MA1114 16/3/22

# Absolute and Conditional Convergence

$\sum a_n \Rightarrow \lim_{n\to\infty} \to \neq 0$ divergent

$\to = 0$

$a_n \geq 0$

- ratio: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \sqrt[n]{a_n}$
- root
- comparison $\sim \frac{1}{n^p}$ $\begin{cases} \text{con. } p>1 \\ \text{div. } p \leq 1 \end{cases}$

$\begin{cases} >1 & \text{divergent} \\ <1 & \text{convergent} \\ =1 & \end{cases}$

$\lim_{n\to\infty} \frac{a_n}{1/n^p} = c$   $c \neq 0$

$\sum a_n$ and $\sum \frac{1}{n^p}$ share the same convergence.

$\sum |a_n|$   $b_n = |a_n|$ $\sum b_n$

logically:

| | | | |
|---|---|---|---|
| $\sum a_n$ | con. | $\sum \vert a_n\vert$ | con. |
| $\sum a_n$ | con. | $\sum \vert a_n\vert$ | div. |
| $\sum a_n$ | div. | $\sum \vert a_n\vert$ | con. |
| $\sum a_n$ | div. | $\sum \vert a_n\vert$ | div. |

× elim by theorems below.

<u>Theorem</u> Let $\{a_n\}$ be a real sequence. Then if $\sum |a_n|$ is convergent, then $\sum a_n$ is convergent.

Proof: $\forall \varepsilon > 0$

$\{S_n\}$ is Cauchy $\Leftrightarrow \forall \varepsilon > 0 \ \exists N \ \forall m, n > N$

$|S_m - S_n| < \varepsilon$

$\forall \varepsilon > 0, \ \exists N \ \forall m > n > N \quad |a_{n+1} + \cdots + a_m| < \varepsilon$

$|a_{n+1}| + \cdots + |a_n| < \varepsilon$

then $|a_{n+1} + \cdots + a_m| \leq |a_{n+1}| + \cdots + |a_m| < \varepsilon$

then $\sum a_n$ is convergent

Definition: Consider $\sum a_n$. If $\sum |a_n|$ is convergent, then we say that $\sum a_n$ is absolutely convergent. If $\sum |a_n|$ is divergent, $\sum a_n$ is convergent, then we say the $\sum a_n$ is conditionally convergent.

Otherwise $\sum a_n$ is divergent

absolutely stronger than conditionally

Example 1. If $a_n \geq 0$ then $|a_n| = a_n$

abs. conv $\Leftrightarrow$ condi. conv.

$\sum \frac{1}{n^2} \quad \sum \frac{1}{2^n} \quad \sum \frac{1}{n(n+1)}$

Example 2 $\sum \frac{(-1)^n}{n^2} \qquad \sum \frac{1}{n^2}$

is also abs. conv.

**Example 3** $\sum \frac{1}{n} \Rightarrow$ divergent

**Theorem** Let $\{a_n\}$ be a real sequence. $a_n = (-1)^{n+1} b_n$ and $\{b_n\}$ satisfies

(1). $\{b_n\}$ is decreasing $\forall n \in \mathbb{N}$ $b_{n+1} \le b_n$

(2) $\lim_{n\to\infty} b_n = 0$ $\quad\quad (a_n \to 0 \ (\text{as } n\to\infty))$

then $\sum_{n=1}^{\infty} a_n$ is convergent. (Leibnitz test.)

**Proof:**

$$S_{2n} = a_1 + a_2 + a_3 + a_4 + \dots + a_{2n-1} + a_{2n}$$

$$= (b_1 - b_2) + (b_3 - b_4) + \dots + (b_{2n-1} - b_{2n}) \geq 0$$

$$b_1 + \underbrace{(b_3 - b_2)}_{\le 0} + \underbrace{(b_5 - b_4)}_{\le 0} + \dots + \underbrace{(b_{2n-1} - b_{2n})}_{\le 0} - b_{2n}$$

$$\le b_1 - b_{2n} \le b_1$$

Then $\{S_{2n}\}$ has a upper bound $b_1$

$$S_{2n+2} - S_{2n} = (b_{2n+1} - b_{2n+2}) \geq 0$$

$\{S_{2n}\} \nearrow$

Then $\{S_{2n}$ is convergent.

