MATH 271.1: Statistical Methods

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The "complete SVD" decomposes the matrix as:

$$\begin{array}{c} \mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^{\mathbf{T}}, \\ \mathsf{nxp} = \mathsf{nxn} \mathsf{nxp} \mathsf{pxp} \end{array}$$
 where
$$\begin{array}{c} \mathbf{U} \in \mathbb{R}^{n \times n} \mathsf{ is orthonormal} \\ \mathbf{\Sigma} \in \mathbb{R}^{n \times p} \mathsf{ is diagonal} \\ \mathbf{V} \in \mathbb{R}^{p \times p} \mathsf{ is orthonormal} \end{array}$$

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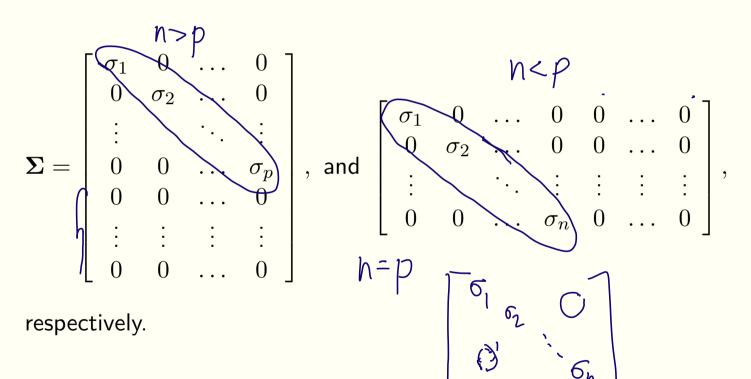
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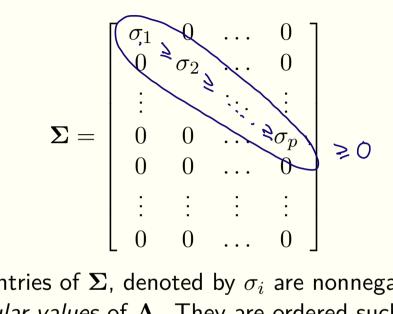
The columns of \mathbf{U} are called the <u>left singular vectors</u> of \mathbf{A} . The columns of \mathbf{V} are called the <u>right singular vectors</u> of \mathbf{A} .

In complete SVD, the Σ is a rectangular diagonal matrix. In cases n>p and n< p, this matrix is in the forms:

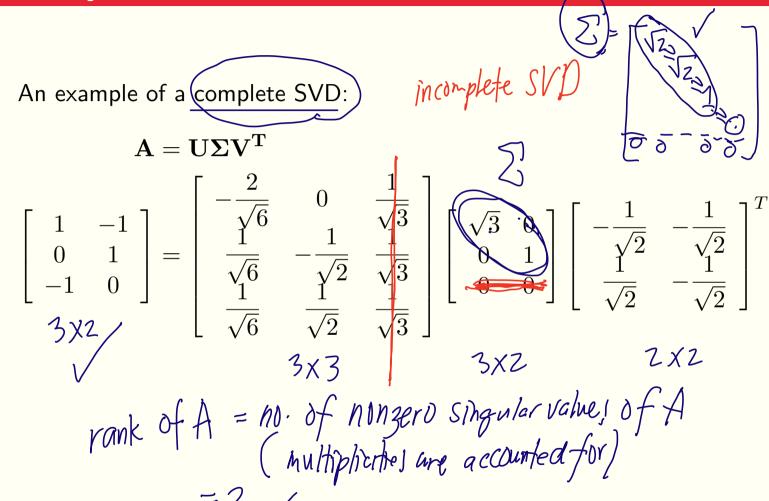


$$oldsymbol{\Sigma} = \left[egin{array}{ccccc} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_p \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{array}
ight]$$

The diagonal entries of Σ , denoted by σ_i are nonnegative and are called the *singular values* of A.



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Incomplete SVD

The "incomplete SVD" decomposes the matrix as:

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^{\mathbf{T}},$$
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where

 $-\mathbf{U} \in \mathbb{R}^{n \times l}$ is orthonormal

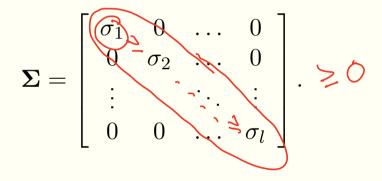
That $\mathbf{\Sigma} \in \mathbb{R}^{l imes l}$ is diagonal

 $\mathbf{V} \in \mathbb{R}^{p imes l}$ is orthonormal,

such that $l := \min(n, p)$. The columns of \mathbf{U} and the columns of \mathbf{V} are still called the *left singular vectors* and the *right singular vectors* of \mathbf{A} , respectively.

