VaR for Bonds

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Assumptions

1. The market is operating under normal conditions (not during a recession, for instance).
2. We have a long position on all bonds in our portfolio.
3. Portfolio composition stays the same for the period over which VaR is being computed.
4. The historical market yields are continuously-compounded.
5. Interest rates for tenors one year or less are zero/money market rates, while interest rates for tenors more than a year are not.
6. One year consists of 360 days.

Delta-Normal Approach

Suppose a financial institution holds a position consisting solely of bond B_1.

Let $y_{1,0}$ be the yield on the bond today,
y_1 be the yield on the bond tomorrow.
P_1 be the current bond price,
D_1^* be the bond's modified duration.

The change in portfolio value ΔP is given by:

$$\begin{aligned}\Delta P &= \Delta P_1 \\ &= -P_1 D_1^* \Delta y_1 \\ &= \frac{-100^2 P_1 D_1^*}{100^2} \Delta y_1 \\ &= -100^2 \mathrm{DV01}_1 \Delta y_1 \\ &= -100^2 \mathrm{DV01}_1 (y_1 - y_{1,0}).\end{aligned}$$

Delta-Normal Approach

Let R_1 be the logarithmic return on the yield between today and tomorrow.

$$\Rightarrow R_1 = \ln\left(\frac{y_1}{y_{1,0}}\right)$$
$$\Rightarrow y_1 = y_{1,0}e^{R_1} \approx y_{1,0}(1 + R_1)$$
$$\Rightarrow \Delta P_1 \approx -100^2 \mathrm{DV01}_1 y_{1,0} R_1$$

Delta-Normal Approach

In the delta-normal approach, $R_1 \sim \mathcal{N}(0, \sigma_1^2)$.
Let $|V|$ be the one-day 99% VaR.

$$\Rightarrow \mathbf{P}(\Delta P \geq V) = 0.99$$

$$\mathbf{P}(-100^2 \mathrm{DV01}_1 y_{1,0} R_1 \geq V) = 0.99$$

$$\mathbf{P}\left(R_1 \leq \frac{V}{-100^2 \mathrm{DV01}_1 y_{1,0}}\right) = 0.99$$

$$\mathbf{P}\left(\frac{R_1 - 0}{\sigma_1} \leq \frac{\frac{V}{-100^2 \mathrm{DV01}_1 y_{1,0}} - 0}{\sigma_1}\right) = 0.99$$

$$\frac{V}{-100^2 \mathrm{DV01}_1 y_{1,0} \sigma_1} = \Phi^{-1}(0.99)$$

$$\therefore |V| = 100^2 \mathrm{DV01}_1 y_{1,0} \sigma_1 \Phi^{-1}(0.99)$$

VaR for a Single Bond B_1

Formula for VaR

The one-day 99% VaR, $|V|$, of a portfolio consisting of a long position on bond B_1 with a current dollar duration of $\mathrm{DV01}$ is

$$|V| = 100^2 \mathrm{DV01}_1 y_{1,0} \sigma_1 \Phi^{-1}(0.99)$$

where $y_{1,0}$ is the current continuously-compounded market yield of B_1 and σ_1 is the one-day volatility of the yield returns.[1]

[1] The formula can be modified for any $100\alpha\%$ VaR.

Delta-Normal Approach

D^* for a Zero-Coupon Bond

Consider a zero-coupon bond with price per hundred P, continuously-compounded yield y, and time to maturity T.

$$P = \frac{100}{e^{yT}} \approx \frac{100}{1+yT}$$
$$\frac{dP}{dy} = \frac{-100}{(1+yT)^2} \cdot T$$

Delta-Normal Approach

D^* for a Zero-Coupon Bond

Consider a zero-coupon bond with price per hundred P, continuously-compounded yield y, and time to maturity T.

$$\begin{aligned} D^* &= -\frac{1}{P}\frac{-100}{(1+yT)^2}\cdot T \\ &= \frac{1}{P}\frac{100^2}{(1+yT)^2}\cdot\frac{T}{100} \\ &= -\frac{1}{P}\cdot P^2\cdot\frac{T}{100} \\ &= \frac{PT}{100} \end{aligned}$$

Delta-Normal Approach

n-bond Portfolio

Consider a portfolio consisting of n bonds. Let $y_{i,0}$ be the current yield on bond i, R_i be the logarithmic return on the yield of bond i between today and tomorrow. The change in portfolio is given by

$$\begin{aligned}
\Delta P &= \sum_{i=1}^{n} \Delta P_i \\
&= \sum_{i=1}^{n} (-100^2 \mathrm{DV01}_i \Delta y_i) \\
&\approx \sum_{i=1}^{n} (-100^2 \mathrm{DV01}_i y_{i,0} R_i) \\
&= \sum_{i=1}^{n} a_i R_i, \text{ where } a_i = -100^2 \mathrm{DV01}_i y_{i,0}
\end{aligned}$$

Delta-Normal Approach

Undiversified VaR

Sum of the individual VaR's of the bonds in the portfolio.

