ATENEO DE MANILA UNIVERSITY

MATH 100.2

Topics in Financial Mathematics II

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Lecture Notes v1.0

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Preface

This document serves as a compilation of the lecture notes used in the MATH 100.2 class (previously coded as MA 195L.2) offered by the Mathematics Department of the Ateneo de Manila University for S.Y. 2021-2022. Through this learning resource, we aim to introduce the students to two broad areas in financial mathematics: (i) financial risk management and (ii) pricing and valuation of financial derivatives. In particular, we will discuss Value-at-Risk (VaR)—a market risk metric widely used among practitioners. We will explore how different approaches in calculating VaR can measure the risk in the equity, foreign exchange, and fixed-income markets. The latter part of this document shifts the focus to pricing and valuation of forward-rate agreements (FRAs)—one of the simplest financial contracts available in the derivatives market.

Accompanying this document is a GitHub repository[1] containing the code and data used in our class lectures and assessments. In particular, the Jupyter notebooks in the repository showcase the Python implementation from scratch of the mathematical models and algorithms presented here.

This document would not be made possible without the wisdom (and thorough documentation) passed down by instructors of this course throughout the years: Dr. Elvie de Lara-Tuprio, Dr. Noel Cabral, Jeric Briones, Carlo Mallari, Len Garces, Lean Yao, Engel Dela Vega, and Luigi Cortez. We truly are standing on the shoulders of giants.

Jakov Dumbrique
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[1] https://github.com/ateneomathdept/math100.2_2021Sem1

Version Releases

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PART I

Value-at-Risk

CHAPTER 1

Value-at-Risk for Stocks

1.1 Introduction to Value-at-Risk and Basic Data Considerations

Basic Definitions

Value-at-risk (VaR) is a market *risk metric*–a measure of the uncertainty in the future value of a portfolio, particularly with regards to its return or profit or loss [**Alexander**; **2008**]. Crucial to understanding this uncertainty is characterizing the movement of the prices of the individual assets in the portfolio and how these movements are dependent on one another. While volatility and correlation are popular measures of a portfolio's risk (and its possible deviation from a specified target value), it is also important to acknowledge that asset prices or risk factors may not necessarily follow a multivariate normal distribution.

For any given portfolio, VaR summarizes the worst loss over a target horizon that will not be exceeded with a given level of confidence. Formally, VaR describes the quantile of the projected distribution of gains and losses over the target horizon [**Jorion**; **2007**].

Let ΔP denote the *loss* in portfolio value over a fixed time horizon ΔT, and denote by $F_\Delta(x) = \mathbb{P}[\Delta P \leq x]$ the cumulative distribution function of ΔP. We note that ΔP is a random variable since the future value of the portfolio is unknown; further, $\Delta P > 0$ indicates a loss, whereas $\Delta P < 0$ indicates a gain in portfolio value.

Given some confidence level $\alpha \in (0, 1)$, the $100\alpha\%$ VaR of the portfolio is the smallest number x such that the probability that the loss ΔP exceeds x is no larger than $1 - \alpha$. In symbols,

VaR definition using portfolio losses

$$\begin{aligned} \mathrm{VaR}_\alpha &= \inf\{x \in \mathbb{R} : \mathbb{P}[\Delta P > x] \leq 1 - \alpha\} \\ &= \inf\{x \in \mathbb{R} : F_\Delta(x) \geq \alpha\}. \end{aligned} \tag{1.1}$$

This definition is visualized in Figure 1.1.

If ΔP indicates the *change* in portfolio value over the same time horizon with cdf $F_\Delta(x)$, then the $(100 - \alpha)\%$ VaR of the portfolio is the absolute value of the largest number $x \in \mathbb{R}$ such that $F_\Delta(x) \geq \alpha$. This is visualized in Figure 1.2. In symbols,

VaR definition using portfolio gains

$$\begin{aligned} \mathrm{VaR}_\alpha &= |\sup\{x \in \mathbb{R} : \mathbb{P}[\Delta P \leq x] \leq 1 - \alpha\}| \\ &= |\sup\{x \in \mathbb{R} : F_\Delta(x) \leq 1 - \alpha\}|\,. \end{aligned} \tag{1.2}$$

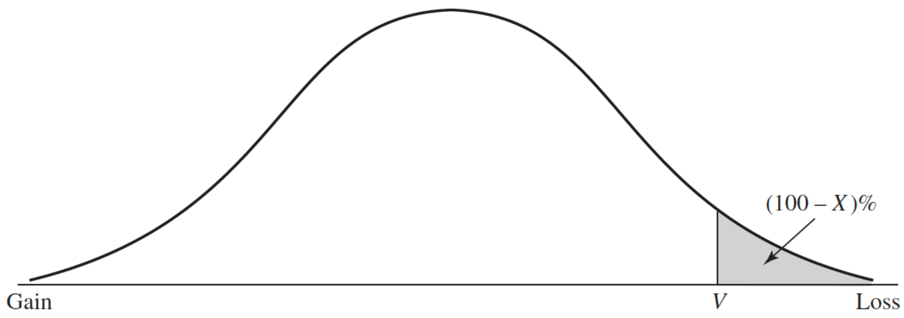

Figure 1.1: Calculation of VaR from the Probability Distribution of the Loss in Portfolio Value
Gains are negative losses; confidence level is $X\%$; VaR level is V. fig:var-loss

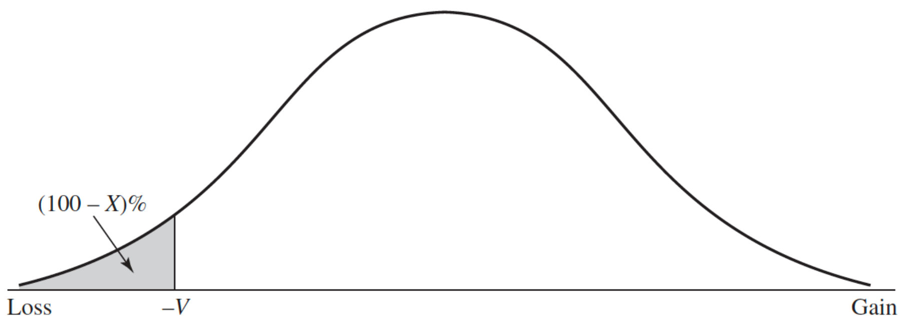

Figure 1.2: Calculation of VaR from the Probability Distribution of the Change (Gain) in Portfolio Value
Losses are negative gains; confidence level is $X\%$; VaR level is V. fig:var-gain

The consideration of a sufficiently well-behaved probability distribution for the gains and losses of the portfolio assumes that market conditions are likewise well-behaved. Thus, the VaR as a risk metric provides the worst possible loss over a given time horizon *provided normal market conditions*.

The notion of VaR can also be extended to other types of financial risk, such as credit risk and operational risk. The extension relies on identifying the appropriate risk driver (e.g. for credit risk, the likelihood of defaults on loans and other credit products and the corresponding financial exposure, and for operational risk, the frequency and severity of operational loss events) and describing its probability structure. VaR can also be used to assess possible losses on derivative products by considering the movement of the underlying asset prices.

In [**Alexander**; **2008**], the following characteristics of VaR as a risk metric are provided:

1. It corresponds to an amount that could be lost with some chosen probability;

2. It measures the risk of the risk drivers as well as the risk factor sensitivities;
3. It can be compared across different markets and different exposures;
4. It is a universal metric that applies to all activities and all types of risk;
5. It can be measured at any level, from an individual trade or portfolio, up to a single enterprise-wide VaR measure covering all the risks in the firm as a whole; and
6. When aggregated or disaggregated, it takes into account the dependencies between the constituent assets or portfolios.

One major criticism of VaR as a risk metric is that it is not subadditive. That is, it can be shown that the VaR of a combination of two or more portfolios (i.e. a diversified portfolio) is *not* bounded above by the sum of the VaR of each component portfolio. Thus, VaR is inconsistent with the notion that there is benefit of decreased risk in diversification. This also means that aggregating the VaR of individual portfolios or business units will provide a bound of the overall risk of the enterprise [**McNeil et al.**; **2005**].[1]

Measuring Returns

Let P_t denote the value of the asset at time t. The arithmetic rate of return is given by

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}, \tag{1.3}$$

which is the gain over the time period divided by the previous asset value. On the other hand, the geometric rate of return is given by

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right). \tag{1.4}$$

This is also known as the continuously compounded rate of return on the asset price. At time $t-1$, r_t and R_t are treated as random variables since P_t is unknown.

In the succeeding analyses, we will use the continuously compounded rate of return. Its advantages are outlined below [**Jorion**; **2007**]:

1. If the distribution of $\{R_t\}$ is normal, then we are ensured positive asset prices. Normally distributed arithmetic return may entail negative asset prices;
2. In the case of returns from exchange rate movements, continuously compounded returns lend themselves easily to changes in currency;
3. Geometric returns can easily be extended to multiple periods. If t is time in days, then the 2-day geometric return can be written as

$$R_{t,2} = \ln\left(\frac{P_t}{P_{t-2}}\right) = \ln\left(\frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}}\right) = R_t + R_{t-1}.$$

[1] Artzner et al. outline four axioms of *coherent* risk measures, namely translation invariance, subadditivity, positive homogeneity, and monotonicity. A discussion of these axioms of the coherence of risk measures can be found in [**McNeil et al.**; **2005**].

