MA1114 2/3/22

Examples for calculating eigenspaces of a matrix by row reduction; Eigenvalues of matrix power

Proposition

If $A, B \in M_n(\mathbb{C})$ are similar (ie $\exists P$ (invertible) with $A = P^{-1}BP$) then $\chi_A(t) = \chi_B(t)$

Proof $A = P^{-1}BP$ (as above)

$$
\begin{aligned}
\chi_A(t) &= \det(tI - A) \\
&= \det(tI - P^{-1}BP) \\
&= \det(tP^{-1}P - P^{-1}BP) \\
&= \det(P^{-1}(tI - B)P) \\
&= \det(P^{-1})\det(tI-B)\det(P) \\
&= \det(P^{-1})\det(P)\chi_B(t) \\
&= \chi_B(t)
\end{aligned}
$$

Finding eigenvectors

Lemma If $T: V \to V$ is linear and λ is an eigenvalue then $V_\lambda = \ker(\lambda 1_V - T)$.

Proof

$$
\begin{aligned}
V_\lambda &= \{v \in V \mid T(v) = \lambda v\} \\
&= \{v \in V \mid \lambda v - T(v) = 0\} \\
&= \{v \in V \mid (\lambda\,\mathrm{id}_V - T)(v) = 0\} \\
&= \ker(\lambda\,\mathrm{id}_V - T)
\end{aligned}
$$

Remark $\ker(\lambda\,\mathrm{id}_V - T) = \mathrm{null}(\lambda I - A)$ for any matrix $A = [T]_B$

Example Find the eigenvectors and eigenvalues of $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ where $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

(clearly $[T_A]_B = A$ B standard basis)

to calculate eigenvalues, solve:

$$\det(\lambda I_3 - A) = \chi_A(t) = 0$$

$$\det\left(\begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix} - \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \right) = \det \begin{pmatrix} t & 0 & 2 \\ -1 & t-2 & -1 \\ -1 & 0 & t-3 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$ expanding along first row

$$\begin{aligned} \chi_A(t) &= t((t-2)(t-3) + 2(t-2)) \\ &= (t-2(t(t-3)+2) \\ &= (t-2)(t^2-3t+2) \\ &= (t-2)^2(t-1) \end{aligned}$$

$\chi_A(t) = 0 \Rightarrow t = 2, t = 1$

$\Rightarrow$ eigenvalues are 1 and 2

we want $V_\lambda = \text{null}(\lambda I_3 - A)$

if $\lambda = 2 \Rightarrow 2I_3 - A = \left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \right)$

$$= \begin{pmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix}$$

<u>Gauss elim</u>

$$\begin{pmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \text{null} \begin{pmatrix} +1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_1 - x_3 = 0 \Rightarrow x_1 = x_3$$

$\Rightarrow$ a general vector looks like $\begin{pmatrix} +\lambda \\ \mu \\ -\lambda \end{pmatrix}$

$$V_2 = \left\{ \begin{pmatrix} \lambda \\ \mu \\ \lambda \end{pmatrix} \middle| \lambda, \mu \in \mathbb{R} \right\}$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

e.g. $v = \begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ -8 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix} \Rightarrow v \in V_2$

$$V_1 = \text{null}\,(I - A) = \text{null} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \right)$$

$$= \text{null} \begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in V_1 = \text{null} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix}$$

$\Rightarrow$ a general vector looks like ($x_3 = \mu$)

$$\begin{pmatrix} -2\mu \\ \mu \\ \mu \end{pmatrix}$$

$$\Rightarrow V_1 = \left\{ \begin{pmatrix} -2\mu \\ \mu \\ \mu \end{pmatrix} \middle| \mu \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$v = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is eigenvector since

$$\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow v_1 \in V_2$

<u>Proposition</u>

If $A \in M_n(\mathbb{C})$ has eigenvalues λ (really $T_A : \mathbb{C}^n \to \mathbb{C}^n$ has eigenvalue λ) then $A^k = \underbrace{A \cdots A}_{k \text{ times}}$

has eigenvalues λ^k and also $(A^{-1})^k$ has eigenvalues λ^{-k} $(k > 0)$

<u>Proof</u>

Suppose $Av = \lambda v$ some $0 \neq v \in \mathbb{C}^n$

prove by induction $A^k v = \lambda^k v$

base case $k = 1$ $A^1 v = \lambda^1 v = \lambda v$ <u>holds</u>

holds suppose the statement holds for $k-1$ then $A^{k-1}v = \lambda^{k-1}v$

then $A^k v = A(A^{k-1}v) = A(\lambda^{k-1}v)$

$= \lambda^{k-1}Av$ (linearity of matrix mult^n)

$= \lambda^{k-1}(\lambda v) = \lambda^{k-1+1}v = \lambda^k$

so true for all $k \geq 1$

(other half) exercise or see notes

<u>Example.</u> Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$

<u>STEP 1:</u> Find eigenvalues by solving $\det(tI - A) = \chi_A(t) = 0$

<u>STEP 2:</u> Calculate null $(I\lambda - A)$ for each eigenvalues

$\chi_A(t) = \det(tI_2 - A) = 0$

$$\det\left(\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}\right) = \begin{pmatrix} t & -3 \\ -4 & t \end{pmatrix}$$

$= t^2 - (-3 \cdot -4)$

$= t^2 - 12 = 0$

$t = \sqrt{12}$

$\Rightarrow$ eigenvalues $t = \sqrt{12}$

$V_\lambda = \text{null}(\lambda I_2 - A)$

if $\lambda = \sqrt{12} \Rightarrow \sqrt{12}\, I_2 - A = \left(\begin{pmatrix} \sqrt{12} & 0 \\ 0 & \sqrt{12} \end{pmatrix} - \right($