A type system for algebraic effects and handlers with dynamic instances

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Chapter 1

Introduction

Side-effects are ubiquitous in programming. Examples include mutable state, exceptions, nondeterminism, and user input. Side-effects often make functions hard to understand, test and debug. This is because every invocation of the same function with the same arguments may yield different results. Furthermore side-effectful programs can also be difficult to optimize, since the compiler does not have much freedom in rearranging parts of the program.

Any function that includes such side-effects is called *impure*, while functions whose only effect is computing a result are called *pure*. Pure functions on the other hand do not rely on any global state and thus can be reasoned about in isolation of the rest of the program. Every time a pure function is called with the same input, it will return the same output. This means those functions are easier to understand, test, and debug.

There has been a lot of work on programming languages that allow more control over the pure and impure parts of a program. Examples include Haskell [14], Eff [1], Koka [5], and Links [3]. These languages, in one way or another, give the programmer more control over which parts of their program are pure and which parts are impure. By factoring out the pure parts from the impure parts, we can still gain the benefits of pure functions for many parts of our programs. In addition these languages allow to keep track of which effects exactly are used by which function. They also allow some side-effects to be encapsulated, meaning that the use of a particular side-effect

can be completely hidden such that the function still appears to be pure to the outside world.

Type systems play an essential role in enforcing the distinction between pure and impure code. By extending type systems to also show which effects a function may use, we can statically enforce which functions are pure and which are not. This gives insight to the user to what a function may do when called, and also allows a compiler to do more interesting optimizations. For example pure function calls may be reordered in any way that the compiler sees fit, while impure function calls may not, since the effects may interact. These *effect systems* can have different levels of granularity. For example one system could only keep track of a single bit per function, whether the function is impure or not. More fine-grained systems are also possible, where each function is annotated with a set of effects that is used, where the set of possible effects is defined by the language. For example in Koka a function which prints something to the console may be given the type:

```
string -> <console> ()
```

Where console shows the use of the console. User-defined effects are also supported in languages such as Koka and Eff. Users, in those systems, can define the effects with which the functions are annotated.

Algebraic effects and handlers [15] are an approach to programming with side-effects that has many of the desirable properties previously described. Algebraic effects provide a way to factor out the pure parts, the operation calls, from the impure parts. Users can define effects and easily use them in functions, with different effects composing without any extra effort. Algebraic effects also easily admit typing, with different type-and-effect already proposed [2, 4, 3]. Each effect is defined as a set of operations, for example nondeterminism can be represented by an operation which takes to values and chooses one. Similarly, state can be defined as two operations, get and put, where get is meant to return the current value of the state and put is meant to change this value. Functions are tagged by the set of effects they may use. These operations can then be called anywhere in a function. Handlers take a program that calls operations and for each operation call defines how to proceed. For example the following piece of code defines an effect called *State* which simulates a single mutable state cell. The function postInc increments the current value in the state cell and returns the

previous value.

```
effect State {
  get : () -> Int
  put : Int -> ()
}

postInc : Int!{State}
postInc =
  x <- get ();
  put (x + 1);
  return x
```

While algebraic effects and handlers have many of the desirable properties we would like, they are is unable to express multiple mutable state cells. In the previous example it can be seen that `postInc` does not refer to any variables, but instead can only manipulate the mutable state using the get and put operations. In Haskell the the so-called "ST monad" [16] can be used to safely implement multiple mutable state cells in such a way that stateful computations can be encapsulated and that the references to the mutable objects are not leaked outside of the function. A feature called dynamic instances was introduced by the Eff programming language [1]. With dynamic instances multiple different instances of the same effect can be dynamically created. Using this multiple state cells can be implemented. Unfortunately there is no type-and-effect for dynamic instances, using them can results in runtime errors when instances do not have an associated handler.

In this thesis we define a calculus based on algebraic effects and handlers which allows for the definition of side-effects such as local references, local exceptions, and the dynamic opening of channels. Using this system we can implement a system similar to the "ST monad" in Haskell. This system gives full control of which parts of a program are pure and impure. Functions also compose easily, irrelevant of which side-effects they use. Using a type-and-effect system every function keeps track of which effects it may use. We also statically ensure that side-effects are encapsulated. We give examples of programs using these side-effects in our system and show how to implement local mutable references in this system. We give a formal description of the syntax, typing rules and semantics of the system.

Contributions

- **Language.** We define a language based on algebraic effects and handlers that can handle a form of dynamic effect instances.
- **Mutable references.** We give examples in our language that would be difficult or impossible to express with just algebraic effects.
- **Operational semantics and type system.** We define a core calculus of our language together with a small-step operational semantics and a type system.
- **Type soundness.** We prove type soundness of our type system with respect to the operational semantics via type preservation and progress for the core calculus.

Thesis structure

The thesis is structured as follows. Chapter 2 gives an introduction to algebraic effects and handlers, and static and dynamic instances. Chapter 3 gives an introduction to our proposed language. Chapter 4 gives formal definitions of systems with algebraic effecst and handlers, and static instances. Chapter 5 gives a formal account of X. Chapter 6 discusses related work. Chapter 7 concludes the thesis and discusses future work.

Chapter 2

An introduction to algebraic effects and handlers

Side-effects are an essential part of a programming language. Without side-effects the program would have no way to print a result to the screen, ask for user input or change global state. We consider a function pure if it does not perform any side-effects and unpure if it does. A pure function always gives the same result for the same inputs. A pure function can be much easier to reason about than an unpure one because you know that it won't do anything else but compute, it won't have any hidden inputs or outputs. Because of this property testing pure functions is also easier, we can just give dummy inputs to the functions and observe the output. As already said programs without side-effects are useless, we would not be able to actually observe the result of a function call without side-effects such as printing to the screen. So we would the benefits of pure functions but still have side-effects. We could give up and simply add some form of side-effects to our language but that would immediately make our function impure, since any function might perform side-effects. This would make us lose the benefits of pure functions.

Algebraic effects and handlers are a structured way to introduce side-effects to a programming language. The basic idea is that side-effects can be described by sets of operations, called the interface of the effect. Operations from different effects can then be called in a program. These operations will stay abstract though, they will not actually do anything. Instead, similar to

exceptions where exceptions can be thrown and caught, operations can be "caught" by handlers. Different from exceptions however the handler also has access to a continuation which can be used to continue the computation at the point where the operation was called.

In this chapter we will introduce algebraic effects and handlers through examples. Starting with simple algebraic effects and handlers (§2.1). After we will continue with static instances (§2.2) which allows for multiple static instances of the same effect to be used in a program. We end with dynamic instances (§2.3) which allows for the dynamic creation of effect instances. The examples are written in a statically typed functional programming language with algebraic effects and handlers with syntax reminiscent to Haskell but semantically more similar to Koka[5].

2.1 Algebraic effects and handlers

We will start with the familiar exceptions. We define an `Exc` effect interface with a single operation `throw`.

```
effect Exc {
  throw : String -> Void
}
```

For each operation in an effect interface we specify a parameter type (on the left of the arrow) and a return type (on the right of the arrow). The parameter type is the type of a value that is given when the operation is called and that the handler also has access too. The return type is the type of a value that has to be given to the continuation in the handler, this will be shown later. This return value is received at the point where the operation was called. In the case of `Exc` we take `String` as the parameter type, this is the error message of the exception. An exception indicates that something went wrong and that we cannot continue in the program. This means we do not want the program to continue at the point where the exception was thrown, which is the point where the `throw` operation was called. So we do not want to be able to call the continuation with any value. To achieve this we specify `Void` as the return type of `throw`. This is a type with no values at all, which means that the programmer will never be able to conjure up a

suitable value when a value of type `Void` is requested. By using `Void` as the return type we can ensure that the continuation cannot be called and so that the program will not continue at the point where `throw` was called. To make the code more readable we assume `Void` implicitly coerces to any other type.

We can now write functions that use the `Exc` effect. For example the following function `safeDiv` which will throw an error if the right argument is 0. We assume here that `Void` is equal to any type.

```
safeDiv : Int -> Int -> Int!{Exc}
safeDiv a b =
  if b == 0 then
    throw "division by zero!"
  else
    return a / b
```

We can call this function like any other function, but no computation will actually be performed. The effect will remain abstract, we still need to give them a semantics.

```
result : Int!{Exc}
result = safeDiv 10 2
```

In order to actually "run" the effect we will need to handle the operations of that effect. For example, for `Exc` we can write a handler that returns 0 as a default value if an exception is thrown.

```
result : Int
result = handle (safeDiv 10 0) {
  throw err k -> return 0
  return v -> return v
} -- results in 0
```

For each operation we write a corresponding case in the handler, where we have access to the argument given at operation call and a continuation, which expects a value of the return type of the operation. There is also a case for values `return`, which gets as an argument the final value of a computation and has the opportunity to modify this value or to do some final computation. In this case we simply ignore the continuation and exit the computation early with a 0, we also return any values without modification.

We can give multiple ways of handling the same effect. For example we can also handle the `Exc` effect by capturing the failure or success in a sum type `Either`.

```
data Either a b = Left a | Right b

result : Either String Int
result = handle (safeDiv 10 0) {
  throw err k -> return (Left err)
  return v -> return (Right v)
} -- results in (Left "division by zero!")
```

Here we return early with `Left` err if an error is thrown, otherwise we wrap the resulting value using the `Right` constructor.

Another effect we might be interested in is non-determinism. To model this we define the `Flip` effect interface which has a single operation `flip`, which returns a boolean when called with the unit value.

```
effect Flip {
  flip : () -> Bool
}
```

Using the `flip` operation and if-expression we can write non-deterministic computations that can be seen as computation trees where `flip` branches the tree off into two subtrees. The following program `choose123` non-deterministically returns either a 1, 2 or 3.

```
choose123 : Bool!{Flip}
choose123 =
  b1 <- flip ();
  if b1 then
    return 1
  else
    b2 <- flip ();
    if b2 then
      return 2
    else
      return 3
```

Here the syntax (`x <- c1; c2`) sequences the computations `c1` and `c2` by

first performing `c1` and then performing `c2`, where the return value of `c1` can accessed in `x`.

Again `choose123` does not actually perform any computation when called, because we have yet to give it a semantics. We could always return `True` when a `flip` operation is called, in the case of `choose123` this will result in the first branch being picked returning 1 as the answer.

```
result : Int
result = handle (choose123) {
  flip () k -> k True
  return v -> return v
} -- returns 1
```

Another handler could try all branches returning the greatest integer of all possibilities.

```
maxresult : Int
maxresult = handle (choose123) {
  flip () k ->
    vtrue <- k True;
    vfalse <- k False;
    return (max vtrue vfalse)
  return v -> return v
} -- returns 3
```

Here we first call the continuation `k` with `True` and then with `False`. The we return the maximum between those results.

We could even collect the values from all branches by returning a list.

```
allvalues : List Int
allvalues = handle (choose123) {
  flip () k ->
    vtrue <- k True;
    vfalse <- k False;
    return vtrue ++ vfalse
  return v -> return [v]
} -- returns [1, 2, 3]
```

Again we call the continuation `k` twice, but we append the two results in-

stead. For the `return` base case we simply wrap the value in a singleton list.

Algebraic effects have the nice property that they combine easily. For example by combining the `Exc` and `Flip` we can implement backtracking, where we choose the first non-failing branch from a computation. For example we can write a function which returns all even sums of the numbers 1 to 3 by reusing `choose123`.

```
evensums123 : Int!{Flip, Exc}
evensums123 =
  n1 <- choose123;
  n2 <- choose123;
  sum <- return (n1 + n2);
  if sum % 2 == 0 then
    return sum
  else
    throw "not even!"
```

We implement backtracking in `backtrack` by handling both the `flip` and `throw` operations. For `flip` and the `return` case we do the same as in `allvalues`, calling the continuation `k` with both `True` and `False` and appending the results together. For `throw` we ignore the error message and continuation and exit early with the empty list, this means that branches that results in a failure will not actually return any values.

```
backtrack : List Int
backtrack () = handle (handle (evensums123) {
  flip () k ->
    vtrue <- k True;
    vfalse <- k False;
    return vtrue ++ vfalse
  return v -> return [v]
}) {
  throw msg k -> return []
  return v -> return v
} -- returns [2, 4, 4, 6]
```

We can also handle the effects independently of each other. For example we could implement a partial version of `backtrack` that only handles the `Flip` effect. Any operation that is not in the handler is just passed through.

```
partlybacktrack : (List Int)!{Exc}
partlybacktrack = handle (evensums123) {
  flip () k ->
    vtrue <- k True;
    vfalse <- k False;
    return vtrue ++ vfalse
  return v -> return [v]
}
```

Now we can factor out the `throw` handler into its own function.

```
fullbacktrack : List Int
fullbacktrack = handle (partlybacktrack) {
  throw msg k -> return []
  return v -> return v
} -- returns [2, 4, 4, 6]
```

Algebraic effects always commute, meaning the effects can be handled in any order. In the backtracking example the order of the handlers does not actually matter, but in general different orders could have different results.

Lastly we introduce the `State` effect, which allows us to implement local mutable state. We restrict ourselves to a state that consists of a single integer value, but in a language with parametric polymorphism a more general state effect could be written.

```
effect State {
  get : () -> Int
  put : Int -> ()
}
```

Our state effect has two operations, `get` and `put`. The `get` operation allows us to retrieve a value from the state and with the `put` operation we can change the value in the state.

We can now implement the familiar "post increment" operation as seen in the C programming language. This function retrieves the current value of the state, increments it by 1 and returns the previously retrieved value.

```
postInc : Int!{State}
postInc =
```

```
x <- get ();
put (x + 1);
return x
```

To implement the semantics of the `State` effect we use parameter-passing similar to how the State monad is implemented in Haskell. We will abstract the implementation of the state handler in a function `runState`.

```
runState : Int!{State} -> (Int -> (Int, Int))
runState comp = handle (comp) {
  get () k -> return (\s -> (f <- k s; return f s))
  put v k -> return (\s -> (f <- k (); return f v))
  return v -> return (\s -> return (s, v))
}
```

`runState` takes a computation that returns an integer and may use the `State` effect, and returns a function that takes the initial value of the state and returns a tuple of the final state and the return value of the computation. Let us take a look at the `return` case first, here we return a function that takes a state value and returns a tuple of this state and the return value. For the `get` case we return a function that takes a state value and runs the continuation `k` with this value, giving access to the state at the point where the `get` operation was called. From this continuation we get back another function, which we call with the current state, continuing the computation without changing the state. The `put` case is similar to the `get` but we call the continuation with the unit value and we continue the computation by calling `f` with the value giving with the `put` operation call.

