Value-at-Risk for Stocks

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Introduction to Financial Risk Management

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- It is useful to define financial risk in terms of a risk horizon, the point at which an asset will be realized, or turned into cash.
- Financial risk is the uncertainty of the future total cash value of an investment on the investor's horizon date.
- This uncertainty arises from many sources.

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4. Operational risk
   - risk arising from all aspects of a firm's business activities

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4. Other market risk
   - volatility risk - affects option traders
   - basis risk

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   - As well as helping to enforce limits, risk managers will work more closely with portfolio managers, adjusting the portfolio as necessary to increase or decrease risk
4. Communicating risk
   - FRMs need to take a large set of financial instruments and market data and reduce them to small number of key statistics and insights

Introduction to Value-at-Risk

Introduction

- Value-at-risk (VaR) is a market *risk metric*–a measure of the uncertainty in the future value of a portfolio, particularly with regards to its return or profit or loss
- For any given portfolio, VaR summarizes the worst loss over a target horizon that will not be exceeded with a given level of confidence, *provided normal market conditions*.
- Formally, VaR describes the quantile of the projected distribution of gains and losses over the target horizon
- Informally, VaR answers the question, "What loss level is such that we are $X\%$ confident it will not be exceeded in N business days?"

An Illustration

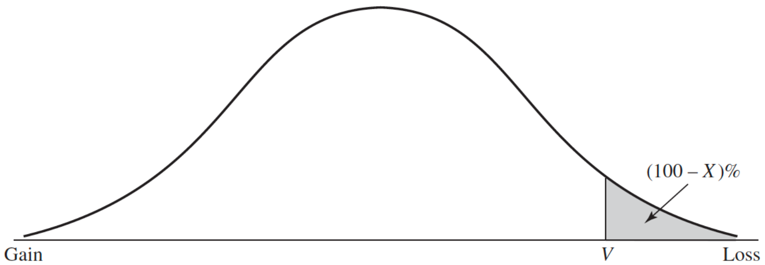

Figure 12.2 Calculation of VaR from the Probability Distribution of the Loss in the Portfolio Value
Gains are negative losses; confidence level is $X\%$; VaR level is V.

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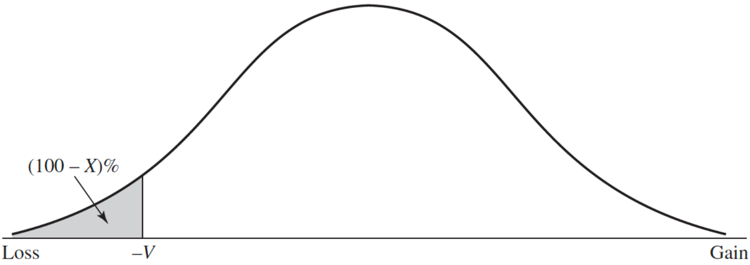

Figure 12.1 Calculation of VaR from the Probability Distribution of the Gain in the Portfolio Value
Losses are negative gains; confidence level is $X\%$; VaR level is V.

Introduction

- Let ΔP denote the *loss* in portfolio value over a fixed time horizon ΔT.
- Denote by $F_{\Delta}(x) = \mathbf{P}[\Delta P \leq x]$ the cumulative distribution function of ΔP.
- We note that ΔP is a random variable since the future value of the portfolio is unknown; further, $\Delta P > 0$ indicates a loss, whereas $\Delta P < 0$ indicates a gain in portfolio value.

Definition (Value-at-Risk)

Given some confidence level $\alpha \in (0,1)$, the $100\alpha\%$ VaR of the portfolio is the smallest number x such that the probability that the loss ΔP exceeds x is no larger than $1-\alpha$. In symbols,

$$\begin{aligned} \mathbf{VaR}_\alpha &= \inf\{x \in \mathbb{R} : \mathbf{P}[\Delta P > x] \leq 1-\alpha\} \\ &= \inf\{x \in \mathbb{R} : F_\Delta(x) \geq \alpha\}. \end{aligned} \tag{1}$$

