MA1114 13/12/21

Linear Independance & Basis

<u>Example</u>

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^2$

observe $\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (y - x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

However for $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

only $\begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is only possible

<u>Question</u> Let V be a vector space

For all $v \in V$ have a unique expression a linear combination of elements of S?

<u>Definition 5.1</u>

A subset $S = \{ v_1, v_2, \ldots, v_k \} \subset V$, a vector space is linearly independant, if whenever $\lambda_1 v_1 + \lambda_2 v_2 + \ldots \lambda_k v_k = 0$ then $\lambda_k = 0$ otherwise, S is dependant

<u>Example</u>

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^2$ is linearly dependant since $2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

but $2 \neq 0$, $1 \neq 0$

Proposition

Suppose $\{v_1, v_2\}$ is a vector space. Then $\{v_1, v_2\}$ is linearly dependant

$\Leftrightarrow$ v_1 and v_2 are parallel

Proof

Suppose $\{v_1, v_2\}$ is linearly dependant $\Rightarrow$ there exists λ_1, λ_2 (not both 0) with $\lambda_1 v_1 + \lambda_2 v_2 = 0$

Suppose (up to reordering lables for v_1 v_2) that $\lambda_1 \neq 0$

$$\frac{1}{\lambda_1}(\lambda_1 v_1 + \lambda_2 v_2) = \underline{0}$$

$$\Rightarrow v_1 + \frac{\lambda_2 v_1}{\lambda_1} = \underline{0}$$

$$v_1 = -\frac{\lambda_2 v_1}{\lambda_1} \Rightarrow v_1 \text{ and } v_2 \text{ are parallel.}$$

Conversly, suppose there exists $\mu \in \mathbb{R}$ such that

$$v_1 = \mu v_2 \Rightarrow v_1 - \mu_2 = 1 v_1 - \mu v_2 = 0 \ , \ 1 \neq 0$$

$$\Rightarrow v_1, v_2 \text{ are linearly dependant}$$

Question

Are the following sets linear independant ?

① $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^2$

② $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^3$

③ $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \subset \mathbb{R}^2$

① Assume $\lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \lambda_1, \lambda_2 = 0$$

$\Rightarrow$ linear independant.

② Assume $\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underline{0}$

$$\begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\lambda_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underline{0}$$

$$\Rightarrow \begin{pmatrix} \lambda_1 - \lambda_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \lambda_1, \lambda_2 \neq 0$$

(3) Assume $\lambda_1 \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \lambda_1 \\ \lambda_1/2 \end{pmatrix} + \begin{pmatrix} \lambda_2 \\ 2\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_1/2 + 2\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lambda_1 + \lambda_2 = 0$$
$$\tfrac{1}{2}\lambda_1 + 2\lambda_2 = 0$$

equivallently $\begin{bmatrix} 1 & 1 \\ 1/2 & 2 \end{bmatrix} \times \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1/2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 3/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda \end{bmatrix} = \underline{0}$$

$$\Rightarrow \lambda_1, \lambda_2 = 0$$

$\Rightarrow$ linear independant

<u>Example</u>

Is $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 4 \end{pmatrix} \right\}$ linear independant?

Assume $\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 4 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_1 \end{pmatrix} + \begin{pmatrix} \lambda_2 \\ 2\lambda_2 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} 4\lambda_3 \\ 5\lambda_3 \\ 4\lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the system is inconsistent so the system is linearly dependant have 1 no

$$\begin{aligned} \lambda_1 + \lambda_2 + 4\lambda_3 &= 0 \\ \lambda_2 + \lambda_3 &= 0 \end{aligned}$$

$\Rightarrow \lambda_1 = 3, \ \lambda_2 = 1, \ \lambda_3 = -1$

<u>Method</u>

$S = \{a_1, a_2, a_3, \dots, a_n\} \subset \mathbb{R}^m$ is linear independant

$\Leftrightarrow A x = \underline{0}$ has a unique solution $(x = 0)$ where $x = \begin{pmatrix} 0 \\ \vdots \\ \lambda_n \end{pmatrix}$

$$\lambda = \begin{pmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ a_1 & a_2 & a_3 & \dots & a_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{pmatrix} \in M_{m,n}$$

(proof) $\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = 0 \ \ (a_1 a_2 \cdots a_n) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \lambda_n \end{pmatrix}$

so if this is a unique solution then $\lambda_1, \lambda_2, \dots, \lambda_n = 0$