MA1014 20/10/21

Proving Composition of Functions is continuous

Theorem:

If $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x = a$ ①
and $g: \mathbb{R} \to \mathbb{R}$ is continuous at $x = f(a)$ ②
then $gf: \mathbb{R} \to \mathbb{R}$ is continuous at $x = a$ ③

Proof

① $\forall \varepsilon' > 0 \quad \exists \delta > 0 : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon'$

② $\forall \varepsilon > 0 \quad \exists \varepsilon' > 0 : |y - f(a)| < \varepsilon' \Rightarrow |g(y) - g(f(a))| < \varepsilon$

Put ① & ② together:

Given any $\varepsilon > 0$, $\exists \underset{②}{\varepsilon'} > 0$, $\exists \underset{①}{\delta} > 0$ such that

$$|x - a| < \delta \overset{①}{\Rightarrow} |f(x) - f(a)| < \varepsilon'$$

$$\underset{②}{\Rightarrow} |g(f(x)) - g(f(a))| < \varepsilon$$