MA1114 8/11/21

Inverse Algorithm ctd.

<u>lemma:</u> elementary matrices are invertible

note: since the products of invertible matrices is invertible suffices to show each E_i invertible.

<u>Proof</u> determine the inverses of

$X(i,\lambda)$, $Y(i,j)$, $Z(i,j,\lambda)$

inverse of: $X(i,\lambda)$ is $X(i,\lambda^{-1})$
$Y(i,j)$ is $Y(i,j)$
$Z(i,j,\lambda)$ is $Z(i,j,-\lambda)$ ⊛

<u>Proposition</u> If A is invertible, the inverse algorithm works.

<u>Theorm 3.20</u>

the following are equivalent

(i) A is invertible
(ii) linear system $Ax=0$ has 1 solution
(iii) the reduced row echelon form of A is the identity matrix I_n
(iv) A is product of elementary matrices

<u>Proof</u> we prove (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i)

(i) ⇒ (ii) suppose A is invertible
suppose $Ax=0 \Rightarrow A^{-1}(xA) = A^{-1}0 = A^{-1}Ax = A^{-1}0 = I_n x = A^{-1}Ax = 0$
(ii) ⇒ (iii) suppose $Ax=0$ has a unique solution
(A) Gauss $\begin{pmatrix} 1 & \cdots & a \\ 0 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \ddots & * \\ & & 1 \end{pmatrix}$ Gauss-jordan $\begin{pmatrix} 1 & \cdots & \\ & & 1 \end{pmatrix}$
no solution without a leading 1

(iii) $\Rightarrow$ (iv) follow from inverse algorithm

$$(A \mid I_n) \to \dots \to (I_n \mid B)$$

by (part 3) hypothesis

know $B = E_1 E_2 \dots E_k$ (elementary matrices)
and $B = A^{-1}$
so $A = B^{-1} = (E_1, \dots E_k)^{-1} = (E_k^{-1} E_{k-1}^{-1}, \dots, E_1^{-1})$
is a product of elementary matrices

(iv) $\Rightarrow$ (i) clear: a product of elementary matrices is invertible

<u>Corollary</u> if $A, B \in M_{n,n}$

then AB is invertible $\Leftrightarrow$ A, B are invertible

<u>Proof</u> "$\Leftarrow$" $(AB)^{-1}$ makes sense and this must be <u>the</u> inverse of AB

"$\Rightarrow$" assume AB is invertible with inverse C
then $A(BC) = (AB)C = I_n$
by previous corollary BC is inverse of A

and $(CA)B = C(AB) = I_n$ so by previous corollary CA is inverse of B

Corollary check one get one free.

if $A, B \in M_{n,n}$ with $BA = I_n$ then $AB = I_n$ and A is invertible.

Proof of check one get one free

it suffices to show that $B = A^{-1}$ then ($AB = AA^{-1} = I_n$)

will show that Ax has a unique solution

suppose $Ax = 0$

then $0 = B0 = B(Ax) = I_n x$
$= x.$

So A is invertible by proposition

$B = I_n B = B(AA^{-1})$
$= (BA)A^{-1}$
$= I_n A^{-1}$
$= A^{-1}$