T- Invariant Direct Sums

$T: V \to V$ linear operator , V vector space

Def: W subspace in V is T- invariant if $T(w) \in W$ $\forall w \in W$ (or $T(W) \subseteq W$)
$T_W : W \to W$

Ex-s : (1) $\{0\}$, V are T- invariant
(2) If U is 1-dim subspace of V. Then
U is T-invariant $\Leftrightarrow$ $U = \text{span}\{v\}$ v evector

Theorem 4.5 Suppose $T: V \to V$ and W is T-invariant subspaces of V. Then $\exists$ basis S of V s.t

$$[T]_S = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \text{ (block } \Delta) \text{ with } A = [T_W]$$

Proof: Choose any S' of W, $S' = \{w_1, \ldots, w_r\}$ and extend to a basis of V, $S = \{w_1, \ldots, w_r, v_1, \ldots, v_n\}$

then

$$[T]_S = \Big[[T(w_1)]_S [T(w_2)]_S \cdots [T(w_r)]_S [T(v_1)]_S \cdots [T(v_r)]_S \Big] \ominus$$

$T(w_1) = \alpha_1 w_1 + \cdots + \alpha_r w_r + 0 \cdot v_1 + \cdots + 0 \cdot v_n$ (as $T(w_1) \in W$)

$\vdots$

$T(w_r) = \gamma_1 w_1 + \cdots + \gamma_r w_r + 0 \cdot v_1 + \cdots + 0 \cdot v_n$

$T(v_1) = * w_1 + \cdots + * w_r + * v_1 + \cdots + * v_n$

$$\begin{array}{c} \swarrow A \\ \ominus \left[\begin{array}{ccc|ccc} \alpha_1 & \cdots & \gamma_1 & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_r & \cdots & \gamma_r & * & \cdots & * \\ \hline 0 & \cdots & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \end{array}\right] \end{array} \begin{array}{l} \swarrow B \\ \\ \\ \leftarrow C \end{array} = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \text{ with } A = [T_W]$$

5 Invariant Direct Sum Decomposition

<u>Definition 5.1</u>: V is a direct sum of subspace $W_1, \dots, W_r$ (we write $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$) if every $v \in V$ can be written <u>uniquely</u> as

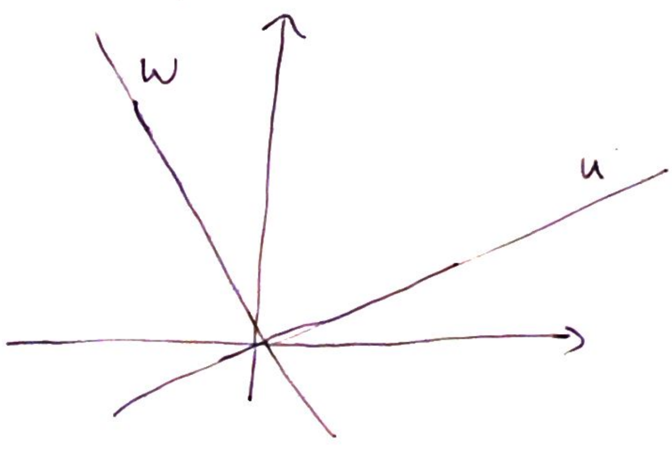
$$v = w_1 + w_2 + \cdots + w_r \quad \text{with } w_i \in W \ \forall i$$

(without "uniquely" is just $V = W_1 + W_2 + \cdots + W_r$)

<u>Exercise 5.3</u> Show that $V = W_1 \oplus W_2 \Leftrightarrow$ if $V = W_1 + W_2$ and $W_1 \cap W_2 = 0$

<u>Example</u>: (1) $V = \mathbb{R}^2$, U, W 1-dim subspace., $U \cap W = \{0\}$
Then $V = U \oplus W$

Note that also $V = U + W$
$V = U + V$
but $V \neq U \oplus V$



$\mathbb{R}^2 = x\text{-axis} \oplus y\text{-axis}$

Theorem 5.4 Suppose $W_1, \dots, W_r$ are subspaces of V and B_i is a basis of W_i.
Then $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r \iff B = B_1 \cup B_2 \cup \cdots \cup B_r$ is a basis of V.

Proof: Excercise (also Schaums Problem 10.7)
Ex of use "$\Leftarrow$": start with V, take any bases B of V, say
$B_1 = \{e_1, e_2, e_4\}$ and $B_2 = \{e_3\}$
Define $W_i = \text{span}\{B_i\}$. Then $V = W_1 \oplus W_2$

Ex of use "$\Rightarrow$": construct basis of V which agrees with subspaces $W_1, \dots, W_r$.

Corollary 5.6 If $V = W_1 \oplus \cdots \oplus W_r$ then $\dim V = \dim W_1 + \cdots + \dim W_r$

Definition 5.7 Let $T: V \to V$. Suppose $V = W_1 \oplus \cdots \oplus W_r$.
Then this direct sum is T-invariant if each W_i is T-invar
(also T-invariant direct sum decomposition)

Theorem 5.8 Suppose $T: V \to V$ and $V = W_1 \oplus \cdots \oplus W_r$ T-invariant direct sum decomposition. Let B_i be a basis in W_i and let $T_i = T_{W_i}$. Set $B = B_1 \cup B_2 \cup \cdots \cup B_r$ (basis of V)
Then
$$[T]_B = \text{diag}([T_1]_{B_1}, \dots, [T_r]_{B_r})$$
(block diagonal)

Proof: Let $B_1 = \{e_1, e_2, \dots\}$,
$B_2 = \{f_1, f_2, \dots\}, \dots, B_r$

Then $B = B_1 \cup B_2 \cup \cdots = \{e_1, e_2, \dots, f_1, f_2, \dots\}$ basis of V (by 5.4)

Then $[T]_B = \Big[[T(e_1)]_B \, [T(e_2)]_B \cdots [T(f_1)]_B \, [T(f_2)]_B \cdots \Big]$ $=$

$$\underbrace{T(e_1)}_{\in W_1} = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + 0 \cdot f_1 + 0 \cdot f_2 + \cdots$$

$$\underbrace{T(f_1)}_{\in W_2} = 0 \cdot e_1 + 0 \cdot e_2 + \cdots + \gamma_1 f_1 + \gamma_2 f_2 + \cdots + 0 \cdots$$

$[T_1]_{B_1}$

$$= \left[\begin{array}{ccc|cc|c}
\alpha_1 & * & \cdots & 0 & & \\
\alpha_2 & * & & \vdots & 0 & 0 \\
\vdots & \vdots & & & & \\
* & * & & 0 & & \\
\hline
0 & & & \gamma_1 & & \\
\vdots & & & \gamma_2 & & 0 \\
 & 0 & & \vdots & & \\
 & & & * & & \\
\hline
\vdots & & & & & \ddots \\
0 & & & 0 & & \\
\end{array}\right]$$

$[T_2]_{B_2} = \mathrm{diag}\left([T_1]_{B_1}, [T_2]_{B_2}, \cdots\right)$

e. basis of V then $[T]_B$ is diag