MA1014 24/11/22

Basic Properties, "antiderivatives" & Fundamental Theorem of Calculus.

① Basic Properties: $f, g : [a,b] \to \mathbb{R}$, two continuous functions with $f(x) \leq g(x)$ $\forall x \in [a,b]$

then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

why? For any $[x_{i-1}, x_i]$ of any partition P

we have m_i^f , m_i^g , M_i^f , M_i^g

$= f(s_i)$ $= g(t_i)$ $= f(u_i)$ $= g(v_i)$

$f(s_i) \leq f(t_i) \leq g(t_i)$ $\qquad$ $f(u_i) \leq f(v_i) \leq g(v_i)$

$m_i^f \leq m_i^g$ $\qquad$ $M_i^f \leq M_i^g$

$L_f(P) \leq L_g(P)$ $\qquad$ $U_f(P) \leq U_g(P)$

$$\Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

$f(x) < g(x) \Rightarrow \int_a^b f(x)\,dx < \int_a^b g(x)\,dx$

f is strictly positive $\Rightarrow \int_a^b f(x)\,dx$ is too

$f < 1 \Rightarrow \int_a^b f(x)\,dx < \int_a^b 1\,dx = b - a$

Easy Examples

$f(x) = 1 \quad \forall x$

$\int_a^b f(x)\, dx = b-a$

why? for any $[x_{i-1}, x_i]$ of any P.

$m_i = 1 \quad M_i = 1$

$L_f(P) = \sum m_i \Delta x_i = \sum 1 \cdot \Delta x_i = b-a$

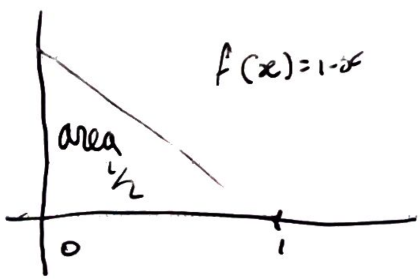
$U_g(P) = \sum M_i \Delta x_i = \sum 1 \cdot \Delta x_i = b-a$

$= \int_a^b f(x)\, dx = b-a$

How about $f(x) = 1-x$, $f: [0,1] \to \mathbb{R}$

$\int_0^1 1-x \, dx = \frac{1}{2}$ why?

consider P_n partition of $[0,1]$ into equal parts



$L_f(P_n) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} \frac{n-i}{n} \cdot \frac{1}{n}$

$U_f(P_n) = \sum M_i \Delta x_i = \sum_{i=1}^{n} \frac{n-i+1}{n} \cdot \frac{1}{n}$

$L_f(P_n) = \frac{1}{n^2} \sum_{i=1}^{n} n-i = \frac{1}{n^2} \sum_{j=0}^{n-1} j = \frac{1}{n^2} \frac{1}{2}(n-1)n = \frac{1}{2}(1-\frac{1}{n}) \to \frac{1}{2}$

$U_f(P_n) \to \frac{1}{2}$ similarly $\int_0^1 (1-x)\, dx = \frac{1}{2}$

"Antiderivative" and Fundamental Theory of Calculus.

Theorm $f : [a, b] \to \mathbb{R}$ integrable

$$\Rightarrow F : [a,b] \longrightarrow \mathbb{R}, \quad F(x) = \int_0^x f(t)\,dt.$$

is uniformaly continuous.

Proof [see p65 of notes 2nd paragraph of section 5.2]

Theorm [Derivative of Integrals]

If $f : [a,b] \to \mathbb{R}$ continuous
then $F [a,b] \to \mathbb{R}$, $F(x) = \int_a^x f(t)\,dt$
is differentiable & $F'(x) = f(x) \quad \forall x$

Proof Given any $\varepsilon > 0$, $\exists \delta > 0 \mid |x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \quad (x>0)$

$$\& \quad \left| \frac{\int_c^x (f(t) - f(c))\,dt}{x-c} \right| \leq \frac{\int_c^x (f(t) - f(c))\,dt}{x-c} < \frac{\int_c^x \varepsilon\,dt}{x-c} = \varepsilon$$

$$\frac{\int_c^x (f(t) - f(c))\,dt}{x-c} = \frac{\int_c^x f(t)\,dx}{x-c} - f(c)$$

$a < c < x$

$$= \underbrace{\frac{F(x) - F(c)}{x-c}}_{\text{difference quotient}} - f(c) \to 0 \quad \text{as } x \to c$$

i.e $\dfrac{F(x) - F(c)}{x-c} \to f(c)$ as $x \to c$

$$F'(c) = f(c)$$

Theorm [Integral of a derivative]

Suppose $f : [a,b] \to \mathbb{R}$ is integrable

Suppose "$F : [a,b] \to \mathbb{R}$ is an antiderivative of f"

① F differentiable ② $F''(x) = f(x) \quad \forall x$

Then $\int_a^b f(x)\,dx = F(b) - F(a)$

Prove By the MVT for F on any $[x_{i-1}, x_i]$

$$m_i \leq \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(\xi_i) \leq M_i$$
$$= f(\xi_i)$$
for some $\xi_i \in [x_{i-1}, x_i]$

$$m_i \Delta x_i \leq F(x_i) - F(x_{i-1}) \leq M_i \Delta x_i$$

$$L_f(P) < F(x_i) - F(x_{i-1}) \leq U_f(P)$$
(other terms canceled)

LUB GLB

$$\int_a^b f(x)\,dx \leq F(x_i) - F(x_{i-1}) \leq \int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx = F(x_i) - F(x_{i-1})$$

$\int_0^1 (1-x)\,dx$ $\qquad$ $\underset{F}{(x - \tfrac{1}{2}x^2)'} = \underset{f}{1-x}$

$$= (1 - \tfrac{1}{2}1^2) - (0 - \tfrac{1}{2}0^2)$$

$\int \sin = -\cos$