MA1114 16/11/21

Determinant of Matrix Product

Proposition

let $A = \begin{bmatrix} a_{11} & * & \cdots & * \\ & a_{22} & * \cdots & * \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$ then $\det(A) = a_{11} a_{22} a_{33} \cdots a_{nn}$

Proof: By induction, base case $n=2$, assume true for $(n-1) \times (n-1)$ matrices

expanding $\det(A)$ using 1^{st} column

$$\det(A) = a_{11}(-1)^2 \det(\hat{A}_{11}) = a_{11}(a_{22} \cdots a_{nn}) \text{ as needed}$$

Proposition (determinants of elementary operations)

(i) $\det(X_n(i, \lambda) A) = \lambda$

(ii) $\det(Y_n(i, j) A) = -1$

(iii) $\det(Z_n(i, j, \lambda) A) = 1$

Proof set $A = I_n$ in proposition 3.47 I

(i) $\det(X_n(i, \lambda) A) = \lambda \det(I_n)$

(ii) and (iii) are similar.

Corollary

If $A \in M_{n,n}$ and E is an elementary matrix then

$$\det(E_i A) = \det(E_i)\det(A)$$

case 1: $E = X_n(i, \lambda)$:

$$\det(EA) = \lambda \det(A) = \det(E)\det(A)$$

case 2: $E = Y_n(i,j)$:

$$\det(EA) = -\det(A) = \det(E)\det(A) = -1 \cdot \det(A)$$

case 3: $E = Z_n(i,j,\lambda)$

$$\det(EA) = \det(A) = \det(E)\det(A)$$

Upshot: if $E_1 E_2 \quad E_r$ are elementary matrices and $A \in M_{n,n}$

$$\det(E_1 E_2 E_3 \cdots E_r A) = \det(E_1)\det(E_2 E_3 \cdots E_r A)$$
$$= \det(E_1)\det(E_2)\det(E_3)\cdots\det(E_r)\det(A)$$

Upshot: we can use reduction to calculate $\det(A)$

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{bmatrix} = \tilde{A}$$

so $A = Y_3(1,2)^{-1} X(1, \frac{1}{3})^{-1} Z(3,1,2)^{-1} Z_3(3,2,-10)^{-1}(\tilde{A})$

$$\det(A) = -1 \cdot 3 \cdot 1 \cdot 1 \cdot -55 = 165$$