VaR for Bonds: Cash Flow Mapping

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October 21, 2021

Cash Flow Mapping

- CFM is an interest risk management procedure in which the cash flows of a specific claim are mapped to a set of **benchmark claims** in order to measure and manage the effects of different-scale changes in interest rates, associated with different maturities.
- This procedure represents a financial instrument as a **portfolio of zero-coupon bonds** for the purpose of calculating its VaR.

Assumptions of CFM

- There is a linear path between standard maturities
- There is a linear path between standard deviation of standard maturities
- Assumes normality of returns

PV Invariant and Volatility Invariant Maps

- For our discussion, we will use the following mappings:
  1. Present Value Invariant Map
  2. Volatility Invariant Map
- Other mappings include:
  1. Duration Invariant Map
  2. DV01 Invariant Map

PV Invariant Map

- Suppose the original cash flow is at time T, where $T_1 < T < T_2$ with T_1 and T_2 being the two adjacent benchmark tenors (standard buckets), and suppose the original cash flow has a present value of $\$1$.
- Let $\$\alpha_1$ of the cash flow be mapped to the T_1-maturity interest rate, and $\$\alpha_2$ be mapped to the T_2-maturity interest rate.
- Assuming we want the present value to be preserved by the mapping, then

$$\begin{aligned} \mathrm{PV}(CF_{T_1}) + \mathrm{PV}(CF_{T_2}) &= \mathrm{PV}(CF_T) \\ \alpha_1 + \alpha_2 &= 1 \end{aligned}$$

Volatility Invariant Map

- Let the volatilities of the interest rates of maturities T_1, T and T_2 be σ_1, σ, and σ_2 and the correlation between the changes in interest rates of maturities T_1 and T_2 be ρ.
- Normally we will know the volatilities at the standard buckets and their correlation, but we would not know the volatility at T. This can be obtained by linearly interpolating between the volatilities at the adjacent buckets.
- If we want the mapped cash flow to have the same volatility as the original cash flow, then

$$\begin{aligned}\sigma^2 &= \boldsymbol{\alpha}^T \boldsymbol{\Sigma} \boldsymbol{\alpha} \\ &= \boldsymbol{\alpha}^T (\boldsymbol{\sigma}^{\boldsymbol{T}} \boldsymbol{\rho} \boldsymbol{\sigma}) \boldsymbol{\alpha} \\ &= \alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2 + 2\alpha_1 \alpha_2 \rho \sigma_1 \sigma_2\end{aligned}$$

Combining the Two Maps

- We assume that the present values of the mapped cash flows at T_1 and T_2 are positive: $\alpha_1, \alpha_2 > 0$.
- Combining the two previous maps we attain a unique solution for α_1, α_2:

$$\begin{aligned} \alpha_1 + \alpha_2 &= 1 \\ \sigma^2 &= \alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + 2\alpha_1\alpha_2\rho\sigma_1\sigma_2 \end{aligned}$$

Let us solve this system of equations.

Combining the Two Maps

- α_1 can be acquired by solving the following quadratic equation:

$$(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)\alpha_1^2 + (2\rho\sigma_1\sigma_2 - 2\sigma_2^2)\alpha_1 + (\sigma_2^2 - \sigma^2) = 0$$

- $\alpha_2 = 1 - \alpha_1$

Cash Flow Mapping

Example

Consider a portfolio consisting of a long position in a T-bond with a principal of $\$1\mathrm{M}$ and with remaining maturity of 0.8 years. Suppose that the bond provides a coupon of 10% p.a. payable semiannually. Find the portfolio's 10-day 99% VaR.

Cash Flow Mapping

Given

Maturity	3 mos	0.3 yrs	6 mos	0.8 yrs	1 yr
Cash Flow		$50,000$		$1,050,000$	
Zero Rate	5.50%		6.00%		7.00%
Bond Volatility	0.06%		0.10%		0.20%

Correlation between returns

$$\begin{array}{cc} & \begin{array}{ccc} 3m & 6m & 1y \end{array} \\ \begin{array}{c} 3m \\ 6m \\ 1y \end{array} & \begin{bmatrix} 1.0 & 0.9 & 0.6 \\ 0.9 & 1.0 & 0.7 \\ 0.6 & 0.7 & 1.0 \end{bmatrix} \end{array}$$

Remark: The 1-day bond volatility is the standard deviation of the 1-day percentage change in the value of the bond.