Notes on Value-at-Risk

- If ΔP indicates the *change* in portfolio value over the same time horizon with cdf $F_\Delta(x)$, then the $100\alpha\%$ VaR of the portfolio is the absolute value of the largest number $x \in \mathbb{R}$ such that $F_\Delta(x) \leq 1 - \alpha$. In symbols,

$$\begin{aligned} \mathbf{VaR}_\alpha &= \left|\sup\{x \in \mathbb{R} : \mathbf{P}[\Delta P < x] \leq 1 - \alpha\}\right| \\ &= \left|\sup\{x \in \mathbb{R} : F_\Delta(x) \leq 1 - \alpha\}\right| . \end{aligned} \tag{2}$$

Measuring Portfolio Returns

Portfolio Returns

- Let P_t denote the value of the asset or portfolio at time t.
- The arithmetic rate of return is given by

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}, \tag{3}$$

  which is the gain over the time period divided by the previous asset value.
- The geometric rate of return is given by

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right). \tag{4}$$

  This is also known as the **continuously compounded rate** of return on the asset price.

Continuous Portfolio Returns

1. If the distribution of $\{R_t\}$ is normal, then we are ensured positive asset prices. Normally distributed arithmetic return may entail negative asset prices;
2. Geometric returns can easily be extended to multiple periods.
3. For sufficiently small returns, there is little difference between the geometric and arithmetic returns.

Time Aggregation

Time Aggregation

- Suppose $P_0, P_1, \ldots, P_N$ denote *daily* asset price data.
- Let $R_1, R_2, \ldots, R_N$ (where $R_t = \ln(P_t/P_{t-1})$) indicate the daily (continuous) asset price returns calculated from the data.
- In order to reflect the assumption of normal market conditions, we assume that the return process contains independent and identically distributed (i.i.d.) variables.
- Thus for any two return variables R_i and R_j, we have

$$\mathbf{E}(R_i + R_j) = \mathbf{E}(R_i) + \mathbf{E}(R_j) = 2\mu \tag{5}$$

$$\mathbf{Var}(R_i + R_j) = \mathbf{Var}(R_i) + \mathbf{Var}(R_j) = 2\sigma^2 \tag{6}$$

where μ and σ^2 represent the common mean and variance of the return process.

I.I.D. Returns

- Let $R_{N,N}$ denote the N-day return on the asset, observed at time N using P_0 and P_N.
- If μ_N and σ_N^2 represent the mean and variance of N-day returns, then assuming that the daily returns are i.i.d. random variables with mean μ and variance σ^2, we have

$$\mu_N = N\mu, \qquad \sigma_N^2 = N\sigma^2. \tag{7}$$

- The standard deviation of the N-day returns is $\sqrt{N}\sigma$. This is known as **square-root-of-time scaling**.

Delta-Normal Approach: Single-Asset Portfolios

Introduction

- Consider a portfolio consisting of a long position on N shares of a stock with current price S_0. Thus, the current value of the portfolio is $P_0 = NS_0$.
- Let S_1 be the price of the stock tomorrow. Assuming that no additional shares of stock are purchased or sold, the portfolio will still contain N shares tomorrow and so its value is $P_1 = NS_1$.
- Since S_1 is unknown, it follows that P_1 is a random variable.

Delta-Normal Approach

- Let ΔP denote the change in portfolio value following the asset price transition from S_0 to S_1. We then have

$$\Delta P = NS_1 - NS_0 = N(S_1 - S_0). \tag{8}$$

- For small values of R_1, we recall that $e^{R_1} \approx 1 + R_1$ and so

$$S_1 - S_0 \approx S_0 R_1 \qquad \text{and} \qquad \Delta P \approx N S_0 R_1.$$

Delta-Normal Approach

- The key assumption in the **delta-normal approach** is the assumption that the daily returns R_1 are normally distributed with mean 0 and variance σ^2.
- Here, σ^2 is taken to be the variance of the daily returns.