That is, the 2-day return is the sum of the 1-day returns in the past two days.

Note further that

$$R_t = \ln\left(\frac{P_t - P_{t-1} + P_{t-1}}{P_{t-1}}\right) = \ln\left(r_t + 1\right).$$

Equivalently, we have $e^{R_t} = 1 + r_t$. Recall that the Maclaurin series expansion for e^x is $1+x+x^2/2!+x^3/3!+\ldots$. Thus for very small values of R_t, $e^{R_t} \approx 1+R_t$. This implies that for sufficiently small returns, there is little difference between the geometric and arithmetic returns. This may not be true, however, when the annual volatility is high or the time horizon is long [**Jorion**; **2007**].

Time Aggregation

Suppose $P_0, P_1, \ldots, P_N$ denote *daily* asset price data. The analysis of asset returns requires the specification of a *time horizon* (i.e. one-day return, one-week return, one-year return) indicating the length of time until the computed return is realized. In practice, daily data is used to obtain more information about the movements of the asset price. We now discuss the procedure of computing asset returns over different time horizons given daily asset price data.

Let $R_1, R_2, \ldots, R_N$ (where $R_t = \ln(P_t/P_{t-1})$) indicate the daily (continuous) asset price returns calculated from the data. In order to reflect the assumption of normal market conditions, we assume that the return process contains independent and identically distributed (i.i.d.) variables. This means that for any two return variables R_i and R_j, we have

$$\mathbb{E}(R_i + R_j) = \mathbb{E}(R_i) + \mathbb{E}(R_j) = 2\mu \tag{1.5}$$

$$\mathrm{Var}(R_i + R_j) = \mathrm{Var}(R_i) + \mathrm{Var}(R_j) = 2\sigma^2 \tag{1.6}$$

where μ and σ^2 represent the common mean and variance of the return process. Note that estimates $\hat{\mu}$ and $\hat{\sigma}^2$ for the mean and variance of the return process can be obtained from the daily data.[2]

We note that the assumption of independence of returns assumes that markets are *efficient*; that is, the current asset price includes all relevant information about the asset. This implies further that asset prices change due to market shocks that, in principle, cannot be anticipated and therefore, asset prices are uncorrelated over time. In other words, asset prices are assumed to follow a random walk.

Recall that the N-day return, which we denote by $R_{N,N}$, is the return observed at time N using the current asset price P_N and the initial asset price P_0. To this end, we have the following computation:

$$R_{N,N} = \ln\left(\frac{P_N}{P_0}\right) = \ln\left(\prod_{t=1}^{N} \frac{P_t}{P_{t-1}}\right) = \sum_{t=1}^{N} \ln\left(\frac{P_t}{P_{t-1}}\right) = \sum_{t=1}^{N} R_t.$$

[2] The sample mean is $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} R_i$ and the sample variance is $\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(R_i - \hat{\mu})^2$. These quantities can easily be computed using any computational software.

In general, the N-day continuous return is the sum of the 1-day continuous returns. Furthermore, we note that

$$\mathbb{E}(R_{N,N}) = \mathbb{E}\left(\sum_{t=1}^{N} R_t\right) = \sum_{t=1}^{N} \mathbb{E}(R_t) = N\mu \tag{1.7}$$

$$\mathrm{Var}(R_{N,N}) = \mathrm{Var}\left(\sum_{t=1}^{N} R_t\right) = \sum_{t=1}^{N} \mathrm{Var}(R_t) = N\sigma^2. \tag{1.8}$$

The N-day return series $R_{t,N}$ can be estimated from daily asset price data $P_0, P_1, \ldots, P_T$, where T is taken to be divisible by N. If $T = kN$ for some positive integer k, then we can derive k N-day returns from the data. Under normal market conditions, the observed sample variance from the N-day returns is approximately equal to $N\sigma^2$, where σ^2 is the variance of the daily asset price returns.

In short, if μ_N and σ_N^2 represent the mean and variance of N-day returns, then assuming that the daily returns are i.i.d. random variables with mean μ and variance σ^2, we have

$$\mu_N = N\mu, \qquad \sigma_N^2 = N\sigma^2. \tag{1.9}$$

Furthermore, the standard deviation of the N-day returns is $\sqrt{N}\sigma$. This is known as **square-root-of-time scaling**.

A simple model for a serially-correlated daily returns process is the autoregressive process of order 1, written as AR(1), with functional specification

$$R_t = \rho R_{t-1} + u_t. \tag{1.10}$$

Here, we assume that the innovations process $\{u_t\}$ are mean- and variance-stationary. If $\rho > 0$, then the process possesses a trend; if $\rho < 0$, then the process exhibits mean-reverting behavior.

Given that the daily returns process follows the AR(1) model, the variance of the 2-day return is [3]

$$\sigma_2^2 = \mathrm{Var}(R_t + R_{t-1}) = \sigma^2 + \sigma^2 + 2\rho\sigma^2 = \sigma^2(2 + 2\rho).$$

In general, it can be shown that the variance of the N-day return is

$$\begin{aligned}\sigma_N^2 &= \mathrm{Var}\left(\sum_{i=1}^{N} R_{t+i}\right) \\ &= \sigma^2\left[N + 2(N-1)\rho + 2(N-2)\rho^2 + \cdots + 2(1)\rho^{N-1}\right].\end{aligned} \tag{1.11}$$

It can be further demonstrated that if square-root-of-time scaling was performed on a returns process that follows an AR(1) model, the resulting scaled N-day return is an underestimation of the true risk in the positive correlation. If the returns process is mean-reverting, the scaled N-day return overstates the true dispersion of the N-day returns.

[3] In general, for two random variables X_1 and X_2, $\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\mathrm{Cov}(X_1, X_2)$ and $\mathrm{Cor}(X_1, X_2) = \mathrm{Cov}(X_1, X_2)/\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)}$.

1.2 Delta-Normal Approach

Portfolio Containing One Stock

Consider a portfolio consisting of a long position on N shares of a stock with current price S_0.[4] Thus, the current value of the portfolio is $P_0 = NS_0$. Let S_1 be the price of the stock tomorrow. Assuming that no additional shares of stock are purchased or sold, the portfolio will still contain N shares tomorrow and so its value is $P_1 = NS_1$. Since S_1 is unknown, it follows that P_1 is a random variable.

Let ΔP denote the change in portfolio value following the asset price transition from S_0 to S_1. We then have

$$\Delta P = NS_1 - NS_0 = N(S_1 - S_0). \tag{1.12}$$

Furthermore, let R_1 denote the (continuous) one-day return on the portfolio. Then R_1 is a random variable, and

$$R_1 = \ln\left(\frac{P_1}{P_0}\right) = \ln\left(\frac{S_1}{S_0}\right).$$

For small values of R_1, we recall that $e^{R_1} \approx 1 + R_1$ and so

$$S_1 - S_0 \approx S_0 R_1$$

and

$$\Delta P \approx NS_0 R_1.$$

The key assumption in the **delta-normal approach** is the assumption that the daily returns R_1 are normally distributed with mean 0 and variance σ^2.[5] Here, σ^2 is taken to be the variance of the daily returns.

The **1-day 99% VaR** of this portfolio is the number $|V|$ such that

$$\mathbb{P}[\Delta P \geq V] = 0.99.$$

To simplify the succeeding calculations, suppose R_1 is small enough so that $\Delta P = NS_0 R_1$. Then,

$$\begin{aligned}
\mathbb{P}[\Delta P \geq V] = 0.99 &\Rightarrow \mathbb{P}[NS_0R_1 \geq V] = 0.99 \\
&\Rightarrow \mathbb{P}\left[R_1 \geq \frac{V}{NS_0}\right] = 0.99 \\
&\Rightarrow \mathbb{P}\left[R_1 \leq \frac{V}{NS_0}\right] = 0.01 \\
&\Rightarrow \mathbb{P}\left[\frac{R_1}{\sigma} \leq \frac{V}{NS_0\sigma}\right] = 0.01 \\
&\Rightarrow \Phi\left(\frac{V}{NS_0\sigma}\right) = 0.01
\end{aligned}$$

[4] A long position implies that $N > 0$. On the other hand, $N < 0$ implies that the position on the stock is short.

