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Unordered Sampling with(out) Replacement

Example

A train with n coaches is boarded by r passengers ($r \leq n$) each entering a coach at random What is the probability of the passengers all ending up in different coaches?

$$\Omega = \{(i_1, \dots, i_r) \mid i_j \in \{1, 2, \dots, n\} \text{ for } 1 \leq j \leq r\}$$

$|\Omega| = n^r$, all outcomes equally likely

$A = \{$ all passengers choose a different coach $\}$

A occurs $\Leftrightarrow i_j \neq i_k$ for $j \neq k$

1st passenger $= n$ 2nd passenger $= n-1$

each coach can only be chosen once: sampling without replacement

$$|A| = n(n-1) \times \dots \times (n-r+1)$$

$$P(A) = \frac{n(n-1) \cdots (n-r+1)}{n^r}$$

Unordered sampling without replacement

we disregard the order then $\{1, 3, 4, 2\} = \{1, 2, 3, 4\}$

Proposition (4)

Suppose we disregard the order of elements in the combinations $[a_{i_1}, \ldots, a_{i_r}]$, $a_{i_j} \neq a_{i_e}$, for $j \neq e$, which are taken from the set of n elements $\{a_1, \ldots, a_n\}$ $(r \leq n)$. Then the number of unordered samples of r elements from n, is given by

$$C^n_r = \frac{n!}{(n-r)!\,r!}$$

Proof (4)

- The number of permutations of r objects or the number of ways r objects can be ordered, is $r!$.

- By prop. (3) we know that to choose an ordered sample of size r from n objects is equal to

$$\frac{n!}{(n-r)!}$$

- Since the order does not matter and since there are $r!$ ways of ordering a sample size r we have

$$\frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!\,r!} = C^n_r,$$

ways of choosing an unordered sample of size r from n objects and prop. (4) follows

Example: Quality Control Problem

A batch of 100 items are manufactured in a factory, of which 10 are defective items and the remaining 90 are non-defective. An inspector examines 10 items selected at random. If none of the 10 items is defective, the batch is accepted. Otherwise the batch is rejected What is the probability the batch is accepted?

$$\Omega = \{ \omega = [a_{i_1}, \dots, a_{i_{10}}] \mid a_i \in \{a_1, \dots, a_{100}\}\}, a_{i_j} \neq a_{i_k} \text{ if } j \neq k \}$$

The number of possible outcomes: number of ways of selecting 10 out of 100 without replacement, disregarding the order

By prop.(4) $|\Omega| = C_{10}^{100} = \dfrac{100!}{90!\,10!}$

All equally likely, since objects selected "at random"

let A be the even that "the batch is accepted by the inspector"

A consist of those outcomes where all 10 outcomes chosen belong to the 90 non-defective items

$$|A| = C_{10}^{90} = \frac{90!}{10!\,80!}$$

finally $P(A) = \dfrac{|A|}{|\Omega|} = \dfrac{90!\,10!\,90!}{10!\,80!\,100!} \approx .33$

Unordered Samples with Replacement

Proposition (5)

Let each element a_i of an unorded sample $[a_{i_1}, \dots, a_{i_r}]$ be selected from the set $\{a_1, \dots, a_n\}$ (we allow the possibility of duplicates). The number of unordered samples with replacement of r objects from n is given by

$$N = C_r^{n+r-1} = \frac{(n+r-1)!}{r!\,(n-1)!}$$

Example:

the set = $\{a, b, c\}$ and we choose 2 samples from the set with replacement. ($n = 3$, $r = 2$). We have:

$$\frac{(n+r-1)!}{r!(n-1)!} = \frac{(3+2-1)!}{2!\,(3-1)!} = \frac{4!}{2!\,2!} = 6 \text{ different ways}$$

which are:

$\rightarrow$ $[a, \{b, c\}]$, $[b, \{a, c\}]$, $[c, \{a, b\}]$ (2 in each)

	ordered	unordered
with repl.	n^r	C_r^{n+r-1}
without repl.	$\frac{n!}{(n-r)!}$	C_r^n