MA1114 8/12/21

Column Space and Null Space ctd.

last time: Proposition 4.29

let $A \in M_{m,n}$, the linear system $Ax=b$ is consistent

$\Leftrightarrow b \in col(A)$

$V_1, V_2, \dots, V_r \in V$

$\langle V_1, V_2, \dots, V_r \rangle = \{\sum \lambda_i v_i \mid \lambda_i \in \mathbb{R}\}$

$\langle S \rangle$ = $(S = \{V_1, V_2, \dots, V_r\})$

$$A = \begin{pmatrix} \uparrow & & \uparrow \\ a_1 & \cdots & a_n \\ \downarrow & & \downarrow \end{pmatrix} \quad col(A) = \left\langle \begin{pmatrix} \uparrow \\ a_1 \\ \downarrow \end{pmatrix} \cdots \begin{pmatrix} \uparrow \\ a_n \\ \downarrow \end{pmatrix} \right\rangle$$

Suppose $S \subset V$ vector space where S is a collection of vectors $= \{a_1, \dots, a_n\}$

$b \in \langle S \rangle$

$b = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n \,,\quad x_i \in \mathbb{R}$

$\Rightarrow Ax = b$ has a solution where:

$$\rightarrow A = \begin{pmatrix} \uparrow & & \uparrow \\ a_1 & \cdots & a_n \\ \downarrow & & \downarrow \end{pmatrix}$$

Example

① Is $b = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$ a linear combination of $a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

i.e. find x_1, x_2 such that $a_1 x_1 + a_2 x_2 = b$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

$x_1 = 2$
$x_2 = 1$

$x_1 + x_2 = 3$
$x_1 + x_2 = 3$
$x_1 - x_2 = 1$

$2x_2 = 2$
$x_2 = 1$
$x_1 = 1$

② $\left\langle \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ -6 \\ -7 \end{pmatrix} \right\rangle \neq \mathbb{R}^3$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & x \\ -4 & 2 & -6 & y \\ -3 & -2 & -7 & z \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 4 & x \\ 0 & 14 & 10 & y+4x \\ 0 & 7 & 5 & z-3x \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 4 & x \\ 0 & 0 & 0 & y+4x-2z-6x \\ 0 & 7 & 5 & z-3x \end{array}\right]$$

inconsistent, column space does not span

if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow y + 4x - 2(z-3x) = 1+4-2(1+4) = -3 \neq 0$

$\Rightarrow col(A) \neq \mathbb{R}^3$

③ Show that $\left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \right\rangle = \mathbb{R}^2$ find λ_1 and λ_2

$Ax = b$ $\left[\begin{array}{cc|c} 1 & -1 & x \\ 2 & 4 & y \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 6 & y-2x \end{array}\right]$

$A\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\to \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & \frac{y-2x}{6} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{y-2x}{6} + x \\ 0 & 1 & \frac{y-2x}{6} \end{array}\right]$

$$\lambda_1 = \frac{y+4x}{6}$$

$$\lambda_2 = \frac{y-2x}{6}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{y+4x}{6}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{y-2x}{6}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

__Theorm__

If $a_1, a_2, \ldots, a_n \in \mathbb{R}^n$

$\langle a_1, a_2, \ldots, a_n \rangle = \mathbb{R}^n$

$\Leftrightarrow A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ a_1 & a_2 & \cdots & a_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$ is invertible

__Proof__

$\langle a_1, a_2, \ldots, a_n \rangle = \mathbb{R}^n$

$Ax = b$ always has a solution and consistent for all $b \in \mathbb{R}^n$

$A = a_1, a_2, \ldots, a_n = (a_{nn})$

$\Leftrightarrow$ A is invertible.

<u>Definition</u> $m, n > 0$, $A \in M_{m,n}$

$null(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$ is called the null space of A

if $n = m$ then A is invertible

$\Leftrightarrow null(A) = \{0\}$