Forward Rates

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Interest Rates

- Yield to Maturity (YTM)
  - Interest rate that discounts a set of cash flows to today
- Zero Rate
  - Interest rate that discounts a single future cash flow to today
- Forward Rate
  - Interest rate that discounts a single future cash flow to another (nearer) future date

Forward Rates

- Let $R(0, T_1)$ and $R(0, T_2)$ be continuously compounded zero rates observed at time $t = 0$ with tenors T_1 and T_2 years, respectively. These are known as **spot rates**.
- The zero rate $R_F(0, T_1, T_2)$ that is applicable to the period $[T_1, T_2]$ is referred to as the **forward rate** with tenor $T_2 - T_1$ years observed at time $t = 0$. The period $[T_1, T_2]$ is referred to as the **forward period**.
- In general, $R_F(t, T_1, T_2)$ is the forward rate for the period $[T_1, T_2]$ determined at time $t < T_1$.

Note that in the next slides, $R_F(T_1, T_2) = R_F(0, T_1, T_2)$.

- Under the **no-arbitrage assumption**, we have the following relationship:

$$e^{R(0,T_1)T_1} e^{R_F(T_1,T_2)(T_2-T_1)} = e^{R(0,T_2)T_2}. \tag{1}$$

Why?

- Under the **no-arbitrage assumption**, we have the following relationship:

$$e^{R(0,T_1)T_1} e^{R_F(T_1,T_2)(T_2-T_1)} = e^{R(0,T_2)T_2}. \tag{2}$$

Let us show this for the case where $LHS > RHS$ under the following assumptions:

- There are no transaction costs.
- The lending rate equals the borrowing rate.
- There are no arbitrage opportunities ("free lunch").

The other case is left as an exercise.

Forward Rates

Getting the forward rate through spot rates

- Under the **no-arbitrage assumption**, we have the following relationship:

$$e^{R(0,T_1)T_1} e^{R_F(T_1,T_2)(T_2-T_1)} = e^{R(0,T_2)T_2}. \tag{3}$$

- It follows that

$$\begin{aligned} R_F(T_1,T_2) &= \frac{R(0,T_2)T_2 - R(0,T_1)T_1}{T_2 - T_1} \\ &= R(0,T_2) + [R(0,T_2) - R(0,T_1)]\frac{T_1}{T_2 - T_1}. \end{aligned} \tag{4}$$

What's the relationship between $R_F(T_1, T_2)$ and $R(0, T_2)$?

Zero Curves and Forward Rates

- On the relationship between $R_F(T_1, T_2)$ and $R(0, T_2)$:
  - If the zero rate is increasing, i.e. $R(0, T_1) < R(0, T_2)$, then $R_F(T_1, T_2) > R(0, T_2)$
  - If the zero rate is decreasing, then $R_F(T_1, T_2) < R(0, T_2)$.
- Note that in term structure theory, it is common for zero rates to have shapes other than strictly upward or downward sloping curves.

What if we have simple rates instead of continuously-compounded rates?

Forward Rates Implied by Simple Spot Rates

- Suppose $R(0, T_1)$ and $R(0, T_2)$ are simple rates. The no-arbitrage forward rate relation then becomes

$$[1+R(0,T_1)T_1][1+R_F(T_1,T_2)(T_2-T_1)] = 1+R(0,T_2)T_2. \quad (5)$$

- It can therefore be shown that the simple forward rate is

$$R_F(T_1, T_2) = \left[\frac{DF(0,T_1)}{DF(0,T_2)} - 1\right] \frac{1}{T_2 - T_1}, \quad (6)$$

where $DF(0, T_i) = [1 + R(0, T_i)]^{-1}$ is the discount factor applicable over the period $[0, T_i]$ ($i = 1, 2$).

Exercise

Suppose $R(0, T_1)$ and $R(0, T_2)$ are compounded m times per year. Under the assumption of no arbitrage, find a formula for the forward rate $R_F(T_1, T_2)$ (also compounded m times per year) in terms of the discount factors implied by $R(0, T_i)$, $i = 1, 2$.

Zero Rates

- Sometimes, the spot zero rates $R(0, T_1)$ and/or $R(0, T_2)$ are not readily available from published benchmark rates.
- The current market practice is to use linear interpolation to estimate the desired zero rates.

References

- De Lara-Tuprio, E. P. (2016). Financial Derivatives: Lecture Notes in AMF 270. Ateneo de Manila University.
- Hull, J. C. (2018). Options, Futures, and Other Derivatives (10th ed). Prentice Hall.