Delta-Normal Approach

Theorem

The **1-day 99% VaR** *of this portfolio is the number* $|V|$ *such that*

$$\mathbf{P}[\Delta P \geq V] = 0.99.$$

This is given by

$$\mathbf{VaR}_{0.99} = |V| = NS_0\sigma\Phi^{-1}(0.99). \tag{9}$$

Delta-Normal Approach: n-Asset Portfolios

Introduction

- Suppose a portfolio contains shares on n stocks, where N_i and $S_{i,0}$ denote the number of shares and the current price of the stock for $i = 1, 2, \ldots, n$.
- The current value of the portfolio is

$$P_0 = N_1 S_{1,0} + N_2 S_{2,0} + \cdots + N_n S_{n,0} = \sum_{i=1}^{n} N_i S_{i,0}. \tag{10}$$

- Letting $S_{i,1}$ denote the price of stock i the following trading day, the one-day (continuously compounded) return on the stock is

$$R_i = \ln\left(\frac{S_{i,1}}{S_{i,0}}\right), \qquad i = 1, 2, \ldots, n.$$

Undiversified Portfolio VaR

- The 99% VaR of the ith stock is given by

$$\mathbf{VaR}_{i,0.99} = N_i S_{i,0}\sigma_i \Phi^{-1}(0.99), \qquad i = 1, 2, \ldots, n,$$

where σ_i^2 is the variance of the daily returns R_i of stock i.

- The **99% undiversified VaR** of the entire portfolio is given by the sum of the individual VaRs,

$$|V_u| = \sum_{i=1}^{n} N_i S_{i,0}\sigma_i \Phi^{-1}(0.99). \tag{11}$$

Diversified Portfolio VaR

- Assume that $R_i \sim N(0, \sigma_i^2)$ for all $i = 1, 2, \ldots, n$ and is sufficiently small so that

$$\Delta P = \sum_{i=1}^{n} N_i S_{i,0} R_i. \tag{12}$$

- Let $\alpha_i = N_i S_{i,0}$. Then,

$$\Delta P = \sum_{i=1}^{n} \alpha_i R_i = \boldsymbol{\alpha}^T \mathbf{R}, \tag{13}$$

where $\boldsymbol{\alpha}^T = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ denotes the vector of asset weights and $\mathbf{R}^T = [R_1, R_2, \ldots, R_n]$ is the vector of one-day returns.

Diversified Portfolio VaR

- Assume that the $\mathbf{R}$ has a multivariate normal distribution, written as $\mathbf{R} \sim N(\mathbf{0}, \mathbf{\Sigma})$, where $\mathbf{0}$ is a vector (of length n) of zeros and $\mathbf{\Sigma}$ is the covariance matrix of $\mathbf{R}$.
- If $\sigma_{i,j} = \mathbf{Cov}(R_i, R_j)$, the covariance matrix is written as

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \dots & \sigma_{1,n} \\ \sigma_{1,2} & \sigma_2^2 & \dots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1,n} & \sigma_{2,n} & \dots & \sigma_n^2 \end{bmatrix}. \tag{14}$$

- Since $\sigma_{i,j} = \sigma_{j,i}$, $\mathbf{\Sigma}$ is symmetric.

Diversified Portfolio VaR

- **Recall:** Let the $p \times 1$ random vector $\mathbf{y}$ be $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and let $\mathbf{a}$ be any $p \times 1$ vector of constants. Then $\mathbf{z} = \mathbf{a}^T\mathbf{y}$ is $N(\mathbf{a}^T\boldsymbol{\mu}, \mathbf{a}^T\boldsymbol{\Sigma}\mathbf{a})$.
- Therefore, ΔP is normally distributed with mean 0 and variance σ_P^2, where

$$\sigma_P^2 = \boldsymbol{\alpha}^T\boldsymbol{\Sigma}\boldsymbol{\alpha}. \tag{15}$$

- The one-day 99% diversified VaR for the portfolio is given by

$$|V_d| = \sigma_P \Phi^{-1}(0.99). \tag{16}$$

Diversification Effects

- The presence of more than one asset effectively diversifies the portfolio.
- If there are adverse movements in the price of one asset, the stability of the other asset prices will dampen the losses observed from the portfolio.
- The difference between the diversified and undiversified VaRs is called the **benefit of diversification**. It can be shown that

$$|V_d| \leq |V_u|.$$

Delta-Normal Approach: N-Day VaR

Two Approaches for Computing the N-Day VaR

1. Square-root-of-time scaling
2. using N-day returns
    1. calculating the N-day returns on a portfolio level (i.e. treating the portfolio as a single stock)
    2. calculating the N-day returns on a stock level (i.e. using the covariance approach)