[5] In general, the delta-normal approach assumes that $R_1 \sim N(\mu, \sigma^2)$. In practice, we remove the mean from the observed returns series to center the distribution at zero.

$$\Rightarrow V = NS_0\sigma\Phi^{-1}(0.01).$$

1-day 99-percent VaR of a single-asset portfolio using Delta-Normal Approach

Here, $\Phi(\cdot)$ denotes the cdf of the standard normal distribution. Since the standard normal distribution is symmetric, $|\Phi^{-1}(0.01)| = \Phi^{-1}(0.99)$, and so

$$\mathbf{VaR_{0.99}} = |\mathbf{V}| = \mathbf{NS_0}\sigma\mathbf{\Phi^{-1}(0.99)}. \tag{1.13}$$

Strictly speaking, the 1-day 99% VaR of the portfolio is obtained by means of the following calculations:

$$\begin{aligned}\mathrm{VaR}_{0.99} = |V| &= -\sup\{x \in \mathbb{R} : \mathbb{P}[\Delta P \le x] \le 0.01\} \\ &= -\sup\left\{x \in \mathbb{R} : \Phi\left(\frac{x}{NS_0\sigma}\right) \le 0.01\right\} \\ &= -\sup\left\{x \in \mathbb{R} : x \le NS_0\sigma\Phi^{-1}(0.01)\right\}.\end{aligned}$$

Since $\Phi(\cdot)$ is continuous and monotonically increasing, so is its inverse. Thus, the upper bound for x, which is $NS_0\sigma\Phi^{-1}(0.01)$, is attainable and is the supremum of the set. Therefore,

$$\mathrm{VaR}_{0.99} = V = -NS_0\sigma\Phi^{-1}(0.01) = NS_0\sigma\Phi^{-1}(0.99).$$

Since N and S_0 are taken as given information, solving for the 1-day 99% VaR for the stock requires estimating the variance σ^2 of the 1-day return.

Two-Asset Portfolios

Now consider a portfolio containing N_1 and N_2 shares of two stocks with current price $S_{1,0}$ and $S_{2,0}$. Again, the position held on the stock is indicated by the algebraic sign of N_1 and N_2. The current value of the portfolio is given by

$$P_0 = N_1S_{1,0} + N_2S_{2,0}. \tag{1.14}$$

Furthermore, denote by σ_i^2 the variance of the daily returns of stock i ($i = 1, 2$). Letting $S_{i,1}$ denote the price of stock i the following trading day, the one-day (continuously compounded) return on the stock is

$$R_i = \ln\left(\frac{S_{i,1}}{S_{i,0}}\right), \qquad i = 1, 2.$$

Undiversified VaR

The presence of more than one asset effectively diversifies the portfolio; for example, if there are adverse movements in the price of one asset, the stability of the other asset prices will dampen the losses observed from the portfolio.

In this light, there are two ways to calculate the VaR for a two-stock portfolio. The first, and simpler, approach is to assume that there are no diversification effects. The **undiversified portfolio VaR** is the sum of the VaR of each individual stock in the portfolio. Let $\mathrm{VaR}_{i,0.99}$ denote the 99% VaR for the ith stock in the portfolio ($i = 1, 2$), where

$$\mathrm{VaR}_{i,0.99} = N_iS_{i,0}\sigma_i\Phi^{-1}(0.99), \qquad i = 1, 2.$$

1-day 99-percent undiversified VaR of a two-asset portfolio using Delta-Normal Approach

The undiversified portfolio VaR, denoted by $|V_u|$ is given by

$$|V_u| = N_1S_{1,0}\sigma_1\Phi^{-1}(0.99) + N_2S_{2,0}\sigma_2\Phi^{-1}(0.99). \tag{1.15}$$

Diversified VaR

Taking into account the benefits from diversification requires looking at the joint distribution of the two return variables R_1 and R_2. Let $S_{1,1}$ and $S_{2,1}$ denote the prices of each stock the next trading day. The change in the value of the portfolio is the sum of the changes in the total value of each stock holding. As we have seen in the previous section, for sufficiently small R_i, $N_i S_{i,0} R_i$ is the change in the total value of holding stock i. Thus, the change in portfolio value is

$$\Delta P = N_1 S_{1,0} R_1 + N_2 S_{2,0} R_2 = \begin{bmatrix} N_1 S_{1,0} & N_2 S_{2,0} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \boldsymbol{\alpha}^T \mathbf{R}. \quad (1.16)$$

In the above equation, $\boldsymbol{\alpha}^T = (\alpha_1, \alpha_2)$, where $\alpha_i = N_i S_{i,0}$ for $i = 1, 2$, is a vector indicating the weight of each stock relative to the portfolio value.

To obtain the diversified portfolio VaR using the Delta-Normal approach, we assume that $\mathbf{R}$ has a bivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$. Let ρ and σ_{12} denote the correlation and covariance, respectively, between R_1 and R_2. On the margins, this implies that $R_i \sim N(0, \sigma_i^2)$ for $i = 1, 2$. It follows from the property of the bivariate normal distribution that $\Delta P \sim N(0, \sigma_P^2)$, where

$$\sigma_P^2 = \boldsymbol{\alpha}^T \boldsymbol{\Sigma} \boldsymbol{\alpha}. \quad (1.17)$$

To see this, we have the following calculations:

$$\begin{aligned}
\sigma_P^2 = \mathrm{Var}(\Delta P) &= \mathbb{E}\left[(\Delta P)^2\right] - [\mathbb{E}(\Delta P)]^2 \\
&= \mathbb{E}\left[(\Delta P)^2\right] \\
&= \mathbb{E}\left[(\alpha_1 R_1 + \alpha_2 R_2)^2\right] \\
&= \alpha_1^2 \mathbb{E}\left(R_1^2\right) + 2\alpha_1\alpha_2 \mathbb{E}\left(R_1 R_2\right) + \alpha_2^2 \mathbb{E}\left(R_2^2\right) \\
&= \alpha_1^2 \mathrm{Var}(R_1) + 2\alpha_1\alpha_2 \mathrm{Cov}(R_1, R_2) + \alpha_2^2 \mathrm{Var}(R_2) \\
&= \alpha_1^2 \sigma_1^2 + 2\alpha_1\alpha_2\sigma_{1,2} + \alpha_2^2\sigma_2^2 \\
&= \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\
&= \boldsymbol{\alpha}^T \boldsymbol{\Sigma} \boldsymbol{\alpha}.
\end{aligned}$$

In terms of the correlation ρ,

$$\begin{aligned}
\sigma_P^2 &= \alpha_1^2\sigma_1^2 + 2\alpha_1\alpha_2\rho\sigma_1\sigma_2 + \alpha_2^2\sigma_2^2 \\
&= \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\
&= \boldsymbol{\alpha}^T \boldsymbol{\sigma} \widetilde{\boldsymbol{\rho}} \boldsymbol{\sigma} \boldsymbol{\alpha} \\
&= (\boldsymbol{\sigma}\boldsymbol{\alpha})^T \widetilde{\boldsymbol{\rho}} \boldsymbol{\sigma} \boldsymbol{\alpha}. \quad (1.18)
\end{aligned}$$

Recall that $\Delta P \sim N(0, \sigma_P^2)$, where σ_P^2 is given above. Let $|V_d|$ denote the 99% diversified portfolio VaR. We then have the following calculations:

$$\mathbb{P}(\Delta P \geq V_d) = 0.99 \Rightarrow \mathbb{P}\left(\frac{\Delta P}{\sigma_P} \geq \frac{V_d}{\sigma_P}\right) = 0.99$$

$$
\begin{aligned}
&\Rightarrow \mathbb{P}\left(\frac{\Delta P}{\sigma_P} \le \frac{V_d}{\sigma_P}\right) = 0.01 \\
&\Rightarrow \Phi\left(\frac{V_d}{\sigma_P}\right) = 0.01 \\
&\Rightarrow V_d = \sigma_P \Phi^{-1}(0.01).
\end{aligned}
$$

By the properties of the standard normal distribution, the 99% diversified portfolio VaR is given by

1-day 99-percent diversified VaR of a two-asset portfolio using Delta-Normal Approach

$$
|V_d| = \left|\sigma_P \Phi^{-1}(0.01)\right| = \sigma_P \Phi^{-1}(0.99). \tag{1.19}
$$

Benefit of Diversification

The difference between the diversified and undiversified VaRs is called the benefit of diversification. Note that a portfolio's diversified VaR should be less than or equal to its undiversified VaR. This is proven below.

