MA1114 14/12/21

Linear Independance & Linear Dependance

last time

$S = \{v_1, v_2, \dots, v_k\} \subset V$, vector space, linear independant if

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \dots + \lambda_k v_k = \underline{0}$$

$$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_k = 0$$

<u>linearly dependant</u> otherwise

Method $(V = \mathbb{R}^n)$

$S = \{a_1, a_2, \dots, a_n\}$ linearly independand

$$\Leftrightarrow \begin{pmatrix} \uparrow & & \uparrow \\ a_1 & \cdots & a_n \\ \downarrow & & \downarrow \end{pmatrix} x = \underline{0} \text{ has a unique solution}$$

<u>Example</u>

$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 4 \end{pmatrix} \right\}$ linearly dependant ✓

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ system $Ax = 0$ has infinilly many solutions

$\Rightarrow S$ is linearly dependant

This method also works for other vector spaces

Are the following sets linearly independant ?

① $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset M_{2\times 2}$

(rember $\underline{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in the vector space $M_{2\times 2}$
$= \{2\times 2$ with $\mathbb{R}$ entries$\}$

② $\{x^2 - 3x + 1,\ 2x^2 + x - 2,\ x^2 + 4x - 3\} \subset P_2$
(remember $\underline{0} = 0$ (zero polynomial) in $P_2 = \{$degree of 2 with real coefficients$\}$.

Consider $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$\lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \underline{0}$$

$$\Leftrightarrow \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \lambda_2 & \lambda_2 \\ \lambda_2 & \lambda_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \lambda_3 \end{pmatrix} = \underline{0}$$

$$\begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_2 \\ \lambda_2 & \lambda_2 + \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(A) $= \lambda_1 + \lambda_2 = 0$

(B) $= \lambda_2 = 0$

(C) $= \lambda_2 = -\lambda_3$ $\qquad \Rightarrow \lambda_1, \lambda_2, \lambda_3 = 0$

$\Rightarrow$ sets in ① is linear independant

could also try and solve.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(2) Assume $\lambda_1(x^2-3x+1)+\lambda_2(2x^2+x-2)+\lambda_3(x^2+4x-3)=0$

$$\Leftrightarrow 0 = x^2(\lambda_1+2\lambda_2+\lambda_3) + x(-3\lambda_1+\lambda_2+4\lambda_3) + 1(\lambda_1-2\lambda_2-3\lambda_3)$$

$$\Leftrightarrow \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & 4 \\ 1 & -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & 4 \\ 1 & -2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 7 \\ 0 & -4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \lambda_1, \lambda_2, \lambda_3 \neq 0 \Rightarrow$ linear dependant

<u>Proposition</u>

Let $S = \{v_1, v_2, ..., v_k\} \subset V$, a vector space

S linearly independant $\Leftrightarrow$ some vectors in S is a linear combination of the others

(aside: saw this yesterday for two vectors)

<u>Proof</u>

"$\Rightarrow$" Suppose S is linearly dependant. There exists $\lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{R}$ not all 0 with

$$\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_k v_k = \underline{0}$$

Assume. $\lambda_l \neq 0$ $(1 \leq l \leq k)$ $\Rightarrow \frac{1}{\lambda_l}(\lambda_1 v_1 + \cdots + \lambda_k v_k) = \underline{0}$

$$\underline{0} = \frac{\lambda_i}{\lambda_l} v_1 + \cdots + \frac{\lambda_{l-1}}{\lambda_l} v_{l-1} + v_l + \frac{\lambda_{l+1}}{\lambda_l} v_{l+1} + \cdots + \frac{\lambda_k}{\lambda_l} v_k$$

$$\Rightarrow v_l = -\frac{\lambda_1}{\lambda_l} v_1 - \cdots - \frac{\lambda_{l-1}}{\lambda_l} v_{l-1} - \frac{\lambda_{l+1}}{\lambda_l} v_{l+1} - \cdots - \frac{\lambda_k}{\lambda_l} v_k$$

$\Rightarrow$ v_l is a linear combination of the remaining vectors

"$\Leftarrow$" Conversly we assume $v_l = \mu_1 v_1, \ldots$

$$\Rightarrow \underline{0} = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_{l-1} v_{l-1} + v_l + \mu_{l+1} v_{l+1} + \cdots + \mu_k v_k$$

$\Rightarrow \{v_1, v_2, \ldots, v_k\}$ is linear dependant

(since coeficient of v_l is $-1 \neq 0$)

<u>Exercise</u>

$S = \{v_1\} \subset V$, a vector space, S is linear independant $\Leftrightarrow v_1 \neq 0$

<u>Proposition</u>

Suppose $S = \{v_1, v_2, \ldots, v_k\} \subseteq \mathbb{R}^n$

if $k > n$ then S is linear dependant.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{0} \qquad \begin{matrix} x + 2z = 0 \\ y - \frac{1}{2}z = 0 \end{matrix} \quad \Rightarrow \begin{pmatrix} -2z \\ \frac{1}{2}z \\ z \end{pmatrix}$$

Proofs of Proposition

$k > n \Rightarrow Ax = 0$ has more variables than equations

where $A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \cdots & v_k \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$

$\Rightarrow$ there must be free variables in the solution space $\Rightarrow \infty$ many solutions

$\Rightarrow s$ is linear dependant.