Square-root-of-time scaling

- Assume that the daily returns series $\{R_t\}$ *for the portfolio* is a collection of independent and identically distributed normal random variables with mean μ and σ^2
- Then the series of N-day returns $\{\eta_t\}$ is a collection of i.i.d. $N(\mu_N, \sigma_N^2)$ variables with

$$\mu_N = N\mu, \qquad \sigma_N^2 = N\sigma^2.$$

Square-root-of-time scaling

- Let ΔP represent the change in portfolio value over N days; that is, $\Delta P = P_N - P_0$.
- Let η denote the continuously compounded rate of return on the portfolio over N days,

$$\eta = \ln\left(\frac{P_N}{P_0}\right).$$

- Then for sufficiently small values of η,

$$\Delta P = P_N - P_0 = P_0 e^{\eta} - P_0 \approx P_0 \eta.$$

For convenience, we assume that $\Delta P = P_0 \eta$.

Square-root-of-time scaling

- The N-day 99% VaR for the portfolio is the number V such that

$$\mathbf{P}[\Delta P \geq V] = 0.99.$$

- Similar to our previous calculations, we obtain

$$|V| = P_0 \sqrt{N} \sigma \Phi^{-1}(0.99). \tag{17}$$

- Since $P_0 \sigma \Phi^{-1}(0.99)$ gives the one-day 99% VaR of the portfolio (which we denote by $|V_1|$), we then have the relation

$$|V| = \sqrt{N} |V_1| \tag{18}$$

Calculating N-day returns on a portfolio level

- The N-day returns can be estimated from the time series of daily asset prices.
- Let $\{P_t\}$ be the time series of *portfolio values* over time. Then the N-day continuously compounded return observed at time t is given by

$$\eta_t = \ln\left(\frac{P_t}{P_{t-N}}\right), \qquad t \geq N. \tag{19}$$

- The variance estimated from $\{\eta_t\}$ is taken to be σ_N^2. The formula

$$|V| = P_0 \sigma_N \Phi^{-1}(0.99)$$

is then used to compute for the N-day 99% VaR of the portfolio.

Remarks on Computing the N-Day VaR

- Note that the process discussed above provides the *diversified* portfolio VaR over N days since the returns of the portfolio are used instead of the returns on the individual asset prices.
- An *undiversified* version of the portfolio VaR is obtained by computing the N-day VaR for each individual stock then adding the resulting VaRs.
- For larger values of N, larger inconsistencies between the VaR obtained from square-root-of-time scaling and that obtained from the use of empirical N-day returns arise.

Calculating N-day returns on a stock level

- An alternative to calculating the **diversified** portfolio VaR using N-day returns is through the covariance approach.
- We calculate the N-day returns on a stock level and get the variance-covariance matrix $\boldsymbol{\Sigma}$ of these returns.
- The variance of the change in portfolio value ΔP is then computed as

$$\sigma_P^2 = \boldsymbol{\alpha}^T \boldsymbol{\Sigma} \boldsymbol{\alpha}. \tag{20}$$

- The N-day 99% diversified VaR for the portfolio is given by

$$|V_d| = \sigma_P \Phi^{-1}(0.99). \tag{21}$$

Estimating Volatilities and Correlations

Definitions and Notation

For the succeeding discussion, let $P_0, P_1, \ldots, P_{n-1}$ denote the sequence of historical asset prices with P_{n-1} being the most current price.

- Let σ_n denote the volatility of the price of the asset on day n, as estimated at the end of day $n-1$.
- Let R_t be the one-day continuously compounded rate of return during day t (measured between the end of day $t-1$ and day t),

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right), \qquad t \geq 1.$$

- Using historical asset prices $P_0, P_1, \ldots, P_{n-1}$, an unbiased estimate for σ_n^2 is given by

$$\hat{\sigma}_n^2 = \frac{1}{n-2}\sum_{i=1}^{n-1}\left(R_{n-i} - \bar{R}\right)^2, \qquad \bar{R} = \frac{1}{n-1}\sum_{i=1}^{n-1} R_{n-i}. \quad (22)$$

Some Assumptions

1. We may replace R_i with r_i, the simple rate of return (percentage change in P_t).
2. We assume that $\bar{R} = 0$.
3. The unbiased estimator for σ_n^2 may be replaced with the MLE for σ_n^2. Taking $\bar{R} = 0$, the MLE for σ_n^2 is given by

$$\tilde{\sigma}_n^2 = \frac{1}{n-1} \sum_{i=1}^{n-1} R_{n-i}^2. \tag{23}$$

Weighting Schemes

- The last equation is an unweighted average of the squared log returns on the asset price.
- Since our goal is to estimate the most recent volatility σ_n, we take note that more recent information may have more weight than movements observed in the far past.
- We consider the alternative model

$$\sigma_n^2 = \sum_{i=1}^{n-1} \alpha_i R_{n-i}^2, \tag{24}$$

where α_i is the weight assigned to the return observed i days ago, with $\alpha_i > 0$, $\alpha_{i-1} > \alpha_i$, and $\sum_{i=1}^{n-1} \alpha_i = 1$.

