MA1014 8/11/21

Application to inverse function

$f(x) = x^2$ $x = g(y) = {}^{+}\sqrt{y}$

$\mathbb{R} \to [0, \infty)$ $[0, \infty) \to [0, \infty)$

$f'(x) = 2x$ $g'(y) = \frac{1}{2}\frac{1}{\sqrt{y}} = \frac{1}{2x}$

<u>Theorem</u> If f is differentiable and 1-1 on some interval then so is the inverse

<u>Theorem</u> If f, f^{-1} are both differentiable then $(f^{-1})'(y) = \frac{1}{f'(x)}$

(assuming $f'(x) \neq 0$)

<u>Proof</u> $y = f(x)$ $f^{-1}(y) = x$

$$f^{-1}(f(x)) = x$$

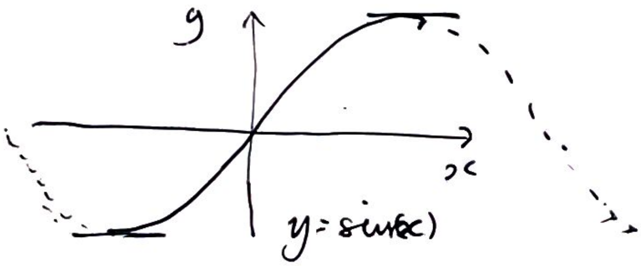
chain $\Rightarrow$ rule $(f^{-1})'(f(x)) \cdot f'(x) = 1$

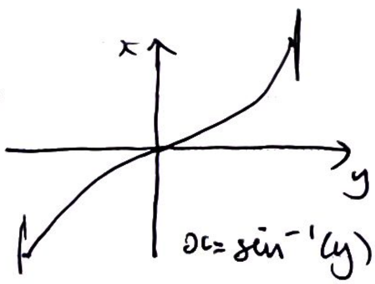
$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

List of Examples

	$f(x)$	$f'(x)$	$f^{-1}(y)$	$(f^{-1})'(y)$
$\frac{1}{\cos(x)}$	$\sin(x)$	$\cos(x)$	$\sin^{-1}(y)$	$\frac{1}{\sqrt{1-y^2}}$
$\sqrt{\sin^2(x)+1}$	$\cos(x)$	$-\sin(x)$	$\cos^{-1}(y)$	$-\frac{1}{\sqrt{1-y^2}}$
	$\tan(x)$	$\frac{1}{\cos^2(x)}$	$\tan^{-1}(y)$	$\frac{1}{1+y^2}$
$\rightarrow$	e^x	e^x	$\ln(y)$	$1/y$
$\frac{e^x-e^{-x}}{2}$	$\sinh(x)$	$\cosh(x)$	$\sinh^{-1}(y)$	$\frac{1}{\sqrt{1+y^2}}$
$\frac{e^x+e^{-x}}{2}$	$\cosh(x)$	$\sinh(x)$	$\cosh^{-1}(y)$	
$\frac{e^x-e^{-x}}{e^x+e^{-x}}$	$\tanh(x)$			

$\sin : (-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1]$ one-to-one & onto

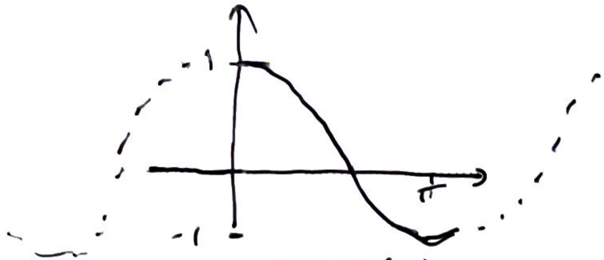




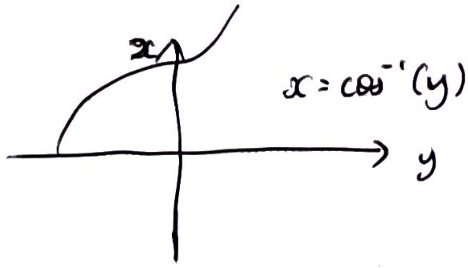
$$\frac{1}{y'} = \frac{1}{\cos(x)} = \frac{1}{\sqrt{1-\sin^2(x)}} = \frac{1}{\sqrt{1-y^2}}$$