Let $S^* = \lim_{n\to\infty} S_{2n}$

$$S_{2n+1} = S_{2n} + a_{2n+1} \to S^* + 0 = S^*$$

$\lim_{n\to\infty} S_n = S$

Example 4 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Set $b_n = \frac{1}{n}$, $\{b_n\}$ is decreasing

$\lim_{n\to\infty} b_n = 0$ by Leibnitz test.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent

Then $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent.

$\sum a_n \Rightarrow \lim_{n=1} (a_n) \to \neq 0$ divergent

$\to = 0$

$\downarrow$

$\sum |a_n|$

ratio: $<1$, $\boxed{=1}$, $>1$

root: $<1$, $\boxed{=1}$, $>1$

$\to$ comparison

con $\Rightarrow$ absolutely

$\triangle$ Remark If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho > 1$ then $\sum a_n$ is divergent.

$$\left| \left|\frac{a_{n+1}}{a_n}\right| - \rho \right| < \varepsilon \qquad \rho - \varepsilon > 1$$

$$\left|\frac{a_{n+1}}{a_n}\right| > \rho - \varepsilon$$

$$|a_{n+1}| > \rho - |a_n| > \dots > (\rho - \varepsilon)^{n-N} |a_N|$$

$$= \underbrace{(\rho-\varepsilon)^n}_{} \underbrace{\frac{|a_N|}{(\rho-\varepsilon)^N}}_{\text{constant}} \not\to 0$$

then $\lim\limits_{n\to\infty} |a_n| \neq 0$

$$\Updownarrow$$

$\lim\limits_{n\to\infty} a_n \neq 0$ so $\sum a_n$ is divergent.

<u>Example 5</u>

$$\sum (-1)^n \cdot \frac{n}{n+1}$$

$\lim\limits_{n\to\infty} (-1)^n \cdot \frac{n}{n+1}$ does not exist $\Rightarrow$ div

<u>Example 6</u>

$$\sum (-1)^n \frac{1}{2^n} \left(1+\frac{1}{n}\right)^{n^2}$$

let $|a_n| = \left| (-1)^n \frac{1}{2^n} \left(1+\frac{1}{n}\right)^{n^2} \right|$

$\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim \frac{1}{2} \left(1+\frac{1}{n}\right)^n = \frac{e}{2} > 1$

then $\sum\limits_{n=1}^{\infty} a_n$ is divergent

<u>Power Series</u>

$\sum\limits_{n=0}^{\infty} a_n \cdot x^n$ , $x \in \mathbb{R}$


<u>Ex</u> $\sum\limits_{n=0}^{\infty} \frac{x^n}{n}$

let $a_n = \frac{x^n}{n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right|$$

$$= |x| \cdot \frac{n}{n+1} \longrightarrow |x|$$

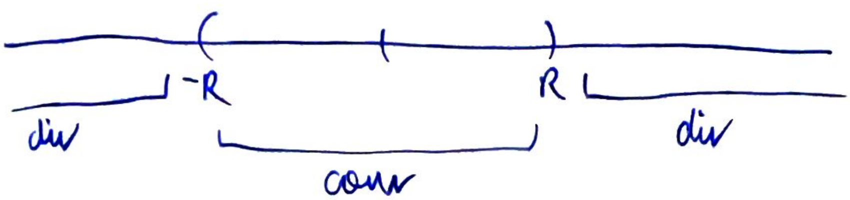
If $|x| < 1$ then $\sum_{n=0}^{\infty} \frac{x^n}{n}$ is conv

$|x| > 1$ then $\sum_{n=0}^{\infty} \frac{x^n}{n}$ is div

$|x| = 1$, $\sum_{n=0}^{\infty} \frac{1}{n}$ is div

$= -1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ is conv.

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ is convergent $x \in [-1, 1)$
divergent elsewhere




R – convergent radius