Let the undiversified VaR be $|V_u|$ and the diversified VaR be $|V_d|$. Recall that $\sigma_P^2 = \alpha_1^2\sigma_1^2 + 2\alpha_1\alpha_2\rho\sigma_1\sigma_2 + \alpha_2^2\sigma_2^2$, and note that $|V_d| = (\alpha_1\sigma_1 + \alpha_2\sigma_2)\Phi^{-1}(0.99)$.

$$
\begin{aligned}
\sigma_P &= \sqrt{\alpha_1^2\sigma_1^2 + 2\alpha_1\alpha_2\rho\sigma_1\sigma_2 + \alpha_2^2\sigma_2^2} \\
&= \sqrt{\alpha_1^2\sigma_1^2 + 2\alpha_1\alpha_2\rho\sigma_1\sigma_2 + \alpha_2^2\sigma_2^2 + 2(1-\rho)\alpha_1\alpha_2\sigma_1\sigma_2 - 2(1-\rho)\alpha_1\alpha_2\sigma_1\sigma_2} \\
&= \sqrt{\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 + 2\alpha_1\alpha_2\sigma_1\sigma_2 - 2(1-\rho)\alpha_1\alpha_2\sigma_1\sigma_2} \\
&= \sqrt{(\alpha_1\sigma_1 + \alpha_2\sigma_2)^2 - 2(1-\rho)\alpha_1\alpha_2\sigma_1\sigma_2}.
\end{aligned}
$$

Note that $2(1-\rho)\alpha_1\alpha_2\sigma_1\sigma_2 > 0$. Since $-1 \le \rho \le 1$,

$$
\begin{aligned}
2(1-\rho) &\le 4 \\
2(1-\rho)\alpha_1\alpha_2\sigma_1\sigma_2 &\le 4\alpha_1\alpha_2\sigma_1\sigma_2.
\end{aligned}
$$

By the AM-GM Inequality,

$$
\begin{aligned}
\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 &\ge 2\alpha_1\alpha_2\sigma_1\sigma_2 \\
\alpha_1^2\sigma_1^2 + 2\alpha_1\alpha_2\sigma_1\sigma_2 + \alpha_2^2\sigma_2^2 &\ge 4\alpha_1\alpha_2\sigma_1\sigma_2 \\
(\alpha_1\sigma_1 + \alpha_2\sigma_2)^2 &\ge 4\alpha_1\alpha_2\sigma_1\sigma_2.
\end{aligned}
$$

Thus, σ_P is defined, and

$$
\begin{aligned}
\sqrt{(\alpha_1\sigma_1 + \alpha_2\sigma_2)^2 - 2(1-\rho)\alpha_1\alpha_2\sigma_1\sigma_2} &\le \sqrt{(\alpha_1\sigma_1 + \alpha_2\sigma_2)^2} \\
\sigma_P &\le \alpha_1\sigma_1 + \alpha_2\sigma_2 \\
\sigma_P\Phi^{-1}(0.99) &\le (\alpha_1\sigma_1 + \alpha_2\sigma_2)\Phi^{-1}(0.99) \\
|V_d| &\le |V_u|.
\end{aligned}
$$

n-Asset Portfolio

Suppose a portfolio contains shares on n stocks, where N_i and $S_{i,0}$ denote the number of shares and the current price of the stock for $i = 1, 2, \ldots, n$. If $N_i > 0$, then the portfolio contains a long position on stock i; otherwise, the portfolio holds a short position on the asset. The current value of the portfolio is

$$P_0 = N_1 S_{1,0} + N_2 S_{2,0} + \cdots + N_n S_{n,0} = \sum_{i=1}^{n} N_i S_{i,0}. \tag{1.20}$$

Letting $S_{i,1}$ denote the price of stock i the following trading day, the one-day (continuously compounded) return on the stock is

$$R_i = \ln\left(\frac{S_{i,1}}{S_{i,0}}\right), \qquad i = 1, 2, \ldots, n.$$

Similar to the case of 2 assets, the n-asset portfolio has an undiversified and diversified VaR.

Undiversified VaR

The 99% VaR of the ith stock is given by

$$\text{VaR}_{i,0.99} = N_i S_{i,0} \sigma_i \Phi^{-1}(0.99)$$

for $i = 1, 2, \ldots, n$. Here, σ_i^2 is the variance of the daily returns R_i of stock i. The 99% undiversified VaR of the entire portfolio is given by the sum of the individual VaRs,

1-day 99-percent undiversified VaR of a n-asset portfolio using Delta-Normal Approach

$$|V_u| = \sum_{i=1}^{n} N_i S_{i,0} \sigma_i \Phi^{-1}(0.99). \tag{1.21}$$

Diversified VaR

Assume that $R_i \sim N(0, \sigma_i^2)$ for all $i = 1, 2, \ldots, n$. Then with the same reasoning as in the case of 2 assets, we assume that R_i is small enough so that the change in portfolio value can be written as

$$\Delta P = \sum_{i=1}^{n} N_i S_{i,0} R_i. \tag{1.22}$$

Let $\alpha_i = N_i S_{i,0}$. Then in vector notation, we can write

$$\Delta P = \sum_{i=1}^{n} \alpha_i R_i = \boldsymbol{\alpha}^T \mathbf{R}, \tag{1.23}$$

where $\boldsymbol{\alpha}^T = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ denotes the vector of asset weights and $\mathbf{R}^T = [R_1, R_2, \ldots, R_n]$ is the vector of one-day returns.

Assume that the $\mathbf{R}$ has a multivariate normal distribution, written as $\mathbf{R} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$, where $\mathbf{0}$ is a vector (of length n) of zeros and $\boldsymbol{\Sigma}$ is the covariance matrix of $\mathbf{R}$. If $\sigma_{i,j} = \text{Cov}(R_i, R_j)$, the covariance matrix is written as

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \ldots & \sigma_{1,n} \\ \sigma_{1,2} & \sigma_2^2 & \ldots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1,n} & \sigma_{2,n} & \ldots & \sigma_n^2 \end{bmatrix}. \tag{1.24}$$

Since $\sigma_{i,j} = \sigma_{j,i}$, $\mathbf{\Sigma}$ is symmetric.

By the properties of the multivariate normal distribution, since ΔP is a linear combination of the components of $\mathbf{R}$, ΔP is normally distributed with mean 0 and variance σ_P^2, where

$$\sigma_P^2 = \boldsymbol{\alpha}^T \mathbf{\Sigma} \boldsymbol{\alpha}. \tag{1.25}$$

(Refer to the discussion of the properties of the multivariate normal distribution in MATH 62.2.[6])

Let $\rho_{i,j}$ denote the correlation of R_i and R_j. It then follows that

$$\begin{aligned}
\sigma_P^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \rho_{i,j} \sigma_i \sigma_j \\
&= \begin{bmatrix} \alpha_1\sigma_1 & \alpha_2\sigma_2 & \dots & \alpha_n\sigma_n \end{bmatrix} \begin{bmatrix} 1 & \rho_{1,2} & \dots & \rho_{1,n} \\ \rho_{2,1} & 1 & \dots & \rho_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n,1} & \rho_{n,2} & \dots & 1 \end{bmatrix} \begin{bmatrix} \alpha_1\sigma_1 \\ \alpha_2\sigma_2 \\ \vdots \\ \alpha_n\sigma_n \end{bmatrix} \\
&= \boldsymbol{\alpha}^T \boldsymbol{\sigma}\boldsymbol{\rho}\boldsymbol{\sigma}\boldsymbol{\alpha}
\end{aligned}$$

1-day 99-percent diversified VaR of a n-asset portfolio using Delta-Normal Approach

With the same set of manipulations in the case of 2 assets, the one-day 99% diversified VaR for the portfolio is given by

$$|V_d| = \sigma_P \Phi^{-1}(0.99). \tag{1.26}$$

Computing N-Day VaR

So far, we have discussed the Delta-Normal approach to compute for the one-day 99% VaR for a given portfolio of stocks. To recall, this metric provides us a number x such that the total portfolio loss *the next trading day* will not exceed x with 99% probability. Now, we consider extending the time horizon to N days.

The Delta-Normal approach for computing the N-day VaR relies on the concept of time aggregation for asset returns. If the daily returns series $\{R_t\}$ *for the portfolio* is a collection of independent and identically distributed normal random variables with mean μ and σ^2, then the series of N-day returns $\{\eta_t\}$ is a collection of i.i.d. $N(\mu_N, \sigma_N^2)$ variables with

$$\mu_N = N\mu, \qquad \sigma_N^2 = N\sigma^2.$$

Let ΔP represent the change in portfolio value over N days. That is, if P_0 and P_N represent the value of the portfolio today and N days from now, respectively, then $\Delta P = P_N - P_0$. Let η denote the continuously compounded rate of return on the portfolio over N days,

$$\eta = \ln\left(\frac{P_N}{P_0}\right).$$

[6]The exact property is the following: Let the $p \times 1$ random vector $\mathbf{y}$ be $N_p(\boldsymbol{\mu}, \mathbf{\Sigma})$ and let $\mathbf{a}$ be any $p \times 1$ vector of constants. Then $\mathbf{z} = \mathbf{a}^T\mathbf{y}$ is $N(\mathbf{a}^T\boldsymbol{\mu}, \mathbf{a}^T\mathbf{\Sigma}\mathbf{a})$.

Then for sufficiently small values of η,

$$\Delta P = P_N - P_0 = P_0 e^{\eta} \approx P_0 \eta.$$

Note that since P_N is unknown, η is likewise unknown. We further assume that $\Delta P = P_0 \eta$.

Since the daily returns are independent and identically normally distributed, then η (which we established to be the sum of the daily returns from day 1 up until day N) is also normally distributed with mean μ_N and variance σ_N^2. Since it is usually assumed that the daily returns have zero mean, then $\mu_N = 0$.

