Interest Rate Swaps

Lecture Notes of E. P. de Lara-Tuprio, AMF 270

An interest rate swap (IRS) is a contractual agreement between two parties under which each party agrees to make periodic payments to the other for a prespecified time period based upon a notional amount of principal. The principal amount is called *notional* because this amount is not exchanged, but is used as a notional figure to determine the cash flows that are exchanged periodically. In a *plain vanilla interest rate swap*, fixed cash flows computed using a fixed interest rate on the notional amount are exchanged for floating cash flows computed using a floating interest rate on the notional amount, also called *fixed-for-floating interest rate swap.*

The stream of floating cash flows in a swap agreement is called the *floating leg*, whereas the stream of fixed cash flows constitute the *fixed leg*. The dates at which the floating rates are observed are called *reset* or *fixing dates*. Note that the number of payments on the floating leg is not necessarily equal to the number of payments on the fixed leg, and cash flows are not necessarily exchanged on the same dates. Also, one must consider the day count conventions used for computing the cash flows corresponding to the floating rates and fixed rates.

When the IRS is contracted today for a period starting at a future date, it is called a *forward IRS* or *forward swap.*

At any time t before maturity, the value of an IRS is the net present (time t) value of all future cash flows from both legs. The fixed interest rate that makes the value of IRS at time t equal to 0 is called the *swap rate* at time t. To price an IRS at inception shall mean to find the swap rate at this time. The swap rate for the forward IRS is referred to as the forward swap rate.

Valuing and Pricing IRS

Let L be the notional of the swap for both legs[1]. Suppose the first reset date of the swap is S_0. Further, suppose that

1. the fixed leg makes N fixed payments at times $S_1, S_2, ..., S_N$ in years, where $S_i < S_{i+1}$ for all i, measured according to the day count convention for the fixed rate.

2. The floating leg makes n floating payments at times $T_1, T_2, ..., T_n$ in years, where $T_i < T_{i+1}$ for all i, measured according to the day count convention for the floating rate, based upon the floating interest rates that are reset at times $S_0 = T_0, T_1, ..., T_{n-1}$, respectively.

3. The last payment dates of both the floating leg and the fixed leg are the same, or $S_N = T_n$.

[1] Note that both legs have cash flows in the same currency.

The swap can be valued at time T_0 $(= S_0)$ by assuming that the floating rate between the i^{th} and $(i+1)^{st}$ reset dates, $i \geq 1$, is the forward rate, denoted by $F_{i,i+1}$, observed at time T_0 for that period. Denote by $DF_{t,s}$ the discount factor that takes a value at time s to time t, where $t \leq s$. Recall that for $i \geq 1$, $F_{i,i+1} = \left(\frac{DF_{T_0,T_i}}{DF_{T_0,T_i+1}} - 1 \right) \frac{1}{T_{i+1} - T_i}$. Let $F_{0,1}$ be the interest rate at time T_0 maturing at time T_1. Then the value at time T_0 of the floating leg is

$$V_{\text{float}\,ing}\left(T_0\right) = \sum_{i=0}^{n-1} L \cdot F_{i,i+1} \cdot \left(T_{i+1} - T_i\right) \cdot DF_{T_0,T_{i+1}}.$$

Since $F_{i,i+1} \cdot \left(T_{i+1} - T_i\right) = \frac{DF_{T_0,T_i}}{DF_{T_0,T_{i+1}}} - 1$, then

$$\begin{aligned} V_{\text{float}\,ing}\left(T_0\right) &= L \sum_{i=0}^{n-1} \left(\frac{DF_{T_0,T_i}}{DF_{T_0,T_{i+1}}} - 1 \right) DF_{T_0,T_{i+1}} \\ &= L \sum_{i=0}^{n-1} \left(DF_{T_0,T_i} - DF_{T_0,T_{i+1}} \right) \\ &= L \left(DF_{T_0,T_0} - DF_{T_0,T_n} \right) \\ &= L \left(1 - DF_{T_0,T_n} \right) \end{aligned}$$

At any time t on or before the first reset date T_0, the value of the floating leg can be obtained by discounting the last expression to time t. That is,

$$\begin{aligned} V_{\text{float}\,ing}\left(t\right) &= L \left(1 - DF_{T_0,T_n} \right) DF_{t,T_0} \\ &= L \left(DF_{t,T_0} - DF_{t,T_n} \right) \end{aligned}$$

since $DF_{T_0,T_n} DF_{t,T_0} = DF_{t,T_n}$. (Verify).

Let K be the fixed rate of the swap. At each payment date S_{i+1}, $i \geq 0$, the fixed-rate payment is

$$L \cdot K \cdot \left(S_{i+1} - S_i\right).$$

The value of of the fixed leg at time S_0 is

$$\begin{aligned} V_{fixed}\left(S_0\right) &= \sum_{i=0}^{N-1} L \cdot K \cdot \left(S_{i+1} - S_i\right) DF_{S_0,S_{i+1}} \\ &= L \cdot K \cdot \sum_{i=0}^{N-1} \left(S_{i+1} - S_i\right) DF_{S_0,S_{i+1}} \end{aligned}$$

At any time t on or before the first reset date S_0, the value of the floating leg can be given by discounting the last expression to time t.

$$\begin{aligned} V_{fixed}(t) &= L \cdot K \cdot \sum_{i=0}^{N-1} (S_{i+1} - S_i) DF_{S_0, S_{i+1}} DF_{t, S_0} \\ &= L \cdot K \cdot \sum_{i=0}^{N-1} (S_{i+1} - S_i) DF_{t, S_{i+1}} \end{aligned}$$