Using state now is as simple as calling `runState`.

```
stateResult : (Int, Int)
stateResult =
  f <- runState postInc; -- returns a function taking the initial state
  f 42 -- post-increments 42 returning (43, 42)
```

Using the state effect we can implement imperative algorithms such as summing a range of numbers. We first implement a recursive function `sumRangeRec` which uses `State` to keep a running sum. After we define `sumRange` which calls `sumRangeRec` and runs the `State` effect with 0 as the initial value.

```
sumRangeRec : Int -> Int -> Int!{State}
```

```
sumRangeRec a b =
  if a > b then
    (_, result) <- get ();
    return result
  else
    x <- get ();
    put (x + a);
    sumRangeRec (a + 1) b

sumRange : Int -> Int -> Int
sumRange a b =
  f <- runState (sumRangeRec a b);
  f 0 -- initial sum value is 0
```

2.2 Static instances

Static instances extend algebraic effects by allowing multiple instances of the same effect to co-exist. These instances be handled independently of each other. Operations in such a system are always called on a specific instance and handlers also have to note instance they are handling. We will write operation calls as `inst##op(v)` where `inst` is the instance. Handlers are modified to take an instance parameter as follows `handle##inst(comp) { ... }`.

As an example let us take another look at the `safeDiv` function.

```
safeDiv : Int -> Int -> Int!{Exc}
safeDiv a b =
  if b == 0 then
    throw "division by zero!"
  else
    return a / b
```

We can rewrite this to use static instances by declaring an instance of `Exc` called `divByZero` and calling the `throw` operation on this instance. Note that in the we now state the instance used instead of the effect, since multiple instances of the same effect could be used and we would like to know which instances exactly.

```
instance Exc divByZero

safeDiv : Int -> Int -> Int!{divByZero}
safeDiv a b =
  if b == 0 then
    divByZero#throw "division by zero!"
  else
    return a / b
```

Imagine we wanted to also throw an exception in the case that the divisor was negative. Using instances we can easily declare another Exc instance, let us call it negativeDivisor, and use it in our function. We also have to modify the type to mention the use of negativeDivisor.

```
instance Exc divByZero
instance Exc negativeDivisor

safeDivPositive : Int -> Int -> Int!{divByZero, negativeDivisor}
safeDivPositive a b =
  if b == 0 then
    divByZero#throw "division by zero!"
  else if b < 0 then
    negativeDivisor#throw "negative divisor!"
  else
    return a / b
```

We can now see from the type what kind of exceptions are used in the function. We can also handle the exceptions independently. For example we could handle divByZero by defaulting to 0, but leave negativeDivisor unhandled.

```
defaultTo0 : Int!{divByZero, negativeDivisor} -> Int!{negativeDivisor}
defaultTo0 c =
  handle#divByZero (c) {
    throw msg -> return 0
    return v -> return v
  }
```

2.3   Dynamic instances

Having to predeclare every instance we are going to use is very inconvenient, especially when we have effects such as reference cells or communication channels. The global namespace would be littered with all references and channels the program would ever use. Furthermore we do not always know how many references we need. Take for example a function which creates a list of reference cells giving a length l. We do not know statically what the length of the list will be and so we do not know ahead how many instances we have to declare. Furthermore because all the instances would be predeclared some information about the implementation of a function would be leaked to the global namespace. This means it is impossible to fully encapsulate the use of an effect when using static instances.

Dynamic instances improve on static instances by allowing instances to be created dynamically. Instances become first-class values, they can be assigned to variables and passed to functions just like any other value. We use `new E` to create a new instance of the `E` effect. The actual implementation of the function can stay exactly the same, as can the handler `defaultTo0`. We can translate the previous example to use dynamic instances by defining the `divByZero` and `negativeDivisor` as top-level variables and assigning newly created instances to them. We omit type annotation, since there does not exist any type system that can type all usages of dynamic instances.

```
divByZero = new Exc
negativeDivisor = new Exc

safeDivPositive a b =
  if b == 0 then
    divByZero#throw "division by zero!"
  else if b < 0 then
    negativeDivisor#throw "negative divisor!"
  else
    return a / b

defaultTo0 c =
  handle#divByZero (c) {
    throw msg -> return 0
```

```
    return v -> return v
  }
```

Using locally created instances we can emulate variables as they appear in imperative languages more easily. We can implement the factorial function in an imperative style using a locally created `State` instance. The `factorial` function computes the factorial of the paramter `n` by creating a new `State` instance named `ref` and calling the helper function `factorialLoop` with `ref` and `n`. The base case of `factorialLoop` retrieves the current value from `ref` and returns it. In the recursive case of `factorialLoop` the value in `ref` is modified by multiplying it by `n` and then we continue by recursing with `n - 1`. The call to `factorialLoop` in `factorial` is wrapped in the `State` handler explained earlier, chosing `1` as the initial value of `ref`. `factorial` thus computes the factorial of a number by using a locally created instance, but the use of this instance or the `State` effect in general never escapes the function, it is completely encapsulated.

```
factorialLoop ref n =
  if n == 0 then
    ref#get ()
  else
    x <- ref#get();
    ref#put (x * n);
    factorialLoop ref (n - 1)

factorial n =
  ref <- new State;
  statefn <- handle#r (factorialLoop ref n) {
    get () k -> return (\s -> (f <- k s; return f s))
    put v k -> return (\s -> (f <- k (); return f v))
    return v -> return (\s -> return v)
  };
  statefn 1 -- use 1 as the initial value of ref
```

Next we will implement references more generally similar to the ones available in Standard ML[13], in our case specialized to `Int`. In the previous example we see a pattern of creating a `State` instance and then calling some function with it wrapped with a handler. This is the pattern we want to use when implementing references. To implement this pattern more generally this we

first introduce a new effect named `Heap`. `Heap` has one operation called `ref` which takes an initial value `Int` and returns a `State` instance. `Heap` can be seen as a collection of references. We then define a handler `runRefs` which takes a `Heap` instance and a computation, and creates `State` instances for every use of `ref`. After we call the continuation with the newly created instance and wrap this call in the usual `State` handler, giving the argument of `ref` as the initial value.

```
effect Heap {
  ref : Int -> Inst State
}

runRefs inst c =
  handle#inst (c) {
    ref v k ->
      r <- new State;
      statefn <- handle#r (k r) {
        get () k -> return (\s -> (f <- k s; return f s))
        put v k -> return (\s -> (f <- k (); return f v))
        return v -> return (\s -> return v)
      };
      statefn v
    return v -> return v
  }
```

By calling `runRefs` at the top-level we will have the same semantics for references as Standard ML. In the following example we create two references and swap their values using a `swap` function. First `main` creates a new `Heap` instance `heap` and then calls `runRefs` with this instance. The computation given to `runRefs` is the function `program` called with `heap`.

```
swap r1 r2 =
  x <- r1#get ();
  y <- r2#get ();
  r1#put(y);
  r2#put(x)

program heap =
  r1 <- heap#ref 1;
```

```
  r2 <- heap#ref 2;
  swap r1 r2;
  x <- r1#get ();
  y <- r2#get ();
  return (x, y)

main =
  heap <- new Heap;
  runRefs heap (program heap) -- returns (2, 1)
```

In the Haskell programming language the ST monad?? can be used to implement algorithms that internally use mutable state. The type system, using the runST function, will make sure that the mutable state does not leak outside of the function. For example the following function fibST implements the Fibonacci function in constant space by creating two mutable references.

```
fibST :: Integer -> Integer
fibST n =
    if n < 2 then
      n
    else runST $ do
        x <- newSTRef 0
        y <- newSTRef 1
        fibST' n x y

    where fibST' 0 x _ = readSTRef x
          fibST' n x y = do
              x' <- readSTRef x
              y' <- readSTRef y
              writeSTRef x y'
              writeSTRef y $! x' + y'
              fibST' (n - 1) x y
```

Using dynamic instances we can implement the same algorithm, named fib below. Our fib takes a parameter n and returns the nth Fibonacci number. First we check if n is smaller than 2, in which case we can return n as the result, since nth Fibonacci number is n, if $n < 2$. Else we create a new Heap instance named heap and use the runRefs function defined earlier to

run a computation on this heap. We create two `State` instances on `heap`, `x` and `y` initialized with `0` and `1` respectively and call the auxillary function `fibRec` with `n` and the two instances `x` and `y`. `fibRec` implements the actual algorithm. It works by (recursively) looping on `n`, subtracting by `1` each recursive call. `x` and `y` store the current and next Fibonacci respectively and each loop they are moved one Fibonacci number to the right. When `n` is `0` we know `x` contains the `n`th (for the initial value of `n`) Fibonacci number and we can just get the current value from `x` and return it. Even though this algorithm uses the `Heap` and `State` effects, their uses are completely encapsulated by the `fib` function. The `fib` function does not leak the fact that it's using those effects to implement the algorithm.

```
fib n =
  if n < 2 then
    n
  else
    heap <- new Heap;
    runRefs heap (
      x <- heap#ref 0;
      y <- heap#ref 1;
      fibRec n x y
    )

fibRec n x y =
  if n == 0 then
    x#get ()
  else
    x' <- x#get ();
    y' <- y#get ();
    x#put(y');
    y#put(x' + y');
    fibRec (n - 1) x y
```

Dynamic instances have one big problem though: they are too dynamic. Similar to how in general it is undecidable to know whether a reference has escaped its scope, it is also not possible to know whether an instance has a handler associated with it. This makes it hard to think of a type system for dynamic instances which ensures that there are no unhandled operations. Earlier versions of the Eff programming language[1] had dynamic instances

but its type system underapproximated the uses of dynamic instances which meant you could still get a runtime error if any operation calls were left unhandled.

Chapter 3

Introduction to X

In Chapter 2.3 we saw how dynamic instances allow us to implement mutable references in a system with algebraic effects. This system is untyped however, meaning that you can get runtime errors if an operation is unhandled. This can happen if an operation is called on a dynamic instance outside of a handler. In this chapter we introduce X, which combines algebraic effects and dynamic instances while still remaining typeable. In order to achieve this we introduce the notion of an *effect scope*. An effect scope groups together instances. When creating an instance we give an effect scope to create the instance in. Every instance belongs to a specific effect scope. Different from the system with dynamic instances from Chapter 2.3, we always have to specify a handler when creating an instance, similar to resources in Eff. Specifying a handler at the moment of creating an instance ensures that each instance always has a handler associated with it. Performing effects is done with a new runscope construct, similar to how the handle construct performed effects in Chapter 2. runscope creates a fresh scope and makes it available for use in a given computation. After all the instances created on this new scope will actually be instantiated and all the operations called on these instances will be handled. In order to allow computations to be polymorphic over effect scopes we also introduce *effect scope polymorpishm* together with *effect scope abstraction* and *effect scope application*.

We start with explaining all the novel concepts in Section 3.1, using the example of mutable references. Then we will show how mutable vectors can

be defined, followed by an implementation of a list shuffling algorithm in Section 3.2.

We build on the language used in Chapter 2.1. We use syntax reminiscent of Haskell with algebraic data types and pattern matching. Type constructors and effect names are uppercase while type variables are lowercase.

3.1   Effects, effect scopes and instances

In Figure 3.1 we give an example containing all the novel constructs of X.

3.1.1   Effects

To start off we define a `State` effect specialized to `Int`s. `State` is meant to represent a mutable reference to a single value of type `Int`. This definition is exactly the same as the `State` effect definition in Chapter 2.1, in the basic algebraic effects system. With the `effect` keyword we declare a new effect called `State` with two operations: `get` and `put`. For each operation we give parameter and return types. For `get` we give the unit type `()` as the parameter type. As the return type we give `Int`, meaning that calling the `get` operation will return a integer value. For the `put` operation it is the other way around. As the parameter type we have `Int`, meaning that `put` requires an integer value when called, and the return type is `()`, calling `put` will give back the unit value.

3.1.2   Effect instances

In Figure 3.1 the function `ref` creates a new *effect instance* of the `State` effect. We can create a fresh instance of an effect using the `new` keyword. The construct `new State@s { ... }` can be read as "Create a fresh `State` instance in the effect scope `s`". Here we have to give a specific scope `s` to create the instance on. The instance can only be used within this given scope. The newly-created instance is available in the body of the `new` construct.

Figure 3.1: Example of all the novel constructs

```
1   effect State {
2     get : () -> Int
3     put : Int -> ()
4   }
5
6   ref : forall s. Int -> (Inst s State)!{s}
7   ref [s] v =
8     new State@s {
9       get () k -> \st -> k st st
10      put st' k -> \st -> k () st'
11      return x -> \st -> return x
12      finally f -> f v
13    } as x in return x
14
15  postInc : forall s. Inst s State -> Int!{s}
16  postInc [s] inst =
17    x <- inst#get();
18    inst#put(x + 1);
19    return x
20
21  result : Int!{}
22  result =
23    runscope(s1 ->
24      r1 <- ref [s1] 10;
25      runscope(s2 ->
26        r2 <- ref [s2] 20;
27        x <- postInc [s2] r2;
28        r1#put(x);
29        return x);
30      y <- r1#get ();
31      return y) -- result is 20
```

When creating an instance we have to give a *handler*. The handler specifies
what should happen when the operations are called. The handler is defined
within curly braces and consists of a case for each operation of the effect,
plus a `return` case and a `finally` case. The handler given in the example is
the same as the handler given in Chapter 2.1, implementing a single mutable
reference. In addition we also have to define a `finally` case. In there we can
perform an extra computation after the handler is performed. In the case of
the handler given this is necessary because the return type of the handler is
`Int -> T` for some return type `T`. The handler transforms a computation in
to a function expected the initial value for the mutable reference. Using the
`finally` case we can call this function and get back the return value of the
computation (of type `T`. In the example we simply call the function `f` with
the argument `v` given to `ref`.

