MA1114 1/2/22

Calculating the intersection of a pair of subspaces.

Proposition 6.22

If $U, W \leq V$, a vector space then $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

Example

Find a basis for the intersection of $U = \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle \leq \mathbb{R}^3$

$$W = \left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\rangle \leq \mathbb{R}^3$$

$v \in U \cap W$

A $\quad v = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \mu_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \lambda_1 + \lambda_2 - 2\mu_1 \\ \lambda_1 - \mu \\ 2\lambda_2 - 2\mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} \lambda_1 = \mu_1 \\ \lambda_2 = \mu_2 \end{matrix}$$

$\Rightarrow$ a general element of $U \cap W$ look like

$$\begin{pmatrix} 2\lambda - 2\mu \\ \lambda - \mu \\ \lambda - 3\mu \end{pmatrix} = \underline{0} \qquad \text{A} \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \mu_1 \\ \mu_2 \end{pmatrix} = \underline{0}$$

$0 = \lambda_1 + \lambda_2 - 2\mu_1 = \lambda_1 + \mu_2 - 2\lambda_1 = \mu_1 - \lambda_1$

$\Rightarrow \lambda_2 = \mu_1$, $\Rightarrow \lambda_1 = \lambda_2 = \mu_2 = \mu_1$

here $\lambda_1 - \mu_1 = 0$ so $\lambda_1 = \lambda_2 = \mu_2 = \mu_1$

By ① $v = \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ \lambda \\ 2\lambda \end{pmatrix}$

$= U \cap W$

$= \left\langle \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rangle$

Proposition 6.28

Every subspace $U \leq \mathbb{R}^n$ is the solution space of some homogeneous systems.

Proof (non-examinable → not in exam)

- idea " $null(null(U)) = U$

Example

$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $U = \langle u_1, u_2 \rangle \leq \mathbb{R}^4$

$A = \begin{pmatrix} u_1^T \\ u_2^T \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & {}^-\!1/2 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & {}^-\!1/2 & 0 \end{pmatrix}$

$$\Rightarrow \text{null}(A) = \left\{ \begin{pmatrix} \lambda \\ \lambda \\ -2\lambda \\ \mu \end{pmatrix} \middle| \lambda, \mu \in \mathbb{R} \right\}$$

$$= \left\{ \lambda \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$B = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ claim $\text{null}(B) = \{ x \in \mathbb{R}^4 \mid Bx = 0 \}$

$x \in \text{null}(B) \Rightarrow x_1 + x_2 - 2x_3 = 0 \ , \ x_4 = 0$

so the general element of null(B looks like (setting
$x_3 = \lambda \ , \ x_1 = \mu \ \Rightarrow \ x_2 = 2\lambda - \mu$)

$\begin{pmatrix} \mu \\ 2\lambda - \mu \\ \lambda \end{pmatrix}$ $\text{null}(B) = \left\{ u \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$