MA1061 20/10/21

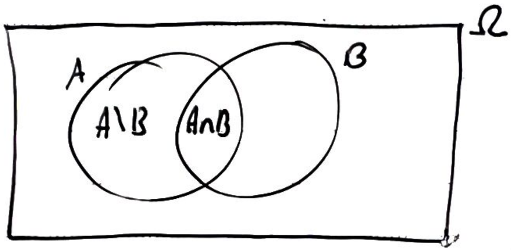
Definition of Probability & Multiplication Principle

⑤ If $A \cap B = \emptyset$ then

$$P(A \cup B) = \sum_{\{i : w_i \in A \cup B\}} P(w_i) = \sum_{\{i : \text{either } w_i \text{ in } A \text{ or in } B\}} P(w_i)$$

$$= \sum_{\{j : w_j \in A\}} P(w_j) + \sum_{\{k : w_k \in B\}} P(w_k) = P(A) + P(B)$$

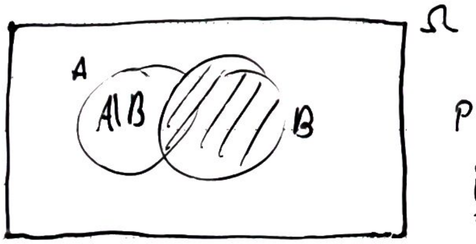
③ Observe that $A = (A \setminus B) \cup (A \cap B)$



Clearly $A \setminus B$ and $A \cap B$ are disjoint. Hence by ⑤

$$P(A) \overset{⑤}{=} P(A \setminus B) + P P(A \cap B)$$

④ Observe that $A \cup B = (A \setminus B) \cup B$, and clearly $A \setminus B$ and B are disjoint



Thus:

$$P(A \cup B) \overset{⑤}{=} P(A \setminus B) + P(B)$$

$$\overset{③}{=} P(A) - P(A \cap B) + P(B)$$

Formal Definition of Probability

A <u>probability</u> P is a function defined on $\mathcal{A}$ such that

① for any event $A \in \mathcal{A}$, $P(A) \geq 0$

② $P(\Omega) = 1$

③ Let $A_1, A_2, \ldots$ be disjoint events,

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

Obviously, $P : \mathcal{A} \to [0,1]$.

Write $(a_1, a_2, a_3, \ldots, a_r)$ for an <u>ordered</u> sample, and $\{a_1, \ldots, a_r\}$ (or $[a_1, \ldots, a_r]$) for an <u>unordered</u> sample

Multiplication Principle

<u>Proposition (1)</u>

Suppose there are r groups of elements where $k = 1, \ldots, r$, the k-th k-th group consists of n_k elements $a_1^{(k)}, a_2^{(k)}, \ldots, a_{n_k}^{(k)}$ We form a combination of r elements, by taking one from each group Then the number of all such combinitions $(a_{i_1}^{(1)}, \ldots a_{i_r}^{(r)})$, $k \in \{1, \ldots, r\}$ is $N = n_1 n_2 \cdots n_r$

Proof

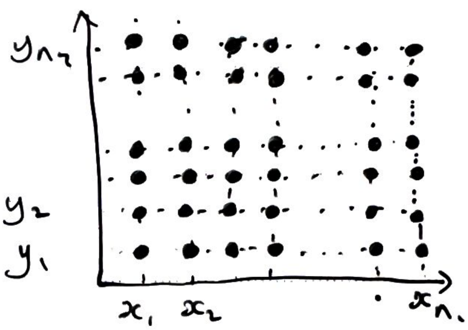
By mathematical induction

- For $r=2$, we represent combinations on the plane

- ① on x-axis represent elements in first group,
  ② on y-axis elements in second group

- Number of possible combinations in number of points in the rectangular lattice;

- hence $N = n_1 \times n_2$



Suppose the result holds for $r=k$, that is $N_k = n_1 \times n_2 \times \dots n_k$

Inductive step: Show that the result holds for $r=k+1$
Represent the $n_1 \times n_2 \times \dots \times n_k$ possible combinations from the first k groups on the axis, and the n_{k+1} choices from the $(k+1)$th group on the y-axis

Count how many points are in the lattice: N_k on the x-axis and n_{k+1} on the y-axis. So there are $N_{k+1} = N_k \times n_{k+1} = n_1 \times n_2 \times n_3 \times \dots \times n_{k+1}$ points in total.

Example

When rolling three fair dice, what is the probability of getting three 6's?

- Let a, b, c be the number of points on the 1st, 2nd, 3rd. die respectively.

- Write individually outcomes as ordered triples (a, b, c), where each entry is $1, \dots 6$;

$$\Omega = \{(a, b, c) : a, b, c = 1, \dots 6\}.$$

- By proposition (1) with $r = 3$ and $n_1 = n_2 = n_3 = 6$ there are precisely

$$|\Omega| = 6^3 = 216 \text{ equiprobable outcomes}$$

- Let A be the event $= \{(6, 6, 6)\}$

- How many outcomes in A? i.e what is $|A|$? In how many ways can this occur?

- Probability of getting three 6's is

$$P(A) = \frac{|A|}{|\Omega|} = \frac{1}{6^3} = \frac{1}{216}.$$