We can now understand the type of `ref`:

```
ref : forall s. Int -> (Inst s State)!{s}
```

We can see from the `forall s.` that `ref` is polymorphic over scopes. That
means that this function works for any scope `s`.

The type variable `s` here is an *effect scope variable*. An effect scope vari-
able can be seen as the name of a collection of instances that we call an
effect scope. Such a scope can contain zero or more instances, where each
instance can be of any effect. A scope restricts instances in such a way that
they cannot escape that scope and instances from one scope cannot be used
in another. This also means that we can never get a runtime error because
of an unhandled operation call.

In order to apply `ref` we have to give an explicit scope `s` using the syn-
tax `ref [s]`. We call this *effect scope application*. In the definition we show
that this function has a scope parameter using angle brackets `[s]`. We call
this *effect scope abstraction*. The second parameter is a value of type `Int`
which we call `v`. This is the initial value that we want our mutable reference
to have. The return type is `Inst s State`. This is an effect instance of the
`State` effect in the scope `s`. From this type we can see that `ref` gives back an
instance on the given scope `s`. The effect annotation of `ref` is `!{s}`, which
shows that we actually perform effects in the scope `s`. We can see from the
function implementation that the only effect we perform is the creation of

an instance on `s`.

Using `ref` we can fully emulate multiple mutable references. We have the added guarantee that the references will not escape their effect scope, they will not escape their corresponding `runscope`. Adding parametric polymorphism to the effects to give `State` `t` for any type `t` will enable us to emulate references of any type. With references of different types coexisting. This is very similar to how the ST monad works in Haskell [16]. Using mutable references have interesting applications such as meta variables in unification algorithms and typed logic variables [17].

Looking at the type of creating mutable references using the ST monad in Haskell (`newSTRef`), we can see that the type of `ref` is very similar (we explicitly wrote down the quantification and specialized `newSTRef` to `Int`):

```
ref : forall s. Int -> (Inst s State)!{s}
newSTRef :: forall s. Int -> ST s (STRef s Int)
```

Here `ST` `s` serves the same purpose as `!{s}` in our system. `STRef` `s` `Int` is the type of a mutable reference in the ST monad in Haskell. `s` is some "state thread", the purpose if this type variable is to statically ensure that references do not escape their scope. This is exactly like the `s` type variable in our system, but we generalize "state threads" to effect scopes, where any algebraic effect may be performed.

3.1.3 Using effect instances

In Figure 3.1 the function `postInc` shows how an effect instance can actually be used:

```
postInc : forall s. Inst s State -> Int!{s}
postInc [s] inst =
  x <- inst#get();
  inst#put(x + 1);
  return x
```

We can see from the type that this function is polymorphic over some effect scope `s`. It expects some scope `s` and some `State` instance on `s` as its arguments. It returns an integer value and may perform some effects on `s`

(`Int`!`{s}`. The type of `postInc` can be read as "For any scope `s`, given a `State` instance in `s`, return a value of type `Int` possibly by calling operations on instances in `s`".

Effects can be used by calling operations. Operations are always called on an effect instance. Without an instance we are unable to perform operations. In the case of `postInc` we get an instance as an argument to the function. Operation can be called on an instance using the syntax `instance\#operation(argument)`. We write `instance\#operation()` to mean `instance\#operation(())`, when the unit value `()` is given as the argument. `postInc` implements the traditional "post increment" operation on a mutable reference. In the C language this is written `x++` for some reference `x`. We first call the `get` operation on `inst`. We get back a value of type `Int`, which we name `x`. Then we call `put` on `inst` with the argument `(x + 1)`. Finally we return the previous value of the mutable reference `x`.

We could define functions wrapping the `get` and `put` operations:

```
performGet : forall s. Inst s State -> Int!{s}
performGet [s] inst = inst#get()

performPut : forall s. Inst s State -> Int -> ()!{s}
performPut [s] inst v = inst#put(v)
```

Again we can compare to the corresponding functions in the ST monad in Haskell, `readSTRef` and `writeSTRef`:

```
readSTRef :: forall s. STRef s Int -> ST s Int
writeSTRef :: forall s. STRef s Int -> Int -> ST s ()
```

We can see that the types are very similar, but that we generalize the ST monad to work for any algebraic effect.

3.1.4   Running scopes

The definition `result` shows how the effects in a computation can be performed:

```
result : Int!{}
result =
  runscope(s1 ->
    r1 <- ref [s1] 10;
    ret <- runscope(s2 ->
      r2 <- ref [s2] 20;
      x <- postInc [s2] r2;
      r1#put(x);
      return x);
    y <- r1#get ();
    return y) -- result is 20
```

From the type we can see that `result` is a computation that returns an integer value. We can see from the effect annotation (`!{}`) that `result` does not have any unhandled effects. In the future we will omit writing `!{}` if a computation does not have any unhandled effects.

The `runscope(s' -> ...)` construct provides a new scope, which we named `s1` and `s2` in our case, which can be used in its body. Inside `runscope` we can create and use instances in this new scope. `runscope` will make sure that any instances that are created on its scope will actually be created and that any operation calls on these instances will be handled.

In `result` we use two nested scope. First we create a scope called `s1`. On this scope we call `ref` to create a mutable reference `r1` with 10 as its initial value. Then we create another scope called `s2`. In `s2` we create another mutable reference `r2` with 20 as its initial value. We then call `postInc` on `r2` and store the return value in `x` (20). Then we call `r1##put(x)`, setting `r1` to 20. We then return `x` as the return value of the `s2` scope, storing this value in `ret` in the `s1` scope. At this point the `s2` scope is gone and any effects in it will be handled. The type system will make sure that no instances created in `s2` can escape `s2`.

Note that we also used `r1` inside `s2`. Since `r1` belongs to s1, all the operations called on it will not be handled inside `s2` but these will be *forwarded* instead. This means that these operations will remain unhandled until `s2` is done. Because of this forwarding behaviour we can combine effects from multiple scopes, giving us fine-grained control over where effects may happen.

Continuing in `result` we get the current value of `r1` and return from `s1`. This value is `20` which was set in scope `s2`. After this the scope `s1` is done and any effect in it will be handled. This leaves us with a computation of type `Int!{}` with no remaining effects.

3.2 Mutable vectors

Figure 3.2: Mutable vectors

```
1   -- list of mutable references
2   data Vector s = VNil | VCons (Inst s State) Vector
3
4   -- get the length of a vector
5   vlength : forall s. Vector s -> Int
6   vlength VNil = 0
7   vlength (VCons _ tail) = 1 + (vlength tail)
8
9   -- get the value at the index given as the first argument
10  -- assumes the index is within range of the vector
11  vget : forall s. Int -> Vector s -> Int!{s}
12  vget [s] 0 (VCons h _) = h#get()
13  vget [s] n (VCons _ t) = vget [s] (n - 1) t
14
15  -- set the value at the index given as the first argument
16  -- to the value given as the second argument
17  -- assumes the index is within range of the vector
18  vset : forall s. Int -> Int -> Vector s -> ()!{s}
19  vset [s] 0 v (VCons h _) = h#put(v)
20  vset [s] n v (VCons _ t) = vset [s] (n - 1) v t
```

In the previous section we have defined mutable references using the `ref` function. We will now build on them to define mutable vectors. In Figure 3.2 we define the `Vector` datatype. `Vector` is a list of `State` instances and is indexed by the scope of instances: `s`. We define three functions on `Vector`: `vlength`, `vget`, and `vset`. With `vlength` we can get the length of a

vector. With `vget` we can retrieve a value from a vector by giving an index. We assume the index is within the range of the vector. With `vset` we can set an element of a vector by giving an index and a value. Again we assume the index is within the range of the vector. In order to allow these functions to work for any vector we have to introduce an effect scope variable `s` again. We define both functions by recursion on the index.

As an example application we will write a shuffling algorithm for vectors. This simple algorithm will shuffle a vector by randomly swapping two random elements of the vector and repeating this some amount of times. In Figure 3.3 we show the algorithm. First we define an effect `Rng` in order to abstract out the generation of random numbers. `Rng` has a single operation `rand` which returns a random integer between `0` and `n` given an integer `n`. We define a function `vlength` to get the length of the vector.

We then define the actually shuffling function `shuffleVector`. This function takes two scope variables, `s` and `s'`, for the vector and `Rng` instance respectively. As arguments we take an instance of `Rng`, in order to generate random numbers, an integer, for the amount of times to shuffle, and the vector we want to shuffle. By taking a seperate scope for the `Rng` instance we are more flexible when handling the computation. We can handle the effects on the vector while leaving the `Rng` effects to be handled higher up.

`shuffleVector` proceeds as follows. If the amount of times we want to shuffle is `0` we stop and return the vector. If not then we first get the length of the vector. Then we generate two random numbers, `i` and `j`, between `0` and this length. These two numbers will be the two elements we will swap. We then get the current values at these indeces. And we swap the values at these indeces in the vector. We then recurse, subtracting the amount of times to shuffle by one.

Figure 3.3: Vector shuffling

```
1   -- random number generation effect
2   -- the operation `rand` gives back a random integer
3   -- between 0..n, where n is the argument given (exclusive)
4   effect Rng {
5     rand : Int -> Int
6   }
7
8   -- shuffles a vector given an instance of Rng
9   -- by swapping two random elements of the vector
10  -- the second argument to shuffleVector is the amount of times
11  -- to swap elements
12  shuffleVector : forall s s'. Inst s' Rng -> Int
13    -> Vector s -> ()!{s, s'}
14  shuffleVector [s] [s'] _ 0 vec = vec
15  shuffleVector [s] [s'] rng n vec =
16    let len = vlength vec;
17    i <- rng#rand(len);
18    j <- rng#rand(len);
19    a <- vget [s] i vec;
20    b <- vget [s] j vec;
21    vset [s] i b vec;
22    vset [s] j a vec;
23    shuffleVector [s] [s'] rng (n - 1) vec
```

Using `shuffleVector` we can implement a function to shuffle a list in Figure 3.4. We first define the usual `List` datatype, with `Nil` and `Cons` cases. Then we define two functions `toVector` and `toList` to convert between lists and vectors. `toVector` simply recurses on the list and creates fresh mutable references for each element of the list, initialized with the value of the element. `toList` converts a vector to a list by getting the current values of each reference in the vector. The function `shuffle` implements the actual shuffling. It takes an effect scope, a `Rng` instance `rng` in this scope and a list `lst`. We first convert the list to a mutable vector. Then we use `shuffleVector` to shuffle the vector 100 times, passing `rng` for generating the random numbers. Finally we convert the vector back to a list and return this result. We wrap

this computation in `runscope` to handle the effects of the mutable vector. The use of mutable vectors is not leaked outside of the function, from the type and behaviour of `shuffleVector` we are unable to find out if mutable vectors are used. We say that the use of the `State` effect is completely *encapsulated*. The type system ensures that `runscope` actually does encapsulate all effects in its scope. Note that we do not handle the scope of `rng`, we leave the `Rng` to be handled higher up by the caller of `shuffle`.

In Figure 3.5 we define the function `runshuffle` which can shuffle a list `lst`. We use a naive implementation of random number generation using a linear congruential generator. Inside of a new scope `s` we first create a mutable reference called `seedref`, which we set to a random intial seed value. This reference will store the current state of the random number generator, which we call the seed. We then create a fresh `Rng` instance called `rng`. We implement the `rand` operation by first getting the current seed value from `seedref`. Then we calculate a new seed value using arbitrarily chosen numbers, store this in `seedref` and call the continuation with it. The return and finally cases do not do anything special. Finally we call `shuffle` with our `Rng` instance and `lst`. In this example we can see how we can use other effects in the handler of an instance. The `Rng` uses an instance of `State` to implement the `rand` operation. Both of these effects exist and are handled in the same scope `s`. From the type of `runshuffle` (`List -> List`) we can see that all the effects are encapsulated and that the function is pure. In the next Chapter we will give a formal account of the system, including a type system and semantics.

In this Chapter we have seen how to program with effect scopes. Like the regular algebraic effects (Chapter 2.1) we can use and combine different effects simply by using their operations in a program. What is different is that handlers are given when creating instances. We have seen that we can abstract over and instantiate effect scopes. Lastly we saw how effect scopes enable use to implement mutable references and vectors while still being safe.

Figure 3.4: List shuffling

```
1   -- (linked) list of integer values
2   data List = Nil | Cons Int List
3
4   -- transform a list to a vector by replacing each value
5   -- in the list by a reference initialized with that value
6   toVector : forall s. List -> (Vector s)!{s}
7   toVector [s] Nil = VNil
8   toVector [s] (Cons h t) =
9     h' <- ref [s] h;
10    t' <- toVector [s] t;
11    return (VCons h' t')
12
13  -- transform a vector back to a list by getting the
14  -- current values from the references in the vector
15  toList : forall s. Vector s -> List!{s}
16  toList [s] VNil = Nil
17  toList [s] (VCons h t) =
18    h' <- h#get();
19    t' <- toList [s] t;
20    return (Const h' t')
21
22  -- shuffles a list given an instance of Rng
23  -- by converting it to a vector
24  -- and shuffling 100 times
25  shuffle : forall s'. Inst s' Rng -> List -> List!{s'}
26  shuffle [s'] rng lst =
27    runscope(s ->
28      let vec = toVector [s] lst;
29      shuffleVector [s] [s'] rng 100 vec;
30      return (toList vec))
```

Figure 3.5: List shuffling

```
1  runshuffle : List -> List
2  runshuffle lst =
3    runscope(s ->
4      seedref <- ref [s] 123456789;
5      rng <- new Rng@s {
6        rand n k ->
7          currentseed <- seedref#get();
8          let newseed = (1103515245 * currentseed + 12345) % n;
9          seedref#put(newseed);
10         k newseed
11       return x -> return x
12       finally x -> return x
13     };
14     shuffle [s] rng lst)
```

Chapter 4

Semantics and types of algebraic effects and handlers

In this chapter we will show the basics of algebraic effects and handlers. We will start with the simply-typed lambda calculus (§4.1) and add algebraic effects (§4.2) and static instances (§4.3 to it.