Exponentially Weighted Moving Average (EWMA)

- The exponentially weighted moving average (EWMA) model is a particular case of the weighted estimate for σ_n^2 in that the weights α_i decrease exponentially as we move back through time:

$$\alpha_i = \lambda \alpha_{i-1}, \qquad 0 < \lambda < 1. \tag{25}$$

- Under these conditions, the variance equation becomes

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1-\lambda) R_{n-1}^2. \tag{26}$$

Exponentially Weighted Moving Average (EWMA)

- Working out the recursion, it can be shown that

$$\sigma_n^2 = (1-\lambda)\sum_{i=1}^{n-1}\lambda^{i-1}R_{n-i}^2 + \lambda^{n-1}\sigma_1^2. \tag{27}$$

- For large n, we can take $\lambda^{n-1} \approx 0$, and so we can write

$$\sigma_n^2 \approx \sum_{i=1}^{n-1}(1-\lambda)\lambda^{i-1}R_{n-i}^2, \tag{28}$$

giving us $\alpha_i = (1-\lambda)\lambda^{i-1}$, $i = 1, 2, \ldots, n-1$.

Remarks on the EWMA

- The parameter λ governs how responsive volatility is to new information.
  - A low λ indicates higher weight is given to new market information.
  - A high λ produces estimates of daily volatility that respond relatively slowly to new information provided by the return on the asset.
- At any given time, only the current estimate of the variance rate and the most recent value of the log return is needed to update the variance estimate.

EWMA and the Single-Asset VaR

- Using our previous notation, the change in portfolio value is given by
$$\Delta P \approx N S_0 R,$$
where S_0 is the most recent stock price and R is the estimated one-day log return.
- The variance estimate σ_n^2 from the EWMA can be used as the variance of the random variable R, and so we assume that $R \sim N(0, \sigma_n^2)$.
- Therefore, the one-day 99% VaR on the portfolio is given by
$$\mathbf{VaR}_{0.99} = |V| = N S_0 \sigma_n \Phi^{-1}(0.99). \tag{29}$$

Estimating Covariances (Two-Asset Case)

- Consider the one-day log returns of two assets,

$$R_{1,i} = \ln\left(\frac{P_{1,i}}{P_{1,i-1}}\right) \qquad R_{2,i} = \ln\left(\frac{P_{2,i}}{P_{2,i-1}}\right). \tag{30}$$

- We introduce the following notation:
  - $\sigma_{1,n}$: one-day volatility for the first asset estimated for day n
  - $\sigma_{2,n}$: one-day volatility for the second asset estimated for day n
  - $\mathbf{Cov}_{1,2,n}$: estimate of covariance between the two log returns calculated for day n
- An estimate for the correlation between the returns of the two assets is given by

$$\rho_n = \frac{\mathbf{Cov}_{1,2,n}}{\sigma_{1,n}\sigma_{2,n}}. \tag{31}$$

Estimating Covariances (Two-Asset Case)

- Under the assumption that mean log returns are zero, estimates (MLE) for the individual volatilities are given by

$$\sigma^2_{1,n} = \frac{1}{n-1}\sum_{i=1}^{n-1} R^2_{1,n-i} \qquad \sigma^2_{2,n} = \frac{1}{n-1}\sum_{i=1}^{n-1} R^2_{2,n-i} \tag{32}$$

- The MLE for the covariance is likewise given by

$$\mathbf{Cov}_{1,2,n} = \frac{1}{n-1}\sum_{i=1}^{n-1} R_{1,n-1}R_{2,n-1}. \tag{33}$$

- These estimators assign equal weights to historically observed returns.

