MA1114 15/11/21

Effect of Elementary Row Operations on Determinants

Proposition (def^n)

$A \in M_{n,n}$ $\quad A_{ij} = a_{ij}$

$$\det(A) = \sum_{j=1}^{n} A_{ij}(-1)^{i+j} \det(\hat{A}_{ij})$$

Proposition 3.47

Recall $X_n(k, \lambda)$
$Y_n(i, j)$
$A \in M_{n,n}$

$$\det(X_n(k,\lambda)A) = \lambda \det(A)$$

$$\det(Y_n(i,j)A) = -\det(A)$$

Proof

(i) $\det(X_n(k,\lambda)A) = \begin{pmatrix} 1 & & \\ & \ddots \lambda & \\ & & \ddots 1 \end{pmatrix} = \sum_{j=1}^{n} \lambda a_{kj}(-1)^{k+j} \det(\hat{A}_{kj})$

$= \lambda \sum_{j=1}^{n} a_{kj}(-1)^{k+j} \det(\hat{A}_{kj})$

(ii) already know 2×2 is already true assume $n \geq 3$ proof by induction
suppose $n = 2$ trues for $(n-1)\cdot(n-1)$ matrices
choose $k \neq i, j$

$$\det(Y_n(i,j)A) = \sum_{l=1}^{n} (-1)^{k+l} a_{kl} \det(Y_{n-1}(i,j)\hat{A}_{kl})$$

$$= \sum_{l=1}^{n} (-1)^{k+l} a_{kl} (-\det(\hat{A}_{kl}) \quad \text{by induction}$$

$$= \det(A)$$

$$Z_n(i,j,\lambda) = \begin{pmatrix} 1 & \ddots & 0 \\ \lambda & & 1 \end{pmatrix}$$

<u>Proposition</u>

$A \in M_{n,n}$ Write $A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ a_1 & a_2 & \cdots & a_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$

i) $\det \begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ a_1 & \cdots & a_k + a_{kb} & \cdots & a_n \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix} = \det \begin{bmatrix} \uparrow & & \uparrow \\ a_1 & \cdots & a_k \\ \downarrow & & \downarrow \end{bmatrix}$

$= \det \begin{bmatrix} \uparrow & & \uparrow \\ a_k & \cdot & ab_k \\ \downarrow & & \downarrow \end{bmatrix}$

ii) if $a_k = a_l$ for some $1 \leq k l \leq n$ then $\det(A) = 0$

iii) $\det(Z_n(i,j,\lambda)A)$

<u>Proof</u>

i) row k expansion of $\det(a_1 \cdots a_k + ab_k \cdots a_n)$

$$= \sum_{i=1}^{n} (-1)^{i+k} (a_{ik} + b_{ik}) \det(\hat{A}_{ik})$$

$$= \left(\sum_{i=1}^{n} (-1)^{i+k} a_{ik} \det(\hat{A}_{ik})\right)\left(\sum_{i=1}^{n} (-1)^{i+k} b_{ik} \det(\hat{A}_{ik})\right)$$

ii) work with A^T k^{th} row $= l^{th}$ row in A^T

$$Y_n(k, l)\, A^T = A^T$$

$$\det(Y_n(k, l)\, A^T) = \det(A^T) \Rightarrow -\det(A^T) = \det(A^T)$$

$$\Rightarrow \det(A^T) = 0$$

$$\Rightarrow \det(A) = 0$$

iii) work with A^T

$$\det(Z_n(i, j, \lambda)\, A^T)$$

$$= \det \begin{bmatrix} \leftarrow a_1 \rightarrow \\ \vdots \\ \leftarrow \lambda a_i + a_j \rightarrow \\ \vdots \\ \leftarrow a_n \rightarrow \end{bmatrix} = \det \begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ a_1 & \cdots & \lambda a_i + a_j & \cdots & a_n \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix}$$

$$= \det \begin{bmatrix} a_1 & \cdots & \lambda a_i & \cdots & a_n \end{bmatrix} + \det \begin{bmatrix} \uparrow & & \uparrow & & \uparrow \\ a_1 & \cdots & a_j & \cdots & a_n \\ \downarrow & & \downarrow & & \downarrow \end{bmatrix}$$

$$= \lambda \det \begin{bmatrix} \uparrow & & \uparrow & & \uparrow & & \uparrow \\ a_1 & \cdots & a_i & \cdots & a_j & \cdots & a_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \end{bmatrix} \det \begin{bmatrix} \uparrow & & \uparrow \\ a_1 & \cdots & a_n \\ \downarrow & & \downarrow \end{bmatrix}$$

$$= \det(A)$$