0$ impossible; as the lemma would f is positive on $(c - \delta, c + \delta)$
But f is negative on $[a, c - \delta/2]$

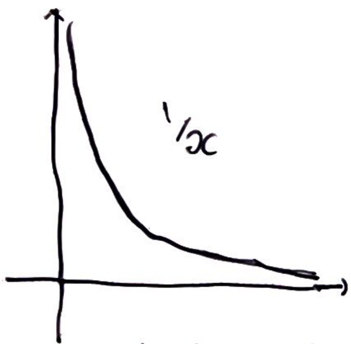
$\boxed{\text{so } \exists c \text{ with } f(c) = 0}$

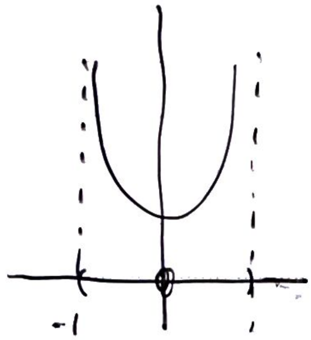
Bounded Functions

function f whose image is a bounded subset of $\mathbb{R}$
$\exists B$ such that $|f(x)| < B$

I interval, $f: I \to \mathbb{R}$ continuous

$(0, \infty)$ $(-1, 1)$ $[-1, 1]$





not bounded
on open interval

$f(x) = 1/(1-x^2)$
unbounded on open interval

lemma $f: [a,b] \to \mathbb{R}$

① continuous from above at $x = a$
② continuous from below at $x = b$
③ continuous at all $x = c \in (a, b)$

Then f is a bounded function

Prove (take $\varepsilon = 1$ in definition of continuity)

① $\exists \delta > 0 : x \in [a, a+\delta) \Rightarrow f(x) \in (f(a) - 1, f(a) + 1)$
② $\exists \delta > 0 : x \in (b - \delta, b] \Rightarrow f(x) \in (f(b) - 1, f(b) + 1)$
③ $\exists \delta > 0 : x \in (c - \delta, c + \delta) \Rightarrow f(x) \in (f(c) - 1, f(c) + 1)$

f bounded on $\overset{1}{[a, a+\delta/2]}$, $\overset{2}{[b+\delta/2, b]}$, $\overset{3}{[c-\delta/2, c+\delta/2]}$

Let $S = \{x \in [a,b] \mid f \text{ is bounded on } [a, x]\}$

1 says $S \neq \emptyset$, $a+\delta/2 \in S$

3 says $\underline{c = LUB(S)} < b$ is $\underline{\text{impossible}}$

it would mean bounded on

$$\underbrace{[a, c-\delta/2]}_{\text{by defn of } S} \cup \underbrace{[c-\delta/2, c+\delta/2]}_{3 \Rightarrow \text{bounded}}$$

$c+\delta/2 \in S \quad = [a, c+\delta/2]$

So $c=b$ & f is bounded on

$$[a, b-\delta/2] \cup \underset{2}{[b-\delta/2, b]} = \underline{[a,b]}$$