Volatilies and Correlations

- If the zero-coupon bond volatilities are not readily available, then they could be estimated by calculating the standard deviation of the historical zero rates.
- If the correlations between the returns of the zero-coupon bonds are not readily available, then a proxy would be the correlations between the historical zero rates.

Interpolation

Let r_a, r_b, and r_c be the zero rates for zero-coupon bonds with maturities a, b, and c, respectively.
Let σ_a, σ_b, and σ_c be 1-day volatilities for zero-coupon bonds prices with maturities a, b, and c, respectively.
Then,

$$r_c = r_a + (c - a)\left(\frac{r_b - r_a}{b - a}\right)$$

$$\sigma_c = \sigma_a + (c - a)\left(\frac{\sigma_b - \sigma_a}{b - a}\right)$$

Cash Flow Mapping

Maturity	3 mos	0.3 yrs	6 mos	0.8 yrs	1 yr
Cash Flow	$\boldsymbol{C_{1,a}}$	$50,000$	$\boldsymbol{C_{1,b}}$		
			$\boldsymbol{C_{2,a}}$	$1,050,000$	$\boldsymbol{C_{2,b}}$
Zero Rate	5.50%	$\mathbf{5.60\%}$	6.00%	$\mathbf{6.60\%}$	7.00%
Bond Volatility	0.06%	$\mathbf{0.068\%}$	0.10%	$\mathbf{0.160\%}$	0.20%

Consider the cash flows $C_{i,a}$ occurring at a and
$C_{i,b}$ occurring at b

Let $P_1 = 50,000$ and $P_2 = 1,050,000$.

Let $\sigma_{i,c}$ be the volatility for cash flow i at its original tenor c.

Find $0 \leq \alpha_1 \leq 1$ and $0 \leq \alpha_2 \leq 1$ such that:

1. The present value is preserved.
   $\mathrm{PV}(C_{i,a}) + \mathrm{PV}(C_{i,b}) = \mathrm{PV}(P_i)$
2. The variance is preserved.
   $\sigma_{i,c}^2 = (\boldsymbol{\sigma\alpha})^T \boldsymbol{\rho}(\boldsymbol{\sigma\alpha})$

Computing VaR

Let $P =$ value of the bond today

$= B_1 + B_2 + \cdots + B_n$ where $B_k = \sum \mathrm{PV}(C_{i,j})$

Maturity	3 mos	0.3 yrs	6 mos	0.8 yrs	1 yr
Cash Flow	$\boldsymbol{C_{1,a}}$	$50,000$	$\boldsymbol{C_{1,b}}$		
			$\boldsymbol{C_{2,a}}$	$1,050,000$	$\boldsymbol{C_{2,b}}$
Present Value	$\boldsymbol{B_1}$		$\boldsymbol{B_2}$		$\boldsymbol{B_3}$
Zero Rate	5.50%	$\mathbf{5.60\%}$	6.00%	$\mathbf{6.60\%}$	7.00%
Bond Volatility	0.06%	$\mathbf{0.068\%}$	0.10%	$\mathbf{0.160\%}$	0.20%

Then,

$$\Delta P = B_1 R_1 + B_2 R_2 + \cdots + B_n R_n$$

where $R_i = \dfrac{\Delta B_i}{B_i}$, and $\Delta P, \Delta B$ are 1-day changes in value.

Computing VaR

To compute for VaR,

$$\begin{aligned} \Delta P &= B_1 R_1 + B_2 R_2 + \cdots + B_n R_n \\ \mathbf{E}(\Delta P) &= 0 \\ \mathbf{Var}(\Delta P) &= \sigma_p^2 = \left(\boldsymbol{\sigma B}\right)^T \boldsymbol{\rho} \left(\boldsymbol{\sigma B}\right) \end{aligned}$$

and the 1-day (diversified) 99% VaR is given by

$$|V| = \sigma_P \Phi^{-1}(0.99).$$

Advantages of CFM

- CFM is necessary when we do not have sufficient data on positions.
- CFM helps us to cut down on the dimensionality of covariance matrices and correlations.

Disadvantage of CFM

- Because the original portfolio is represented as portfolio of zero-coupon bonds, some precision of the calculated VaR is lost.

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