Forward Rate Agreements

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Exercise

Suppose $R(0, T_1)$ and $R(0, T_2)$ are compounded m times per year. Under the assumption of no arbitrage, find a formula for the forward rate $R_F(T_1, T_2)$ (also compounded m times per year) in terms of the discount factors implied by $R(0, T_i)$, $i = 1, 2$.

It can be shown that the forward rate derived from zero rates compounded m times per year is

$$R_F(T_1, T_2) = m \left[\left(\frac{DF(0, T_1)}{DF(0, T_2)} \right)^{\frac{1}{m(T_2 - T_1)}} - 1 \right],$$

where $DF(0, T_i) = \left(1 + \frac{R(0, T_i)}{m} \right)^{mT_i}$.

Financial Derivatives

Definition

A **financial derivative** is a financial instrument whose value depends on, or is derived from, the values of other, more basic, underlying variables.

- Forward Rate Agreements
- Forward contracts
- Futures (Treasury Bond Futures, Commodity Futures)
- Swaps (Interest Rate Swaps, Currency Swaps, Foreign Exchange (FX) Swaps)
- Options (Stock Options, Bond Options, Options on Futures, Swaptions, Exotic options)
- Interest Rate Derivatives

Forward Rate Agreement

Definition

A forward rate agreement (FRA) is a cash-settled forward contract in which one party (the fixed-rate payer, long FRA) pays a fixed interest rate R_{fixed} and receives a floating interest rate R_{float} while the other party (the floating-rate payer, short FRA) receives the fixed rate and pays the floating rate over a specified period of time in the future.

The agreement is based on a notional principal amount L but the payment is based only on the differential or net interest amount. There is no exchange of principal.

Forward Rate Agreement

Let us try to understand this by drawing a timeline.

Cash Settlement of FRAs

- Consider an FRA on a principal amount L over a future period $[T_1, T_2]$, $T_1 < T_2$ (expressed in years).
- The life of the FRA is the period $[0, T_1]$, the **expiration date** of the FRA is T_1, and $[T_1, T_2]$ is referred to as the **FRA period or forward period**. The FRA is said to have tenor $T_2 - T_1$ years.
- Let R_{fixed} be the fixed interest rate of the FRA agreed upon at time $t = 0$.
- Let R_{float} be the interest rate observed at $t = T_1$ applicable over the period $[T_1, T_2]$. That is, R_{float} is the spot zero rate with tenor $T_2 - T_1$ years observed at $t = T_1$. Note that R_{float} is unknown for $t < T_1$.

Cash Settlement of FRAs

- Assume that R_{fixed} and R_{float} are simple rates.
- Below are the (net) cash flows at time T_2:
  - The cash flow to the fixed rate payer is $L(R_{float} - R_{fixed})(T_2 - T_1)$.
  - The cash flow to the floating rate payer is $L(R_{fixed} - R_{float})(T_2 - T_1)$

Why are these the cash flows?

Cash Settlement of FRAs

- Assume that R_{fixed} and R_{float} are simple rates.
- Below are the cash flows at time T_2:
  - The cash flow to the fixed-rate payer is $L(R_{float} - R_{fixed})(T_2 - T_1)$.
  - The cash flow to the floating-rate payer is $L(R_{fixed} - R_{float})(T_2 - T_1)$
- Cash settlements are usually done at time T_1[1]. Discounting the cash flows to time T_1 using R_{float}, we have

$$\begin{aligned}
\text{Value at } T_1 \text{ of CF (Fixed)} &= \frac{L(R_{float} - R_{fixed})(T_2 - T_1)}{1 + R_{float}(T_2 - T_1)} \\
\text{Value at } T_1 \text{ of CF (Floating)} &= \frac{L(R_{fixed} - R_{float})(T_2 - T_1)}{1 + R_{float}(T_2 - T_1)} \\
&= -\text{Value at } T_1 \text{ of CF (Fixed)}
\end{aligned}$$

[1] In practice, settlement is done 2 to 3 days after time T_1.

