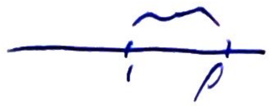
MA1014 15/5/22

Ratio Test ctd & Root Test & Comparison Test

Proof: (ratio test ctd)



If $\rho > 1$ then $\exists\, \varepsilon > 0$, s.t. $\rho - \varepsilon > 1$

$$\exists N \quad \forall n > N \quad \left| \frac{a_{n+1}}{a_n} - \rho \right| < \varepsilon$$

$$\frac{a_{n+1}}{a_n} > \rho - \varepsilon$$

$$a_{n+1} > (\rho - \varepsilon) a_n > \cdots > (\rho - \varepsilon)^{n-N} a_N$$

$$= (\rho - \varepsilon)^n \cdot \underbrace{\frac{a_N}{(\rho - \varepsilon)^N}}_{\text{constant.}}$$

$\sum (\rho - \varepsilon)^n$ is divergent

by comparison test $\sum a_n$ is div.

Example 1 $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

$$\lim_{n \to \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{n+1} = 1$$

Example 2 $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent

$$\lim_{n \to \infty} \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} = \lim_{n \to \infty} \frac{n\cancel{(n+1)}}{\cancel{(n+1)}(n+2)} = 1$$

Theorem 7.16.2 (root test)

If $\{a_n\}$ is non-negative. $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$

then $\sum a_n$ is $\begin{cases} \text{conv.}, & \text{when } \rho < 1 \\ \text{div.}, & \text{when } \rho > 1 \\ \times, & \text{when } \rho = 1 \end{cases}$

$\cancel{\sum \frac{1}{3^n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{3^n}} = \frac{1}{3} < 1}$

Example 3 $\sum_{n=1}^{\infty} \frac{1}{n!}$ $\qquad n! = n(n-1)\cdots 2\cdot 1$

since $\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$

by ratio test $\sum_{n=1}^{\infty} \frac{1}{n!}$ is conv.

$\boxed{\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}}$ beyond.

Example 4 $\sum_{n=1}^{\infty} \left(\frac{n}{3n-1}\right)^n$

$a_n = \left(\frac{n}{3n-1}\right)^n$, $\lim_{n\to\infty} \sqrt[n]{a_n}$

$= \lim_{n\to\infty} \frac{n}{3n-1} = \lim_{n\to\infty} \frac{1}{3-\frac{1}{n}} = \frac{1}{3} < 1$

by root test is conv

Example 5 $\sum_{n=1}^{\infty} \frac{2n}{3^n}$ $\lim_{n\to\infty} \sqrt[n]{n}$ $\lim_{n\to\infty} \sqrt[n]{2}$

Lemma

when $a>0$ then $\lim_{n\to\infty} (\sqrt[n]{a}) = 1$

$\lim_{n\to\infty} (\sqrt[n]{n}) = 1$

Proof

$\beta_n = \sqrt[n]{a} - 1$

Case 1: $a>1$ $\beta_n > 0$

$a = (1+\beta_n)^n > n\beta_n^1 \cdot 1^{n-1}$

$0 < \beta_n < \frac{a}{n}$ $\beta \to 0$ as $n \to \infty$

$\downarrow$ $\downarrow$

0 0

Case 2: $a=1$ ✓

Case 3: $0<a<1$ $\frac{1}{a} > 1$

by case 1: $1 = \lim_{n\to\infty} \sqrt[n]{\frac{1}{a}}$

$= \lim_{n\to\infty} \frac{1}{\sqrt[n]{a}}$

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = 1$

Solution to exe 5

$$\lim_{n\to\infty} \sqrt[n]{\frac{2n}{3^n}} = \frac{\lim_{n\to\infty} \sqrt[n]{2n}}{n} = \frac{1}{3} < 1$$

$p=1$ $\quad \sum \frac{1}{n}$ $\quad \sum \frac{1}{n(n+1)}$

$\sum a_n$ $\quad$ 1. anticipate the conv.

comparison test to know. 2. find the comparison series

a_n b_n $\quad$ 3. $a_n \leq b_n$

Example 6 $\sum \frac{1}{n^p}$ $\quad (p \in \mathbb{R})$ [p-series]

> when $p \leq 0$ $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$ so div

> when $p \leq 1$ $\quad \frac{1}{n^p} \geq \frac{1}{n}$

by comparison test is div.