4.1 Simply-typed lambda calculus

As our base language we will take the fine-grained call-by-value simply-typed lambda calculus (FG-STLC) [10]. This system is a version of the simply-typed lambda calculus with a syntactic distinction between values and computations. Because of this distinction there is exactly one evaluation order: call-by-value. In a system with side effects the evaluation order is very important since a different order could have a different result. Having the evaluation order be apparent from the syntax is thus a good choice for a system with algebraic effects. Another way to look at FG-STLC is to see it as a syntax for the lambda calculus that constrains the program to always be in A-normal form [11].

The terms are shown in Figure 4.1. The terms are split in to values and computations. Values are pieces of data that have no effects, while computations are terms that may have effects.

Figure 4.1: Syntax of the fine-grained lambda calculus

$$\nu ::= x, y, z, k \mid \lambda x.c \mid ()$$
$$c ::= \textsf{return } \nu \mid \nu\ \nu \mid x \leftarrow c; c$$

Values   We have x, y, z, k ranging over variables, where we will use k for variables that denote continuations later on. Lambda abstractions are denoted as $\lambda x.c$, note that the body c of the abstraction is restricted to be a computation as opposed to the ordinary lambda calculus where the body can be any expression. To keep things simple we take unit $()$ as our only base value, this because adding more base values will not complicate the theory. Using the unit value we can also delay computations by wrapping them in an abstraction that takes a unit value.

Computations   For any value ν we have $\textsf{return } \nu$ for the computation that simply returns a value without performing any effects. We have function application $(\nu\ \nu)$, where both the function and argument have to be values. Sequencing computations is done with $(x \leftarrow c; c)$. Normally in the lambda calculus the function and the argument in an application could be any term and so a choice would have to be made in what order these have to be evaluated or whether to evaluate the argument at all before substitution. In the fine-grained calculus both the function and argument in $(\nu\ \nu)$ are values so there's no choice of evaluation order. The order is made explicit by the sequencing syntax $(x \leftarrow c; c)$.

Semantics   The small-step operational semantics is shown in Figure 4.2. The relation $\rightsquigarrow$ is defined on computations, where the $c \rightsquigarrow c'$ means c reduces to c' in one step. These rules are a fine-grained approach to the standard reduction rules of the simply-typed lambda calculus. In S-App we apply a lambda abstraction to a value argument, by substituting the value for the variable x in the body of the abstraction. In S-SeqReturn we sequence a

Figure 4.2: Semantics of the fine-grained lambda calculus

S-APP

$$(\lambda x.c)\ \nu \rightsquigarrow c[x := \nu]$$

S-SEQRETURN

$$(x \leftarrow \mathsf{return}\ \nu; c) \rightsquigarrow c[x := \nu]$$

S-SEQ

$$\frac{c_1 \rightsquigarrow c_1'}{(x \leftarrow c_1; c_2) \rightsquigarrow (x \leftarrow c_1'; c_2)}$$

Figure 4.3: Types of the fine-grained simply-typed lambda calculus

$$\tau ::= ()\ |\ \tau \rightarrow \underline{\tau}$$
$$\underline{\tau} ::= \tau$$

computation that just returns a value in another computation by substituting the value for the variable x in the computation. Lastly, in S-SEQ we can reduce a sequence of two computations, c_1 and c_2 by reducing the first, c_1.

We define $\rightsquigarrow^*$ as the transitive-reflexive closure of $\rightsquigarrow$. Meaning that c in $c \rightsquigarrow^* c$ can reach c' in zero or more steps, while c in $c \rightsquigarrow c'$ reaches c' in exactly on step.

Types Next we give the *types* in Figure 4.3. Similar to the terms we split the syntax into value and computation types. Values are typed by value types and computations are typed by computation types. A value type is either the unit type () or a function type with a value type τ as argument type and a computation type $\underline{\tau}$ as return type.

For the simply-typed lambda calculus a computation type is simply a value type, but when we add algebraic effects computation types will become more meaningful by recording the effects a computation may use.

Figure 4.4: Typing rules of the fine-grained simply-typed lambda calculus

$$
\frac{\text{T-VAR}}{\Gamma \vdash x : \tau} \qquad \frac{\text{T-UNIT}}{\Gamma \vdash () : ()} \qquad \frac{\text{T-ABS}}{\Gamma, x : \tau_1 \vdash c : \underline{\tau_2}}{\Gamma \vdash \lambda x.c : \tau_1 \to \underline{\tau_2}}
$$

$$
\frac{\text{T-RETURN}}{\Gamma \vdash \nu : \tau}{\Gamma \vdash \text{return } \nu : \underline{\tau}} \qquad \frac{\text{T-APP}}{\Gamma \vdash \nu_1 : \tau_1 \to \underline{\tau_2} \qquad \Gamma \vdash \nu_2 : \tau_1}{\Gamma \vdash \nu_1\ \nu_2 : \underline{\tau_2}}
$$

$$
\frac{\text{T-SEQ}}{\Gamma \vdash c_1 : \underline{\tau_1} \qquad \Gamma, x : \tau_1 \vdash c_2 : \underline{\tau_2}}{\Gamma \vdash (x \leftarrow c_1; c_2) : \underline{\tau_2}}
$$

Typing rules   Finally we give the typing rules in Figure 4.4. We have a typing judgment for values $\Gamma \vdash \nu : \tau$ and a typing judgment for computations $\Gamma \vdash c : \underline{\tau}$. In both these judgments the context Γ assigns value types to variables.

The rules for variables (T-VAR), unit (T-UNIT), abstractions (T-ABS) and applications (T-APP) are the standard typing rules of the simply-typed lambda calculus. For return ν (T-RETURN) we simply check the type of ν. For the sequencing of two computations $(x \leftarrow c_1; c_2)$ (T-SEQ) we first check the type of c_1 and then check c_2 with the type of c_1 added to the context for x.

Type soundness   In order to prove type soundness for the previously defined calculus we first have define what it means for a computation to be a value. We define a computation c to be a value if c is of the form return ν for some value ν.

$$
\text{value}(c) \text{ if } \exists \nu.c = \text{return } \nu
$$

Using this definition we can state the following type soundness theorem for the fine-grained simply typed lambda calculus.

Theorem 1 (Type soundness)**.**

$$
\text{if } \cdot \vdash c : \underline{\tau} \ \wedge \ c \rightsquigarrow^* c' \text{ then } \text{value}(c') \vee (\exists c''. \ c' \rightsquigarrow c'')
$$

This states that given a well-typed computation c and taking some amount of steps then the resulting computation c' will be of either a value or another step can be taken. In other words the term will not get "stuck". Note that this is only true if the computation c is typed in the empty context. If the context is not empty then the computation could get stuck on free variables.

We can prove this theorem using the following lemmas:

Lemma 1 (Progress).

$$\text{if } \cdot \vdash c : \underline{\tau} \text{ then } \mathsf{value}(c) \vee (\exists c'.\ c \rightsquigarrow c')$$

Lemma 2 (Preservation).

$$\text{if } \Gamma \vdash c : \underline{\tau} \ \wedge \ c \rightsquigarrow c' \text{ then } \Gamma \vdash c' : \underline{\tau}$$

Where the progress lemma states that given a well-typed computation c then either c is a value or c can take a step. The preservation lemma states that given a well-typed computation c and if c can take a step to c' then c' is also well-typed. We can prove both these by induction on the typing derivations. Note again that the context has to be empty for the Progress lemma, again because the computation could get stuck on free variables. For the Preservation lemma the context can be anything however, since the operational semantics will not introduce any new free variables that are not already in the context.

We formalized the fine-grained simply-typed lambda calculus and have proven the type soundness theorem in the Coq proof assistant.

Figure 4.5: Syntax of algebraic effects

$$\nu ::= x, y, z, k \mid \lambda x.c \mid ()$$

$$c ::= \text{return } \nu \mid \nu \; \nu \mid x \leftarrow c; c \mid op(\nu) \mid \text{handle}(c)\{h\}$$

$$h ::= op \; x \; k \rightarrow c; \; h \mid \text{return } x \rightarrow c$$

4.2 Algebraic effects

We now extend the previous calculus with algebraic effects and handlers. We assume there is a set of effect names $\mathsf{EffName}$ with $E \subseteq \mathsf{EffName}$, for example $E = \{\mathsf{Flip}, \mathsf{State}, ...\}$. For each effect ϵ there assume there is a non-empty set of operations O^ϵ. For example $O^{\mathsf{Flip}} = \{\mathsf{flip}\}$ and $O^{\mathsf{State}} = \{\mathsf{get}, \mathsf{put}\}$.

Syntax The syntax for the extended system is shown in Figure 4.5, additions are highlighted with a gray background. Values stay the same. We add two forms of computations, operation calls $op(\nu)$ where $op \in O^\epsilon$ for some effect ϵ and we can handle computations using $\mathsf{handle}(c)\{h\}$. Handlers h are lists of operation cases $op \; x \; k \rightarrow c$; h ending in the return case $\mathsf{return } \; x \rightarrow c$. We assume that operations are not repeated within a handler.

Semantics We give a small-step operational semantics in Figure 4.6. S-APP, ALGEFF-S-SEQRETURN and S-SEQ are the same as in the fine-grained system and are left out of the figure. To be able to handle a computation we first transform the computation to the form $\mathsf{return } \; \nu$ or $(x \leftarrow op(\nu); c)$. S-FLATTEN and S-OP are used to get a computation to those forms. The last four rules are used to handle a computation. S-HANDLERETURN handles a computation of the form $\mathsf{return } \; \nu$ by substituting ν in the body of the return case of the handler. S-HANDLEOP and S-HANDLEOPSKIP handle computations of the form $(x \leftarrow op(\nu); c)$. If the operation op is contained in the handler h then the rule S-HANDLEOP substitutes the value ν of the operation call in the body of the matching operation case c'. We also substitute

Figure 4.6: Semantics of algebraic effects

S-FLATTEN

$$(x \leftarrow (y \leftarrow c_1; c_2); c_3) \rightsquigarrow (y \leftarrow c_1; (x \leftarrow c_2; c_3))$$

S-OP

$$op(\nu) \rightsquigarrow (x \leftarrow op(\nu); \mathsf{return}\ x)$$

S-HANDLERETURN

$$\mathsf{handle}(\mathsf{return}\ \nu)\{h; \mathsf{return}\ x \to c\} \rightsquigarrow c[x := \nu]$$

S-HANDLEOP

$$\frac{op\ x\ k \to c' \in h}{\mathsf{handle}(y \leftarrow op(\nu); c)\{h\} \rightsquigarrow c'[x := \nu, k := (\lambda y.\mathsf{handle}(c)\{h\})]}$$

S-HANDLEOPSKIP

$$\frac{op \notin h}{\mathsf{handle}(x \leftarrow op(\nu); c)\{h\} \rightsquigarrow (x \leftarrow op(\nu); \mathsf{handle}(c)\{h\})}$$

S-HANDLE

$$\frac{c \rightsquigarrow c'}{\mathsf{handle}(c)\{h\} \rightsquigarrow \mathsf{handle}(c')\{h\}}$$

Figure 4.7: Types of algebraic effects

$$\tau ::= () \mid \tau \to \underline{\tau}$$
$$\underline{\tau} ::= \tau \mathbin{!} r$$
$$r ::= \{\epsilon_1, ..., \epsilon_n\}$$

Figure 4.8: Subtyping rules of algebraic effects

$$\text{SUB-UNIT} \over () <: ()$$

$$\text{SUB-ARR} \\ \frac{\tau_3 <: \tau_1 \qquad \underline{\tau}_2 <: \underline{\tau}_4}{\tau_1 \to \underline{\tau}_2 <: \tau_3 \to \underline{\tau}_4}$$

$$\text{SUB-ANNOT} \\ \frac{\tau_1 <: \tau_2 \qquad r_1 \subseteq r_2}{\tau_1 \mathbin{!} r_1 <: \tau_2 \mathbin{!} r_2}$$

a continuation in c', which continues with the computation c wrapped by the same handler h. If the operation op is not contained in the handler then we float out the operation call $op(\nu)$ and wrap the handler h around the continuing computation c. Lastly, S-HANDLE is able to reduce a computation in the handle computation.

Type syntax   We now give a type system which ensures that a program reduced by the given semantics will not get "stuck" meaning that the result will be a computation of the form return ν for some value ν. In Figure 4.7 we give the syntax of the types. Value types τ are the same as in the fine-grained system. Computation types $\underline{\tau}$ are now of the form $\tau \mathbin{!} r$ for some value type τ. An annotation $r \subseteq E$ is a set of effect names.