The value of the swap to the fixed-rate payer, denoted by $V_{swap-fixed}$, is

$$V_{swap-fixed} = V_{\text{float}\,ing}(t) - V_{fixed}(t)$$

and the value to the floating-rate payer, denoted by $V_{swap-\text{float}\,ing}$, is

$$V_{swap-\text{float}\,ing} = V_{fixed}(t) - V_{\text{float}\,ing}(t).$$

The swap rate at time t, denoted by $s(t)$, is the fixed rate K that makes the value of the swap equal to 0. Thus,

$$\begin{aligned} V_{fixed}(t) &= V_{\text{float}\,ing}(t) \\ L \cdot s(t) \cdot \sum_{i=0}^{N-1} (S_{i+1} - S_i) DF_{t, S_{i+1}} &= L \left(DF_{t,T_0} - DF_{t,T_n} \right) \\ s(t) &= \frac{DF_{t,T_0} - DF_{t,T_n}}{\sum_{i=0}^{N-1} (S_{i+1} - S_i) DF_{t, S_{i+1}}} \end{aligned}$$

To value the IRS after the first reset date, future cash flows must be considered. The formulas above do not apply immediately because the assumption is that there is no cash flow at time T_0. The formulas may still be used after making the necessary adjustments.

Examples.

1. A 2-year IRS is initiated on June 25, 2014 for semi-annual exchange of fixed rate and floating rate payments beginning 6 months from initiation date. The continuously compounded interest rates with day count convention Actual/365 are as follows.

tenor	6-mo	1-yr	18-mo	2-yr
rate	2.5%	3.25%	4.1%	5%

   Determine the price of the swap (the swap rate) at initiation.

2. Today is June 25, 2015. Consider a \$10 million 1-year fixed-for-floating swap with quarterly payments where the floating rate is 3-month LIBOR. Suppose the swap rate is 6% and that we are 1 month into the swap. The LIBOR rates (simple, Actual/360) are currently at 5%, 5.5%, 6% and 6.5% at the 2-, 5-, 8- and 11-month maturities, respectively. The first payment on the floating leg was fixed 1 month ago, based on the (then) current 3-month forward rate of 5.5%. Find the value of the swap now. If we were to fix the swap rate now, what would it be?

3. A contract is initiated today, May 6, 2014, for a 2-year fixed-for-floating swap on a notional of 5 million pesos to begin on June 10, 2014 (first reset date). The fixed leg will make payments every 3 months after June 10, 2014 (so a total of 8 payments). The floating leg will make payments every 6 months after June 10, 2014 (4 payments). Hence, the last payment date is June 10, 2016 for both legs. The simple interest rates with day count convention Actual/365 are as follows.

tenor	O/N	1-mo	3-mo	6-mo	1-yr	2-yr	3-yr
interest rate	3.25%	4.5%	5.75%	6.25%	7.5%	8.75%	9.25%

   (a) What is the current swap rate?

   (b) What is the value of the swap to the fixed-rate payer on June 10, 2014? The simple interest rates on June 10, 2014 are as follows.

tenor	1-mo	3-mo	6-mo	1-yr	2-yr
interest rate	4.35%	5.65%	6.35%	7.45%	8.65%

   (c) What is the swap rate on June 10, 2014?

4. Do no. 3 with the floating leg paying floating interest rate + 2 bps.

Valuing IRS using bonds

An interest rate swap can also be regarded as an exchange of a fixed-rate bond and a floating-rate bond with face values L of both bonds equal to the notional principal of the swap. To see this, the notional L must be added to both the fixed leg and the floating leg at maturity $S_N = T_n$ to convert them to a fixed-coupon bond and a floating-rate bond, respectively. At time $t \leq T_0$, the price of the floating-rate bond is given by

$$\begin{aligned} B_{\text{float}-bond}(t) &= V_{\text{float}\,ing}(t) + L \cdot DF_{t,T_n} \\ &= L\left(DF_{t,T_0} - DF_{t,T_n}\right) + L \cdot DF_{t,T_n} \\ &= L \cdot DF_{t,T_0} \end{aligned}$$

Note that the price of the floating-rate bond converges to its face value L at the reset date T_0.

On the other hand, the price of the fixed-rate bond at time t is

$$\begin{aligned} B_{fixed-bond}(t) &= V_{fixed}(t) + L \cdot DF_{t,S_N} \\ &= L \cdot K \cdot \sum_{i=0}^{N-1} (S_{i+1} - S_i)\, DF_{t,S_{i+1}} + L \cdot DF_{t,S_N} \\ &= L\left(K \cdot \sum_{i=0}^{N-1} (S_{i+1} - S_i)\, DF_{t,S_{i+1}} + DF_{t,S_N}\right). \end{aligned}$$

Obviously, since the additional terms $L \cdot DF_{t,T_n}$ and $L \cdot DF_{t,S_N}$ are equal, then $B_{\text{float}-bond}(t) - B_{fixed-bond}(t)$ gives the value of the IRS to the fixed-rate payer and $B_{fixed-bond}(t) - B_{\text{float}-bond}(t)$ gives the value of the IRS to the floating-rate payer.

(A note on Bloomberg's IRS pricing)

Non-standard IRS - see Hull, pp. 174-175; Alexander, p. 41