The N-day 99% VaR for the portfolio is the number V such that

$$\mathbb{P}[\Delta P \geq V] = 0.99.$$

We then have

$$\begin{aligned} \mathbb{P}[\Delta P \geq V] = 0.99 &\Rightarrow \mathbb{P}\left[\eta \geq \frac{V}{P_0}\right] = 0.99 \\ &\Rightarrow \mathbb{P}\left[\frac{\eta}{\sigma_N} \geq \frac{V}{P_0 \sigma_N}\right] = 0.99 \\ &\Rightarrow \Phi\left(\frac{V}{P_0 \sigma_N}\right) = 0.01 \\ &\Rightarrow V = P_0 \sigma_N \Phi^{-1}(0.01) \\ &\Rightarrow |V| = P_0 \sigma_N \Phi^{-1}(0.99). \end{aligned}$$

Since $\sigma_N = \sqrt{N}\sigma$, where σ is the standard deviation of the daily returns, we have

$$|V| = P_0 \sqrt{N} \sigma \Phi^{-1}(0.99). \tag{1.27}$$

Since $P_0 \sigma \Phi^{-1}(0.99)$ gives the one-day 99% VaR of the portfolio (which we denote by $|V_1|$), we then have the relation

N-day 99-percent VaR of a portfolio using square-root-of-time scaling

$$|V| = \sqrt{N} |V_1| \tag{1.28}$$

that relates the one-day VaR and the N-day VaR. This relation is known as **square-root-of-time scaling**.

At this point, we highlight some remarks on the computations above.

1. The N-day returns can be estimated from the time series of daily asset prices. Let $\{P_t\}$ be the time series of *portfolio values* over time. Then the N-day continuously compounded return observed at time t is given by

   $$\eta_t = \ln\left(\frac{P_t}{P_{t-N}}\right), \qquad t \geq N. \tag{1.29}$$

   N-day 99-percent VaR of a portfolio using empirical N-day returns

   The variance estimated from $\{\eta_t\}$ is taken to be σ_N^2. The formula

   $$|V| = P_0 \sigma_N \Phi^{-1}(0.99) \tag{1.30}$$

   is then used to compute for the N-day 99% VaR of the portfolio.

2. The process outlined above works for getting the N-day VaR of a portfolio consisting of one stock.

3. Note that the process discussed above provides the *diversified* portfolio VaR over N days since the returns of the portfolio are used instead of the returns on the individual asset prices. An *undiversified* version of the portfolio VaR is obtained by computing the N-day VaR for each individual stock then adding the resulting VaRs.

4. For larger values of N, larger inconsistencies between the VaR obtained from square-root-of-time scaling and that obtained from the use of empirical N-day returns arise. This is because for longer time horizons, the i.i.d. assumption placed on the daily returns process is no longer realistic due to higher chances of adverse market movements.

1.3 Estimating Volatilities and Correlations

Exponentially Weighted Moving Average Model

Let $P_0, P_1, \ldots, P_{n-1}$ denote the sequence of historical asset prices, with P_{n-1} being the most current price. Let σ_n denote the volatility of the price of the asset on day n as estimated at the end of day $n-1$. Let R_t be the one-day continuously compounded rate of return during day t (measured between the end of day $t-1$ and day t), so $R_t = \ln(P_t/P_{t-1})$ for all $t \geq 1$.

Using historical asset prices, an unbiased estimate for σ_n^2 is given by

$$\hat{\sigma}_n^2 = \frac{1}{n-2} \sum_{i=1}^{n-1} \left(R_{n-i} - \bar{R}\right)^2, \qquad \bar{R} = \frac{1}{n-1} \sum_{i=1}^{n-1} R_{n-i}. \tag{1.31}$$

Note that the same can be done with the arithmetic returns r_t. It is also possible to assume that $\bar{R} = 0$ (to keep consistent with the assumption that R has a zero-mean normal distribution). Furthermore, it is also possible to use the MLE estimator for σ_n^2 with $\bar{R} = 0$, giving us

$$\tilde{\sigma}_n^2 = \frac{1}{n-1} \sum_{i=1}^{n-1} R_{n-i}^2. \tag{1.32}$$

This means that volatility can be estimated from the unweighted average of the squared log returns on the asset prices.

Since our goal is to estimate the most recent volatility, we take note that more recent information may have more weight than movements observed in the far past. Thus, we consider the alternative model

$$\sigma_n^2 = \sum_{i=1}^{n-1} \alpha_i R_{n-i}^2, \tag{1.33}$$

where α_i is the weight assigned to the return observed i days ago, with $\alpha_i > 0$ and $\alpha_i > \alpha_{i-1}$ for all $i \geq 1$ and $\sum_{i=1}^{n-1} \alpha_i = 1$.

Note that if R_t itself follows a time series model (referred to as the mean equation), such as the ARMA model, then the mean equation and the above equation constitute an autoregressive conditional heteroskedasticity (ARCH) model. Furthermore, the variance estimate may also take into consideration a long-run variance estimate ν as reflected in the following equations:

$$\sigma_n^2 = \gamma\nu + \sum_{i=1}^{n-1} \alpha_i R_{n-i}^2, \qquad \gamma + \sum_{i=1}^{n-1} \alpha_i = 1. \tag{1.34}$$

A particular weighted estimation equation for σ_n^2 is in the form of an **exponentially weighted moving average (EWMA) model**. Here, we assume that the weights α_i decrease exponentially as we move back through time, as given by the equation

$$\alpha_i = \lambda\alpha_{i-1}, \qquad 0 < \lambda < 1. \tag{1.35}$$

Under this condition, it can be shown that the variance equation becomes

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)R_{n-1}^2. \tag{1.36}$$

Working out the recursion, it can be shown further that

$$\sigma_n^2 = (1-\lambda)\sum_{i=1}^{n-1} \lambda^{i-1} R_{n-i}^2 + \lambda^{n-1}\sigma_1^2. \tag{1.37}$$

For large n, we note that $\lambda^{n-1} \approx 0$, and so

$$\sigma_n^2 \approx \sum_{i=1}^{n-1} (1-\lambda)\lambda^{i-1} R_{n-i}^2. \tag{1.38}$$

Therefore, the sequence of weights is given by

$$\alpha_i = (1-\lambda)\lambda^{i-1}, \qquad i = 1, 2, \ldots, n-1. \tag{1.39}$$

The parameter λ governs how responsive volatility is to new information. A low value of λ indicates that higher weight is given to new market information. On the other hand, a high λ produces estimates of daily volatility that respond relatively slowly to new information provided by the return on the asset. As such, at any given time, only the current estimate of the variance rate and the most recent value of the log return is needed to update the variance estimate.

A more general form of the volatility equation is given by

$$\sigma_n^2 = \omega + \alpha R_{n-1}^2 + \beta\sigma_{n-1}^2, \tag{1.40}$$

where $\omega + \alpha + \beta = 1$. This is known as the GARCH(1,1) model. The EWMA is a special case of the GARCH(1,1) model with $\omega = 0$, $\alpha = \lambda$, and $\beta = 1 - \lambda$. The GARCH model may be extended to consider the past p variance estimates and the past q returns. Such a model is known as the GARCH(p, q) model.

EWMA and the Delta-Normal Approach

At this juncture, we now combine the results of the EWMA model for volatility and the Delta-Normal approach in estimating the VaR for a portfolio containing a single asset. This approach uses historical stock prices $S_0, S_1, \ldots, S_{n-1}$, where S_{n-1} is the most recent stock price.

From the historical stock prices, we first compute the one-day continuously compounded rate of return $R_1, R_2, \ldots, R_{n-1}$, where $R_t = \ln(S_t/S_{t-1})$ for all $t = 1, 2, \ldots, n-1$.

We assume that $\lambda \in (0,1)$ is given.[7] Thus, an estimate for σ_n^2, the variance of the price of the asset on day n as estimated at the time of computing the VaR, is given by the equation

$$\sigma_n^2 = \sum_{i=1}^{n-1} (1-\lambda)\lambda^{i-1} R_{n-i}^2.$$

Let $R = \ln(S_n/S_{n-1})$ be a random variable representing the one-day continuously compounded return between day $n-1$ and day n (here, note that day n is the forecast horizon of the VaR). With the variance estimate σ_n^2, we assume that R has a normal distribution with zero mean and variance σ_n^2. This is similar to the central assumption made in the original discussion of the Delta-Normal approach.

In the discussion of the single-asset VaR, the change in portfolio value is estimated to be $\Delta P = N\tilde{S}_0 R$, where $\tilde{S}_0 = S_{n-1}$ is the most recent stock price.[8] Therefore, ΔP is *approximately* normally distributed with mean 0 and variance $(N\tilde{S}_0\sigma_n)^2$. Through the same procedure described prior, the one-day $100\alpha\%$ value-at-risk for the portfolio is given by

$$|V| = N\tilde{S}_0\sigma_n\Phi^{-1}(\alpha), \qquad \alpha \in \{0.95, 0.99, 0.999\}. \tag{1.41}$$

Estimating Covariances

Consider the one-day log returns of two assets,

$$R_{1,i} = \ln\left(\frac{P_{1,i}}{P_{1,i-1}}\right) \qquad R_{2,i} = \ln\left(\frac{P_{2,i}}{P_{2,i-1}}\right). \tag{1.42}$$

Furthermore, we consider the following notation.