Subtyping   It is always valid in the system to weaken a type by adding more effects to an annotation. This is done using subtyping judgments $\tau <: \tau$ and $\underline{\tau} <: \underline{\tau}$. In Figure 4.13 we give the subtyping rules for the system. Subtyping proceeds structurally on the value and computation types. In SUB-ARR we compare function arguments contravariantly. To compare two annotated types we compare the value types and then check that the annotation on the

Figure 4.9: Typing rules of algebraic effects

$$\text{T-Var} \quad \frac{\Gamma[x] = \tau}{\Gamma \vdash x : \tau \,!\, \varnothing}$$

$$\text{T-Unit} \quad \frac{}{\Gamma \vdash () : ()}$$

$$\text{T-Abs} \quad \frac{\Gamma, x : \tau_1 \vdash c : \underline{\tau_2}}{\Gamma \vdash \lambda x.c : \tau_1 \to \underline{\tau_2}}$$

$$\text{T-SubVal} \quad \frac{\Gamma \vdash \nu : \tau_1 \qquad \tau_1 <: \tau_2}{\Gamma \vdash \nu : \tau_2}$$

$$\text{T-Return} \quad \frac{\Gamma \vdash \nu : \tau}{\Gamma \vdash \mathsf{return}\ \nu : \tau \,!\, \varnothing}$$

$$\text{T-App} \quad \frac{\Gamma \vdash \nu_1 : \tau_1 \to \underline{\tau_2} \qquad \Gamma \vdash \nu_2 : \tau_1}{\Gamma \vdash \nu_1\ \nu_2 : \underline{\tau_2}}$$

$$\text{T-Seq} \quad \frac{\Gamma \vdash c_1 : \tau_1 \,!\, r \qquad \Gamma, x : \tau_1 \vdash c_2 : \tau_2 \,!\, r}{\Gamma \vdash (x \leftarrow c_1; c_2) : \tau_2 \,!\, r}$$

$$\text{T-Op} \quad \frac{op \Rightarrow (\epsilon, \tau_{op}^1, \tau_{op}^2) \qquad \Gamma \vdash \nu : \tau_{op}^1}{\Gamma \vdash op(\nu) : \tau_{op}^2 \,!\, \{\epsilon\}}$$

$$\text{T-Handle} \quad \frac{\Gamma \vdash c : \tau_1 \,!\, r_1 \qquad op \in h \Leftrightarrow op \in O^\epsilon \qquad \Gamma \vdash^{\tau_1} h : \tau_2 \,!\, r_2}{\Gamma \vdash \mathsf{handle}(c)\{h\} : \tau_2 \,!\, ((r_1 \setminus \{\epsilon\}) \cup r_2)}$$

$$\text{T-SubComp} \quad \frac{\Gamma \vdash c : \underline{\tau_1} \qquad \underline{\tau_1} <: \underline{\tau_2}}{\Gamma \vdash c : \underline{\tau_2}}$$

$$\text{T-HOp} \quad \frac{op \Rightarrow (\epsilon, \tau_{op}^1, \tau_{op}^2) \qquad \begin{array}{c} \Gamma \vdash^{\tau_1} h : \tau_2 \,!\, r \\ \Gamma, x : \tau_{op}^1, k : \tau_{op}^2 \to \tau_2 \,!\, r \vdash c : \tau_2 \,!\, r \end{array}}{\Gamma \vdash^{\tau_1} (op\ x\ k \to c;\ h) : \tau_2 \,!\, r}$$

$$\text{T-HReturn} \quad \frac{\Gamma, x : \tau_1 \vdash c : \tau_2 \,!\, r}{\Gamma \vdash^{\tau_1} (\mathsf{return}\ x \to c) : \tau_2 \,!\, r}$$

left is a subset of the annotation on the right.

Typing rules   Finally we give the typing rules in Figure 4.14.   We have three judgements:

1. $\Gamma \vdash \nu : \tau$, which types the value ν with the value type τ
2. $\Gamma \vdash c : \underline{\tau}$, which types the computation c with the computation type $\underline{\tau}$
3. $\Gamma \vdash^{\tau} h : \underline{\tau}$ which types the handler h with the computation type $\underline{\tau}$ given some value type τ

We can get the type of a variable from the context using $\Gamma[x] = \tau$. For each operation op we have a parameter type τ_{op}^1 and a return type τ_{op}^2. We use the syntax $op \Rightarrow (\epsilon, \tau_{op}^1, \tau_{op}^2)$ to retrieve the effect, parameter and return type given an operation op.

T-VAR, T-UNIT, T-ABS, T-APP, and T-SEQ are the same as in the fine-grained system. We can weaken the type of values and computations using subtyping using the rules T-SUBVAL and T-SUBCOMP. For return computations return ν we type the value and annotate it with the empty effect set using the rule T-RETURN. T-OP shows that for operation calls we first lookup the operation in the context to find the effect, parameter and return types. We then check that the argument of the operation call is of the same type as the paramter type of the operation. Finally we type the operation call as an annotated type of the return type and a singleton effect set of the effect of the operation.

For handling we use the rule T-HANDLE. First we typecheck the type of the computation we are handling as having the computation type $\tau_1 \ ! \ r_1$. Then we check that all operations in the handler h are in the set of operations of some effect ϵ, this means that handlers always have to contain exactly the operations of some effect. We then typecheck the handler h, giving it the type of the computation we are handling τ_1 and getting the return type $\tau_2 \ ! \ r_2$. The return type of the handling computation is then τ_2 annotated with the effects from the handled computation minus the effect ϵ we handled together with the effects from the handler.

Finally the rules T-HOP and T-HRETURN type the two cases of a handler. T-HRETURN checks that the computation c of the return case types

as τ_2 ! r after adding x to Γ with the given type τ_1. τ_2 ! r is the return type of the handler. T-HOP first checks the rest of the handler. Then the parameter and return types of the operation op are retrieved. Finally we add the parameter x of the operation and the continuation k to Γ and check that the type of the computation c agrees with the return type of the rest of the handler.

Type soundness   We can defined a type soundness theorem very similar to the one for the simply-typed lambda calculus.

Theorem 2 (Type soundness)**.**

$$\text{if } \cdot \vdash c : \tau \, ! \, \varnothing \, \wedge \, c \rightsquigarrow^* c' \text{ then } \mathsf{value}(c') \vee (\exists c''. \, c' \rightsquigarrow c'')$$

The theorem states that if a computation c is typed in the empty context with an empty effect annotation and we take some amount of steps, then the computation is value or we can take another step. This theorem only works with the empty effect annotation, because if an effect is in the annotation then there may be an unhandled operation call and the computation would not be able to proceed.

We can proof the type soundness theorem using progress and preservation lemmas. We first give the preservation lemma, which is exactly the same as the one from the simply-typed lambda calculus.

Lemma 3 (Preservation)**.**

$$\text{if } \Gamma \vdash c : \underline{\tau} \, \wedge \, c \rightsquigarrow c' \text{ then } \Gamma \vdash c' : \underline{\tau}$$

The progress lemma differs only in that the computation cannot have any effects. Again because we could get stuck if any operation calls are unhandled.

Lemma 4 (Progress)**.**

$$\text{if } \cdot \vdash c : \tau \, ! \, \varnothing \text{ then } \mathsf{value}(c) \vee (\exists c'. \, c \rightsquigarrow c')$$

We need a more general version of the progress lemma in order to prove it. We define a computation c to be effectful if c has unhandled operations. Formally we define this as follows.

$$\mathsf{effectful}(c) \text{ if } (\exists op \, \nu.c = op(\nu)) \vee (\exists x \, op \, \nu \, c'.c = x \leftarrow op(\nu); c')$$

We can now give the generalized version of the progress lemma:

Lemma 5 (Progress effectful)**.**

$$\text{if } \cdot \vdash c : \tau \mathbin{!} r \text{ then } \mathsf{value}(c) \vee \mathsf{effectful}(c) \vee (\exists c'.\ c \rightsquigarrow c')$$

This lemma allows c to be typed with any effect annotation, by adding the possibility that c has unhandled operations.

We formalized the algebraic effect and handler system and have proven the type soundness theorem in the Coq proof assistant. The system formalized slightly differs in that we follow the syntax from Bauer and Pretnar [2]. Operation calls are of the form $op(\nu, x.c)$, carrying around a continuation. Also handlers are first-class values of type $\underline{\tau} \Rightarrow \underline{\tau}$. These changes make the proofs easier. Having continations inside operation calls means we can move operation calls over sequencing, so the **effectful** predicate only has one form.

Figure 4.10: Syntax of algebraic effects with static instances

$$\nu ::= x, y, z, k \mid \lambda x.c \mid () \mid \boxed{\iota}$$

$$c ::= \mathsf{return}\ \nu \mid \nu\ \nu \mid x \leftarrow c; c \mid \boxed{\nu\#op(\nu) \mid \mathsf{handle}^{\nu}(c)\{h\}}$$

$$h ::= op\ x\ k \rightarrow c;\ h \mid \mathsf{return}\ x \rightarrow c$$

4.3 Static instances

Finally extend algebraic effects with static instances. We assume there exists a set of instances $I = \{\iota_1, ..., \iota_n\}$, where each instance belongs to a single effect ϵ, written as $E[\iota] = \epsilon$.

Syntax The syntax of the system with algebraic effects and handlers is extended in Figure 4.10, changes and new additions are shown in gray. For values we add instances, these are taking from the set of instances I. We also change the operation call and handle computations to take an extra value term, which is the instance they are operating on.

Semantics The semantics for static instances are shown in Figure 4.11. The rules from algebraic effects that did not change are left out of this figure. The rules S-OP, S-HANDLERETURN and S-HANDLE are, except for the change in syntax with the addition of the value term, identical to the corresponding rules in the previous system. For static instances the only important change is in the S-HANDLEOP and S-HANDLEOPSKIP rules. In S-HANDLEOP the instance in the handle and the instance in the operation call have to be the same, besides this the rule is the same as the corresponding rule in the previous system. If the instances do not match or if the operation is not in the handler then the rule S-HANDLEOPSKIP is used to lift the operation call over the handler, also like in the previous system.

Type syntax The updated syntax for types is shown in Figure 4.12. We add instances types, which are just instance names from the set I. The effect

Figure 4.11: Semantics of algebraic effects with static instances

S-Op

$$\overline{\nu_1 \# op(\nu_2) \rightsquigarrow (x \leftarrow \nu_1 \# op(\nu_2); \mathsf{return}\ x)}$$

S-HandleReturn

$$\overline{\mathsf{handle}^{\nu_1}(\mathsf{return}\ \nu_2)\{h; \mathsf{return}\ x \rightarrow c\} \rightsquigarrow c[x := \nu_2]}$$

S-HandleOp

$$\frac{op\ x\ k \rightarrow c' \in h}{\mathsf{handle}^{\iota}(y \leftarrow \iota \# op(\nu); c)\{h\} \rightsquigarrow c'[x := \nu, k := (\lambda y.\mathsf{handle}^{\iota}(c)\{h\})]}$$

S-HandleOpSkip

$$\frac{op \notin h \wedge \iota_1 \neq \iota_2}{\mathsf{handle}^{\iota_1}(x \leftarrow \iota_2 \# op(\nu); c)\{h\} \rightsquigarrow (x \leftarrow \iota_2 \# op(\nu); \mathsf{handle}^{\iota_1}(c)\{h\})}$$

S-Handle

$$\frac{c \rightsquigarrow c'}{\mathsf{handle}^{\nu}(c)\{h\} \rightsquigarrow \mathsf{handle}^{\nu}(c')\{h\}}$$

Figure 4.12: Types of algebraic effects with static instances

$$\tau ::= () \mid \boxed{\mathsf{inst}(\iota)} \mid \tau \to \underline{\tau}$$

$$\underline{\tau} ::= \tau \mathbin{!} r$$

$$\boxed{r ::= \{\iota_1, ..., \iota_n\}}$$

Figure 4.13: Subtyping rules of algebraic effects with static instances

$$\textsc{Sub-Inst}$$
$$\overline{\mathsf{inst}(\iota) <: \mathsf{inst}(\iota)}$$

annotation on the computation types are now sets of instance names instead effect names.

Subtyping   For subtyping we keep the rules from the previous system but we add a rule for the instance types (Figure 4.13).

Typing rules   The typing rules from the previous system mostly stay the same except for the rules T-OP and T-HANDLE, they are showin in Figure 4.14. We also had a rule to type instances (T-INST), this rule simply types an instance as a instance type with the same name. For both T-OP and T-HANDLE we just have to check that the added value term is an instance and that the effect of that instance matches the operations.

Type soundness   We give a type soundness theorem for the system with static instances.

Theorem 3 (Type soundness)**.**

$$\text{if } \cdot \vdash c : \tau \mathbin{!} \varnothing \ \wedge \ c \rightsquigarrow^* c' \text{ then } \mathsf{value}(c') \vee (\exists c''.\ c' \rightsquigarrow c'')$$

Figure 4.14: Typing rules of algebraic effects with static instances

T-OP

T-INST

$$\dfrac{}{\Gamma \vdash \iota : \mathsf{inst}(\iota)}$$

$$\dfrac{\Gamma \vdash \nu_1 : \mathsf{inst}(\iota) \qquad}{E[\iota] = \epsilon \qquad \Gamma[op] = (\epsilon, \tau_{op}^1, \tau_{op}^2) \qquad \Gamma \vdash \nu_2 : \tau_{op}^1}{\Gamma \vdash \nu_1 \# op(\nu_2) : \tau_{op}^2 \ ! \ \{\iota\}}$$

T-HANDLE

$$\dfrac{\Gamma \vdash c : \tau_1 \ ! \ r_1 \qquad \Gamma \vdash \nu : \mathsf{inst}(\iota) \qquad E[\iota] = \epsilon}{op \in h \Leftrightarrow op \in O^\epsilon \qquad \Gamma \vdash^{\tau_1} h : \tau_2 \ ! \ r_2}{\Gamma \vdash \mathsf{handle}^\nu(c)\{h\} : \tau_2 \ ! \ ((r_1 \setminus \{\iota\}) \cup r_2)}$$

This theorem is exactly the same as the one for algebraic effects in Section 4.2. We can reuse the notion of a computation being a value **value** without any modifications.

Again we use progress and preservation lemmas to prove the type soundness theorem. These lemmas are exactly the same as the ones in Section 4.2. We only need to slightly update our **effectful** notion, because operation calls are now called on instances.

$$\mathsf{effectful}(c) \text{ if } (\exists \iota \ op \ \nu.c = \iota \# op(\nu)) \vee (\exists \iota \ x \ op \ \nu \ c'.c = x \leftarrow \iota \# op(\nu); c')$$

We formalized the system with static instances and have proven the type soundness theorem in the Coq proof assistant. The formalization follows closely the one for algebraic effects without static instances. We only formalized the system with handlers with one operation. Again we use a system that follows the one used by Bauer and Pretnar [2]. Static instances do not add much complexity compared to algebraic effects.

Chapter 5

Semantics and types of X

In this chapter we give a formal account of X. We give the syntax, typing rules and a small-step operation semantics. We end the chapter with a type soundness theorem. The system builds on the formal system with algebraic effects, handlers and static instances of Chapter 4.3. We add constructs to handle effect scope polymorphism, to create new instances, and to handle effect scopes, as informally described in Chapter 3.

In Section 5.1 we give the syntax of the terms and types of X. In Section 5.2 we give the environments and judgments used in the typing rules and semantics. In Section 5.3 we give subtyping rules for the types. In Section 5.4 we give well-formedness rules for the types. In Section 5.5 we give the typing rules. In Section 5.6 we give a small-step operation semantics for X. Finally in Section 5.7 we give a type-soundness theorem.