Diversified VaR

Let $\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, Σ be the variance-covariance matrix, and $|V|$ be the one-day 99% VaR. We have

$$|V| = \sigma_P \Phi^{-1}(0.99)$$

where $\sigma_P^2 = \boldsymbol{a}^T \Sigma \boldsymbol{a}$.

Computing the N-Day VaR

1. Using N-day logarithmic returns $R_t = \ln\left(\frac{y_t}{y_{t-N}}\right)$.
2. Using square-root-of-time scaling
$$|V_{N-\text{day}}| = \sqrt{N}|V|.$$

EWMA Model

Single Bond Case

We can approximate σ_1^2 as follows

$$\sigma_1^2 \approx \sum_{j=1}^{m} (1-\lambda_1)\lambda_1^{j-1} R_{1,j}^2$$

Formula for VaR

The one-day 99% VaR, $|V|$, of a portfolio consisting of a long position on bond B_1 with a current dollar duration of $\mathrm{DV01}$ is

$$|V| = 100^2 \mathrm{DV01}_1 y_{1,0} \sigma_1 \Phi^{-1}(0.99)$$

where $y_{1,0}$ is the current continuously-compounded market yield of B_1 and σ_1 is the one-day volatility of the yield returns.[2]

[2] The formula can be modified for any $100\alpha\%$ VaR.

EWMA Model

n-Bond Case

$$\sigma_i^2 \approx \sum_{j=1}^{m} (1-\lambda_i)\lambda_i^{j-1} R_{i,j}^2$$

$$\sigma_{p,q} \approx \sum_{j=1}^{m} (1-\lambda_{p,q})\lambda_{p,q}^{j-1} R_{p,j} R_{q,j}$$

From these variances and covariances, we can form the variance-covariance matrix Σ which we'll use in calculating the diversified VaR.

Historical Simulation

Assumptions

1. Historical interest rates for different tenors are readily available.
2. One year consists of 360 days. (Sometimes actual day counts are used.)
3. No parametric distribution is assumed for the random variable. The distribution of the random variable is uniform.
4. The random variable takes the values of the historical data.
5. Scenarios generated from historical data are equally probable.

Historical Simulation

Let's look at a sample spreadsheet with VaR calculations.

Bootstrapping

What if we have to work with zero rates, not market yields?

Bootstrapping

Exercise

Suppose today is October 12, 2021. A 10-year corporate bond has $1,000 par value, 10% coupon rate with interest paid annually, and maturity date on December 31, 2025. Use the following table of zero rates to calculate the value of the bond:

Tenor (years)	1	2	3	4	5
Rate	10.25%	10.33%	10.50%	10.75%	11%

Bootstrapping

Interest rates quoted in the market for tenors more than 1 year are yields for coupon-bearing bonds and are, therefore, not zero rates. Estimating zero rates from these yields is done using a method called **bootstrapping**.

Bootstrapping

Example

Consider a peso-denominated bond, with face value PHP 100,000, with coupon payment of PHP 5,000 every quarter, and maturity two years. Compute the bond's 10-day 99% VaR using the rates in `3.3 VaR for Bonds Example - Bootstrapping.xlsx`.

Bootstrapping

Assumption: Interest rates for tenors less than a year are zero rates, while interest rates for tenors more than a year are not.

Let $r_{t,k}$ = interest rate for tenor t, k trading days ago,
$r'_{t,j}$ = interest rate for tenor t, 10 days from now, under scenario j,
$R_{t,j}$ = the 10-day log return for the interest rate with tenor t, under scenario j.

Then

$$R_{t,j} = \ln\left(\frac{r_{t,j}}{r_{t,j+10}}\right)$$
$$r'_{t,j} = r_{t,0}e^{R_{t,j}}$$

Bootstrapping

Getting Discount Factors

If $t \leq 1$ year, then

$$
\begin{aligned}
DF_{t,j} &= \exp\left\{-r'_{t,j}\left(\frac{t}{360}\right)\right\} \\
&= \frac{1}{\exp\left\{r'_{t,j}\left(\frac{t}{360}\right)\right\}} \\
&\approx \frac{1}{1 + r'_{t,j}\left(\frac{t}{360}\right)}
\end{aligned}
$$

Bootstrapping

Getting Discount Factors

If $t = 2$ years:

Consider a par bond with face value 1, maturity 2 years, coupon rate $r'_{2y,j}$, coupons paid annually.

$$1 = r'_{2y,j} DF_{1y,j} + \left(r'_{2y,j} + 1\right)\left(DF_{2y,j}\right)$$
$$DF_{2y,j} = \frac{1 - r'_{2y,j} DF_{1y,j}}{1 + r'_{2y,j}}$$

This is called bootstrapping.