$\frac{dy}{dx}$ ✓ $\quad \frac{dx}{dy} = \frac{1}{(dx/dy)}$ $\qquad \frac{1}{-\sin(x)} = -\frac{1}{\sqrt{1-y^2}}$

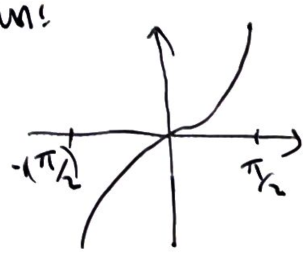
$\cos : [0, \pi) \to [-1, 1]$ $\quad$ $\cos^{-1} : [-1, 1] \to [0, \pi)$



$y = \cos(x)$



tan:

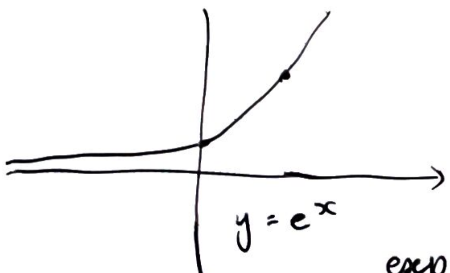


$y = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$ $\quad$ $\dfrac{d}{dx} \tan^{-1}(y) = \dfrac{1}{1+y^2}$

$\dfrac{dy}{dx} = \dfrac{1}{\cos^2 x} = \sec^2 x$

$= 1 + \tan^2(x) = \dfrac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$

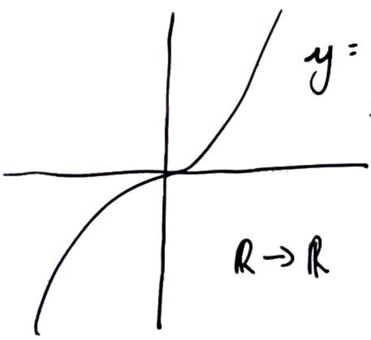
$= 1 + y^2$



$\dfrac{dy}{dx} = e^x = y$ $\quad$ $\dfrac{dx}{dy} = \dfrac{1}{y}$

$\dfrac{d}{dy} (\ln(y)) = \dfrac{1}{y}$

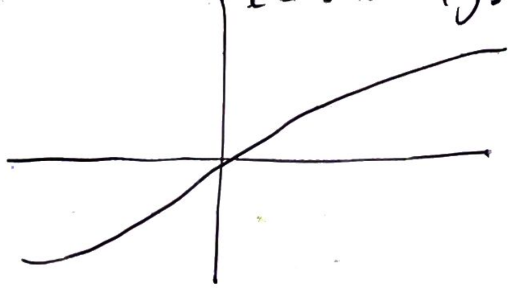
$\exp : \mathbb{R} \to (0, \infty)$ $\quad$ $\ln : (0, \infty) \to \mathbb{R}$



$y = \sinh(x) = \dfrac{e^x - e^{-x}}{2}$

$\mathbb{R} \to \mathbb{R}$

$x = \sinh^{-1}(y)$



$\cos^2 + \sin^2 = 1$

$\cosh^2 - \sinh^2 = 1$

$\dfrac{dx}{dy} = \dfrac{1}{\cosh x} = \dfrac{1}{\sqrt{1 + \sinh^2 x}} = \dfrac{1}{\sqrt{1+y^2}}$

When theory / exact answers fail = find good numerical approximations.

Bolzano's theorm

Problem: Solve $x^6 - x - 2 = 0$

numerically $f(x)$ cts

$[0, 2]$ $f(2) = 60$ $f(0) = -2$

different signs

problem has a solution between 0 & 2

$[0, 1]$ ① $f(1)$ — +ve $\Rightarrow$ between 0 & 1

$[1, 2]$ $f(1)$ — -ve $\Rightarrow$ between 1 & 2

Repeat: find $\exists$ a solution between

$[c_n, d_n]$ $f(c_n), f(d_n)$
different signs

$\Rightarrow f\left(\frac{c_n + d_n}{2}\right)$ tells us it solution

is in $\left[c_n, \frac{c_n + d_n}{2}\right]$ or $\left[\frac{c_n + d_n}{2}, d_n\right]$

$\rightsquigarrow$ converge on a solution