MA1114 16/3/22

Pf of distinct e.values → LI e.vectors ; G.mult ≤ A.mult ; diagable iff n.

Definition $T: V \to W$ is diagonalisable if there exists a bases $\mathcal{B}$ of V of eigenvectors of T

Proposition If $v_1, v_2, \ldots, v_k$ are eigenvectors with distinct eigenvalues then $\{v_1, v_2, \ldots, v_k\}$ is linearly independant.

Proof

Suppose $\{v_1, v_2, \ldots, v_k\}$ is linearly dependant. Can choose $m < k$ maximal such that $\{v_1, v_2, \ldots, v_k\}$ is linearly dependant

claim $\{v_1, \ldots, v_{m+1}\}$ is linearly independant

(contradictory maximality of m)

Sps. $\sum_{i=1}^{m+1} \mu_i v_i = 0$

$$0 = T(0) = T\left(\sum_{i=1}^{m+1} \mu_i v_i\right)$$

$$= \sum_{i=1}^{m+1} \mu_i T(v_i) = \sum_{i=1}^{m+1} \mu_i \lambda_i v_i$$

(where λ_i is an eigenvalue for v_i)

$$0 = \lambda_{m+1} \cdot 0 - 0$$

$$= \lambda_{m+1} \left(\sum_{i=1}^{m+1} \mu_i v_i\right) - \sum_{i=1}^{m+1} \mu_i \lambda_i v_i$$

$$= \sum_{i=1}^{m+1} v_i (\lambda_{m+1} \mu_i - \mu_i \lambda_i)$$

$$= \sum_{i=1}^{m+1} \mu_i v_i (\lambda_{m+1} \mu_i - \lambda_i)$$

$$= \sum_{i=1}^{m} \mu_i v_i (\lambda_{m+1} - \lambda_i)$$

$$\Rightarrow \mu_i \underbrace{(\lambda_{m+1} - \lambda_i)}_{\neq 0 \text{ since } \lambda_i \text{ distinct}} = 0 \quad (\text{since } \{v_1, \dots, v_k\} \text{ LI})$$

$\neq 0$ since λ_i distinct $\Rightarrow \mu_i = 0 \ \forall 1 \le i \le m$

$$\Rightarrow 0 = \sum_{i=1}^{m+1} \mu_i v_i = \mu_{m+1} v_{m+1}$$

eigenvectors

$$\Rightarrow \mu_{m+1} = 0 \quad \text{since } v_{m+1} \neq 0$$

$\Rightarrow \{v_1, \dots, v_k\}$ is linearly independant

contradiction

$\times$ since m chose maximally with this property

so $\{v_1, \dots, v_k\}$ is LI

$\boxed{}$

Corollary

If $\dim(V) = n$ and $T: V \to V$ has n distinct eigenvalues then T is diagonalisable

Proof

Suppose $\lambda_1, \dots, \lambda_n$ are distinct let $0 \neq v_i$ be an eigenvector for λ_i then $B = \{v_1, \dots, v_n\}$ is linearly independant

So B is a basis by COGOF for basis

actually can do better..

Theorem

T is diagonalisable $\iff n = \sum_{\lambda \text{ distinct}} \dim(V_\lambda)$

geometric multiplicity of A

(just done case $\dim(V_\lambda) = \quad \forall \lambda$ distinct.)

Proposition

If $T: V \to V$ and λ is an eigenvalue of T. Then geometric multiplicity of $\lambda \leq$ algebraic . . . .

$\dim(V_\lambda)$

max k such that $(t-\lambda)^k$

divides $\chi_T(t)$

Recall

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -1 & -5 & 6 \\ 2 & -2 & 0 \end{bmatrix} \quad \chi_A(t) = t^2(t-4)$$

λ	alg.	geom.
0	2	1
-4	1	1

V_0, V_{-4}

both with a dim of 1.

Proof of Proposition

Let $\{v_1, \dots, v_k\}$ be a basis for $V_\lambda \leq V$. Extend to a basis $\mathcal{B} =$
$= \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ $(n = \dim(V))$

Represent T using $\mathcal{B}$, $A = [T]_{\mathcal{B}}$

$$A = [T]_{\mathcal{B}} = \left([T(v_1)]_{\mathcal{B}} [T(v_2)]_{\mathcal{B}} \cdots [T(v_n)]_{\mathcal{B}} \right)$$

$$= \left([T(v_1)]_{\mathcal{B}} \cdots [T(v_k)]_{\mathcal{B}} [T(v_{k+1})]_{\mathcal{B}} \cdots [T(v_n)]_{\mathcal{B}} \right)$$

$$\chi_T(t) = \chi_A(t) = \det(A - It)$$

$$\begin{array}{c} \text{kth column} \\ = \det \begin{bmatrix} \lambda-t & & & & & \\ & \lambda-t & & & & \\ & & \lambda-t & 0 & & \\ & & & \lambda-t & & \\ & & & & \ddots & \boxed{?} \end{bmatrix} \end{array}$$

using column expansion of determinant then is $(\lambda - t)^k p(t)$

for some $p(t)$

so algebraic multiplicity of λ is at least k = geom multiplicity

$\blacksquare$

Example

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$ two eigenvalues

$2, 3$ $\quad \dim(V_2) = \dim(V_3) = 1$

$V_2 = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle \quad V_3 = \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle$

$1 + 1 = 2$

$\sum_{\lambda \text{ distinct}} \dim(V_\lambda) = 2$

Exercise

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

1. eigenvalues?
2. eigenvectors?
3. show $\sum \dim(V_\lambda) = 3 \quad (\lambda \text{ distinct} = n)$

eigenvalue $2 \quad 1$

$$\det(tI - A) \Rightarrow \det \begin{bmatrix} t-0 & & +2 \\ -1 & t-2 & -1 \\ -1 & 0 & t-3 \end{bmatrix} = t(t-2)(t-3) + 2(t-2)$$

$$= (t-2)(t(t-3)+2)$$

$$= (t-2)(t^2 - 3t + 2)$$

$$= (t-2)^2 (t-1)$$

$$= 2, 1$$