MA1114 6/12/21

Subspaces ctd...

$(V, +, \cdot) \leftarrow$ vector space

VA0-4, SM0-4

Subspaces

$W \in V$ (W is a vector space with $+, \cdot$)

$\Rightarrow$
- $\underline{0} \in W$
- $u + w \in W$
- $\lambda \cdot w \in W$ for any $w \in W$, $\lambda \in \mathbb{R}$

Examples

$n \in \mathbb{N}$ $\leftarrow$ diagonal

- $\left\{ \begin{pmatrix} \bullet & & 0 \\ & \ddots & \\ 0 & & \bullet \end{pmatrix} \right\} \leq M_{n,n}(\mathbb{R})$

  symmetric matrices
- $\{ A \in M_{n,n}(\mathbb{R}) \mid A^T = A \} \leq M_{n,n}(\mathbb{R})$ (exercise)

  anti-symmetric matrices
- $\{ A \in M_{n,n}(\mathbb{R}) \mid A^T = -A \} \leq M_{n,n}(\mathbb{R})$
- $\left\{ \begin{pmatrix} \bullet & \cdots & \bullet \\ & \ddots & \vdots \\ 0 & & \bullet \end{pmatrix} \right\} \leq M_{n,n}$

  $\leftarrow$ upper triangle matrices

Proposition

Suppose V is a vector space and $U, W \leq V$

(i) $U \cap W \leq V$

(ii) $U \cup W \leq V \Leftrightarrow$ either $U \subset W$ or $W \subset U$

Proof (exercises)

Definition 4.17

Suppose $U, W \leq V$ a vector space

$$U + W = \{u + w \mid u \in U, w \in W\}$$

Proposition 4.18

Let $U, W \leq V$, a vector space

Then $U, W \leq U + W \leq V$

$U + W$

Examples

① $U = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\} \leq \mathbb{R}$

$W = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \mid y \in \mathbb{R} \right\} \leq \mathbb{R}$

$U + W = \underbrace{\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{R} \right\}}_{\mathbb{R}^2}$

$U \cap W = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \leq \mathbb{R}^2$

Proof (of proposition 4.18)

- $\underline{0} = \underline{0} + \underline{0} \in U + W$ (since $\underline{0} \in U, W$)
- $u_1, u_2 \in U$, $w_1, w_2 \in W$

  $(u_1 + w_1) + (u_2 + w_2) = u_1 + u_2 + w_1 + w_2$ by VA3-4 $\in U + W$

  (since $u_1 + u_2 \in U$, $w_1, w_2 \in W$)
- $u + w \in U + W$, $\lambda \in \mathbb{R}$

  $\lambda(u + w) = \lambda u + \lambda w \in U + W$ by SM (since $\lambda u \in U$, $\lambda w \in W$)
- $U = \{u \mid u \in U\}$

  $= \{u + \underline{0} \mid u \in U\} \leq U + W$ (also subspace) similarly for W

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