- $\sigma_{1,n}$: one-day volatility for the first asset estimated for day n
- $\sigma_{2,n}$: one-day volatility for the second asset estimated for day n
- $\mathrm{Cov}_{1,2,n}$: estimate of covariance between the two log returns calculated for day n

Thus, an estimate for the correlation between the two returns for day n is

$$\rho_n = \frac{\mathrm{Cov}_{1,2,n}}{\sigma_{1,n}\sigma_{2,n}}. \tag{1.43}$$

Under the assumption that mean log returns are zero, estimates (MLE) for the individual volatilities are given by

$$\sigma_{1,n}^2 = \frac{1}{n-1}\sum_{i=1}^{n-1} R_{1,n-i}^2 \qquad \sigma_{2,n}^2 = \frac{1}{n-1}\sum_{i=1}^{n-1} R_{2,n-i}^2 \tag{1.44}$$

[7] In practice, λ must be estimated from historical data. Standard texts on time series models may be consulted for the estimation procedures for the EWMA and GARCH models.

[8] Be mindful of the notation adopted in this section. In the previous discussion, S_0 represents the current stock price. In this section, S_{n-1} or $\tilde{S}_0$ represents the current stock price.

The MLE for the covariance is likewise given by

$$\mathrm{Cov}_{1,2,n} = \frac{1}{n-1}\sum_{i=1}^{n-1} R_{1,n-1}R_{2,n-1}. \tag{1.45}$$

These estimators assign equal weights to historically observed returns.

As an alternative to these estimates, we can use individual exponentially weighted moving average models to generate forecasts for $\sigma_{1,n}^2$, $\sigma_{2,n}^2$, and $\mathrm{Cov}_{1,2,n}$. Let λ_1, λ_2, and λ, all constants from the interval $(0,1)$ be the decay parameters of the individual EWMA processes for $\sigma_{1,n}^2$, $\sigma_{2,n}^2$, and $\mathrm{Cov}_{1,2,n}$, respectively. We thus have the following estimating equations,

$$\sigma_{1,n}^2 = \lambda_1\sigma_{1,n-1}^2 + (1-\lambda_1)R_{1,n-1}^2 \tag{1.46}$$

$$\sigma_{2,n}^2 = \lambda_2\sigma_{2,n-1}^2 + (1-\lambda_2)R_{2,n-1}^2 \tag{1.47}$$

$$\mathrm{Cov}_{1,2,n} = \lambda\mathrm{Cov}_{1,2,n-1} + (1-\lambda)R_{1,n-1}R_{2,n-1} \tag{1.48}$$

The variance equations will have the form

$$\sigma_{j,n}^2 = \sum_{i=1}^{n-1}(1-\lambda_j)\lambda_j^{i-1}R_{j,n-i}^2, \qquad j=1,2 \tag{1.49}$$

under the EWMA specification. Similarly, the covariance equation can be expressed as

$$\mathrm{Cov}_{1,2,n} = \sum_{i=1}^{n-1}(1-\lambda)\lambda^{i-1}R_{1,n-i}R_{2,n-i}. \tag{1.50}$$

Let $\mathbf{R} = (R_1, R_2)^T$ be the vector of one-day continuously compounded returns on the two assets measured between days $n-1$ and n. That is, each return is given by $R_{j,n} = \ln(S_{j,n}/S_{j,n-1})$, where $S_{j,t}$ is the price of the asset on day t, $j=1,2$, and $S_{j,n-1}$ is the most recent price available for stock j. With the prior discussion on covariance estimates, we assume that $\mathbf{R}$ has a bivariate normal distribution with mean $\mathbf{0}$ and covariance matrix

$$\mathbf{\Sigma}_n = \begin{bmatrix} \sigma_{1,n}^2 & \mathrm{Cov}_{1,2,n} \\ \mathrm{Cov}_{1,2,n} & \sigma_{2,n}^2 \end{bmatrix}. \tag{1.51}$$

In the two-asset case, the change in portfolio value ΔP is approximated as

$$\Delta P = N_1S_{1,n-1}R_1 + N_2S_{2,n-1}R_2 = \boldsymbol{\alpha}^T\mathbf{R},$$

where $\boldsymbol{\alpha} = (N_1S_{1,n-1}, N_2S_{2,n-1})^T$. It therefore follows that ΔP is approximately normally distributed with mean 0 and variance σ_P^2, where

$$\sigma_P^2 = \boldsymbol{\alpha}^T\mathbf{\Sigma}_n\boldsymbol{\alpha}.$$

Consequently, the diversified one-day $100\alpha\%$ VaR for the portfolio is given by

$$|V_d| = \sigma_P\Phi^{-1}(\alpha).$$

Not all covariance matrices are internally consistent; that is, estimates for individual volatilities and covariances may not always provide positive semi-definite covariance matrices. To ensure that a positive semi-definite covariance matrix is produced, variances and covariances should be calculated consistently. That is, if variances are calculated by giving equal weight to historical log returns, then the same should be done to estimate covariances. Likewise, if EWMA models are used for the variances, then an EWMA model must also be used for the covariance.

1.4 Historical Simulation

The primary basis of the Delta-Normal approach in the computation of the VaR is the assumption that asset price returns (or portfolio returns) are normally distributed. The parameters of the distribution of the return series may be estimated by means of looking at historical asset prices.

Historical data can also be used to estimate VaR for stock portfolios without assuming a particular parametric distribution for the asset return series. This particular procedure is known as **historical simulation**.

Single-Asset Portfolios

Consider a financial institution with a position of N shares on a stock E with price S_0 today. Let S_1 be the price per share of E_1 tomorrow. If P_0 and P_1 denote the value of the portfolio today, with

$$P_0 = NS_0 \qquad \text{and} \qquad P_1 = N_1 S_1,$$

then the change in portfolio value, ΔP, is given by

$$\Delta P = P_1 - P_0 = N(S_1 - S_0).$$

Note that the stock price tomorrow is unknown, and so ΔP is a random variable. Let $R_1 = \ln(S_1/S_0)$ be the continuously compounded return observed tomorrow. Recalling that for small x, $e^x \approx 1 + x$, we have

$$S_1 = S_0 e^{R_1} \approx S_0(1 + R_1).$$

This further implies that

$$\Delta P = N(S_1 - S_0) \approx NS_0 R_1.$$

In historical simulation, it is assumed that R_1 is a random variable whose values consist of the one-day asset returns observed historically. Thus, we require information on the stock prices over the last M trading days. We denote this time series by $\{S_{-t}\}$, $t = 0, 1, 2, \ldots, M$, where S_{-t} indicates the stock price t trading days ago and $M + 1$ is the length of the time series. Define

$$R_{1,j} = \ln\left(\frac{S_{-j}}{S_{-(j+1)}}\right), j = 0, 1, \ldots, M - 1 \tag{1.52}$$

and so the series $\{R_{1,j}\}$ contains all possible values of the random variable R_1, with $R_{1,j}$ representing the jth scenario for R_1. In total, there are M possible scenarios for R_1.

In scenario j, the stock price tomorrow is approximated as

$$S_{1,j} = S_0 e^{R_{1,j}}. \tag{1.53}$$

Consequently, the change in portfolio value under scenario j is given by the approximation

$$\Delta P_j = N(S_{1,j} - S_0) \approx NS_0 R_{1,j}. \tag{1.54}$$

Like the random variable R_1, ΔP can be represented as a random variable with M possible scenarios denoted by $\{\Delta P_j\}$.

The one-day 99% VaR of the portfolio thus corresponds to the scenario j^*, given by ΔP_{j^*}, in the 1st percentile of the scenario space $\{\Delta P_j\}$. With ΔP_{j^*} in the 1st percentile, 99% of the other possible changes in portfolio value; as such, it represents the worst possible movement in portfolio value with probability 99%. If the ΔP_js are arranged in ascending order, the one-day 99% VaR for the portfolio is the $(0.01M)$th smallest value in the array.

n-Asset Portfolios

Now suppose there are n stocks $E_1, E_2, \ldots, E_n$ in the portfolio, with positions $N_1, N_2, \ldots, N_n$ respectively. Computing for the one-day 99% VaR for the portfolio using historical simulation is similar to the calculations discussed in the previous section.

