Kai Lee 413930 ECBM 6040 HW1 Theory section

ci $F_Y(y) = P(Y \le y) = P(-\frac{1}{\lambda}\ln X \le y) = P(\ln X \ge -\lambda y) = P(X \ge e^{-\lambda y}) = 1 - P(X \le e^{-\lambda y})$

Differentiate $F_Y(y)$ wrt to y, $\frac{d}{dy} F_Y(y) = -(-\lambda e^{-\lambda y}) F_X(e^{-\lambda y})$

$$\therefore f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & \text{if } -\infty \le y \le 0 \\ 0 & \text{otherwise} \end{cases}$$

cii $p(X=x, Y=y) = \begin{cases} 3(xy^2 + yx^2) & \text{for } x,y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

$p(X=x) = \int_0^1 3(xy^2 + yx^2)\, dy$

$= 3[x y^3/3 + y^2/2\, x^2]_0^1 = 3[x/3 + \frac{1}{2}x^2] = x + \frac{3}{2}x^2$

$p(y=y) = \int_0^1 3(xy^2 + yx^2)\, dx$

$= 3[x^2/2\, y^2 + y\, x^3/3]_0^1 = 3[\frac{1}{2}y^2 + \frac{1}{3}y] = y + \frac{3}{2}y^2$

$E(X) = \int_0^1\int_0^1 3x(xy^2 + yx^2)\, dy\, dx = 3\int_0^1 x(\frac{x}{3} + \frac{1}{2}x^2)\, dx = 3\int_0^1 \frac{x^2}{3} + \frac{1}{2}x^3\, dx = 3[\frac{1}{9}x^3 + \frac{1}{8}x^4]_0^1 = 17/24$

$E(y) = \int_0^1 3y \int_0^1 xy^2 + yx^2\, dx\, dy = 17/24$

$E(XY) = \int_0^1 3x \int_0^1 xy^3 + y^2x^2\, dy\, dx = \int_0^1 3x[x y^4/4 + y^3/3\, x^2]\, dx = \int_0^1 3x^2/4 + x^4\, dx = [3x^3/12 + x^4/4]_0^1$
$= 3/12 + 1/4 = 1/2$

X and Y are not independent as $E(XY) \ne E(X)E(Y)$

di $L(\mu, \Sigma) = \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{|\Sigma|^{n/2}} \exp(\sum_{i=1}^n -\frac{1}{2}(x_i - \mu)^T \Sigma^{-1} (x_i - \mu))$

$\ln(\mu, \Sigma) = -\frac{n}{2}\ln|\Sigma| + \sum_{i=1}^n (-\frac{1}{2}(x_i - \mu)^T \Sigma^{-1} (x_i - \mu)) + \text{constant}$

$d\ln/d\mu = \frac{d}{d\mu}\sum_{i=1}^n -\frac{1}{2} \underbrace{(x_i - \mu)^T \Sigma^{-1} (x_i - \mu)}_{\text{scalar } 1\times 1}$

$= -\frac{1}{2}\sum_{i=1}^n (x_i - \mu)^2 \Sigma^{-1}$

$= \sum_{i=1}^n (x_i - \mu)\Sigma^{-1}$

equate to 0, $\hat{\mu} = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$

$d\ln/d\Sigma = \frac{d}{}(-\frac{n}{2}\ln|\Sigma| + -\frac{1}{2}\sum_{i=1}^n (x_i - \bar{x})^T \Sigma^{-1} (x_i - \bar{x}) / d\Sigma$

$= -\frac{n}{2}\, \frac{d}{d\Sigma}|\Sigma| / |\Sigma| - d(\frac{1}{2}\sum_{i=1}^n ((x_i - \bar{x})^T \Sigma^{-1}(x_i - \bar{x})) / d\Sigma$

$= -\frac{n}{2}\, \cancel{|\Sigma|}(\Sigma^{-1})^T / \cancel{|\Sigma|} - \frac{1}{2}\sum_{i=1}^n (-\Sigma^{-T}(x_i - \bar{x})(x_i - \bar{x})^T \Sigma^{-T})$ matrix cookbook (63)

equate to 0, $(\Sigma^{-T}) = \frac{1}{n}\sum_{i=1}^n \Sigma^{-T}(x_i - \bar{x})(x_i - \bar{x})^T \Sigma^{-T}$

$\Sigma^T = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x})^T$

since Σ is symmetrical, $\Sigma^T = \Sigma = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x})^T$

d ii $E(\hat{\mu}) = E(1/n \sum_{i=1}^{n} x_i) = 1/n \sum_{i=1}^{n} E(x_i) = 1/n \sum_{i=1}^{n} \mu = n\mu/n = \mu$ $\therefore$ unbiased as $= \mu$

$E(\hat{\Sigma}) = E(1/n \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^T)$

$= 1/n\, E(\sum_{i=1}^{n} (x_i - \bar{x})(x_i^T - \bar{x}^T))$

$= 1/n\, E(\sum_{i=1}^{n} ((x_i - \mu) - (\bar{x} - \mu))((x_i^T - \mu^T) - (\bar{x}^T - \mu^T)))$

$= 1/n\, E(\sum_{i=1}^{n} ((x_i - \mu)(x_i - \mu)^T - (x_i - \mu)(\bar{x} - \mu)^T - (\bar{x} - \mu)(x_i - \mu)^T + (\bar{x} - \mu)(\bar{x} - \mu)^T)$

$= \underbrace{1/n\, E(\sum_{i=1}^{n} (x_i - \mu)(x_i - \mu)^T)}_{\Sigma.} + 1/n\, E(\sum_{i=1}^{n} (\bar{x} - \mu + \mu - x_i)(\bar{x} - \mu)^T - (\bar{x} - \mu)(x_i - \mu)^T)$

$= \Sigma + 1/n\, E(\sum_{i=1}^{n} \bar{x}\bar{x}^T - \cancel{\bar{x}\mu^T} - x_i \bar{x}^T + x_i \mu^T - \bar{x} x_i^T + \cancel{\bar{x}\mu^T} + \mu x_i^T - \mu\mu^T)$

$= \Sigma + 1/n \sum_{i=1}^{n} E(\bar{x}\bar{x}^T) - E(\frac{1}{n}\sum_{i=1}^{n} x_i \bar{x}^T) + E(\frac{1}{n}\sum_{i=1}^{n} x_i)\mu^T - E(\bar{x}\frac{1}{n}\sum_{i=1}^{n} x_i^T) + \mu E(\frac{1}{n}\sum_{i=1}^{n} x_i^T) - \mu\mu^T$

$= \Sigma + \cancel{E(\bar{x}\bar{x}^T)} - \cancel{E(\bar{x}\bar{x}^T)} + \mu\mu^T - E(\bar{x}\bar{x}^T) + \cancel{\mu\mu^T} - \cancel{\mu\mu^T}$

$= \Sigma - E(\bar{x}\bar{x}^T) + \mu\mu^T$

$= \Sigma - \underbrace{(E(\bar{x}\bar{x}^T) - \mu\mu^T)}_{cov(\bar{x})}$

$= \Sigma - \Sigma/n$

$= n-1/n\, \Sigma$ $\therefore$ biased