Bootstrapping

Getting Discount Factors

If $t = 3$ years:

Consider a par bond with face value 1, maturity 3 years, coupon rate $r'_{3y,j}$, coupons paid annually.

$$1 = r'_{3y,j} DF_{1y,j} + r'_{3y,j} DF_{2y,j} + \left(r'_{3y,j} + 1\right)\left(DF_{3y,j}\right)$$

$$DF_{3y,j} = \frac{1 - r'_{3y,j} DF_{1y,j} - r'_{3y,j} DF_{2y,j}}{1 + r'_{3y,j}}$$

Bootstrapping

Getting Discount Factors

If $t = n$ years, $n > 1$:

Consider a par bond with face value 1, maturity i years, coupon rate $r'_{i,j}$, coupons paid annually.

$$\begin{aligned} 1 &= r'_{i,j} DF_{1y,j} + r'_{i,j} DF_{2y,j} + \cdots + \left(r'_{i,j} + 1\right)\left(DF_{i,j}\right) \\ &= r'_{i,j} \sum_{k=1}^{i-1} DF_{k,j} + \left(1 + r'_{i,j}\right) DF_{i,j} \end{aligned}$$

$$DF_{i,j} = \frac{1 - r'_{i,j} \sum_{k=1}^{i-1} DF_{k,j}}{1 + r'_{i,j}}$$

Bootstrapping

Interpolation

Example

Consider a peso denominated bond, with face value PHP 100,000, with coupon payment of PHP 5,000 every quarter, and maturity two years. Compute the bond's 10-day 99% VaR using the rates in `3.3 VaR for Bonds Example - Bootstrapping.xlsx`.

How do we do interpolation for tenors not covered by our data?

Bootstrapping

Interpolation

Suppose $T_{i-1} < c < T_i$.

Let $n_c, n_{i-1}, n_i =$ number of days for tenors c, T_{i-1}, and T_i

$r_c, r_{i-1}, r_i =$ continuously-compounded zero rates for the said tenors

Then

$$\frac{r_i - r_{i-1}}{n_i - n_{i-1}} = \frac{r_c - r_{i-1}}{n_c - n_{i-1}}$$

$$r_c = r_{i-1}\left(\frac{n_i - n_c}{n_i - n_{i-1}}\right) + r_i\left(\frac{n_c - n_{i-1}}{n_i - n_{i-1}}\right)$$

Bootstrapping

Interpolation

$$
\begin{aligned}
DF_{n_c} &= \exp\left[-r_c\left(\frac{n_c}{360}\right)\right] \\
&= \left\{\exp\left[-r_{i-1}\left(\frac{n_{i-1}}{360}\right)\right]\right\}^{\left(\frac{n_i-n_c}{n_i-n_{i-1}}\right)\left(\frac{n_c}{n_{i-1}}\right)} \\
&\quad \times \left\{\exp\left[-r_i\left(\frac{n_i}{360}\right)\right]\right\}^{\left(\frac{n_c-n_{i-1}}{n_i-n_{i-1}}\right)\left(\frac{n_c}{n_i}\right)} \\
&= \left(DF_{n_{i-1}}\right)^{\frac{n_c}{n_{i-1}}\left(\frac{n_i-n_c}{n_i-n_{i-1}}\right)} \times \left(DF_{n_i}\right)^{\frac{n_c}{n_i}\left(\frac{n_c-n_{i-1}}{n_i-n_{i-1}}\right)}
\end{aligned}
$$

Bootstrapping

Computing for n-day 99% VaR with Bootstrapping

1. Get the n-day logarithmic returns of the given rates.
2. Establish scenarios using the returns acquired in (1).
3. Compute for the discount factors via bootstrapping.
4. Do interpolation if necessary (if cash flows have tenors not covered by the data).
5. Use the discount factors acquired in the steps above to compute for portfolio value changes.
6. The n-day 99% VaR is the $(0.01m)^{\text{th}}$ smallest change in portfolio value, where m is the number of scenarios.

Bootstrapping

Example

Consider a peso denominated bond, with face value PHP 100,000, with coupon payment of PHP 5,000 every quarter, and maturity two years. Compute the bond's 10-day 99% VaR using the rates in `3.3 VaR for Bonds Example - Bootstrapping.xlsx`.

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