Incomplete SVD

In incomplete SVD, the $\underline{\Sigma}$ is a square diagonal matrix whose main diagonal includes the *singular values* of \mathbf{A} . The matrix Σ is in the form:



Incomplete SVD

An example of an incomplete SVD:

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^{\mathbf{T}}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^{T}$$

$$322$$

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$$322$$

$$222$$

Singular vectors in SVD

Note that in both complete and incomplete SVD, the left singular vectors are orthonormal and the right singular vectors are also orthonormal;

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 $\mathbf{V}^{\mathbf{T}}\mathbf{V} = \mathbf{I}.$

If these orthogonal matrices are not truncated and thus are square matrices, e.g., for complete SVD, we also have:

$$\mathbf{U}\mathbf{U}^{\mathbf{T}} = \mathbf{I},$$

$$\mathbf{V}\mathbf{V}^{\mathbf{T}} = \mathbf{I}.$$

Singular Values and Rank of A

If $\bf A$ has r non-zero singular values, counted according to their multiplicity (i.e. if $\bf \Sigma$ has r non-zero diagonal entries), then $\bf A$ has rank r.

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Theorem 1

In both complete and incomplete \underline{SVD} , the singular values of an $n \times p$ matrix $\underline{\mathbf{A}}$ are the square roots of the eigenvalues of $\underline{\mathbf{A}^T}\underline{\mathbf{A}}$.

 ${}^{\textcircled{3}}$ The right singular vectors are the eigenvectors of ${f A^TA}$.

Eigenvalues & Eigenvectors of A^T**A**

Given the complete SVD of A:
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What are the eigenvalues and eigenvectors of
$$\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$$
. A substitution of $\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$ and $\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}$ and $\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}$ and $\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}$ and $\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}$ and $\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}\mathbf{A}$ and $\mathbf{A}^{\mathbf{T}}\mathbf$

Theorem 2

Given an $n \times p$ matrix \mathbf{A} of rank r, then the left and right singular vectors in its complete singular value decomposition

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^{\mathbf{T}}$$

form orthonormal bases for the four fundamental subspaces.

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form orthonormal bases for the four fundamental subspaces. In particular,

 $\mathbf{u_1}, \dots, \mathbf{u_r}$ is an orthonormal basis for the **column space** $Col(\mathbf{A})$ $\mathbf{u_{r+1}}, \dots, \mathbf{u_n}$ is an orthonormal basis for the **left nullspace** $N(\mathbf{A^T})$ $\mathbf{v_1}, \dots, \mathbf{v_r}$ is an orthonormal basis for the **row space** $Col(\mathbf{A^T})$ $-\mathbf{v_{r+1}}, \dots, \mathbf{v_p}$ is an orthonormal basis for the **nullspace** $N(\mathbf{A})$

This is the third part of the Fundamental Theorem of LinAlg!

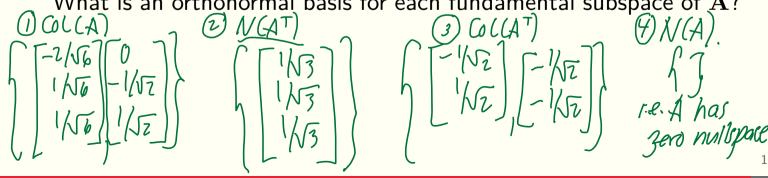
Finding bases for the subspaces of A

Given the complete SVD of **A**:

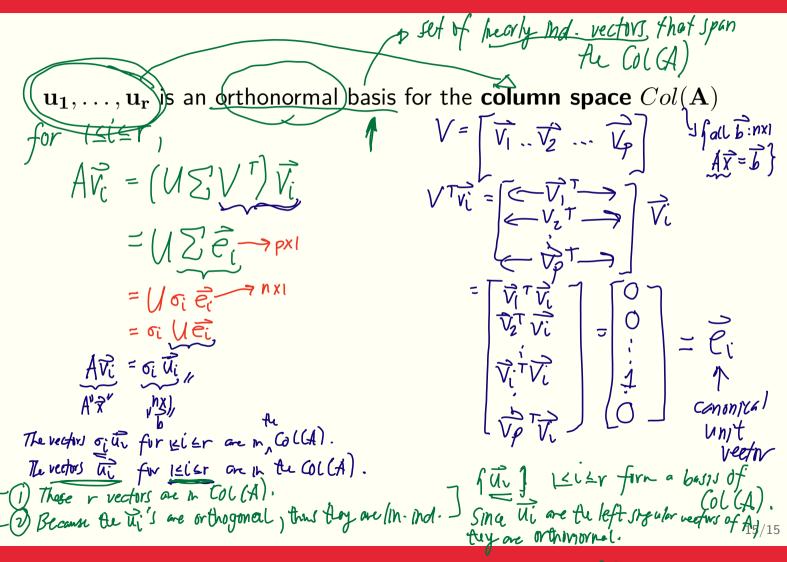
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What is an orthonormal basis for each fundamental subspace of A?



Proof: Theorem 2 Part 1



$$\begin{array}{c}
U_{e_{i}} = [\overline{u_{i}} \ u_{z} \overline{u_{i}} \overline{u_{n}}] \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} \\
= \overline{u_{i}}$$