MA1114 2/11/21

Non-Invertible Matrix

Proposition 3.10

If a square matrix has a zero column then is is not invertible

Proof

$A \in M_{n,n}$

$$A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow & & \uparrow \\ a_1 & a_2 & \cdots & a_k & \cdots & a_n \\ \downarrow & \downarrow & & \downarrow & & \downarrow \end{bmatrix} \text{ where } a_k = 0 \text{ , some } k$$

Suppose A is invertible then there exists $B \in M_{nn}$ such that $BA = I_n$

$$I_n = BA = B \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ a_1 \cdots & a_k \cdots & a_n \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ Ba_1 \cdots & Ba_k \cdots & Ba_n \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & & 0 & & \uparrow \\ & & 0 & & \\ a_1 & \cdots & \vdots & \cdots & a_n \\ \downarrow & & 0 & & \downarrow \end{bmatrix} \neq I_n$$

Contradiction, So A is not invertible.

Corollary

If square matrix has a row of zeros it's not invertible

Proof

Suppose $A \in M_{n,n}$ is invertible and has a row of zeros Then A^T has a column of zeros

$\Rightarrow A^T$ is not invertible

But A is invertible $\Rightarrow \exists B$ such that

$$AB = I_n = B^T A^T = (AB)^T = I_n^T = I_n$$

similarly $BA = I_n \Rightarrow A^T B^T = I_n$
so A^T is invertible (see proof from yesterday's lecture)

a contradiction, so A is not invertible

Functions
Definition 3.12

Let x, y be sets and $f: x \to y$ be a function ~~An inverse~~. An inverse of f (if it exists) is a function $f^{-1}: y \to x$
such that $f(f^{-1}(x)) = x$ for all $x \in X$
$f(f^{-1}(y)) = y$ for all ~~x~~ $y \in Y$

Equivalently, $f^{-1} \circ f = \mathrm{Id}_x$
$f \circ f^{-1} = \mathrm{Id}_y$

here "$\circ$" means composition of functions and Id_x & Id_y are the identity functions

$\mathrm{Id}_x : x \to x$
$X \to X$

$\mathrm{Id}_y : y \to y$
$y \to y$

Proposition 3.13

If $A \in M_{n,n}$ is invertible matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is given by
$v \to Av$ for $v \in \mathbb{R}$ for $v \in \mathbb{R}^n$

then $T_A^{-1} = T_{A^{-1}}$

Proof

Have to check $T_{A^{-1}} \circ T_A = Id_{\mathbb{R}^n} = T_A \circ T_{A^{-1}}$

$$(T_{A^{-1}} \circ T_A)(v) = T_{A^{-1}}(T_A(v)) = T_{A^{-1}}(Av)$$
$$= A^{-1}(Av)$$
$$= (A^{-1}A)v$$
$$= I_n v$$
$$= v$$

similarly $(T_A \circ T_{A^{-1}})(v) = v$

Proposition 3.14

Suppose $A \in M_{n,n}$ and $b \in \mathbb{R}^n$

If A is invertible, the linear system $Ax = b$ has a unique solution

Proof 3.14

$b = Ax$

A^{-1} = A is invertible $\Rightarrow$ there exists $A^{-1} \in M_{nn}$ such that $A^{-1}A = I_n$

$$A^{-1}b = A^{-1}(Ax)$$
$$= A^{-1}Ax$$
$$= I_n \cdot x \quad = x$$

Exercise

Show that $20x - y = 23$

$x + 20y = -59$

has a unique solution by showing $\begin{pmatrix} 20 & -1 \\ 1 & 20 \end{pmatrix}$ is invertible find the solution by calculating $A^{-1}\begin{pmatrix} 23 \\ -59 \end{pmatrix}$.

$$A = \begin{pmatrix} 20 & -1 \\ 1 & 20 \end{pmatrix} \qquad A^{-1} = \frac{1}{400+1}\begin{pmatrix} 20 & -1 \\ 1 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{20}{401} & -\frac{1}{401} \\ \frac{1}{401} & \frac{20}{401} \end{pmatrix}$$

$$A^{-1}\begin{pmatrix} 23 \\ -59 \end{pmatrix} = \begin{pmatrix} \frac{20}{401} & -\frac{1}{401} \\ \frac{1}{401} & \frac{20}{401} \end{pmatrix} = \begin{pmatrix} \frac{460}{401} & -\frac{59}{401} \\ -\frac{23}{401} & -\frac{1180}{401} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$