Valuation of FRAs

- FRAs can be valued at any time during its life $(t \in [0, T_1])$. At $t = T_1$, the value of the FRA is the amount of the cash settlement discussed previously.
- Since the actual floating rate R_{float} to be used in settlement is unknown before time T_1, an FRA is valued at $t < T_1$ by **assuming that the forward rate during the FRA period $[\mathbf{T_1}, \mathbf{T_2}]$ is realized as the floating rate $\mathbf{R_{float}}$.**
- The fixed rate is set at $t = 0$ so that the value of the FRA is 0. Thus, at $t = 0$, R_{fixed} is set equal to the forward rate over the period $[T_1, T_2]$ determined at time $t = 0$.

Valuation of FRAs

- Assume that $R(0, T_1)$ and $R(0, T_2)$ are the simple zero rates at time $t = 0$ with tenors T_1 and T_2 years, respectively.
- We set R_{fixed} to equal $R_F(0, T_1, T_2)$, and so

$$R_{fixed} = R_F(0, T_1, T_2) = \left(\frac{DF(0, T_1)}{DF(0, T_2)} - 1 \right) \frac{1}{T_2 - T_1}, \quad (1)$$

  where $DF(0, T_i)$ is the discount factor implied by $R(0, T_i)$, $i = 1, 2$.

Valuation of FRAs

- At any time $t \in (0, T_1)$, the value of the FRA to the fixed rate payer is

$$V_t = L[R_F(t, T_1, T_2) - R_{fixed}](T_2 - T_1)DF(t, T_2), \qquad (2)$$

  where $R_F(t, T_1, T_2)$ is the forward rate for $[T_1, T_2]$ obtained using the zero rates at time t and $DF(t, T_2)$ is the discount factor implied by $R(t, T_2)$.
- Since the CF of the floating rate payer is the negative of the CF of the fixed rate payer, the value of the FRA to the floating rate payer is $-V_t$.

Extending to Continuous Compounding

- If R_{float} and R_{fixed} follow continuous compounding, then the cash flow at time T_2 to the fixed-rate payer is

$$L\left[e^{R_{float}(T_2-T_1)} - e^{R_{fixed}(T_2-T_1)}\right]. \tag{3}$$

- The value of the cash flow at T_1 (the cash settlement) is

$$L\left[e^{R_{float}(T_2-T_1)} - e^{R_{fixed}(T_2-T_1)}\right] e^{-R_{float}(T_2-T_1)}. \tag{4}$$

- If $R(0,T_1)$ and $R(0,T_2)$ are continuously-compounded zero rates, then the continuously compounded forward rate over $[T_1,T_2]$ at time zero is given by

$$R_F(0,T_1,T_2) = \frac{R(0,T_2)T_2 - R(0,T_1)T_1}{T_2 - T_1}. \tag{5}$$

This is set equal to R_{fixed}.

Extending to Continuous Compounding

- At any time $t \in (0, T_1)$, the value of the FRA to the fixed-rate payer is

$$V_t = L \left[e^{R_F(t,T_1,T_2)(T_2-T_1)} - e^{R_{fixed}(T_2-T_1)} \right] DF(t, T_2), \quad (6)$$

  where $R_F(t, T_1, T_2)$ is the forward rate for $[T_1, T_2]$ obtained using the continuously compounded zero rates at time t.
- Here, the discount factor is given by

$$DF(t, T_2) = \exp\left\{ -R(t, T_2)(T_2 - t) \right\}. \quad (7)$$

Other Compounding Frequencies

Suppose R_{float} and R_{fixed} of a FRA are compounded m times per year.

- The cash flow at time T_1 to the fixed-rate payer is given by

$$L\left[\left(1+\frac{R_{float}}{m}\right)^{m(T_2-T_1)}-\left(1+\frac{R_{fixed}}{m}\right)^{m(T_2-T_1)}\right]DF(T_1,T_2),$$

  where $DF(T_1,T_2)=\left(1+\dfrac{R(T_1,T_2)}{m}\right)^{-m(T_2-T_1)}$.

- The value of the FRA to the fixed-rate payer at a time t is given by

$$L\left[\left(1+\frac{R_F(t,T_1,T_2)}{m}\right)^{m(T_2-T_1)}-\left(1+\frac{R_{fixed}}{m}\right)^{m(T_2-T_1)}\right]DF(t,T_2),$$

  where $R_F(t,T_1,T_2)=m\left[\left(\dfrac{DF(t,T_1)}{DF(t,T_2)}\right)^{\frac{1}{m(T_2-T_1)}}-1\right]$.