Let ΔP_i denote the change in portfolio value brought about by movements in the price of stock E_i. Let $S_{i,0}$ and $S_{1,i}$ denote the price of E_i today and tomorrow, respectively. We thus have

$$\Delta P_i = N_i(S_{i,1} - S_{i,0}).$$

If $R_{1,i}$ indicates the continuously compounded one-day return on stock E_i from today until tomorrow, i.e.

$$R_{1,i} = \ln\left(\frac{S_{i,1}}{S_{i,0}}\right),$$

then we have the approximation

$$\Delta P_i = N_i(S_{i,1} - S_{i,0}) \approx N_i S_{i,0} R_{1,i}.$$

From historical stock prices (using the methods in the previous section), we can construct M scenarios for $R_{1,i}$, which we assume to be a random variable. Denote by $\{R_{1,i,j}\}$ the scenario space for $R_{1,i}$. Thus, M scenarios can also be constructed for ΔP_i, which we denote by $\{\Delta P_{i,j}\}$, where

$$\Delta P_{i,j} = N_i S_{i,0} R_{1,i,j}. \tag{1.55}$$

Let $\Delta P = P_1 - P_0$ be the change in the portfolio value, where P_1 and P_0 denote the portfolio value tomorrow and today, respectively. We note that the change in portfolio value can be written as

$$\begin{aligned}
\Delta P &= P_1 - P_0 \\
&= (N_1 S_{1,1} + \cdots + N_n S_{n,1}) - (N_1 S_{1,0} + \cdots + N_n S_{n,0}) \\
&= N_1(S_{1,1} - S_{1,0}) + \cdots + N_n(S_{n,1} - S_{n,0}) \\
&\approx N_1 S_{1,0} R_{1,1} + \cdots + N_n S_{n,0} R_{1,n} \\
&= \sum_{i=1}^{n} N_i S_{i,0} R_{1,i}
\end{aligned}$$

$$= \sum_{i=1}^{n} \Delta P_i$$

We can also construct a distribution of ΔP based on the scenarios on the daily returns of each stock. In particular, the jth scenario for ΔP, denoted by $\Delta P^{(j)}$, is obtained by using the jth scenario of each ΔP_i. Thus, the scenario space for ΔP is given by $\{\Delta P^{(j)}\}$, where

$$\Delta P^{(j)} = \sum_{i=1}^{n} \Delta P_{i,j} = \sum_{i=1}^{n} N_i S_{i,0} R_{1,i,j}. \tag{1.56}$$

Since there are M scenarios for ΔP, the one-day 99% VaR for the portfolio is given by the $(0.01M)$th smallest value in the collection $\{\Delta P^{(j)}\}$.

Alternative Historical Simulation Approach for the n-Asset Portfolio

The statement $\Delta P^{(j)} = \sum_{i=1}^{n} \Delta P_{i,j}$ implies that the change in portfolio value can be decomposed linearly and additively as changes in the value of each individual stock. This may not always be the case. As an alternative, instead of running scenarios on the return of each individual stock, we consider scenarios on the returns of the *entire portfolio*. In this analysis, we would like to simulate possible values of the value P_1 of the portfolio the next day using the equation

$$P_1 = P_0 \exp(R^{(P)}), \tag{1.57}$$

where $R^{(P)}$ is the (unknown) one-day log-return on the portfolio and P_0 is the current portfolio value.

Let $R_j^{(P)}$ denote the one-day log-return on the portfolio observed j days ago. From $M+1$ historical portfolio values, we can therefore make M scenarios for one-day log-return on the portfolio. The jth scenario for the value of the portfolio in the next trading day is given by

$$P_1^{(j)} = P_0 e^{R_j^{(P)}}. \tag{1.58}$$

The jth scenario for the change in portfolio value is given by

$$\Delta P^{(j)} = P_1^{(j)} - P_0. \tag{1.59}$$

Since there are M scenarios for ΔP, the one-day 99% VaR for the portfolio is given by the $(0.01M)$th smallest value in the collection $\{\Delta P^{(j)}\}$.

1.5 Hybrid Historical Approach: the Boudoukh-Richardson-Whitelaw (BRW) Approach

Introduction and Motivation

The historical simulation approach assumes that historical returns are uniformly distributed (i.e. they are all equally likely to occur) so quantiles are estimated based on the empirical distribution of returns.[9] The problems, among others, that arise from this approach are as follows:

- Extreme percentiles of the distribution are difficult to estimate with a small amount of data;
- Returns are assumed to be independent and identically distributed and hence does not allow for time-varying volatility.

To solve these problems, we may extend the historical window (the length of the historical time series data used for the simulation) to include more observations. However, lengthening the historical window diminishes the weight assigned to more recent market phenomenon.

Thus, an alternative to the ordinary historical simulation approach is the Boudoukh-Richardson-Whitelaw (BRW) Approach, also known as the hybrid historical simulation approach. This procedure entails the use of exponentially-declining weights on historical data to estimate the quantiles of the distribution of portfolio or asset returns, with bigger weights assigned to more recent data. Under this approach, the empirical distribution of historical returns now rely on the assigned weights. That is, quantiles are calculated based on the cumulative weights after arranging historical returns in increasing order.

Assignment of Weights

Let $R_0, R_1, \ldots, R_{M-1}$ denote the observed one-day continuously compounded returns on the stock or portfolio, where $R_i = \ln(S_i/S_{i-1})$ is the rate of return observed i days ago. Let $0 < \lambda < 1$ be constant and let w_i be the weight assigned to R_i for all $i = 1, 2, \ldots, M-1$.[10] The weights are assumed to decay exponentially, so we assume that $w_{i+1} = \lambda w_i$. Making sure that the weights add up to 1, it can be shown that

$$w_i = \left(\frac{1-\lambda}{1-\lambda^M}\right)\lambda^i, \qquad i = 1, 2, \ldots, M-1. \tag{1.60}$$

Consider the single-asset case. Note that each historical return can be mapped to a scenario for the change in portfolio value; that is, the collection $\{\Delta P^{(i)}\}$ where $\Delta P^{(i)} \approx N S_0 R_i$, where N is the number of shares of the stock and S_0 is the current price, represents the scenario space for ΔP.[11] Hence, the weights $w_0, w_1, \ldots, w_{M-1}$ also therefore correspond to the scenarios $\Delta P^{(0)}, \Delta P^{(1)}, \ldots, \Delta P^{(M-1)}$, respectively.

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[9] Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution. Assuming that all elements of the sample are assigned equal probabilities, the **empirical distribution** of this random sample is the function F given by

$$F(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i \le x), \qquad x \in \mathbb{R},$$

where $\mathbf{1}(X_i \le x) = 1$ if $X_i \le x$ and is equal to 0 otherwise.

[10] Note that the decay parameter λ in the BRW approach is different from the λ in the EWMA model. In the BRW approach, λ governs the decay on the weights assigned to asset or portfolio returns, whereas λ in the EWMA model corresponds to the weights on historical volatilities. However, the correspondence of λ to the influence of recent information is the same between the BRW approach and the EWMA model.

[11] In the case of two or more assets, we employ the concepts in the alternative historical simulation approach and assume that R_i represents the rate of return on the entire portfolio.

Constructing the Empirical Distribution and Finding the VaR

After assigning the weights to each element of the scenario space for ΔP, we then arrange the $\Delta P^{(i)}$'s and w_i's in increasing order of $\Delta P^{(i)}$. We denote the resulting sequence as follows:

ΔP **Scenario**	$\Delta\tilde{P}^{(0)}$	$\Delta\tilde{P}^{(1)}$	$\dots$	$\Delta\tilde{P}^{(i)}$	$\dots$	$\Delta\tilde{P}^{(M-1)}$
Weight	$w_{(0)}$	$w_{(1)}$	$\dots$	$w_{(i)}$	$\dots$	$w_{(M-1)}$

Here, $\Delta\tilde{P}_{(i)} \leq \Delta\tilde{P}_{(i+1)}$ for all i.

The empirical cumulative distribution at $\Delta\tilde{P}^{(i)}$, which we denote by ψ_i, is determined by the accumulation of weights

$$\psi_i = \sum_{k=0}^{i} w_{(k)}, \qquad i = 0, 1, \dots, M-1. \tag{1.61}$$

Alternatively, for any $x \in \mathbb{R}$, the continuous version of the empirical distribution is written as

$$\psi(x) = \sum_{k=0}^{M-1} w_{(k)} \mathbf{1}\left(\Delta\tilde{P}^{(k)} \leq x\right). \tag{1.62}$$

Finding the $100\alpha\%$ VaR is then a matter of finding quantiles using ψ. Since $1-\alpha$ may not be a value in the sequence $\psi_0, \psi_1, \dots, \psi_{M-1}$, we first find successive ordered pairs $(\Delta\tilde{P}^{(k)}, \psi_k)$ and $(\Delta\tilde{P}^{(k+1)}, \psi_{k+1})$ such that

$$\psi_k < 1-\alpha < \psi_{k+1}. \tag{1.63}$$

Next, we find V such that V is the linearly interpolated value of ΔP from the ordered pairs $(\Delta\tilde{P}^{(k)}, \psi_k)$ and $(\Delta\tilde{P}^{(k+1)}, \psi_{k+1})$ corresponding to the value $\psi = 1-\alpha$. **The $100\alpha\%$ value-at-risk is therefore given by $|V|$.**

1.6 Monte Carlo Simulation

Whereas historical simulation makes use of past data to construct scenarios, Monte Carlo simulation relies on random number generation to simulate possible stock price paths.

Single-Asset Portfolios

For a portfolio containing shares in one stock E, we assume that the stock price process $\{S_t\}$ follows a geometric Brownian motion (GBM), given by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.64}$$

where $\{W_t\}$ is a Wiener process.[12] In the above equation, μ is interpreted as the stock's annual expected return and σ^2 is the stock's annual volatility rate.