EWMA Models for Covariance

- Separate EWMA equations may be used to estimate $\sigma_{1,n}$, $\sigma_{2,n}$, and $\mathbf{Cov}_{1,2,n}$:

$$\sigma^2_{j,n} = \lambda_j \sigma^2_{j,n-1} + (1-\lambda_j) R^2_{j,n-1} \qquad (j = 1, 2) \tag{34}$$

$$\mathbf{Cov}_{1,2,n} = \lambda \mathbf{Cov}(1, 2, n-1) + (1-\lambda) R_{1,n-1} R_{2,n-1}. \tag{35}$$

- The previous analysis can be used to show that

$$\mathbf{Cov}_{1,2,n} \approx \sum_{i=1}^{n-1} (1-\lambda)\lambda^{i-1} R_{1,n-i} R_{2,n-i}. \tag{36}$$

Covariance Matrices and VaR

- Using our standard notation, the change in portfolio value is given by
$$\Delta P = \boldsymbol{\alpha}^T \mathbf{R}$$
where $\boldsymbol{\alpha} = (N_1 S_{1,0}, N_2 S_{2,0})^T$ and $\mathbf{R} = (R_1, R_2)^T$.
- Under the Delta-Normal approach, we assume that $\mathbf{R} \sim MVN(\mathbf{0}, \boldsymbol{\Sigma})$, implying that $\Delta P \sim N(0, \sigma_P^2)$, where
$$\sigma_P^2 = \boldsymbol{\alpha}^T \boldsymbol{\Sigma} \boldsymbol{\alpha}.$$
- An estimate for the covariance matrix for the return vector $\mathbf{R}$ is given by
$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{1,n}^2 & \mathbf{Cov}_{1,2,n} \\ \mathbf{Cov}_{1,2,n} & \sigma_{2,n}^2 \end{bmatrix}. \tag{37}$$

A Note on Estimating Covariances

- Not all covariance matrices are internally consistent; that is, estimates for individual volatilities and covariances may not always provide positive semi-definite covariance matrices.
- To ensure that a positive semi-definite covariance matrix is produced, variances and covariances should be calculated consistently.
  - If variances are calculated by giving equal weight to historical log returns, then the same should be done to estimate covariances.
  - If EWMA models are used for the variances, then an EWMA model must also be used for the covariance.

Historical Simulation: Single-Asset Portfolios

Introduction to Historical Simulation

- The primary basis of the Delta-Normal approach in the computation of the VaR is the assumption that asset price returns (or portfolio returns) are normally distributed.
- Historical data can also be used to estimate VaR for stock portfolios without assuming a particular parametric distribution for the asset return series. This particular procedure is known as **historical simulation**.

Notation

- Consider a financial institution with a position of N shares on a stock E with price S_0 today. Let S_1 be the price per share of E_1 tomorrow.
- Let $R_1 = \ln(S_1/S_0)$ be the continuously compounded return on the asset observed tomorrow.
- For sufficiently small R_1, we assume that

$$\Delta P = P_1 - P_0 = N(S_1 - S_0) \approx N S_0 R_1.$$

Historical Simulation

- In historical simulation, it is assumed that R_1 is a random variable whose values consist of the one-day asset returns observed historically.
- Consider the time series $\{S_{-t}\}$, $t = 0, 1, 2, \ldots, M$, where S_{-t} indicates the stock price t trading days ago and $M + 1$ is the length of the time series.
- Define

$$R_{1,j} = \ln\left(\frac{S_{-j}}{S_{-(j+1)}}\right), j = 0, 1, \ldots, M-1 \tag{38}$$

and so the series $\{R_{1,j}\}$ contains all possible values of the random variable R_1, with $R_{1,j}$ representing the jth scenario for R_1.

Historical Simulation

- In scenario j, the stock price tomorrow is approximated as

$$S_{1,j} = S_0 e^{R_{1,j}}. \tag{39}$$

- Consequently, the change in portfolio value under scenario j is given by the approximation

$$\Delta P_j = N(S_{1,j} - S_0) \approx N S_0 R_{1,j}. \tag{40}$$

- Like the random variable R_1, ΔP can be represented as a random variable with M possible scenarios denoted by $\{\Delta P_j\}$.