Market Convention

- An FRA is quoted as an $(M \times N)$ FRA, where $M < N$ and are (typically) expressed in months.
- In this notation, the FRA expires in M months and settled based on interest rates with tenor $N - M$ months applicable over the period $[M, N]$.
- The interest differential computed at the end of the FRA period is usually settled a few days after M months.
- Day count conventions are important when the inception date, expiration date, and valuation date are specified as calendar dates.

Example 1

Suppose a (3×9) FRA is set today with notional PHP 5 million. Suppose the 3-month simple rate now is 3% and the 9-month simple rate is 4.5%.

(a) Determine the fixed rate for the FRA.

(b) Suppose that, after 2 months, the 1-month simple rate is 2% and the 7-month simple rate is 4.25%. What is the value of the FRA to the fixed-rate payer? To the floating-rate payer?

(c) Suppose the 6-month simple rate at the expiration date of the FRA is 4.5%. Which party pays the other party? How much is the settlement amount?

Example 2

Suppose a (6×18) FRA is set today with notional PHP 10 million. Suppose the 6-month zero rate with continuous compounding is 2.75% and the 18-month zero rate with continuous compounding is 5%.

(a) Determine the fixed rate for the FRA.

(b) Suppose that, after 3 months, the 3-month zero rate with continuous compounding is 2.25% and the 15-month zero rate with continuous compounding is 4.25%. What is the value of the FRA to the fixed-rate payer? To the floating-rate payer?

(c) Suppose the 1-year zero rate with continuous compounding at the expiration date of the FRA is 6.25%. Which party pays the other party? How much is the settlement amount?

Recall: Equivalent Interest Rates

- To convert an interest rate R_m compounded m times per year to its equivalent interest rate R_n compounded n times per year (and vice versa),

$$\left(1 + \frac{R_n}{n}\right)^n = \left(1 + \frac{R_m}{m}\right)^m \tag{8}$$

- To convert from an interest rate R_m compounded m times per year to its equivalent interest rate R_c compounded continuously (and vice versa),

$$e^{R_c} = \left(1 + \frac{R_m}{m}\right)^m \tag{9}$$

Converting Forward Rates to Simple Rates

- When the given zero rates are not simple, **compute for the forward rates first** before converting them to simple rates—that is, don't convert the given zero rates to simple rates and then use those to compute the forward rates.
- When using converted forward rates to find V_t, the discount factor multiplied at the end—$DF(t, T_2)$—must be the discount factor for the rate with the **original compounding frequency**.
- In summary, in our FRA valuation process, **only the forward rates (not the zero rates)** can be converted to simple rates.

Converting between Different Compounding Frequencies

- It is advisable to keep the compounding frequencies of all rates the same.
- Unlike conversion to simple rates which is restricted only to forward rates, converting between compounding frequencies **may be done for zero rates as well**.

Exercise 1

Suppose a (6×18) FRA is set today with notional PHP 10 million. Suppose the 6-month zero rate with semi-annual compounding is 2.75% and the 18-month zero rate with semi-annual compounding is 5%.

(a) Determine the fixed rate for the FRA.

(b) Suppose that, after 3 months, the 3-month zero rate with quarterly compounding is 2.25% and the 15-month zero rate with quarterly compounding is 4.25%. What is the value of the FRA to the fixed-rate payer? To the floating-rate payer?

(c) Suppose the 1-year zero rate with semi-annual compounding at the expiration date of the FRA is 6.25%. Which party pays the other party? How much is the settlement amount?

Exercise 2

A (3×6) FRA with notional PHP 8 million was set one month ago with a fixed rate of 5%. Suppose the simple zero rates now are as follows:

tenor	1 month	3 months	6 months
interest rates in hundred bps	3%	4.5%	6%

(a) Use linear interpolation to estimate the 2-month and 5-month zero rates.

(b) Determine the value of the FRA now to the fixed-rate payer.

References

- De Lara-Tuprio, E. P. (2016). Financial Derivatives: Lecture Notes in AMF 270. Ateneo de Manila University.
- Hull, J. C. (2018). Options, Futures, and Other Derivatives (10th ed). Prentice Hall.