5.1 Syntax

Just like in the formal systems of algebraic effects of Chapter 4.2 and Chapter 4.3 we assume there is set of effect names $\mathsf{EffName}$ with $E \subseteq \mathsf{EffName}$. For example $E = \{\mathsf{Flip}, \mathsf{State}, \mathsf{Exc}, ...\}$. There is also a set of operation names O. Each effect $\varepsilon \in E$ has a non-empty set of operation names $O^\varepsilon \in O$. For example $O^{\mathsf{Flip}} = \{\mathsf{flip}\}$ and $O^{\mathsf{State}} = \{\mathsf{get}, \mathsf{put}\}$. Every operation name only

Figure 5.1: Syntax

$$
\begin{aligned}
s &::= s_{var} \mid s_{loc} \\
\tau &::= \mathsf{Inst}\ s\ \varepsilon \mid \tau \to \underline{\tau} \mid \forall s_{var}.\underline{\tau} \\
\underline{\tau} &::= \tau\ !\ r \\
\nu &::= x, y, z, k \mid \mathsf{inst}(l) \mid \lambda x.c \mid \Lambda s_{var}.c \\
c &::= \mathsf{return}\ \nu \mid \nu\ \nu \mid x \leftarrow c;\ c \mid \nu\#op(\nu) \mid \nu\ [s] \\
&\quad \mid\ \mathsf{new}\ \varepsilon@s\ \{h; \mathsf{finally}\ x \to c\}\ \mathsf{as}\ x\ \mathsf{in}\ c \\
&\quad \mid\ \mathsf{runscope}(s_{var} \to c) \\
&\quad \mid\ \mathsf{runscope}^{s_{loc}}(c) \\
&\quad \mid\ \mathsf{runinst}^{l}\{h\}(c) \\
h &::= op\ x\ k \to c; h \mid \mathsf{return}\ x \to c
\end{aligned}
$$

corresponds to a single effect. Each operation op has a parameter type τ^1_{op} and a return type τ^2_{op}.

In Figure 5.1 we show the syntax of the types and terms of X. An effect scope s is either a scope variable s_{var} or a scope location s_{loc}. Effect scope variables s_{var} and effects scope locations s_{loc} are both modeled by countable infinite sets.

Like in the systems in Chapter 4 terms and types are both split between values and computations, and value types and computation types. Values are typed by value types and computations are typed by computation types.

Value types τ are either an instance type $\mathsf{Inst}\ s\ \varepsilon$, indexed by an effect scope s and an effect ε. Or a function type $\tau \to \underline{\tau}$ where the parameter type is a value type and the return type is a computation type. Or an universally quantified computation type $\forall s_{var}.\underline{\tau}$, where the domain of quantification are effect scopes.

A computation type $\underline{\tau}$ is always an annotated value type of the form $\tau\ !\ r$. Annotations r are sets of effect scopes $\{s_1, ..., s_n\}$.

Values are either variables x, k, where we always use k do denote variables that refer to continuations. Or instances $\mathsf{inst}(l)$, indexed by some instance location l, which are modeled by some countable infinite set. Instances would not appear in the surface language, but are introduced by the semantics. Or lambda abstractions $\lambda x.c$, where the body is a computation. Or effect scope abstractions $\Lambda s_{var}.c$, where we abstract over a computation c, with the domain of the quantification being effect scopes.

As usual, for computations we have $\mathsf{return}\ \nu$, to lift a value ν in to a computation. We have application $\nu\ \nu$ and sequencing $x \leftarrow c;\ c$. We have operation calls $\nu\#op(\nu)$. The new constructs are as follows. We have effect scope application $\nu\ [s]$. We can create new instances with $\mathsf{new}\ \varepsilon@s\ \{h; \mathsf{finally}\ x \rightarrow c\}\ \mathsf{as}\ x\ \mathsf{in}\ c$, where h is a handler. We can handle computations with $\mathsf{runscope}(s_{var} \rightarrow c)$. Finally we have two more constructs which would not appear in the surface language, but are introduced by the semantics. Effect scope handlers $\mathsf{runscope}^{s_{loc}}(c)$ handle a specific scope s_{loc} in the computation c. Instance handlers $\mathsf{runinst}^l\{h\}(c)$ handle the operations of a single instance of the location l in the computation c.

Finally we have handlers h which are lists of operation cases ending with a return case. Operation cases are of the form $op\ x\ k \rightarrow c; h$, where h is the rest of the handler. Return cases are of the form $\mathsf{return}\ x \rightarrow c$.

5.2 Environments and judgments

We will now give the environment and judgments used in the typing rules and semantics.

Environments The syntax for the environments is shown in Figure 5.2. As usual we have Γ, like in the systems of Chapter 4. Δ is introduced to handle the effect scope polymorphism. Σ is introduced to handle the fresh generation of scope and instance locations.

- Γ is the typing enviroment which assigns variables x to value types τ.

Figure 5.2: Environments

$$\Gamma ::= \cdot \mid \Gamma, x : \tau$$
$$\Delta ::= \cdot \mid \Delta, s_{var}$$
$$\Sigma ::= \cdot \mid \Sigma, s_{loc} \mid \Sigma, l := (s_{loc}, \varepsilon)$$

- Δ is the effect scope variable environment which keeps track of the scope variables s_{var} that are in use.
- Σ is the dynamic environment which keeps track of scope location s_{loc} and instance locations l. Instance locations are assigned a tuple of a scope location and an effect (s_{loc}, ε). Σ is used in both the typing rules and the operational semantics. We do not need a separate environment for the semantics because instances are not assigned dynamic values like references are. Σ enables us to type instances in the typing rules, and to generate fresh scope and instance locations in the semantics. Because there is a dependence of instance locations on scope locations we also introduce a well-formedness judgment for Σ.

Judgments   There are four kinds of judgments: subtyping (Section 5.3), well-formedness (Section **??**) and typing (Section 5.5).

The subtyping judgments are used to weaken the effect annotation of a computation type. Weakening the effect annotation is sometimes necessary in order to type a program. For example when typing the sequencing of two computations $x \leftarrow c_1;\ c_2$, if the two computations do not agree on the effects then subtyping can be used to weaken both the computations such that the effect annotations agree. There is a subtyping judgment for both the value types τ and the computation types $\underline{\tau}$ these mutually depend on one another:

- $\tau <: \tau'$ holds when the value type τ is a subtype of τ'.

- $\underline{\tau} <: \underline{\tau}'$ holds when the computation type $\underline{\tau}$ is a subtype of $\underline{\tau}'$.

We have a well-formedness judgment $\vdash \Sigma$ which makes sure instance locations

in Σ do not depend on an non-existant scope locations and that locations are unique. The well-formedness judgments for types assert that scopes used in the types are valid under the scope variable and dynamic environments. The well-formedness judgments have the following form:

- $\vdash \Sigma$ asserts that all instance locations in Σ refer to valid scope locations and that all locations are unique.

- $\Delta; \Sigma \vdash s$ asserts that the scope s is either in Δ if it is a scope variable or else in Σ if it is a scope location.

- $\Delta; \Sigma \vdash \tau$ asserts that all the scopes in the value type τ are valid under the environments Δ and Σ.

- $\Delta; \Sigma \vdash \underline{\tau}$ asserts that all the scopes in the computation type $\underline{\tau}$ are valid under the environments Δ and Σ. This judgment is only used in the theorems at the end of the Chapter.

Lastly there are three typing judgments:

- $\Delta; \Sigma; \Gamma \vdash \nu : \tau$ asserts that the value ν has the value type τ under the Δ, Σ and Γ environments.

- $\Delta; \Sigma; \Gamma \vdash c : \underline{\tau}$ asserts that the computation c has the computation type $\underline{\tau}$ under the Δ, Σ and Γ environments.

- $\Delta; \Sigma; \Gamma \vdash^{\tau} h : \underline{\tau}$ asserts that the handler h transform a return value of type τ to the computation type $\underline{\tau}$.

5.3 Subtyping

In Figure 5.3 we give the subtyping rules for both the value and the computation types. The subtyping checks that that the effects mentioned in the type on the right are the same or more general than the type on the left. An instance type Inst s ε is a subtype of another instance type if they are structurally equal, shown in the rule SUB-INST. Function types $\tau \rightarrow \underline{\tau}$ are compared by subtyping the parameter types contravariantly and subtyping the return types covariantly, shown in the rule SUB-ARR. Universally quantified types $\forall s_{var}.\underline{\tau}$ are structurally recursed upon, given they the quantified variables are equal (SUB-FORALL). Lastly annotated types $\tau \mathbin{!} r$ are com-

Figure 5.3: Subtyping

$$
\begin{array}{c}
\text{SUB-INST} \\[4pt]
\hline
\textsf{Inst } s \ \varepsilon <: \textsf{Inst } s \ \varepsilon
\end{array}
\qquad
\begin{array}{c}
\text{SUB-ARR} \\[2pt]
\tau_2 <: \tau_1 \qquad \underline{\tau}_1 <: \underline{\tau}_2 \\[2pt]
\hline
\tau_1 \to \underline{\tau}_1 <: \tau_2 \to \underline{\tau}_2
\end{array}
$$

$$
\begin{array}{c}
\text{SUB-FORALL} \\[2pt]
\underline{\tau}_1 <: \underline{\tau}_2 \\[2pt]
\hline
\forall s_{var}.\underline{\tau}_1 <: \forall s_{var}.\underline{\tau}_2
\end{array}
\qquad
\begin{array}{c}
\text{SUB-ANNOT} \\[2pt]
\tau_1 <: \tau_2 \qquad r_1 \subseteq r_2 \\[2pt]
\hline
\tau_1 \, ! \, r_1 <: \tau_2 \, ! \, r_2
\end{array}
$$

pared by comparing the value types and checking that the annotation on the left type is a subtype of the annotation on the right type (SUB-ANNOT).

5.4   Well-formedness

Figure 5.4: Well-formedness for Σ

$$
\begin{array}{c}
\text{WFS-EMPTY} \\[2pt]
\hline
\vdash \cdot
\end{array}
\qquad
\begin{array}{c}
\text{WFS-SCOPELOC} \\[2pt]
\vdash \Sigma \qquad s_{loc} \notin \Sigma \\[2pt]
\hline
\vdash \Sigma, s_{loc}
\end{array}
\qquad
\begin{array}{c}
\text{WFS-INSTANCELOC} \\[2pt]
\vdash \Sigma \qquad l \notin \Sigma \qquad s_{loc} \in \Sigma \\[2pt]
\hline
\vdash \Sigma, l := (s_{loc}, \varepsilon)
\end{array}
$$

In Figure 5.4 we give the well-formedness rules for Σ. If Σ is empty then it is well-formed (WFS-EMPTY). For scope locations s_{loc} we check that the rest of Σ is well-formed and that s_{loc} does not occur in it, meaning that s_{loc} is unique (WFS-SCOPELOC). For instance locations l we also first check that the rest of Σ is well-formed. Then we check that l does not occur in Σ. Lastly we check that the scope location s_{loc} used by l is defined in Σ.

In Figure 5.5 we give the well-formedness rules for the value and computation types. Well-formedness asserts that the effect scopes in the type are accounted for in the environments. The rules WF-SVAR and WF-SLOC as-

Figure 5.5: Well-formedness for types

$$
\text{WF-SVAR} \quad \frac{s_{var} \in \Delta}{\Delta; \Sigma \vdash s_{var}}
\qquad
\text{WF-SLOC} \quad \frac{s_{loc} \in \Sigma}{\Delta; \Sigma \vdash s_{loc}}
\qquad
\text{WF-INST} \quad \frac{\Delta; \Sigma \vdash s}{\Delta; \Sigma \vdash \mathsf{Inst}\ s\ \varepsilon}
$$

$$
\text{WF-ARR} \quad \frac{\Delta; \Sigma \vdash \tau \qquad \Delta; \Sigma \vdash \underline{\tau}}{\Delta; \Sigma \vdash \tau \to \underline{\tau}}
\qquad
\text{WF-FORALL} \quad \frac{\Delta, s_{var}; \Sigma \vdash \underline{\tau}}{\Delta; \Sigma \vdash \forall s_{var}.\underline{\tau}}
$$

$$
\text{WF-ANNOT} \quad \frac{\Delta; \Sigma \vdash \tau \qquad \forall (s \in r) \Rightarrow \Delta; \Sigma \vdash s}{\Delta; \Sigma \vdash \tau\ !\ r}
$$

sert that the effect scope variables s_{var} and locations s_{loc} are valid by checking that they are contained in the scope and dynamic environments respectively. For instance types we check that the mentioned effect scope is valid (WF-INST). For function types $\tau \to \underline{\tau}$ we check that both the paramter and return type is valid. For universally quantified types $\forall s_{var}.\underline{\tau}$ we check that the computation type $\underline{\tau}$ is valid, after adding the variable s_{var} to the environment. Lastly for annotated types $\tau\ !\ r$ we first check that that the value type τ is valid. Then we check that each effect scope in the annotation r is valid.

5.5 Typing rules

The typing rules for the values are given in Figure 5.6. The rules T-VAR, T-ABS and T-SUBVAL are practically unchanged from the corresponding rules in the algebraic effects type system from Chapter 4.2. Instances $\mathsf{inst}(l)$ are assigned instance types, with the scope location and effect looked up in the dynamic environment Σ using the location l (T-INST). Similar to abstractions, effect scope abstractions are assigned a universally quantified type $\forall s_{var}.\underline{\tau}$, by typing the body with $\underline{\tau}$ after adding s_{var} to the scope environment Δ (T-TABS).

