MA1014 1/11/21

Definition of the Derivative

Idea: derivatives are limits

derivative: difference quotient

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$f(x) = x^2$

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c}$$

if $x \neq c$ $\quad \dfrac{x^2 - c^2}{x - c} = \dfrac{(x+c)(x-c)}{x-c}$

$= x + c$

$f'(c) = 2c$

$f(x) = x^2 \Rightarrow f'(x) = 2x$

other notation $\quad y = f(x)$

$Dy \qquad \dfrac{dy}{dx} = f'(x)$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \left(\frac{\Delta y}{\Delta x}\right)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Same example as above

$f(x) = x^2$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x+h) - f(x)$
$= (x+h)^2 - x^2$
$= 2xh + h^2$
$= (2x+h)h$

$$= \lim_{h \to 0} (2x+h) = 2x$$

Example using Difference Quotient def^n

✓ i) $f(x) = x^2$ $\qquad f'(x) = 2x$
✓ ii) $f(x) = x^3$
✓ iii) $f(x) = x^n$ $\qquad n = 1, 2, 3, ..$
iv) $f(x) = 1/x$
v) $f(x) = \sqrt{x} = x^{1/2}$

i) ✓

ii) $$\frac{x^3 - c^3}{x-c} \overset{\text{if } x \neq c}{=} \frac{\cancel{(x-c)}(x^2 + xc + c^2)}{\cancel{x-c}}$$

so $$\lim_{x \to c} \frac{x^3 - c^3}{x-c} = \lim_{x \to c} (x^2 + xc + c^2) = 3c^2$$

$f'(c) = 3c^2, \quad f'(x) = 3x^2$
$y = x^3, \quad \frac{dy}{dx} = 3x^2$

iii) $(x-c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \ldots + c^{n-2}x + c^{n-1}) = x^n - c^n$
All other terms appear twice with opposite signs & cancel

$$\lim_{x \to c} \frac{x^n - c^n}{x-c} = \lim_{x \to c} \left(\sum_{i=0}^{n-1} c^i x^{n-1-i} \right) = nc^{n-1}$$

$y = x^n \qquad \boxed{\frac{dy}{dx} = nx^{n-1}}$

Alternative formula

If $h \neq 0$, $\dfrac{(x+h)^n - x^n}{h} = \dfrac{\cancel{x^n} + nhx^{n-1} + \binom{n}{2}h^2x^{n-2} + \ldots + \cancel{-x^n}}{h}$

$= nx^{n-1} + \binom{n}{2}hx^{n-2} + \ldots$ higher powers of h

$\lim_{h \to 0} \dfrac{(x+h)^n - x^n}{h} = nx^{n-1}$

iv) $f(x) = \frac{1}{x}$, $x, c > 0$

$\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} = \lim_{x \to c} \dfrac{\frac{1}{x} - \frac{1}{c}}{x - c}$

$= \lim_{x \to c} \dfrac{\frac{c - x}{xc}}{x - c} = \lim_{x \to x} -\dfrac{1}{xc}$

$= -\frac{1}{c^2} = -c^{-2}$

$y = x^{-1}$, $\frac{dy}{dx} = -x^{-2}$

v) $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, $x, c \geq 0$

If $x \neq c$

$\dfrac{\sqrt{x} - \sqrt{c}}{x - c} = \dfrac{\cancel{\sqrt{x} - \sqrt{c}}}{\cancel{(\sqrt{x} - \sqrt{c})}(\sqrt{x} + \sqrt{c})}$

$f^{-1}(c) = \lim_{x \to c} \dfrac{1}{\sqrt{x} + \sqrt{c}} = \frac{1}{2} \frac{1}{\sqrt{c}} = \frac{1}{2} c^{-\frac{1}{2}}$

$y = x^{\frac{1}{2}}$, $\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$

(-1) $f(x) = 7$ $\quad f'(x) = 0$

(0) $f(x) = x$ $\quad \left(\frac{x-c}{x-c}\right)$ $\quad f'(x) = 1$