Historical Simulation

- The one-day 99% VaR of the portfolio thus corresponds to the scenario j^*, given by ΔP_{j^*}, in the 1st percentile of the scenario space $\{\Delta P_j\}$.
- With ΔP_{j^*} in the 1st percentile, 99% of the other possible changes in portfolio value; as such, it represents the worst possible movement in portfolio value with probability 99%.
- If the ΔP_js are arranged in ascending order, the one-day 99% VaR for the portfolio is the $(0.01M)$th smallest value in the array.

Historical Simulation: N-Asset Portfolios

Notation

- Now suppose there are n stocks $E_1, E_2, \ldots, E_n$ in the portfolio, with positions $N_1, N_2, \ldots, N_n$ respectively.
- Let ΔP_i denote the change in portfolio value brought about by movements in the price of stock E_i.
- Let $S_{i,0}$ and $S_{1,i}$ denote the price of E_i today and tomorrow, respectively. We thus have

$$\Delta P_i = N_i(S_{i,1} - S_{i,0}).$$

Notation

- Suppose $R_{1,i}$ indicates the continuously compounded one-day return on stock E_i from today until tomorrow, i.e.

$$R_{1,i} = \ln\left(\frac{S_{i,1}}{S_{i,0}}\right).$$

- Then we have the approximation

$$\Delta P_i = N_i(S_{i,1} - S_{i,0}) \approx N_i S_{i,0} R_{1,i}.$$

Historical Simulation

- From historical stock prices, we can construct M scenarios for $R_{1,i}$. Denote by $\{R_{1,i,j}\}$ the scenario space for $R_{1,i}$.
- Thus, M scenarios can also be constructed for ΔP_i, which we denote by $\{\Delta P_{i,j}\}$, where

$$\Delta P_{i,j} = N_i S_{i,0} R_{1,i,j}. \tag{41}$$

Historical Simulation

- Let $\Delta P = P_1 - P_0$ be the change in the portfolio value, where P_1 and P_0 denote the portfolio value tomorrow and today, respectively.
- It follows that $\Delta P = \sum_{i=1}^{n} \Delta P_i$.
- We can also construct a distribution of ΔP based on the scenarios on the daily returns of each stock.

Historical Simulation

- The jth scenario for ΔP, denoted by $\Delta P^{(j)}$, is obtained by using the jth scenario of each ΔP_i.
- Thus, the scenario space for ΔP is given by $\{\Delta P^{(j)}\}$, where

$$\Delta P^{(j)} = \sum_{i=1}^{n} \Delta P_{i,j} = \sum_{i=1}^{n} N_i S_{i,0} R_{1,i,j}. \tag{42}$$

- Since there are M scenarios for ΔP, the one-day 99% VaR for the portfolio is given by the $(0.01M)$th smallest value in the collection $\{\Delta P^{(j)}\}$.

"Alternative" Historical Simulation Method

- The statement $\Delta P^{(j)} = \sum_{i=1}^{n} \Delta P_{i,j}$ implies that the change in portfolio value can be decomposed linearly and additively as changes in the value of each individual stock. This may not always be the case.
- Instead of running scenarios on the return of each individual stock, we consider scenarios on the returns of the *entire portfolio*.
- In this analysis, we would like to simulate possible values of the value P_1 of the portfolio the next day using the equation

$$P_1 = P_0 e^{R^P}, \tag{43}$$

where R^P is the (unknown) one-day log-return on the portfolio and P_0 is the current portfolio value.

"Alternative" Historical Simulation Method

- Let R_j^P denote the one-day log-return on the portfolio observed j days ago. From $M+1$ historical portfolio values, we can therefore make M scenarios for one-day log-return on the portfolio.
- The jth scenario for the value of the portfolio in the next trading day is given by

$$P_1^{(j)} = P_0 e^{R_j^P}. \tag{44}$$

The jth scenario for the change in portfolio value is given by

$$\Delta P^{(j)} = P_1^{(j)} - P_0. \tag{45}$$

- Since there are M scenarios for ΔP, the one-day 99% VaR for the portfolio is given by the $(0.01M)$th smallest value in the collection $\{\Delta P^{(j)}\}$.

Hybrid Historical Approach: the Boudoukh-Richardson-Whitelaw (BRW) Approach

Introduction

- The historical simulation approach assumes that historical returns are uniformly distributed and so quantiles are estimated based on the empirical distribution of returns.
- Problems of the historical simulation approach:
  1. Extreme percentiles of the distribution are difficult to estimate with a small amount of data.
  2. Returns are assumed to be i.i.d. and hence does not allow for time-varying volatility.

