MA1114 23/2/22

Base Change Madness; Relate Matrix Rep of same LM w.r.t Different Basis via Base Change Matrices.

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "T"] \arrow[d, "E"', "v \mapsto {[v]_{\mathcal{B}}}"] & \mathcal{C} \arrow[d, "E'", "w \mapsto {[w]_{\mathcal{C}}}"'] \\
\mathbb{R} \arrow[r, "A = {}_{\mathcal{C}}[T]_{\mathcal{B}}"'] & \mathbb{R}
\end{tikzcd}$$

$w = T(v)$

$A = {}_{\mathcal{C}}[T]_{\mathcal{B}}$ matrix representing T

$$A = \Big[[T(v_1)]_{\mathcal{C}} \ [T(v_2)]_{\mathcal{C}} \cdots [T(v_n)]_{\mathcal{C}} \Big]$$

$$\{v_1, \dots, v_n\} = \mathcal{B}$$

Yesterday, suppose $V = W$ and $T = id$ suppose $\mathcal{B}_1, \mathcal{B}_2$ are 2 basis for V.

$$P_{\mathcal{B}_1 \to \mathcal{B}_2} = {}_{\mathcal{B}_2}[id]_{\mathcal{B}_1}$$

<u>Proposition</u> $v \in V$, $[v]_{\mathcal{B}_2} = P_{\mathcal{B}_1 \to \mathcal{B}_2} [v]_{\mathcal{B}_1}$

<u>Proof</u>

$$[v]_{\mathcal{B}_2} = [id(v)]_{\mathcal{B}_2} \overset{def}{=} {}_{\mathcal{B}_2}[id]_{\mathcal{B}_1} [v]_{\mathcal{B}_1}$$

$$= P_{\mathcal{B}_1 \to \mathcal{B}_2} [v]_{\mathcal{B}_1} \quad \blacksquare$$

Example $V = \mathbb{R}^2$

$$\mathcal{B} = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \right\} \qquad \mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{B}_1 = \mathcal{B} \quad , \quad \mathcal{B}_2 = \mathcal{E}$$

$$P_{\mathcal{B} \to \mathcal{E}} = {}_{\mathcal{E}}[id]_{\mathcal{B}}$$

$$= \left[\left[[id]\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_{\mathcal{E}} \left[[id]\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_{\mathcal{E}} \right]$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$ does it change basis ?

$$v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad [v]_{\mathcal{E}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad [v]_{\mathcal{B}} = \begin{pmatrix} x_2 \\ x_1 - x_2 \end{pmatrix}$$

since $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (x_1 - x_2) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$[v]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_2 \\ x_1 - x_2 \end{pmatrix} = \begin{bmatrix} x_2 + (x_1 - x_2) \\ x_2 \end{bmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = [v]_{\mathcal{E}}$$

what about $P_{\mathcal{E} \to \mathcal{B}}$?

$$= {}_{\mathcal{B}}[id]_{\mathcal{E}} = \left[\left[id\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_{\mathcal{B}} \left[id\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_{\mathcal{B}} \right]$$

$$= \left[\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_{\mathcal{B}} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_{\mathcal{B}} \right]$$

$\begin{pmatrix}1\\0\end{pmatrix} = 0v_1 + 1v_2$

$\begin{pmatrix}0\\1\end{pmatrix} = 1v_1 - 1v_2$ $\qquad P_{E\to B} = \begin{pmatrix}0 & 1\\1 & -1\end{pmatrix}$

$$P_{E\to B}\,[v]_E = \begin{pmatrix}0 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_2\\x_1 - x_2\end{pmatrix} = [v]_B$$

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & -1\end{pmatrix} = P_{E\to B}\,P_{B\to E}$$
$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

<u>lemma</u>

Let V be a finite dimensional vector space. Suppose B_1 and B_2 are bases.

$$P_{B_1\to B_2} = P_{B_2\to B_1}^{-1}$$

$$P_{B_2\to B_1}\,P_{B_1\to B_2} = {}_{B_1}[id]_{B_2}\;{}_{B_2}[id]_{B_1}$$
$$= {}_{B_1}[id \circ id]_{B_1}$$
$$= {}_{B_1}[id]_{B_1}$$
$$= I_n \text{ by proposition 8.47.}$$

<u>Example</u>

$\mathbb{R}^2$, $\mathcal{B} = \left\{ \underset{v_1}{\begin{pmatrix}1\\1\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}1\\2\end{pmatrix}} \right\}$ $\qquad \mathcal{E} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$

$$P_{B\to E} = {}_E[id]_B = \big[[id(v_1)]_E \; [id(v_2)]_E\big]$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$P_{E\to B} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

does this work? $P_{E\to B}[v]_B = [v]_E$?

<u>Proposition</u>

Let $V \to W$ be a linear map between vectors spaces B_1, B_2 basecs for V.
E_1, E_2 basis for W
Then ${}_{e_1}[T]_{B_1} = P^{-1}_{e_1\to e_2}\; {}_{e_2}[T]_{B_2}\; P_{B_1\to B_2}$

<u>Proof</u>

$$\begin{tikzcd} V \arrow[r, "{}_{e_1}[T]_{B_1}"] \arrow[d, "id"', "P_{B_1\to B_2}"'] & W \arrow[d, "id"', "P_{e_1\to e_2}"] \\ V \arrow[r, "{}_{e_2}[T]_{B_2}"'] & W \end{tikzcd}$$

<u>Corollary.</u>

$T: V \to V$, B_1, B_2 bases for v. then $P_{B_2\to B_1}\; {}_{B_2}[T]_{B_2}\; P_{B_1\to B_2}$
$= {}_{B_1}[T]_{B_2}$

$T: V \to V$
B_1 $T(v_1)$ $T(v_2)$

Example

$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \quad \mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$_{\mathcal{B}}[T]_{\mathcal{B}} = \left[\left[T\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\mathcal{B}} \left[T\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_{\mathcal{B}} \right]$$

$$= \left[\left[\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right]_{\mathcal{B}} \left[\begin{pmatrix} 3 \\ 6 \end{pmatrix}\right]_{\mathcal{B}} \right]$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$_{\mathcal{E}}[T]_{\mathcal{E}} = \left[\left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_{\mathcal{E}} \left[T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_{\mathcal{E}} \right]$$

$$= \left[\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right]_{\mathcal{E}} \left[\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right]_{\mathcal{E}} \right]$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$ wanna relate them two

$$P_{\mathcal{B}\to\mathcal{E}} = {}_{\mathcal{E}}[id]_{\mathcal{B}} = \left[\left[id\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\mathcal{E}} \left[id\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_{\mathcal{E}} \right] = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$P_{\mathcal{E}\to\mathcal{B}} = {}_{\mathcal{B}}[id]_{\mathcal{E}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

corollary says

$$P_{E\to B}\ {}_E[T]_E\ P_{B\to E} = {}_B[T]_E$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \checkmark$$