MA1061 1/11/21

Total Probability Theorm

Definition (Decomposition)

We say that a collection of subsets of Ω, $D = \{A_1, \ldots, A_n\}$ is a decomposition of Ω if

① the A_i are not empty, $A_i \neq \emptyset$;

② pairwise disjoint, ie for $i = j$, $A_i \cap A_j \neq \emptyset$; and

③ their sum (disjoint union) is the whole sample space,

$$A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = \Omega$$

A decomposition $D = \{A_1, \ldots, A_n\}$ with $P(A_i) > 0$ for all $i = 1, \ldots, n$ is often called a partition

Theorm (total probability theorm)

Consider a partition $D = \{A_1, A_2, \ldots, A_n$, and an event $B \in \mathcal{A}$. Then

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i)\, P(A_i).$$

* essential for calculating probabilities of complicated events.

Proof of total probability theorm

- clearly $B = (B \cap A_1) \cup \cdots \cup (B \cap A_n)$

since A_i is disjoint so are $B \cap A_i$.

thus $P(B) = P(B \cap A_1) \cup \cdots \cup P(B \cap A_n)$.

but by the multiplication formula

$$P(B \cap A_i) = P(B \mid A_i) \times P(A_i)$$

theorm is proved.

Problem

On Sundays you either stay in or go play football, depending on the weather. If it rains, the probability of you playing football is 0.2, while the corresponding probability is 0.6 if it doesnt. It is forcasted that on Sunday is 0.2. What is the probability of you playing football this Sunday?

A be the event it will rain on Sunday then A & $\bar{A}$ form a partition of Ω: $\Omega = A \cup \bar{A}$

B be the event that you go play football. We want $P(B)$

$P(A) = 0.2$, $P(\bar{A}) = 0.8$, $P(B|A) = 0.2$, $P(B|\bar{A}) = 0.6$

$$P(B) = P(B|A)\,P(A) + P(B|\bar{A})\,P(\bar{A})$$
$$= 0.2 \times 0.2 + 0.6 \times 0.8 = 0.52.$$

Application of total probability theorm

A box contains M balls, m of which are blue and the rest are white. Draw two balls without replacement. Using the TP theorm and conditional probability find the probabilities that the first ball is blue and that the 2nd ball is blue, assuming that the balls are drawn at random.

$\Omega = A \cup \bar{A}$

where A is the event that the 1st ball is blue

B is the event that the second ball is blue.

we want $P(B)$

$P(A) = m/M$ $\quad$ $P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A}) = P(B \cap A) + P(B \cap \bar{A})$

then $P(B \cap A) = \dfrac{m(m-1)}{M(M-1)}$ and $P(B \cap \bar{A}) = \dfrac{(M-m)m}{M(M-1)}$

thus $P(B) = m/M$

it is interesting to observe that $P(A) = P(B) = m/M$

thus when the result of the 1st ball is unknown, doesn't affect 2nd being lucky.

Generalised Multiplication Theorm

$$P(A_1 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1) \cdots P(A_n|A_1 \cap \cdots A_{n-1})$$