MA1061 2/11/21

Bayes' Theorm

Suppose A, B are events with $P(A), P(B) > 0$

Using the definition of conditional probability.

$$P(A \cap B) = P(A)\, P(B|A),$$
$$P(A \cap B) = P(B)\, P(A|B),$$

gives Bayes' formula

$$P(A|B) = \frac{P(A)\, P(B|A)}{P(B)}$$

Theorm (Bayes' Theorm)

Let $\mathcal{D} = \{A_1, \dots, A_n\}$ be a partition of Ω, and B be an event. Then

$$P(A_i|B) = \frac{P(A_i)\, P(B|A_i)}{\sum_{j=1}^{n} P(A_j)\, P(B|A_j)} \quad (2)$$

if $P(A_i) > 0$ for all $1 \le i \le n$ and $P(B) > 0$

Proof of Bayes Theorm

from bayes formula (3)

from total probability theorm (4)

$$P(B) = \sum_{j=1}^{n} P(B|A_j)\, P(A_j)$$

Substitute eq 4 into eq. 3 to obtain the theorm.

Example

Let a bag contain two coins:

> A_1 - a fair coin with $P(H) = P(T) = 1/2$; and

> A_2 - a biased coin with $P(H) = 1/3$.

A coin is drawn at random and tossed. Determined the sample space and assign probabilities to each elementary event in the sample space. Supposed it falls heads, find the probability that the fair coin was selected.

- not equiprobable

$\Omega = \{A_1H, A_1T, A_2H, A_2T\}$

$P(A_1) = P(A_2) = 1/2 \qquad P(H|A_1) = 1/2 \qquad P(H|A_2) = 1/3$

we want $P(A_1|H)$

> $P(A_1H) = P(A_1 \cap H) = P(H|A_1)\,P(A_1) = 1/4$;
> $P(A_1T) = P(A_1 \cap T) = P(T|A_1)\,P(A_1) = 1/4$;
> $P(A_2H) = P(A_2 \cap H) = P(H|A_2)\,P(A_2) = 1/3 \times 1/2 = 1/6$;
> $P(A_2T) = P(A_2 \cap T) = P(T|A_2)\,P(A_2) = 2/3 \times 1/2 = 2/6 = 1/3$;

apply Bayes' Theorm, we need a partition $\mathcal{D} = \{A_1, A_2\}$

$$P(A_1|H) = \frac{P(A_1)P(H|A_1)}{P(A_1)P(H|A_1) + P(A_2)P(H|A_2)} = \frac{1/2 \times 1/2}{1/2 \times 1/2 + 1/2 \times 1/3} = 3/5$$

Example

A computer center has three printers: A. B, and C.
Documents are routed to printer A, B, and C with probability 0.5, 0.3 and, 0.2 respectively.
Printers A, B, and C, jam with probability 0.04, 0.05, and 0.03, respectively.
Now you find your program crashes because of printer jamming. What is the probability that printer A is the culprit?

Partition $\{A, B, C\}$.

$P(A) = 0.5 \quad P(B) = 0.3 \quad P(C) = 0.2$

$P(\text{jam} \mid A) = 0.04 \quad P(\text{jam} \mid B) = 0.05 \quad P(\text{jam} \mid C) = 0.03$

we want $P(A \mid \text{jam})$

$$P(A \mid \text{jam}) = \frac{P(\text{jam} \mid A)\, P(A)}{\sum_{i \in \{A,B,C\}} P(\text{jam} \mid i)\, P(i)}$$

$$= \frac{0.04 \times 0.5}{0.04 \times 0.5 + 0.05 \times 0.3 + 0.03 \times 0.2} = \frac{20}{41}$$

Example

0.5 % of the population have breast cancer. If the breast screening programme is 0.95 accurate, how likely is an invidual to have breast cancer if she has a positive test result?

> 0.9

> 0.5

> 0.1

let c be the event that individual has cancer & T is the event test is positive

Partition $\{C, \bar{C}\}$

$P(C) = 0.005 \qquad P(\bar{C}) = 1 - 0.005 = 0.995$

$P(T|C) = 0.95 \qquad P(T|\bar{C}) = 0.05$

we want $P(C|T)$

$$P(C|T) = \frac{P(T|C)\,P(C)}{P(T|C)\,P(C) + P(T|\bar{C})\,P(\bar{C})}$$

$$= \frac{.95 \times .005}{.95 \times 0.005 + 0.05 \times 0.995} = 0.087$$