2.3 Interval Estimation

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Point estimators do not give us information about their precision!

For an interval estimator of a single parameter θ, we use the random sample to find two L and U, $L \leq \theta \leq U$ with some probability. This interver $[L, U]$ should have two zeros:

- $P(L \leq \theta \leq U)$ is high
- the length of the interval $[L, U]$ should be relatively narrow on the average.

Definition:

The problem of confidence estimation is that finding a family of random sets $S(X)$ for a parameter θ such that for a given α, $0 < \alpha < 1$.

$$P_\theta(\theta \in S(X)) > 1 - \alpha, \text{ for all } \theta \in \Theta$$

Interval estimators are called confidence intervals(CI). The limits L and U are called the lower and the upper confidence limits respectively.

Definition:

The probability $1 - \alpha$ that a confidence interval contains the true parameter θ is the confidence coefficient.

Interval Estimation using Pivots

Definition:

Let X $P()$. A random variable $T(X, \theta)$ is known as a pivot if the distribution of $T(X, \theta)$ does not depend on θ.

The pivotal method relies on our knowledge of smapling distributions.
The pivotal quantity should have the following two characteristics:

- It is a function fo the random sample (a statistic or an estimator $\hat{\theta}$) and the unknown parameter θ, where θ is the only unknown quantity, and
- It has a probability distribution that does not depend on the parameter θ

Suppose that $\hat{\theta} = g(x)$ is a point estimator of θ, and let $T(\hat{\theta}, \theta)$ be the pivotal quantity. Let a and b be constants with $(a < b)$ such that

$$P(a \leq T(\hat{\theta}, \theta) \leq b) = 1 - \alpha$$

for a given value of α, $(0 < \alpha < 1)$

$$T(\bar{X}, \mu) = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}},\ T\ N(0, 1)$$

Procedure for finding CI for θ using pivot

1. Find an estimator of $\hat{\theta} = g(x)$ of θ (usually MLE of θ works)
2. Find a function of $\hat{\theta}$ and θ, $T(\hat{\theta}, \theta)$(pivot), such that the probability distribution of $T(\hat{\theta}, \theta)$ does not depend on θ
3. Find $a < b$ such taht $P_\theta(a < T(\hat{\theta}, \theta) < b) = 1 - \alpha$.

   choose a and b such that $P_\theta(a \geq T(\hat{\theta}, \theta)) = \frac{\alpha}{2} = P_\theta(T(\hat{\theta}, \theta) \geq b)$
4. Transform the pivot confidence interval to a confidence interval for the parameter θ: $P_\theta(L < \theta < U) = 1 - \alpha$, where L and U is the lower and upper confidence limit respectively.