  To solve these problems, we may extend the historical window to include more observations.
- However, longer historical windows diminish the weight assigned to more recent market phenomenon.

The BRW Approach

- The BRW/Hybrid approach entails the use of exponentially-declining weights on historical data to estimate percentiles, with bigger weights assigned to more recent data.
- The empirical distribution of historical returns now rely on the assigned weights. Quantiles are then calculated based on the cumulative weights after ordering historical returns in increasing order.

Implementing the BRW Approach

1. (Assignment of Weights) Let $\{R_0, R_1, \ldots, R_{M-1}\}$ denote the observed one-day log-returns on the stock or portfolio, where R_i is the one-day log-return observed i days ago.

   Let $0 < \lambda < 1$ be constant and let w_i be the weight assigned to R_i, $i = 0, 1, \ldots, M-1$. The weights decay exponentially, so we assume that $w_{i+1} = \lambda w_i$. To ensure that the weights sum to 1, we have

$$w_i = \left(\frac{1-\lambda}{1-\lambda^M}\right)\lambda^i, \qquad i = 0, 1, \ldots, M-1. \tag{46}$$

Implementing the BRW Approach: Some Remarks

- Consider the single-asset case. Note that each historical return can be mapped to a scenario for the change in portfolio value

$$\Delta P^{(i)} \approx N S_0 R_i,$$

  where N is the number of shares of the stock, S_0 is the current price of the stock per share, and R_i is the one-day log-return observed i days ago.
- The weights $w_0, w_1, \ldots w_{M-1}$ therefore also correspond to the collection $\Delta P^{(0)}, \Delta P^{(1)}, \ldots \Delta P^{(M-1)}$, the scenario space for the random variable ΔP.

Implementing the BRW Approach

2. (Constructing the Empirical Distribution) Arrange the ΔP scenario space in increasing order. We denote the resulting sequence as follows:

ΔP Scenario	$\Delta\tilde{P}^{(0)}$	$\Delta\tilde{P}^{(1)}$	$\ldots$	$\Delta\tilde{P}^{(i)}$	$\ldots$	$\Delta\tilde{P}^{(M-1)}$
Weight	$w_{(0)}$	$w_{(1)}$	$\ldots$	$w_{(i)}$	$\ldots$	$w_{(M-1)}$

Here, $\Delta\tilde{P}_{(i)} \leq \Delta\tilde{P}_{(i+1)}$ for all i.

The empirical cumulative distribution at $\Delta\tilde{P}_{(j)}$ is determined by the accumulation of weights ψ_j defined as

$$\psi_j = \sum_{i=0}^{j} w_{(i)}, \qquad j = 0, 1, \ldots, M-1. \tag{47}$$

Implementing the BRW Approach

3. (Calculating the VaR) The $100\alpha\%$ value-at-risk is obtained by first identifying successive ordered pairs $(\Delta\tilde{P}^{(k)}, \psi_k)$ and $(\Delta\tilde{P}^{(k+1)}, \psi_{k+1})$ such that

$$\psi_k < 1 - \alpha < \psi_{k+1}. \tag{48}$$

(This is because $1 - \alpha$ may not be a value assumed in the sequence $\psi_0, \psi_1, \ldots, \psi_{M-1}$.)

Let V be the linearly interpolated value of ΔP from the ordered pairs $(\Delta\tilde{P}^{(k)}, \psi_k)$ and $(\Delta\tilde{P}^{(k+1)}, \psi_{k+1})$ corresponding to the value $\psi = 1 - \alpha$. **The $100\alpha\%$ value-at-risk is therefore given by $|V|$.**

The BRW Approach: N-Asset Case

- The BRW approach can be extended to the N-asset case by taking the R_i's as the one-day log-return on the *entire portfolio*. To R_i we assign the weight w_i (as defined before).
- If P_0 is the current value of the portfolio, then scenarios for the future one-day change in portfolio value ΔP can be obtained using the equation

$$\Delta P_i = P_0(e^{R_i} - 1). \tag{49}$$

- We then resume the BRW approach at Step 2 (Constructing the Empirical Distribution) using $\{\Delta P_i\}$ as the ΔP scenarios.

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