MA1014 22/3/22

Continuous Multivariable functions & Derivatives

$f(x,y)$ $\qquad \lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

$\forall \varepsilon>0\ \exists \delta>0,\ \forall x : x \in D$

$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

$|f(x,y)-L| < \varepsilon$

<u>Definition</u> Let $f(x,y)$ be a 2-variable function defined on $D \subseteq \mathbb{R}^2$. $x_0 \in D$ an interior point.

we say that f is continous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \underline{f(x_0,y_0)}$$

- Functions of poly, exp, sin, cos, ln, $\pm$ $\times$ $\div$ $\circ$ are continuous in their domain

$f(x,y) = (\cos x)e^{(x^2+y^2)\arctan x}$ is continuous on $\mathbb{R}^2$

$f(x,y)\ \frac{x^2y}{x^2+y^2}$ then $f(x,y)$ is continuous on $\mathbb{R}^2$, $\mathbb{R}^2\setminus\{(0,0)\}$

$f(x,y) = \ln(x-y^2)$ then $f(x,y)$ is continuous on $\{(x,y) \mid x-y^2>0\}$

Example 1 $f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2} & , (x,y)\neq(0,0)\\ 0 & , (x,y)=(0,0)\end{cases}$

$$\lim_{\substack{x\to0\\ y\to0\\ (x,y)\to(0,0)}}\frac{x^2y}{x^2+y^2}\overset{?}{=}0=f(0,0)$$

$\lfloor\, x^2+y^2\geq 2|xy|$

$$0\leq\left|\frac{x^2y}{x^2+y^2}\right|=\frac{x^2|y|}{x^2+y^2}\leq\frac{x^2|y|}{2|xy|}=\frac{|x|}{2}$$

$\to 0$

then $f(x,y)$ is continuous on $\mathbb{R}^2$

Example 2 $g(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & , (x,y)\neq(0,0)\\ 0 & , (x,y)=(0,0)\end{cases}$

$\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$ does not exists

(($y=x$ $\qquad$ $y=2x$

$\lim_{x\to0}\frac{x^2}{x^2+x^2}=\frac{1}{2}$ $\qquad$ $\lim_{x\to0}\frac{2x^2}{x^2+4x^2}=\frac{2}{5}$

then $g(x,y)$ is not continuous at $(0,0)$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{\partial f(x,y)}{\partial x}=\lim_{h\to0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial f(x,y)}{\partial y}=\lim_{h\to0}\frac{f(x,y+h)-f(x,y)}{h}$$

partial derivatives

Example 3 $f(x,y) = x \arctan(xy) + e^{2y}$

$$\frac{\partial f}{\partial x} = 1 \cdot \arctan(xy) + x \cdot \frac{1}{1+(xy)^2} \cdot y$$

$$\frac{\partial f}{\partial y} = x^2 \frac{1}{1+(xy)^2} + 2e^{2y}$$

$f'(x)$ $\frac{df}{dx}$ $\dot{f}$

$\boxed{f'_x \quad f'_1}$ $\boxed{\frac{\partial f}{\partial x}}$ $D_1 f$ $D_x f$ notations

Example 4 $f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

compute $\frac{\partial f}{\partial x}$

solution: $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{2xy(x^2+y^2) - 2x^2y^2}{(x^2+y^2)^2}$$

$$\lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \cdot 0}{h^2 + 0^2} \cdot \frac{h}{h} = 0$$

Example 5 $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

is continuous at $(0,0)$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0 \quad \text{exists}$$
$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$$

Example 6 $f(x,y) = |x| \qquad (x,y) \in \mathbb{R}^2$

does not have derivative on $\{(x,y) \mid x = 0\}$

$$\frac{\partial}{\partial (x)}\left(\frac{\partial f}{\partial (x)}\right) \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x^2} \quad \frac{\partial^2 f}{\partial y \partial x} \quad \frac{\partial^2 f}{\partial x \partial y} \quad \frac{\partial f}{\partial y^2}$$

$$f''_{xx} \quad f''_{xy} \quad f''_{yx} \quad f''_{yy}$$

$$f''_{11} \quad f''_{12} \quad f''_{21} \quad f''_{22}$$

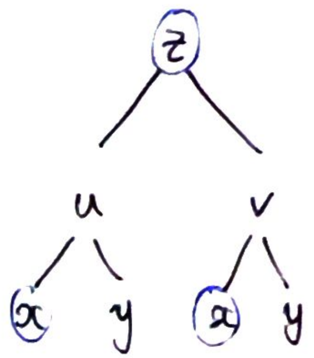
Theorem $f(x,y) : D \to \mathbb{R}$ $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ exists and continuous on D then $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$ $(x,y) \in D$

Chain Rule

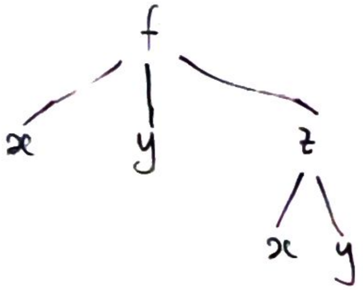
$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

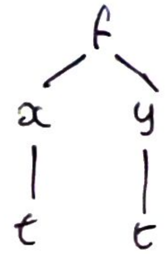
$$z = e^{u} \operatorname{sen} v \qquad u = xy \qquad v = x+y$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$





$$\frac{\partial f}{dx} = f_1' + f_3' \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial f}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dx}{dt}$$

$$z = e^{xy} \operatorname{sen}(x+y)$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

$s = x+y$

$t = x-y$