> when $p > 1$ when $x \leq k$

$$\frac{1}{k^p} \leq \frac{1}{x^p}$$

integrating on the interval $[k-1, k]$

$$\frac{1}{k^p} = \int_{k-1}^{k} \frac{1}{k^p}\,dx \leq \int_{k-1}^{k} \frac{1}{x^p}\,dx$$

$$s_n = 1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p}$$

$$= 1 + \sum_{k=2}^{n} \frac{1}{k^p}$$

$$\leq 1 + \sum_{k=2}^{n} \int_{k-1}^{k} \frac{1}{x^p}\,dx$$

$$= 1 + \int_1^n \frac{1}{x^p}\,dx$$

$$\frac{1}{1-p} x^{1-p}$$

$$= 1 + \frac{1}{1-p} \left. x^{1-p} \right|_1^n$$

$$= 1 + \frac{1}{p-1}\left(1 - n^{1-p}\right) < 1 + \frac{1}{p-1}$$

$p \leq 1$: div

$p > 1$: conv

<u>Example 7</u> $\sum \frac{n-3}{n^3+4n}$

$$\frac{n-3}{n^3+4n} < \frac{n}{n^3} = \frac{1}{n^2}$$

$$\frac{n+3}{n^3-4n} < \times$$

$\{\delta_n\}$ $\exists N$ $\forall n > N$

$$\frac{n+3}{n^3-4n} \sim \frac{n'}{n^3} \sim \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{\left(\frac{n+3}{n^3-4n}\right)}{\left(1/n^2\right)} = \lim_{n\to\infty} \frac{n^3+3n^2}{n^3-4n} = 1$$

$\forall \varepsilon > 0$ $\exists N$ $\forall n > N$

$$\left| \frac{\frac{n+3}{n^3-4n}}{1/n^2} - 1 \right| < \varepsilon$$

$$(1-\varepsilon)\frac{1}{n^2} < \frac{n+3}{n^3-4n} < (1+\varepsilon)\cdot 1/n^2$$

$$(1-\varepsilon)\frac{1}{n} < \frac{n+3}{n^3-4n} < (1+\varepsilon)\frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} \simeq \frac{n^0}{n^1}$$

$$\lim_{n\to\infty} \frac{1/(\sqrt{n(n+1)})}{1/n} = 1$$

setting $\varepsilon = \frac{1}{2}$ $\exists N$ $\forall n > N$

$$\left| \frac{1/(\sqrt{n(n+1)})}{1/n} - 1 \right| < \frac{1}{2}$$

$$\frac{1}{\sqrt{n(n+1)}} > \frac{1}{2} \frac{1}{n}$$

since $\sum \frac{1}{2} \cdot \frac{1}{n}$ is div by comp test

$\sum \frac{1}{\sqrt{n(n+1)}}$ is div.

$x > 0$

$\sin x < x$

$\sin \frac{1}{n} < \frac{1}{n}$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Example 8 $\sum_{n=1}^{\infty} \sin \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{1/n} = 1$$

Example 9 $\sum \sqrt{n+1}\,(1-\cos(\pi/n))$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$$

$\sum \frac{n!}{3^n n!}$

$\sum a_n \rightarrow \lim_{n\to\infty} a_n \neq 0 \rightarrow$ div

$\downarrow$

$\lim a_n = 0$ —— $a_n \geq 0$

ratio $(n!)$ | root $(\ )^n$ | comparison

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \sqrt[n]{a_n} \qquad \sum a_n \sim \sum \frac{1}{n^p}$$