MA1114 26/1/22

E/R to a Basis Examples.

<u>Theorem</u> "extend/reduce to a basis"

V finite dimensional vectorspace and $S \subset V$, finite

(i) if S is linearly independent then there is a basis $\hat{S}$ of V with $\hat{S} \supseteq S$

(ii) if $\text{span}(S) = V$ then there exists a basis $\bar{S}$ of V with $\bar{S} \subseteq S$

<u>Example</u>

Complete / extend $S = \left\{ \begin{pmatrix} 1 \\ 3 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -4 \\ 2 \end{pmatrix} \right\}$ to a basis of $\mathbb{R}^4$

$$? \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \text{span}(S)$$

recall $S = \{v_1, v_2, \dots, v_n\} \subset V$ a vector space

$$\text{span}(S) = \{ \Sigma \lambda_i v_i \mid \lambda_i \in \mathbb{R} \}$$

$$= \text{col}(A), \text{ where } A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \cdots & v_k \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

(recall $A \in M_{n,n}(\mathbb{R})$, $\text{col}(A) = \text{span}(\{\text{columns of } A\})$

we can perform column operations to preserve $\text{col}(A)$

A column operation on A row operation on A)

Examples of column operations on A as above: $v_i \mapsto v_i + \lambda v_j \quad 1 \le i, j \le k$

$v_i \mapsto v_j \quad 1 \le j \le k$

$v_i \mapsto \lambda_i v_i \quad 0 \ne \lambda \in \mathbb{R}$
$1 \le i \le k$

e.g. $v = 1$, $\lambda = 2$, $k = 2$

$$A = \begin{bmatrix} \uparrow & \uparrow \\ v_1 & v_2 \\ \downarrow & \downarrow \end{bmatrix}$$

$$\xrightarrow{c_1 \mapsto c_1 + 2c_2} \begin{bmatrix} \uparrow & \uparrow \\ v_1 + 2v_2 & v_2 \\ \downarrow & \downarrow \end{bmatrix}$$

$$\operatorname{span}(\{v_1 + 2v_2, v_2\})$$
$$= \{\lambda(v_1 + 2v_2) + \mu \lambda_2 \mid \lambda, \mu \in \mathbb{R}\}$$
$$= \{\lambda v_1 + \lambda' v_2 \mid \lambda, \lambda' \in \mathbb{R}\} = \operatorname{span}(v_1, v_2)$$

<u>Back to example</u>

$$S = \left\{ \begin{pmatrix} 1 \\ 3 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -4 \\ 2 \end{pmatrix} \right\}$$

$$\rightsquigarrow A = \begin{bmatrix} 1 & -1 \\ 3 & -2 \\ 6 & -4 \\ -3 & 2 \end{bmatrix} \xrightarrow[c_2 \mapsto c_2 + c_1]{} \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 6 & 2 \\ -3 & -1 \end{bmatrix} \xrightarrow[c_1 \mapsto c_1 - 3c_2]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & -1 \end{bmatrix}$$

$$\mathrm{span}(S) = \mathrm{span}\left(\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\\-1\end{pmatrix}\right\}\right)$$

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \notin \mathrm{span}(S)$$

so $S \cup \left\{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$ is linearly independent by plus/minus theorem.

How to extend to a basis of $\mathbb{R}^n$ using standard basis vector?

Assume $S = \{v_1, v_2, \ldots, v_k\} \subset \mathbb{R}^n$

1) Form a matrix, $A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \cdots & v_k \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$

2) Perform column operations on A to reduce A as much as possible
A' $\quad \mathrm{col}(A) = \mathrm{col}(A')$

3) Choose $c_i = \begin{pmatrix}0\\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0\end{pmatrix}$ such that $c_i \notin \mathrm{col}(A)$
ith pos.

4) Reduce $A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow & \uparrow \\ v_1 & v_2 & \cdots & v_k & c_i \\ \downarrow & \downarrow & & \downarrow & \downarrow \end{pmatrix}$

5) Return to step (2)

6) Stop when A has n columns. (L.I. set of n vectors is a basis) by COGOF of basis.

Reducing to a basis

Problem: $S \subset V$ finite $\text{span}(S) = V$ sok $J \subseteq S$ linearly independant

to do this, find a basis for col(A) where $A = \begin{pmatrix} \uparrow & & \uparrow \\ v_1 & \cdots & v_n \\ \downarrow & & \downarrow \end{pmatrix}$

$(S = \{v_i \cdots v_n\})$

Example

Calculate a basis $\text{span}\left(\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}\right)$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$W \Rightarrow \text{span}\left(\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\}\right)$