As such, the scenario space $\{\Delta P^{(i)}\}$ is given by

$$\Delta P^{(i)} = P_1^{(i)} - P_0 = P_0 e^{R_i^{(i)}} - P_0.$$

[12] The Wiener process $\{W_t\}$ is a continuous-time stochastic process characterized by the following properties:

A discretization of the GBM is given by

$$\Delta S_t = \mu S_t \Delta t + \sigma S_t \Delta W_t, \tag{1.65}$$

where $\Delta W_t = W_{t+\Delta t} - W_t$. It follows from the properties of GBM that $\Delta W_t \sim N(0, \Delta t)$ or that ΔW_t can be written as $\epsilon\sqrt{\Delta t}$, where $\epsilon \sim N(0,1)$.

The arithmetic return on the asset is given by $\Delta S_t / S_t$. From the discretization above, we note that

$$\frac{\Delta S_t}{S_t} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t},$$

and so $\Delta S_t / S_t$ is normally distributed with mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$.

In computing for the VaR, we assume that $\mu \Delta t \ll \sigma \Delta W_t$ (or simply $\mu \Delta t \approx 0$). Furthermore, let Δt denote one trading day or 1/260 years if there are 260 trading days in a year. Thus, if R represents the arithmetic asset return, we then have

$$R = \frac{\Delta S_t}{S_t} \approx 0 + \sigma \epsilon \sqrt{\Delta t} = \frac{\sigma \epsilon}{\sqrt{260}} = \sigma_1 \epsilon, \tag{1.66}$$ {one_asset_MC}

where $\sigma_1 = \sigma / \sqrt{260}$ represents the daily volatility of the asset price process. Thus, R is normally distributed with mean 0 and variance σ_1^2.

Two-Asset Portfolios

Consider two stocks whose daily returns R_1 and R_2 have mean both 0 and variance σ_1^2 and σ_2^2, respectively. Suppose the daily returns are correlated with correlation coefficient ρ. Thus, $\mathrm{Cov}(R_1, R_2) = \rho\sigma_1\sigma_2$. Furthermore, we assume that the vector of returns, $\mathbf{R}^T = [R_1, R_2]$ has a multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}$, where

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

Let $\mathbf{Z}^T = [\epsilon_1, \epsilon_2]$ be a random vector containing i.i.d. standard normal random variables ϵ_1 and ϵ_2. Thus, $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I})$ where $\mathbf{I}$ is the 2×2 identity matrix. In order to simulate the daily returns $\mathbf{R}$, we need to find a transformation of $\mathbf{Z}$ via a matrix $\mathbf{A}$; that is, we want to find a matrix $\mathbf{A}$ such that $\mathbf{R} = \mathbf{AZ}$.

Recall that if $\mathbf{X} \sim N(\boldsymbol{\mu}, \mathbf{\Sigma})$ is a $p \times 1$ random vector and if $\mathbf{A}$ is a $p \times p$ matrix, then $\mathbf{AX} \sim N(\mathbf{A}\boldsymbol{\mu}, \mathbf{A\Sigma A}^T)$. Thus, if $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I})$, it follows that $\mathbf{R} = \mathbf{AZ} \sim N(\mathbf{A0}, \mathbf{AIA}^T)$ or $N(\mathbf{0}, \mathbf{AA}^T)$. Thus, in order to generate such a matrix $\mathbf{A}$, we need to decompose $\mathbf{\Sigma}$ into the form $\mathbf{AA}^T$.[13]

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1. $W_0 = 0$ almost surely (i.e. with probability 1);
2. $W_{t+\Delta t} - W_t$ is independent of $\sigma(W_s : s \leq t)$ for $\Delta t \geq 0$;
3. $W_{t+\Delta t} - W_t$ is normally distributed with mean 0 and variance Δt (i.e. $W_{t+\Delta t} - W_t \sim N(0, \Delta t)$);
4. W_t is a continuous function of t almost surely.

[13]Note that the matrix $\mathbf{A}$ is not unique.

One such decomposition is provided by the Cholesky decomposition of $\mathbf{\Sigma}$. Recall that $\mathbf{\Sigma}$ is a symmetric and positive definite matrix, and so there exists a lower triangular matrix $\mathbf{A}$ such that $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$.

Thus, there exist constants a_{11}, a_{21}, a_{22} such that

$$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{11}a_{21} \\ a_{11}a_{21} & a_{21}^2 + a_{22}^2 \end{bmatrix}. \tag{1.67}$$

Solving the system for the unknown constants, we find that

$$\mathbf{A} = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix}. \tag{1.68}$$

Therefore,

$$\mathbf{R} = \mathbf{A}\mathbf{Z} = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} \sigma_1\epsilon_1 \\ \rho\sigma_2\epsilon_1 + \sigma_2\epsilon_2\sqrt{1-\rho^2} \end{bmatrix}. \tag{1.69}$$ {two_asset_MC}

The last equation above expresses the (correlated) daily returns of each asset in terms of standard normal random variables which can be generated randomly.

n-Asset Portfolios

The process of simulating returns for a portfolio of n assets closely mirrors that of the two-asset portfolio. In general, consider $\mathbf{R}^T = [R_1, R_2, \ldots, R_n]$, where R_i is the daily return of the ith stock, with $R_i \sim N(0, \sigma_i^2)$. Assume further that $\mathbf{R} \sim N(\mathbf{0}, \mathbf{\Sigma})$, where $\mathbf{\Sigma}$ is the covariance matrix of $\mathbf{R}$.

Let $\mathbf{Z}^T = [\epsilon_1, \epsilon_2, \ldots, \epsilon_n]$ denote a vector of i.i.d. $N(0,1)$ variables; i.e. $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I})$. As before, we are looking for a matrix $\mathbf{A}$ such that $\mathbf{R} = \mathbf{A}\mathbf{Z}$. Using the Cholesky decomposition of $\mathbf{\Sigma}$, we can find a lower triangular matrix $\mathbf{A}$ such that $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$. Therefore, for some matrix $\mathbf{A}$,

$$\mathbf{R} = \mathbf{A}\mathbf{Z} = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} = \begin{bmatrix} a_{11}\epsilon_1 \\ a_{21}\epsilon_1 + a_{22}\epsilon_2 \\ \vdots \\ a_{n1}\epsilon_1 + a_{n2}\epsilon_2 + \cdots + a_{nn}\epsilon_n \end{bmatrix}. \tag{1.70}$$ {n_asset_MC}

Simulation Procedure

To perform a full Monte Carlo simulation to obtain the one-day 99% VaR for a portfolio of n stocks, we do the following:

1. Generate an $m \times n$ grid of random numbers from the standard normal distribution. This generates n random numbers for each of m scenarios. Let $\epsilon_{1,j}, \epsilon_{2,j}, \ldots, \epsilon_{n,j}$ denote the set of random numbers for the jth scenario, $j = 1, 2, \ldots, m$.

2. For the jth scenario, use $\epsilon_{1,j}, \epsilon_{2,j}, \ldots, \epsilon_{n,j}$ to calculate the associated daily returns using equation (1.70). In the case of one and two assets, use equation (1.66) and (1.69), respectively. Let $R_{1,j}, R_{2,j}, \ldots, R_{n,j}$ denote the daily returns generated in the jth scenario.

3. Compute for the stock price tomorrow. The price under scenario j is given by
$$S_{i,j} = S_{i,0}e^{R_{i,j}}, \tag{1.71}$$
where $S_{i,0}$ is the latest available stock price for the ith stock.

4. The change in portfolio value under scenario j is given by
$$\Delta P_j = \sum_{i=1}^{n}(N_i S_{i,j} - N_i S_{i,0}) = \sum_{i=1}^{n} N_i(S_{i,j} - S_{i,0}). \tag{1.72}$$

5. The one-day 99% portfolio VaR is the ΔP_{j^*} in the 1st percentile position when the ΔP_js are arranged in ascending order.

As the number of assets n and the number of scenarios m increase, the full Monte Carlo simulation slows down. An alternative process, sometimes called the *partial* Monte Carlo simulation, involves bypassing the simulation of stock prices tomorrow and instead uses the returns to estimate the change in portfolio value. We outline this alternative procedure below:

1. Generate an $m \times n$ grid of random numbers from the standard normal distribution. This generates n random numbers for each of m scenarios. Let $\epsilon_{1,j}, \epsilon_{2,j}, \ldots, \epsilon_{n,j}$ denote the set of random numbers for the jth scenario, $j = 1, 2, \ldots, m$.

2. For the jth scenario, use $\epsilon_{1,j}, \epsilon_{2,j}, \ldots, \epsilon_{n,j}$ to calculate the associated daily returns using equation (1.70). In the case of one and two assets, use equation (1.66) and (1.69), respectively. Let $R_{1,j}, R_{2,j}, \ldots, R_{n,j}$ denote the daily returns generated in the jth scenario.

3. Note that under scenario j, the change in portfolio value from asset i is approximated by $N_i S_{i,0} R_{i,j}$. Thus, the total change in portfolio value under scenario j, denoted by ΔP_j, is estimated by
$$\Delta P_j = \sum_{i=1}^{n} N_i S_{i,0} R_{i,j}. \tag{1.73}$$

4. The one-day 99% portfolio VaR is the ΔP_{j^*} in the 1st percentile position when the ΔP_js are arranged in ascending order.

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