Figure 5.6: Value typing rules

T-VAR
$$\frac{\Gamma[x] = \tau}{\Delta; \Sigma; \Gamma \vdash x : \tau}$$

T-INST
$$\frac{\Sigma(l) = (s_{loc}, \varepsilon)}{\Delta; \Sigma; \Gamma \vdash \mathsf{inst}(l) : \mathsf{Inst}\ s_{loc}\ \varepsilon}$$

T-ABS
$$\frac{\Delta; \Sigma; \Gamma, x : \tau \vdash c : \underline{\tau}}{\Delta; \Sigma; \Gamma \vdash \lambda x.c : \tau \to \underline{\tau}}$$

T-TABS
$$\frac{\Delta, s_{var}; \Sigma; \Gamma \vdash c : \underline{\tau}}{\Delta; \Sigma; \Gamma \vdash \Lambda s_{var}.c : \forall s_{var}.\underline{\tau}}$$

T-SUBVAL
$$\frac{\Delta; \Sigma; \Gamma \vdash \nu : \tau_1 \qquad \Delta; \Sigma \vdash \tau_2 \qquad \tau_1 <: \tau_2}{\Delta; \Sigma; \Gamma \vdash \nu : \tau_2}$$

The typing rule for the computations are given in Figure 5.7. The rules T-RETURN, T-APP, T-SEQ and T-SUBCOMP are practically unchanged from the corresponding rules in the algebraic effects type system from Chapter 4.2.

The rule for effect scope application (T-TAPP) asserts that, in the application $\nu\ [s]$, the scope s is well-formed. Then we check that ν is a universally quantified type $\forall s'.\underline{\tau}$. Finally we substitute the given scope s for the quantified variable s' in $\underline{\tau}$.

For operation calls (T-OP) $\nu_1 \# op(\nu_2)$ we first check that the type of ν_1 is an instance type $\mathsf{Inst}\ s\ \varepsilon$. We then check that the operation op is an operation of the effect of the instance ε ($op \in O^\varepsilon$). Lastly we check that the given value ν_2 matches the parameter type of the operation op (τ_{op}^1). The type given to the operation call is the return type of the operation τ_{op}^2 with the scope of the instance, s, in the annotation.

Figure 5.7: Computation typing rules

T-Return
$$\frac{\Delta; \Sigma; \Gamma \vdash \nu : \tau}{\Delta; \Sigma; \Gamma \vdash \text{return } \nu : \tau \,!\, \varnothing}$$

T-App
$$\frac{\Delta; \Sigma; \Gamma \vdash \nu_1 : \tau \to \underline{\tau} \qquad \Delta; \Sigma; \Gamma \vdash \nu_2 : \tau}{\Delta; \Sigma; \Gamma \vdash \nu_1 \, \nu_2 : \underline{\tau}}$$

T-TApp
$$\frac{\Delta; \Sigma \vdash s \qquad \Delta; \Sigma; \Gamma \vdash \nu : \forall s'_{var}.\underline{\tau}}{\Delta; \Sigma; \Gamma \vdash \nu \,[s] : \underline{\tau}[s'_{var} := s]}$$

T-Seq
$$\frac{\Delta; \Sigma; \Gamma \vdash c_1 : \tau_1 \,!\, r \qquad \Delta; \Sigma; \Gamma, x : \tau_1 \vdash c_2 : \tau_2 \,!\, r}{\Delta; \Sigma; \Gamma \vdash (x \leftarrow c_1; \; c_2) : \tau_2 \,!\, r}$$

T-Op
$$\frac{\Delta; \Sigma; \Gamma \vdash \nu_1 : \text{Inst } s \; \varepsilon \qquad op \in O^\varepsilon \qquad \Delta; \Sigma; \Gamma \vdash \nu_2 : \tau^1_{op}}{\Delta; \Sigma; \Gamma \vdash \nu_1 \# op(\nu_2) : \tau^2_{op} \,!\, \{s\}}$$

T-New
$$\frac{\Delta; \Sigma \vdash s \qquad op \in O^\varepsilon \iff op \in h \qquad \Delta; \Sigma; \Gamma, x : \text{Inst } s \; \varepsilon \vdash c : \tau_1 \,!\, r \qquad \Delta; \Sigma; \Gamma \vdash^{\tau_1} h : \tau_2 \,!\, r \qquad s \in r \qquad \Delta; \Sigma; \Gamma, y : \tau_2 \vdash c' : \tau_3 \,!\, r}{\Delta; \Sigma; \Gamma \vdash \text{new } \varepsilon@s \; \{h; \text{finally } y \to c'\} \text{ as } x \text{ in } c : \tau_3 \,!\, r}$$

T-Handle
$$\frac{\Delta, s_{var}; \Sigma; \Gamma \vdash c : \tau \,!\, r \qquad s_{var} \notin \tau}{\Delta; \Sigma; \Gamma \vdash \text{runscope}(s_{var} \to c) : \tau \,!\, (r \setminus \{s_{var}\})}$$

T-HandleScope
$$\frac{s_{loc} \in \Sigma \qquad \Delta; \Sigma; \Gamma \vdash c : \tau \,!\, r \qquad s_{loc} \notin \tau}{\Delta; \Sigma; \Gamma \vdash \text{runscope}^{s_{loc}}(c) : \tau \,!\, (r \setminus \{s_{loc}\})}$$

T-HandleInst
$$\frac{\Sigma(l) = (s_{loc}, \varepsilon) \qquad op \in O^\varepsilon \iff op \in h \qquad \Delta; \Sigma; \Gamma \vdash c : \tau_1 \,!\, r \qquad \Delta; \Sigma; \Gamma \vdash^{\tau_1} h : \tau_2 \,!\, r}{\Delta; \Sigma; \Gamma \vdash \text{runinst}^l\{h\}(c) : \tau_2 \,!\, r}$$

T-SubComp
$$\frac{\Delta; \Sigma; \Gamma \vdash c : \underline{\tau}_1 \qquad \Delta; \Sigma \vdash \underline{\tau}_2 \qquad \underline{\tau}_1 <: \underline{\tau}_2}{\Delta; \Sigma; \Gamma \vdash c : \underline{\tau}_2}$$

The rule T-New types the creation of new instance: new $\varepsilon@s\ \{h;$ finally $y \to c'\}$ as x in c. We are creating a new instance x of effect ε in the effect scope s. First we check that the given effect scope s is valid. Then we check that the operations in the handler h are exactly the operations of the effect of the new instance ε. This means it is not valid to either omit operations or to have operations of other effects in the handler. This way we can ensure that every operation is accounted for. We then typecheck the computation c with the new instance added the to the environment as x, with the type $\tau_1\ !\ r$. Then we typecheck the handler h, passing the type τ_1 of c to the handler typing judgement. The handler can transform the return type τ_1 to another type τ_2, but note that the effects annotation r has to be the same. We check that the scope s is contained in r. The reason for this is because the creation of an instance in s is also an effect and so we have to account for this effect in the annotation by adding s to r. Lastly we check the finally case. We add τ_2 to the environment as y and typecheck the computation c' as $\tau_3\ !\ r$. The finally case is allowed to transform the return type of the handler τ_2 to another type τ_3. Finally the type of the whole instance creation is $\tau_3\ !\ r$. Where τ_3 is the final type of the finally case and r is the effects from c, the handler h and the finally case.

The rule T-Handle types handling computations: runscope($s_{var} \to c$). We typecheck the body c with the effect scope variable s_{var} added to the scope environment. Then we check that the effects do not escape their scope by checking that that s_{var} is not contained in the return type τ. We then type the whole computation as $\tau\ !\ (r \setminus \{s_{var}\})$. Knowing that s_{var} does not escape we can safely remove it from the effect annotation.

The rule T-HandleScope deals with the handling of a specific scope s_{loc} and is very similar to the previous rule T-Handle. Instead of the effect scope *variable* s_{var} we now deal with an effect scope *location* s_{loc}. We have to check that the location actually exists, by checking that it is contained in the dynamic environment Σ. We can then proceed like in T-Handle, checking that s_{loc} does not escape.

T-HandleInst typechecks the handling of an instance at location l using a handler h. First we check that the *location* is contained in the dynamic environment Σ We retrieve the scope location s_{loc} and effect ε of the instance from Σ. We check that the operations in the handler match the operations

of the effect. We then typecheck the computation c and the handler h like in T-New.

Figure 5.8: Handler typing rules

T-HandlerOp
$$\frac{\Delta; \Sigma; \Gamma \vdash^{\tau_1} h : \tau_2 ! r \qquad \Delta; \Sigma; \Gamma, x : \tau_{op}^1, k : \tau_{op}^2 \to \tau_2 ! r \vdash c : \tau_2 ! r}{\Delta; \Sigma; \Gamma \vdash^{\tau_1} (op\ x\ k \to c; h) : \tau_2 ! r}$$

T-HandlerReturn
$$\frac{\Delta; \Sigma; \Gamma, x : \tau_1 \vdash c : \tau_2 ! r}{\Delta; \Sigma; \Gamma \vdash^{\tau_1} (\mathsf{return}\ x \to c) : \tau_2 ! r}$$

Finally we discuss the typing rules for the handlers, given in Figure 5.8. T-HandlerReturn types the return case of a handler. We typecheck the body c after adding the variable x with type τ_1 to the environment. The type τ_1 is passed with the handler typing judgement in the typing rules T-New and T-HandleInst. It is the return type of the computation we are handling. The computation c can transform this type to another type τ_2 with some effect annotation r, which is the return type of the return case.

The rule T-HandlerOp shows the typing of an operation case. We first typecheck the rest of handler h, passing along the return type τ_1 of the computation we are handling. We typecheck the rest of the handler as $\tau_2 ! r$, this is the return type of the whole handler. Then we typecheck the body of the operation case c. We add the operation call argument x with the parameter type τ_{op}^1 of the operation to the environment. We also add the continuation k to the environment. This is a function from the return type τ_{op}^2 of the operation, to the return type of the whole handler $\tau_2 ! r$. We check that the body of the case c returns a computation of the same type.

5.6   Semantics

Finally we give a small-step operational semantics for X.

The judgment $c_1 \mid \Sigma_1 \rightsquigarrow c_2 \mid \Sigma_2$ takes a step from the computation c_1 to c_2 in the dynamic environment Σ_1. Σ_1 might be updated with new effect scope locations or instance locations resulting in Σ_2. Σ will always be monotically extended by $\rightsquigarrow$.

Figure 5.9: Semantics

S-APP

$$\overline{(\lambda x.c)\, \nu \mid \Sigma \rightsquigarrow c[x := \nu] \mid \Sigma}$$

S-TAPP

$$\overline{(\Lambda s_{var}.c)\, [s'] \mid \Sigma \rightsquigarrow c[s_{var} := s'] \mid \Sigma}$$

S-SEQ

$$\frac{c_1; \Sigma \rightsquigarrow c_1'; \Sigma'}{(x \leftarrow c_1;\ c_2) \mid \Sigma \rightsquigarrow (x \leftarrow c_1';\ c_2) \mid \Sigma'}$$

S-SEQRETURN

$$\overline{(x \leftarrow (\mathsf{return}\ \nu);\ c) \mid \Sigma \rightsquigarrow c[x := \nu] \mid \Sigma}$$

S-FLATTEN

$$\overline{(y \leftarrow (x \leftarrow c_1;\ c_2);\ c_3 \mid \Sigma \rightsquigarrow (x \leftarrow c_1;\ y \leftarrow c_2;\ c_3) \mid \Sigma}$$

S-LIFTNEW

$$\overline{\begin{array}{c}(x \leftarrow (\mathsf{new}\ \varepsilon@s\ \{h; \mathsf{finally}\ z \to c_3\}\ \mathsf{as}\ y\ \mathsf{in}\ c_1);\ c_2) \mid \Sigma \rightsquigarrow \\ \mathsf{new}\ \varepsilon@s\ \{h; \mathsf{finally}\ z \to c_3\}\ \mathsf{as}\ y\ \mathsf{in}\ (x \leftarrow c_1;\ c_2) \mid \Sigma\end{array}}$$

S-FRESHSCOPE

$$\frac{s_{loc} \notin \Sigma}{\mathsf{runscope}(s_{var} \to c) \mid \Sigma \rightsquigarrow \mathsf{runscope}^{s_{loc}}(c[s_{var} := s_{loc}]) \mid \Sigma, s_{loc}}$$

In Figure 5.9 we give the semantics for every construct except the effect scope and instance handlers. The rules S-APP, S-SEQ, S-SEQRETURN and

S-Flatten are the same as the corresponding rules in the algebraic effects system of Chapter 4.2. The rule S-TApp handles an effect scope application similarly to a normal application, by substituting the effect scope s' for the scope variable s_{var} of the scope abstraction. S-LiftNew lifts the creation of an instance out and over sequencing. By repeatedly applying this rule we can bubble up the instance creation until we hit an effect scope handler The rule S-FreshScope creates a fresh scope location with which to handle instances. A fresh scope location s_{loc} is created and added to the environment Σ. This new scope location is substituted in the body c. We wrap the computation with an effect scope handler for new scope location s_{loc}.

In Figure 5.10 we give the semantics for the effect scope handlers. An effect scope handler for a specific effect scope location will create fresh instances when a **new** construct is encountered (S-HandleScopeNew). An instance handler with the handler of the **new** construct is wrapped around the computation and the **finally** case is wrapped around that. The newly created location l is added to the dynamic environment together with the scope location and the effect. If a **new** is encountered with a different effect scope we skip it and nest the scope handler inside (T-HandleScopeNewSkip). Using the rule S-HandleScopeCong to reduce a computation inside a scope handler. We can remove a scope handler if we encounter either a return or operation call (S-HandleScopeReturn and S-HandleScopeOp). Effect scope handlers can be pushed inside sequencing, lifting an operation call over it (S-HandleScopeSeqOp).

Lastly in Figure 5.11 we give the semantics for the instance handlers. Instance handlers handle operation calls on instances with the same location as the handler. To be able to handle operation calls with one rule we first have to transform operation calls that are not being sequenced to the sequencing form (S-HandleInstOpPrepare). When an operation call is encountered on an instance with the same location as the instance handler, the operation is handled (S-HandleInstOp). The operation is looked up in the handler h and the computation c_{op} in the operation case is performed. If a **return** is encountered the computation c_r in the **return** case is performed (S-HandleInstReturn). Operation calls on instances with a different location l' are skipped, nesting the instance handler inside (S-HandleInstOpSkip). Similarly **new** calls are also skipped, again nesting the instance handler inside (S-HandleInstNew). Lastly, computations in-

side instance handlers can be reduced further (S-HANDLEINSTCONG).

