MA1114 15/2/22

Composition of Injective and Surjective Functions

Reminder

A function $f: X \to Y$ between sets X and Y is injective if for any $a, b \in X$

$$f(a) = f(b) \Rightarrow a = b$$

f is surjective if $\operatorname{im}(f) = X$

Proposition 8.23

Let X, Y, Z be sets and $f: X \to Y$, $g: Y \to Z$ be functions

(i) if f, g are injective then $g \circ f$ is injective

(ii) " " " " surjective then $g \circ f$ is surjective

Proof

(i) $g \circ f: X \to Z$ let $a, b \in X$

suppose $(g \circ f)(a) = (g \circ f)(b)$

$$\Rightarrow g(f(a)) = g(f(b))$$

g injective $\Rightarrow f(a) = f(b)$

f injective $\Rightarrow a = b$

$\Rightarrow g \circ f$ is injective.

(ii) suppose f, g are surjective

if $z \in Z$ Want to show there is $x \in X$ such that $(g \circ f)(x) = z$

since g is surjective, there is $y \in Y$ such that $g(y) = z$

since f is surjective, there is $x \in X$ such that $f(x) = y$

now $(g \circ f)(x) = g(f(x))$

$= g(y) = z$ so $g \circ f$ is surjective $\boxtimes$

<u>Proposition</u> Let V, W be the vector spaces and suppose $T: V$ is linear.

T is surjective $\Leftrightarrow \ker(T) = \{0\}$

<u>Proof</u>

"$\Rightarrow$" Suppose T is injective and let $v \in \ker(T)$

$\Rightarrow T(v) = 0$
$\Rightarrow T(v) = T(0)$
$\Rightarrow 0 = 0$ since T is injective
$\Rightarrow \ker(T) = \{0\}$

Suppose $\ker(T) = \{0\}$
suppose $v, w \in V$
and $T(v) = T(w)$
$\Rightarrow T(v) - T(w) = 0$
$T(v - w) = 0$
$\Rightarrow v - w \in \ker(T) = \{0\} \Rightarrow v = w$, T is injective $\boxtimes$

$T(v - w)$
$T(1 \cdot v - 1 \cdot w)$
$1\, T(v) - 1\, T(w)$
$T(v) - T(w)$