MA1014 17/1/22

Riemann Integration & Integrable Functions

Riemann Integration based on:

Area:

Approx = $\sum$ width × height of thin rectangles.

lower approx. upper approx

$\lim_{\text{width} \to 0}$ = $\lim_{\text{width} \to 0}$

<u>Formally</u>: Consider a partition D

$x_0 = a < x_1 < x_2 < \dots < x_n = b$ of the interval $[a, b]$

on $[x_i, x_{i+1}]$ let

infimum

$m_i = $ <u>inf</u> / <u>GLB</u> of $\{f(x) : x_i \leq x \leq x_{i+1}\}$

M_i : <u>sup</u> / <u>LUB</u> of $\{f(x) : x_i \leq x \leq x_{i+1}\}$

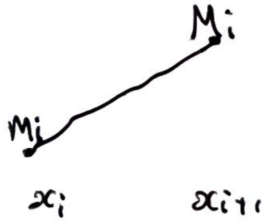
$\Delta x_i = x_{i+1} - x_i$ (width)

Thin rectangle $m_i \, \Delta x_i$
or $M_i \, \Delta x_i$

Lower approx $L_f(P) = \sum_{i=0}^{n-1} m_i \Delta x_i$

Upper approx. $U_f(P) = \sum_{i=1}^{n-1} M_i \Delta x_i$

<u>Example</u> Suppose $f: [a, b] \to \mathbb{R}$
is monotonic <u>increasing</u>



$$L_f(P) = \sum m_i \Delta x_i$$

$x_0 = a$
$$= \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i)$$

$$= f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \dots + f(x_{n-1})(x_n - x_{n-1})$$

$$= f(a) \cdot a + f(a) \cdot x_1 - f(x_1) \cdot x_1 + f(x_1) x_2 + \dots$$

$$= f(a) \cdot a + \sum (f(x_{i-1}) - f(x_i)) \cdot x_i - f(b) \cdot b$$

Similarly for $U_f(P)$ we have $M_i = f(x_{i+1})$

<u>Definition</u> The function $f : [a, b] \to \mathbb{R}$ is integrable if

$$\text{LUB}\{L_f(P) : \text{all partitions } P\} = \text{GLB}\{U_f(P) : \text{all partition } P\}$$

& then this is value is the definite integral $\int_a^b f(x)\,dx$

Consider a special type of partition where all the widths are the same.
$\Delta x_i = \Delta x$ for all i

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1}$$

$$\Delta x_0 \qquad\qquad\qquad\qquad \Delta x_{n-1}$$

So $L_f(P) = \Sigma(m_i)\,\Delta x$

$U_f(P) = \Sigma(M_i)\,\Delta x$

$a = x_0, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad x_3 = a + 3\Delta x, \ldots$

Consider $\lim\limits_{\Delta x \to 0}$ of $L_f(P), U_f(P)$

<u>Theorm 5.6</u> $f : [a,b] \to \mathbb{R}$ (assume this is bounded so that m_i, M_i exist) is <u>integrable</u> if and only if

$$\forall \varepsilon > 0 \ \exists P \text{ such that } U_f(P) - L_f(P) < \varepsilon$$

In more detail to prove f is integrable consider

$$\Sigma M_i \Delta x_i - \Sigma m_i \Delta x_i = \Sigma (M_i - m_i)\Delta x_i$$

If width are all the same $= (\Sigma(M_i - m_i))\Delta x_i$

Let's return to our example:

$f : [a,b] \to \mathbb{R}$ montonic increasing $m_i = f(x_i)$ $M_i = f(x_{i+1})$

Then (*) $= \left(\sum_{i=0}^{n-1} f(x_{(i+1)}) - f(x_i) \right) \Delta x$

$$\cancel{f(x_1)} - f(x_0) + \cancel{f(x_2)} - \cancel{f(x_1)} + f(x_3) - \cancel{f(x_2)} + \dots + f(x_n) - f(x_{n-1})$$

$$= -f(x_0) + f(x_n)$$

so for increasing functions f and partitions with constant widths Δx

$$U_f(P) - L_f(P) = (f(x_1) - f(x_0))\,\Delta x \leq B \cdot \frac{b-a}{n}$$

so $n \to \infty$, $U_f(P) - L_f(P) \to 0$

$\forall \varepsilon > 0 \; \exists P$ width $\Delta x < \frac{\varepsilon}{B}$

$$\text{an } n > \frac{B \cdot (b-a)}{\varepsilon} \Rightarrow U_f(P) - L_f(P) < \varepsilon$$

We just proved:

> Any bounded monotonic increasing functions $f : [a, b] \to \mathbb{R}$ is integrable

Later: any continuous $f : [a, b] \to \mathbb{R}$ is integrable.