Problem 1

Metropolis: The Bounded Normal Mean. Suppose that we have information that the normal mean θ is bounded between m and m, for some known number m. In this case it is natural to elicit a prior on θ with the support on interval [m, m].
A prior with interesting theoretical properties supported on [m, m] is Bickel-Levit prior.

$$\pi(\theta) = \frac{1}{m}\cos^2(\frac{\pi\theta}{2m}), -m \leq \theta \leq m \tag{1}$$

Assume that a sample [-2, -3, 4, -7, 0, 4] is observed from normal distribution

$$f(y|\theta) \propto \sqrt{\tau}\exp\{-\frac{\tau}{2}(y-\theta)^2\} \tag{2}$$

with a known precision $\tau = 1/4$. Assume also that the prior on θ is Bickel-Levit, with m = 2.

This combination likelihood/prior does not result in an explicit posterior (in terms of elementary functions). Construct a Metropolis algorithm that will sample from the posterior of θ.

(a) Simulate 10,000 observations from the posterior, after discarding first 500 observations (burn-in), and plot the histogram of the posterior.
We know that there are 6 observations from the normal distribution. Thus we need to calculate the product of the likelihood.

$$f(y|\theta) \propto \prod_{i=1}^{6}\sqrt{\tau}\exp\{-\frac{\tau(y_i-\theta)^2}{2}\} \tag{3}$$

$$f(y|\theta) \propto \sqrt{\tau}^6\exp\{-\sum_{i=1}^{6}\frac{\tau(y_i-\theta)^2}{2}\} \tag{4}$$

Thus the posterior distribution is as follow:

$$\pi(\theta|y) = f(y|\theta)\pi(\theta) \tag{5}$$

$$\pi(\theta|y) \propto \sqrt{\tau}^6\exp\{-\sum_{i=1}^{6}\frac{\tau(y_i-\theta)^2}{2}\} * \frac{1}{m}\cos^2(\frac{\pi\theta}{2m}) \tag{6}$$

$$\pi(\theta|y) \propto \exp\{-\sum_{i=1}^{6}\frac{\tau(y_i-\theta)^2}{2}\} * \cos^2(\frac{\pi\theta}{2m}) \tag{7}$$

$$\tag{8}$$

Since we know that $\tau = 1/4$ and m= 2. Thus the posterior becomes:

$$\pi(\theta|y) \propto \exp\{-\sum_{i=1}^{6}\frac{(y_i-\theta)^2}{8}\} * \cos^2(\frac{\pi\theta}{4}) \tag{9}$$

$$\pi(\theta|y) \propto \exp\{-\sum_{i=1}^{6}\frac{(y_i-\theta)^2}{8}\} * (1+\cos(\frac{\pi\theta}{2}) \tag{10}$$

$$\tag{11}$$

Then

$$\rho = \frac{exp\{-\sum_{i=1}^{6}\frac{(y_i-\theta')^2}{8}\} * (1+\cos(\frac{\pi\theta'}{2})}{exp\{-\sum_{i=1}^{6}\frac{(y_i-\theta)^2}{8}\} * (1+\cos(\frac{\pi\theta}{2})} \tag{12}$$

We can sample θ' from the Uniform distribution $\mathcal{U}(-2,2)$.

$$q(\theta'|\theta) = \mathcal{U}(-2,2) \tag{13}$$

The initial value for theta is set to be 0. Then we can run the Metropolis algorithm. After 10,000 observations from the posterior, and discarding first 500 observations. The histogram of the posterior is:

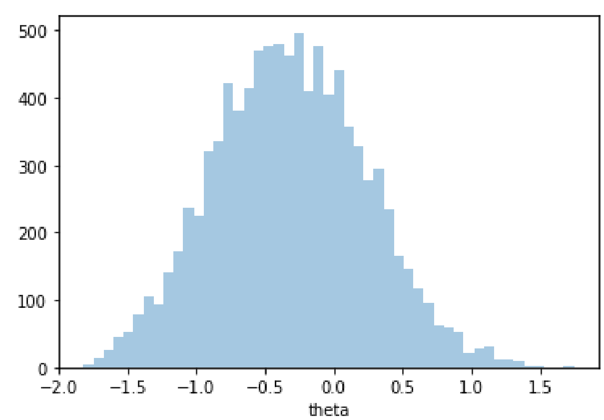

Figure 1: The histogram of the posterior

(b) Find Bayes estimator of θ, and 95% equitailed Credible Set based on the simulated observations.
The Bayes estimator of θ is **-0.3121**.
The 95% equitailed Credible Set are: [**-1.391, 0.766**]

Suggestions: (i) Take uniform distribution on [m, m] as a proposal distribution since it is easy to sample from. This is an independence proposal, the proposed θ' does not depend on the current value of the chain, θ.

(ii) You will need to calculate $\sum_{i=1}^{n}(y_i - \theta)^2$ for current θ and $\sum_{i=1}^{n}(y_i - \theta')^2$ for the proposed θ' prior to calculating the Metropolis ratio.