Figure 5.10: Semantics of effect scope handlers

S-HANDLESCOPECONG
$$\frac{c \mid \Sigma \leadsto c' \mid \Sigma'}{\mathsf{runscope}^{s_{loc}}(c) \mid \Sigma \leadsto \mathsf{runscope}^{s_{loc}}(c') \mid \Sigma'}$$

S-HANDLESCOPERETURN
$$\frac{}{\mathsf{runscope}^{s_{loc}}(\mathsf{return}\ \nu) \mid \Sigma \leadsto \mathsf{return}\ \nu \mid \Sigma}$$

S-HANDLESCOPEOP
$$\frac{}{\mathsf{runscope}^{s_{loc}}(\nu_1 \# op(\nu_2)) \mid \Sigma \leadsto \nu_1 \# op(\nu_2) \mid \Sigma}$$

S-HANDLESCOPESEQOP
$$\frac{}{\begin{array}{l}\mathsf{runscope}^{s_{loc}}(x \leftarrow \nu_1 \# op(\nu_2);\ c) \mid \Sigma \leadsto \\ (x \leftarrow \nu_1 \# op(\nu_2);\ \mathsf{runscope}^{s_{loc}}(c)) \mid \Sigma\end{array}}$$

S-HANDLESCOPENEWSKIP
$$\frac{s_{loc} \neq s'_{loc}}{\begin{array}{l}\mathsf{runscope}^{s_{loc}}(\mathsf{new}\ \varepsilon @ s'_{loc}\ \{h; \mathsf{finally}\ y \rightarrow c'\}\ \mathsf{as}\ x\ \mathsf{in}\ c) \mid \Sigma \leadsto \\ \mathsf{new}\ \varepsilon @ s'_{loc}\ \{h; \mathsf{finally}\ y \rightarrow c'\}\ \mathsf{as}\ x\ \mathsf{in}\ \mathsf{runscope}^{s_{loc}}(c) \mid \Sigma\end{array}}$$

S-HANDLESCOPENEW
$$\frac{l \notin \mathsf{Dom}(\Sigma)}{\begin{array}{l}\mathsf{runscope}^{s_{loc}}(\mathsf{new}\ \varepsilon @ s_{loc}\ \{h; \mathsf{finally}\ y \rightarrow c'\}\ \mathsf{as}\ x\ \mathsf{in}\ c) \mid \Sigma \leadsto \\ \mathsf{runscope}^{s_{loc}}(y \leftarrow \mathsf{runinst}^{l}\{h\}(c[x := \mathsf{inst}(l)]);\ c') \mid \Sigma, l := (s_{loc}, \varepsilon)\end{array}}$$

Figure 5.11: Semantics of instance handlers

S-HANDLEINSTCONG

$$\frac{c \mid \Sigma \leadsto c' \mid \Sigma'}{\mathsf{runinst}^l\{h\}(c) \mid \Sigma \leadsto \mathsf{runinst}^l\{h\}(c') \mid \Sigma'}$$

S-HANDLEINSTNEW

$$\frac{}{\begin{array}{l}\mathsf{runinst}^l\{h\}(\mathsf{new}\ \varepsilon@s\ \{h'; \mathsf{finally}\ y \to c'\}\ \mathsf{as}\ x\ \mathsf{in}\ c) \mid \Sigma \leadsto \\ \quad \mathsf{new}\ \varepsilon@s\ \{h'; \mathsf{finally}\ y \to c'\}\ \mathsf{as}\ x\ \mathsf{in}\ \mathsf{runinst}^l\{h\}(c) \mid \Sigma\end{array}}$$

S-HANDLEINSTOPPREPARE

$$\frac{}{\begin{array}{l}\mathsf{runinst}^l\{h\}(\nu_1\#op(\nu_2)) \mid \Sigma \leadsto \\ \mathsf{runinst}^l\{h\}(x \leftarrow \nu_1\#op(\nu_2);\ \mathsf{return}\ x) \mid \Sigma\end{array}}$$

S-HANDLEINSTOPSKIP

$$\frac{l \neq l'}{\begin{array}{l}\mathsf{runinst}^l\{h\}(x \leftarrow \mathsf{inst}(l')\#op(\nu);\ c) \mid \Sigma \leadsto \\ (x \leftarrow \mathsf{inst}(l')\#op(\nu);\ \mathsf{runinst}^l\{h\}(c)) \mid \Sigma\end{array}}$$

S-HANDLEINSTOP

$$\frac{h[op] = (x, k, c_{op})}{\begin{array}{l}\mathsf{runinst}^l\{h\}(y \leftarrow \mathsf{inst}(l)\#op(\nu);\ c) \mid \Sigma \leadsto \\ c_{op}[x := \nu, k := (\lambda y.\mathsf{runinst}^l\{h\}(c))] \mid \Sigma\end{array}}$$

S-HANDLEINSTRETURN

$$\frac{}{\mathsf{runinst}^l\{h; \mathsf{return}\ x_r \to c_r\}(\mathsf{return}\ \nu) \mid \Sigma \leadsto c_r[x_r := \nu] \mid \Sigma}$$

TODO: add example derivation

5.7 Type soundness

Theorem 4 (Semantics does valid monotonic extension)**.**

$$\text{if } (c \mid \Sigma \rightsquigarrow c' \mid \Sigma') \wedge \ \vdash \Sigma \ \text{then} \ \vdash \Sigma' \ \wedge \ \Sigma \subseteq \Sigma'$$

TODO
We define a computation c to be a value if c is of the form $\mathsf{return}\ \nu$ for some value ν.

$$\mathsf{value}(c) \ \text{if} \ \exists \nu . c = \mathsf{return}\ \nu$$

Theorem 5 (Type soundness)**.**

$$\text{if } (\cdot; \Sigma; \cdot \vdash c : \tau \mathbin{!} \varnothing) \wedge (c \mid \Sigma \rightsquigarrow^* c' \mid \Sigma') \ \text{then} \ \mathsf{value}(c') \vee (\exists c''\ \Sigma''.\ c' \mid \Sigma' \rightsquigarrow c'' \mid \Sigma'')$$

The type soundness theorem states that if a computation typechecks with no effects in the annotation and we take some amount of steps, then either the computation is a value or we can take another step. This means that if a computation typechecks with no effects then we cannot get stuck on an operation call.

Lemma 6 (Progress)**.**

$$\text{if } \cdot; \Sigma; \cdot \vdash c : \tau \mathbin{!} \varnothing \ \text{then} \ \mathsf{value}(c) \vee (\exists c'\ \Sigma'.\ c \mid \Sigma \rightsquigarrow c' \mid \Sigma')$$

Lemma 7 (Preservation)**.**

$$\text{if } (\Delta; \Sigma; \Gamma \vdash c : \underline{\tau}) \ \wedge \ (c \mid \Sigma \rightsquigarrow c' \mid \Sigma') \ \text{then} \ \Delta; \Sigma'; \Gamma \vdash c' : \underline{\tau}$$

Chapter 6

Related work

The Eff programming language **??** was one of the first languages with support for algebraic effects and handlers. It featured fully dynamic instances allowing for dynamic effects. In order to allow for the implementation of mutable references resources were introduced. Resources are handlers given at the creation of a dynamic instance which get wrapped around the whole execution of the program. This way an operation on such an instance can never escape since the handler is around the whole program. These handlers are also called *default handlers*, which have their own semantics apart from regular handlers. This allows Eff to fully implement mutable references. In our case we insist that users provide a handler when creating a new instance. This handler is not around the whole program though but instead only to the matching runscope construct. This means we can implement mutable references like Eff, but not to the same degree since we do not allow them to escape their scope. We cannot implement fully encapsulated memoization for example, since such a function would only be valid within the scope of its corresponding reference. Eff did not feature an effect typing system initially, meaning that unhandled operations would not be caught statically. An effect typing system was proposed **??** but it did not feature dynamic instances, only static instances. The current version of Eff does not have dynamic instances since they were considered too difficult in the theory.

An embedding of Eff in OCaml was proposed **??**. This embedding featured algebraic effects and handlers and dynamic instances, again without an ef-

fect typing system. The implementation relies on multi-prompt delimited control in OCaml. Default handlers are not implemented, instead dynamic effects such as mutable references are possible by treating the creation of instances as just another effect. Creating a new instance can be done using the NEW effect. This operation receives as arguments which effect to create an instance of, and the handler to use. The handler for the NEW operation then actually allocates the new instance and give it to the continuation of the operation, wrapping the entire continuation call in the give handler. We use a very similar technique to implement creation of instances. Our runscope construct is similar to the handler of NEW in the Eff embedding. In essence we ban the normal creation of dynamic instances and force users to always use the equivalent of the NEW effect. This restriction allows use to give a type system which make sure no instance escape their handler.

Koka is a programming language with effect inference using row polymorphism **??**. Later algebraic effects and handler were also added **??**. Extending the language with an LIFT (also called INJECT) operation to inject effects, skipping a handler, some notion of mutable references can be implemented **??**. These references are very tedious to work with though and they are unable to escape even single functions. Daan Leijen proposed an extension for Koka with dynamic effect handlers **??**. This extension introduces *umbrella effects*, which are effects that can contain dynamic effects. For example an umbrella effect HEAP can be defined which contains dynamic effects of type REF (mutable references). Values of type REF can then created using by defining a dynamic handler. Similar to how we have give a handler when creating a dynamic instance. Using these dynamic handlers we can implement polymorphic heaps. In order to let references escape the function in which they are created, a NEW-REF operation is defined on the umbrella effect HEAP. The HEAP handler then creates the dynamic REF handlers for each time NEW-REF is called, installing these handlers under the HEAP handler. This way the references can escape functions that define them as long as these functions are called under the HEAP handler. This is very similar to the NEW effect of the OCaml Eff embedding and to our RUNSCOPE construct. Koka does not statically check that the dynamic effects do not escape their handler. Instead an exception effect is added to the effect annotation each time a dynamic handler is created. This means that you are always forced to handle the exceptions even if you know that none will be thrown. Similar to our system safe references are proposed using higher-ranked types, like in

the ST monad in Haskell, in order to ensure that no unhandled operations will happen. These definitions still do not remove the exception effect in the effect annotation though. In our system instead we do not allow instances to escape their scope at all. We trade in some expressiveness for safety.

Algebraic effects and handlers have an abstraction problem. A function parameter of a higher-order function may interfere with the effects used inside of the function. If the function argument is used inside of handled computation then the effects in the function may be handled, even if the function itself is polymorphic with regards to these effects. In Eff dynamic instances solve this problem because one cannot handle an instance if they do not have access to this instance. In our system this is also true, a handler is defined when creating an instance and so we cannot accidentally handle an effect we are unaware of. The two papers "Abstracting Algebraic Effects" **??** and "Abstraction-safe effect handlers via tunneling" **??** both address the same problem in different ways. The first of these allows for the definition of local effects. These local effects are guaranteed to not interfere with any other effects. This allows for abstraction-safe functions. The local effects do not rise to the same level as dynamic instances though, they do not allow for the definition of dynamic mutable references. "Abstraction-safe effect handlers via tunneling" gives another solution by explicitly naming handlers. Operations are then called on the named handler. Functions are allowed to be polymorphic over a handler using *handler variables*. The technique of explictly naming handlers as similar expressivity to static instances and the INJECT operation in Koka. It does not allow for the definition of dynamic mutable references.

Chapter 7

Conclusion and future work

7.1 Conclusion

Blablabla

7.2 Future work

Formalization. We currently have formalized the system with static instances of Chapter 2.2 and have proven type soundness in Coq. Due to time constraints we were unable to also provide a formalization for X. It would be beneficial to also formalize X to gain more certainty the system is safe.

Parametric polymorphism over any type. In order to keep the system simple and to only focus on the novel elements, X only supports parametric polymorphism over effect scopes. It would be very useful in practice to allow quantification over any type. We do not think that adding this will interfere with the other elements of the system.

Polymorphic effects. Having added polymorphism over any type it makes sense to also allow for polymorphic effects. In our examples we have defined a `State` effect with `Int` values. In order to avoid having to define a seperate effect for each type we would like to keep in our state, we could allow for

effects to have type parameters. For example, we could define a polymorphic State effect like:

```
effect State t {
  get : () -> t
  put : t -> ()
}
```

Using this effect we can have fully polymorphic mutable references. The type of an reference holding an integer value would be `Inst s (State Int)`. After adding full parametric polymorphic this change would not be difficult to add.

Polymorphic operations. Polymorphism in operations can also be useful for some effects. For example an exception effect:

```
effect Exception {
  throw : forall t. String -> t
}
```

Here, because we do not expect to continue a computation after an exception has been throw, we can let the `throw` operation return any type. This makes it convenient to throw errors, for example in an if-expression where the type of both branches have to match. Adding polymorphic operations takes some more thought because in the case of operations like `throw` we cannot allow a handler to call the continuation. The type of the continuation has to be something like `(forall t1. t1) -> t2!r`, so it is impossible to actually invoke it.

Improved effect annotations. Currently the effect annotation of a computation type is a set of effect scopes. We could make these annotations more useful by also noting which effects are used on each scope. For example, from `Int!{s1, s2}` to `Int!{{State, Flip}@s1, {Rng}@s2}`. Of course this would be an over-approximation of the actual effects used. We do not see any difficulty in extending the annotations in this way.

Combine with regular algebraic effects and handlers. In our system handlers are given when creating instances. This is necessary in order to make sure that every instance has a handler, which completely handles the effects of that specific instance. Sometimes it might be useful to change the

handler of an instance though. For example we might want to add logging to a specific `State` instance, to see what values are being stored in it. To allow for this we would just have to add a $\mathsf{handle}(c)\{h\}$ computation like in the system of Chapter 4.2. In order to make sure all operation calls are handled we would still need to use the handler given when creating the instance. The handler h in $\mathsf{handle}(c)\{h\}$ would have to be applied before the handler given when creating the instance, since `h` might reinvoke the operations on that instance. The handler h would not need to complete, it would not have to handle every operation of a specific effect. The handler could also be allowed to be *shallow*, meaning that the handler is not automatically inserted in the continuation. Lastly h could be a *multi-handler*, handling operations of different effects and instances at once. We are not completely sure if this idea would work in practice or if the system would remain safe.

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