MA1114 14/2/22

Proof of Rank-Nullity Theorem

Theorem

Let V, W be vector spaces $T: V \to W$ a linear map. Then
$\text{rank}(T) + \text{nullity}(T) = \dim(V)$

Proof $\dim(V) = n$

$\ker(T) \leq V$, $\ker(T) < v_1, \ldots, v_n >$ some $v_i \in V$ extend to a basis of V. $V = \langle v_1, \ldots, v_k, v_{k+1}, \ldots, v_n \rangle$

set $X = \{ v_{k+1}, \ldots, v_n \}$

then claim $T(X) = \{ T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n) \}$ is a basis for $\text{im}(T)$

Then done since $\text{rank}(T) = n-k$
$\text{nullity}(T) = k$
and $(n-k) + k = n = \dim(V)$

Proof of claim

First check $T(X)$ spans $\text{im}(T)$ $w \in \text{im}(T) \Rightarrow \exists\, v \in V$ such that $T(v) = w$ but $v = \sum_{i=1}^{n} \lambda_i v_i$, some $\lambda_i \in \mathbb{R}$

$$w = T(v) = T\left(\sum_{i=1}^{n} \lambda_i v_i\right)$$

$$= \sum_{i=1}^{n} \lambda_i T(v_i)$$

$$= \sum_{i=1}^{k} \lambda_i T(v_i) + \sum_{i=k+1}^{n} \lambda_i T(v_i)$$

$= \sum_{i=k-1}^{n} \lambda_i (T(v_i))$, since $v_i \in \ker(T)$

$\Rightarrow$ w is a linear combination of elements of $T(X)$ so $T(x)$ spans the image.

assume, $\sum_{i=k+1}^{n} \mu_i T(v_i) = 0$ some $\mu_i \in \mathbb{C}$

$\Rightarrow T(\sum_{i=k+1}^{n} \mu_i v_i) = 0$ (since T is linear)

$\Rightarrow \sum_{i=k+1}^{n} \mu_i v_i \in \ker(T)$

$\Rightarrow \sum_{i=k+1}^{n} \mu_i v_i = \sum_{j=1}^{n} \mu_j v_j$ some $\mu_j \in \mathbb{C}$

$\Rightarrow \sum_{i=k+1}^{n} \mu_i v_i = \sum_{j=1}^{k} \mu_j v_j$ some $\mu_j \in \mathbb{C}$ $1 = j = n$

$\Rightarrow \sum_{i=k+1}^{n} \mu_i v_i - \sum_{j=1}^{k} \mu_j v_i = 0$

$\Rightarrow \sum_{i=k+1}^{n} \mu_i v_i + \sum_{j=1}^{k} (-\mu_j) v_j = 0$

$\Rightarrow \mu_i = 0$ for $1 \leq i \leq n$ since $\{v_1, \dots, v_n\}$ is a basis

In particular $\mu_{n+1} = \mu_{n+2} = \dots = 0$

$\Rightarrow$ $T(x)$ is linearly independant

so nullity(T) + rank(T) = dim(V)

since $T(k)$ is a bases for Im(T)