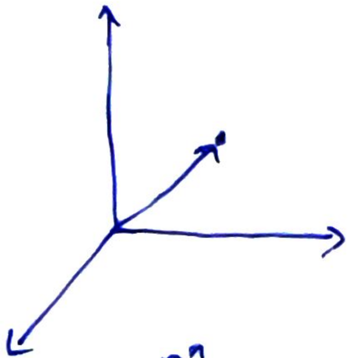
MA1014 21/3/22

Multivariable Calculus.



$(x_0, y_0, z_0) \Leftrightarrow$ a vector

$\mathbb{R}^n$ $\qquad\qquad\qquad\qquad$ $\vec{x}$

$x = (x_1, \ldots, x_n)$

$y = (y_1, \ldots, y_n)$

$x + y = (x_1 \pm y_1, x_2 \pm y_2, \ldots, x_n \pm y_n)$

let $\lambda \in \mathbb{R}$

$\lambda x = (\lambda x_1, \ldots, \lambda x_n)$

Define the norm of a vector $x = (x_1, \ldots, x_n)$

as $||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

For $x \in \mathbb{R}^n$. $||x|| \geq 0$

$||x|| = 0 \Leftrightarrow x = (0, 0, \ldots, 0)$

$||\lambda x|| = |\lambda| \cdot ||x||$

$||x+y|| \leq ||x|| + ||y||$

For $x, y \in \mathbb{R}^n$, defines the <u>distance</u> between x and y

as $\|x-y\|$

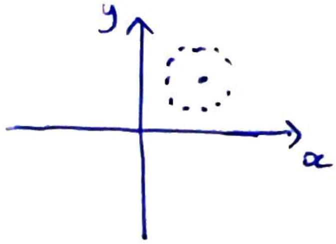
$$= \sqrt{(x_1-y_2)^2 + \cdots + (x_n-y_n)^2}$$

- Relations between a point and a domain in $\mathbb{R}^n$

<u>Definition 1</u> (neighborhood in $\mathbb{R}^n$) Let $x_0 \in \mathbb{R}^n$

Define $B_\delta(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < \delta\}$

$B(x_0, \delta)$ ---



$x_0 = (a_1, a_2)$

$x = (x_1, x_2)$

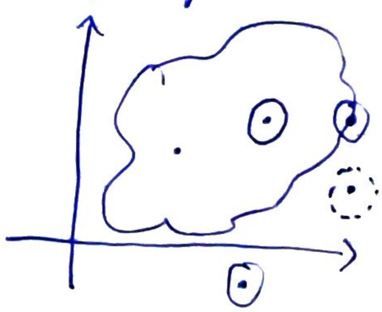
$$\|x - x_0\| = \sqrt{(x_1-a_1)^2 + (x_2-a_2)^2} < \delta$$

$$(x_1-a_1)^2 + (x_2-a_2)^2 < \delta$$

<u>Definition 2</u> (interior point) Let $S \subseteq \mathbb{R}^n$, $x_0 \in S$

we say that x_0 is an interior point of S if

$\exists \delta > 0$, s.t. $B(x_0, \delta) \subseteq S$



$\mathrm{int}(S) = \{x_0 :$
x_0 is an interior point of S

- $\mathrm{int}(S) \subseteq S$

Definition 3 (boundary point) Let $S \subseteq \mathbb{R}^n$. $x_0 \in \mathbb{R}^n$

we say that x_0 is a boundary point of S if

$\forall \delta > 0 \quad B(x_0, \delta) \cap S = \phi$ and
$B(x_0, \delta) \cap S^c = \phi$

$\partial S = \{x_0 : x_0 \text{ is a boundary points of } S\}$.

Definition 4 (isolated point) Let $S \subseteq \mathbb{R}^n$. $x_0 \in S$.

we say that x_0 is an isolated point of S if

$\forall \delta > 0 \quad (B(x_0, \delta) \setminus \{x_0\}) \cap S = \phi$

Definition 5 (open set) Let $S \subseteq \mathbb{R}^n$.

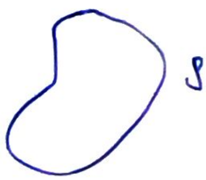
we say that S is an open set if

$\forall \delta > 0$ s.t. $B(x, \delta) \subseteq S$

(a, b)

Definition 6 (closed set) Let $S \subseteq \mathbb{R}^n$

we say that S is closed if $S = S \cup \partial S$



use these 3 diagrams to remember and key points e.g. $f: (x,y) = \frac{2xy}{x^2+y^2}$

<u>Definition 7</u> Let $D \subseteq \mathbb{R}^n$, $f: D \to \mathbb{R}$ is said to be a multivariable function.

<u>Ex</u> $f(x_1, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2$

$D = \{(x_1, \dots, x_n) : x_i > 0 \quad \forall i\}$

$f(x,y) = \frac{2xy}{x^2+y^2} \qquad (x,y) \neq (0,0)$

$g(x,y) = \ln(x+y^2) \qquad\qquad \ln = \log_e$

$D = \{(x,y) : x+y^2 > 0\} \qquad\qquad x \in \mathbb{R}^n \quad x_0 \in \mathbb{R}^n$

<u>Definition 8</u> Let $D \subseteq \mathbb{R}^n$,

$f: D \to \mathbb{R}$ we say that $\lim_{x \to x_0} f(x) = L$ or $\lim_{h \to 0} f(x_0 + h) = L$ if

(i) every neighbourhood of x_0 contains points of the domain of f different from x_0 that is,

$\forall \delta > 0, \exists x \in B_\delta(x_0)$ s.t. $x \in D$

(ii) for each positive number ε there exists a positive number $\delta = \delta(\varepsilon)$ such that $|f(x) - L| < \varepsilon$ holds whenever x is in the domain of f and satisfies $0 < \|x - x_0\| < \delta$, that is

$\forall \varepsilon, \exists \delta = \delta(\varepsilon)$ s.t $\forall x \in D \cap B_\delta(x_0) \setminus \{x_0\}, |f(x) - L| < \varepsilon$