Problem 2

Gibbs Sampler and High/Low Protein Diet in Rats. Armitage and Berry (1994, p. 111)[2] report data on the weight gain of 19 female rats between 28 and 84 days after birth. The rats were placed in randomized manner on diets with high (12 animals) and low (7 animals) protein content.

High protein	Low protein
134	70
146	118
104	101
119	85
124	107
161	132
107	94
83	
113	
129	
97	
123	

Figure 2: Log likelihood vs Iteration

We want to test the hypothesis on dietary effect: Did a low protein diet result in a significantly lower weight gain? The classical t test against the one sided alternative will be significant at 5% significance level, but we will not go in there. We will do the test Bayesian way using Gibbs sampler. Assume that high-protein diet measurements $y_{1i}, i = 1, ..., 12$ are coming from normal distribution $N(\theta_1, 1/\tau_1)$, where τ_i is the precision parameter,

$$f(y_{1i}|\theta_1, \tau_1) \propto \tau_1^{1/2} \exp\{-\frac{\tau_1}{2}(y_{1i} - \theta_1)^2\}, i = 1, ..., 12 \tag{14}$$

The low-protein diet measurements $y_{2i}, i = 1, ..., 7$ are coming from normal distribution $N(\theta_2, 1/\tau_2)$

$$f(y_{2i}|\theta_2, \tau_2) \propto \tau_2^{1/2} \exp\{-\frac{\tau_2}{2}(y_{2i} - \theta_2)^2\}, i = 1, ..., 7 \tag{15}$$

Assume that θ_1 and θ_2 have normal priors $N(\theta_{10}, 1/\tau_{10})$ and $N(\theta_{20}, 1/\tau_{20})$, respectively. Take prior means as $\theta_{10} = \theta_{20} = 110$ (apriori no preference) and precisions as $\tau 10 = \tau 20 = 1/100$.
Assume that τ_1 and τ_2 have the gamma $\Gamma(a_1, b_1)$ and $\Gamma(a_2, b_2)$ priors with shapes $a_1 = a_2 = 0.01$ and rates $b_1 = b_2 = 4$.

(a) Construct Gibbs sampler that will sample θ_1, τ_1, θ_2, and τ_2 from their posteriors.

The joint distributions are below:

$$f(y_1 i, \theta_1, \tau_1) \propto \tau_1^{n/2} \exp\{-\frac{\tau_1}{2}\sum_{i=1}^{n}(y_{1i} - \theta_1)^2\} * \tau_{10}^{1/2} \exp\{-\frac{\tau_{10}}{2}(\theta_1 - \theta_{10})^2\} * \tau_1^{a_1 - 1} \exp\{-b_1\tau_1\} \quad (16)$$

$$f(y_2 i, \theta_2, \tau_2) \propto \tau_2^{n/2} \exp\{-\frac{\tau_2}{2}\sum_{i=1}^{n}(y_{2i} - \theta_2)^2\} * \tau_{20}^{1/2} \exp\{-\frac{\tau_{20}}{2}(\theta_2 - \theta_{20})^2\} * \tau_2^{a_2 - 1} \exp\{-b_2\tau_2\} \quad (17)$$

Thus the conditional distributions are:

$$\pi(\theta_1|\tau_1, y_{1i}) \propto \exp\{-\frac{\tau_1}{2}\sum_{i=1}^{n}(y_{1i} - \theta_1)^2\} \exp\{-\frac{\tau_{10}}{2}(\theta_1 - \theta_{10})^2\} \quad (18)$$

$$\pi(\theta_1|\tau_1, y_{1i}) \propto \exp\{-\frac{1}{2}(\tau_{10} + n\tau_1)(\theta_1 - \frac{\tau_1 \sum_{i=1}^{n} y_{1i} + \tau_{10}\theta_{10}}{\tau_{10} + n\tau_1})^2\} \quad (19)$$

$$\pi(\theta_2|\tau_2, y_{2i}) \propto \exp\{-\frac{\tau_2}{2}\sum_{i=1}^{n}(y_{2i} - \theta_2)^2\} \exp\{-\frac{\tau_{20}}{2}(\theta_2 - \theta_{20})^2\} \quad (20)$$

$$\pi(\theta_2|\tau_2, y_{2i}) \propto \exp\{-\frac{1}{2}(\tau_{20} + n\tau_2)(\theta_2 - \frac{\tau_2 \sum_{i=1}^{n} y_{2i} + \tau_{20}\theta_{20}}{\tau_{20} + n\tau_2})^2\} \quad (21)$$

$$(22)$$

We can see that it is a normal distribution:

$$\pi(\theta_1|\tau_1, y_{1i}) \propto \mathcal{N}(\frac{\tau_1 \sum_{i=1}^{n} y_{1i} + \tau_{10}\theta_{10}}{\tau_{10} + n\tau_1}, \frac{1}{\tau_{10} + n\tau_1}) \quad (23)$$

$$\pi(\theta_2|\tau_2, y_{2i}) \propto \mathcal{N}(\frac{\tau_2 \sum_{i=1}^{n} y_{2i} + \tau_{20}\theta_{20}}{\tau_{20} + n\tau_2}, \frac{1}{\tau_{20} + n\tau_2}) \quad (24)$$

plus in $a_1 = a_2 = 0.01$ and rates $b_1 = b_2 = 4$, $\theta_{10} = \theta_{20} = 110$ and $\tau 10 = \tau 20 = 1/100$, $N_1 = 12$, and $N_2 = 7$.

