MA1114 10/11/21

Determinants by Cofactor Expansions & Properties of Determinants.

Proposition

$A \in M_{2,2}$ is invertible $\iff \det(A) \neq 0$

Formulae

① $\det\begin{pmatrix} ra & b \\ rc & d \end{pmatrix} = r \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

② $\det\begin{pmatrix} a+x & b \\ c+z & d \end{pmatrix} = \det\begin{pmatrix} x & b \\ z & d \end{pmatrix} + \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

③ $\det\begin{pmatrix} b & a \\ d & c \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

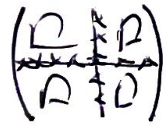
④ $\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

Spoiler $A \in M_{n,n}$ can detect invertibility of using $\det(A)$

Definition "determinant by cofactor expansion"

let $A \in M_{n,n}$ have (i,j) entry a_{ij}

$$\det(A) = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det(\widehat{A_{ij}})$$



where $\widehat{A_{ij}}$ is a matrix obtained A given by deleting row i and column j

$\det(A \in M_{n,n}) \Rightarrow \det(A \in M_{n-1\ n-1}) \Rightarrow \det(A \in M_{n-2\ n-2})$

$\Downarrow$

$\det(A \in M_{22}) \Leftarrow \ldots$

Example

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

$\det(A)$

$a_{11}(-1)^2 \det(\hat{A}_{11}) + a_{12}(-1)^3 \det(\hat{A}_{12}) + a_{13}(-1)^4 \det(\hat{A}_{13})$

where:

	det
$\hat{A}_{11} = \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix}$	-2
$\hat{A}_{12} = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$	0
$\hat{A}_{13} = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$	-4

so $1 \cdot 1 \cdot (-2) + 5 \cdot (-1) \cdot 0 + 0 \cdot 1 \cdot -4 = -2$

Exercise

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

calculate $\det(A)$

$\det(A) = 3 \cdot 1 \cdot -4 + 1 \cdot -1 \cdot -11 = -12 + 11$
$= -1$

Proposition 3.41 (General Cofactor Expansion)

$A \in M_{nn}$ $\quad n \geq 2$

a_{ij} : (i, j) entry of A

$$\det(A) - \sum_{j=1}^{n} (a_{ij})(-1)^{i+j} \det(\widehat{A_{ij}})$$

(row i expansion)

and

$$\det(A) = \sum_{i=1}^{n} (a_{ij})(-1)^{i+j} \det(\widehat{A_{ij}})$$

(column j expansion)

Proof: omitted - too hard for current state

For $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$ we calculate $\det(A)$ using a column 3 expansion

$$\begin{aligned} \det(A) &= a_{13}(-1)^4 \det(\widehat{A_{13}}) \\ &+ a_{23}(-1)^5 \det(\widehat{A_{23}}) \\ &+ a_{33}(-1)^6 \det(\widehat{A_{33}}) \\ &= -1 \cdot -1 \cdot -2 \\ &= -2 \end{aligned}$$

Note: can calculate determinants quickly using a row or column with lots of 0's $\det(A) = 0$ if A contains a row or column of 0's

in fact $\det(A) \neq 0 \Rightarrow A$ is invertible

Properties of the Determinant

<u>Proposition</u> if $A \in M_{n,n}$ then $\det(A) = \det(A^T)$

<u>Proof</u>

$\det(A)$ using row 1 expansion. This is the same as calculating $\det(A^T)$ using column expansion.