MA1114 9/2/22

Examples for Linear Maps on a Basis.

Proposition

Suppose V, W vector spaces. If $\{v_1, \dots, v_n\}$ is a basis for V and $T(v_i)$ is specified then the function

$T: V \to W$ given by

$T(v) = \sum_{i=1}^{n} \lambda_i T(v_i)$ is linear where $v = \sum_{i=1}^{n} \lambda_i v_i$

Recall $T: V \to W$ is linear if $T(\lambda v + \mu u) = \lambda T(v) + \mu T(u)$ for all $\lambda, \mu \in \mathbb{C}$ $v, u \in U$

Example

$T: \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto 3$ $\qquad$ $T: \begin{pmatrix} 4 \\ 6 \end{pmatrix} \mapsto 36$

$T: \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto 4$

$\begin{pmatrix} 4 \\ 6 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{aligned}
T: \begin{pmatrix} 4 \\ 6 \end{pmatrix} &= T\left(4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\
&= 4T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= (4 \cdot 3) + (6 \cdot 4) \\
&= 12 + 24 \\
&= 36
\end{aligned}$$

Example $\mathcal{B}$

$\mathbb{R}^3 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ with v_1, v_2, v_3 respectively

$T(v_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ $T(v_2) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$T(v_2) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\qquad$ where does a general vector get sent?

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\qquad [x]_{\mathcal{B}} = ?$

$\mathcal{B} = \{v_1, v_2, v_3\}$, $\quad x = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$x_1 = \lambda_1 + \lambda_2 + \lambda_3$$
$$x_2 = \lambda_1 + \lambda_2$$
$$x_3 = \lambda_1$$

$$\Rightarrow \lambda_1 = x_3$$
$$\lambda_2 = x_2 - x_3$$
$$\lambda_3 = x_1 - x_3 - (x_2 - x_3)$$
$$= x_1 - x_2$$

$$T(x) = x_3\, T(v_1) + (x_2 - x_3)\, T(v_2) + (x_1 - x_2)\, T(v_3)$$

$$= x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (x_2 - x_3) \begin{pmatrix} 2 \\ -1 \end{pmatrix} + (x_1 - x_2) \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$