Similarly, we can get the conditional distribution for $\pi(\tau_1|\theta_1, y_{1i})$ and $\pi(\tau_2|\theta_2, y_{2i})$. They are:

$$\pi(\tau_1|\theta_1, y_{1i}) \propto \mathcal{G}a(a_1 + n/2, b_1 + \frac{1}{2}\sum_{i=1}^{n}(y_{1i} - \theta_1)^2) \tag{25}$$

$$\pi(\tau_2|\theta_2, y_{2i}) \propto \mathcal{G}a(a_2 + n/2, b_2 + \frac{1}{2}\sum_{i=1}^{n}(y_{2i} - \theta_2)^2) \tag{26}$$

We can use those to construct Gibbs sampler.

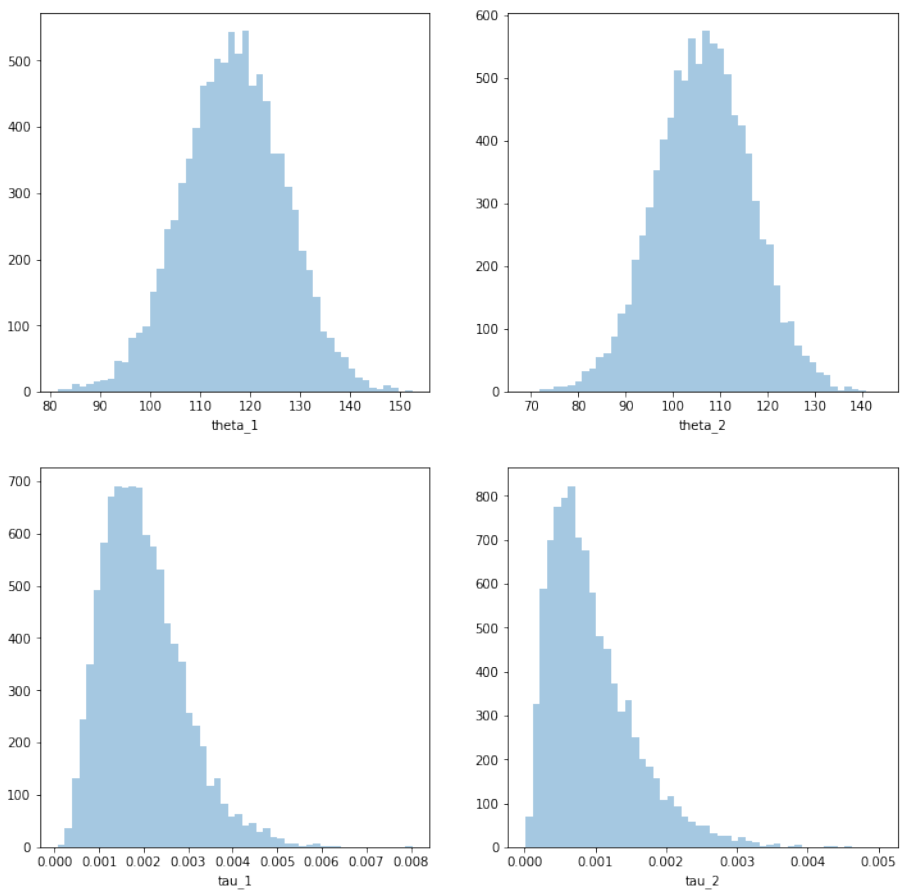

Figure 3: Distribution of $\theta_1, \theta_2, \tau_1, \tau_2$

(b) Find sample differences θ_1 - θ_2 . Proportion of positive differences approximates the posterior probability of hypothesis H0 : $\theta_1 > \theta_2$. What is this proportion if the number of simulations is 10,000, with burn-in of 500?

After running of Gibbs sampler, we can obtain both θ_1 and θ_2, then we can get both means and variances for those two normal random variables. After that, we can generate normal random variable $\mathcal{N}(mean_{\theta_1} - mean_{\theta_2}, var_{\theta_1} + var_{\theta_2})$. Finally, we can get the proportion is **0.7958**.

(c) Using sample quantiles find the 95% equitailed credible set for $\theta_1 - \theta_2$. Does this set contain 0?

The 95% credible set is [**-15.709, 39.738**]. It contains 0!