

# BIOSTATS

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Last updated on 2023-07-06

## Contents

<b>Introduction</b>	<b>2</b>
<b>1 Descriptive Statistics</b>	<b>2</b>
1.1 Central Tendency . . . . .	3
1.2 Variation . . . . .	5
1.3 Laboratory . . . . .	7
<b>2 Sampling</b>	<b>8</b>
2.1 The Unknown: Garden Plant Example . . . . .	8
2.2 Linking Part to the Whole . . . . .	10
2.3 Laboratory . . . . .	15
<b>3 Probabilistic View</b>	<b>16</b>
3.1 Probability Distribution of a Continuous Variable . . . . .	16
3.2 Probability Distribution of a Discrete Variable . . . . .	21
3.3 Why Probability Distributions? . . . . .	27
3.4 Laboratory . . . . .	27
<b>4 Two-Group Comparison</b>	<b>27</b>
4.1 Explore Data Structure . . . . .	27
4.2 Test Statistic . . . . .	29
4.3 t-test in R . . . . .	34
4.4 Laboratory . . . . .	35
<b>5 Multiple-Group Comparison</b>	<b>35</b>
5.1 Partition the Variability . . . . .	36
5.2 Test Statistic . . . . .	40
5.3 ANOVA in R . . . . .	42
5.4 Post-hoc Tests . . . . .	42
5.5 Laboratory . . . . .	43
<b>6 Regression</b>	<b>43</b>
6.1 Explore Data Structure . . . . .	43
6.2 Drawing the “fittest” line . . . . .	45
6.3 Unexplained Variation . . . . .	50
6.4 Laboratory . . . . .	53
<b>7 Linear Model</b>	<b>53</b>
7.1 The Frame . . . . .	53
7.2 Combine Multiple Types of Variables . . . . .	58
7.3 Laboratory . . . . .	61

<b>8 Generalized Linear Model</b>	<b>61</b>
8.1 Count Data . . . . .	61
8.2 Proportional Data . . . . .	66
8.3 The GLM Framework . . . . .	73
8.4 Laboratory . . . . .	74
<b>9 Likelihood</b>	<b>74</b>
9.1 Finding the “More Likely” . . . . .	74
9.2 Maximum Likelihood Method . . . . .	79
9.3 Laboratory . . . . .	82
<b>10 Appendix: Project Management</b>	<b>82</b>
10.1 R Project . . . . .	82
10.2 Robust coding . . . . .	84
<b>11 Appendix: Git &amp; GitHub</b>	<b>84</b>
11.1 Git & GitHub . . . . .	84
11.2 Commit & Push . . . . .	86
<b>12 Appendix: Data Structure</b>	<b>92</b>
12.1 Overview . . . . .	92
12.2 Vector . . . . .	92
12.3 Matrix . . . . .	96
12.4 Data Frame . . . . .	99
12.5 Exercise . . . . .	100
<b>13 Appendix: Tidyverse</b>	<b>100</b>
13.1 Overview . . . . .	100
13.2 Data Manipulation . . . . .	101
13.3 Visualization . . . . .	107
<b>14 Appendix: Base Plot</b>	<b>116</b>
14.1 Overview . . . . .	116
14.2 Plot . . . . .	116
14.3 Boxplot . . . . .	123

## Introduction

This textbook is designed to provide a comprehensive introduction to fundamental statistical techniques and their practical applications in the context of biological data analysis. What sets this book apart is its “flipped-order” approach to teaching. Unlike traditional statistics courses that often begin with theoretical concepts, this book takes a different approach by initially presenting real-world data. By grounding the material in practical scenarios, it aims to make the learning process more accessible, particularly for those who are new to statistics.

Through the “flipped-order” introduction, readers will be able to engage with concrete examples of statistical methods in use before delving into the underlying theories and concepts that support them. This pedagogical approach aims to provide a solid foundation for readers to comprehend and apply statistical techniques effectively in the biological sciences.

## 1 Descriptive Statistics

Descriptive statistics are a set of summary measures that provide a concise overview of a dataset. They help us understand the characteristics and properties of the data without delving into complex statistical

analyses. Some commonly used descriptive statistics include the mean, standard deviation, and median.

To illustrate the concept, let's consider fish length measurement  $x$ . Each fish is identified by a subscript, and their lengths are denoted as follows:

$$x_1 = 15.9, x_2 = 15.1, x_3 = 21.9, x_4 = 13.3, x_5 = 24.4$$

Often times, we use subscript  $i$  (or any character you like) instead of actual number to indicate a given data point. For example, we write fish length  $x_i$  for individual  $i$ . Alternatively, instead of writing each individual data point, we can represent them as a vector using boldface, denoted as  $\mathbf{x}$ . In this case, the vector  $\mathbf{x}$  can be expressed as:

$$\mathbf{x} = \{15.9, 15.1, 21.9, 13.3, 24.4\}$$

Now let's see how we represent summary statistics of vector  $\mathbf{x}$ .

## 1.1 Central Tendency

### 1.1.1 Measures of Central Tendency

Central tendency measures (Table 1) provide insights into the typical or central value of a dataset. There are three commonly used measures:

- **Arithmetic Mean.** This is the most commonly used measure of central tendency. It represents the *additive* average of the data. To calculate the arithmetic mean, you sum up all the values and divide the total by the number of data points. It can be heavily influenced by extreme values, known as outliers.
- **Geometric Mean.** The geometric mean is a *multiplicative* average. It is always smaller than the arithmetic mean and less sensitive to unusually large values. However, it is not applicable when the data contain negative values.
- **Median.** The median is the value that separates the higher half from the lower half of the dataset. It represents the 50th percentile data point. The median is less affected by outliers compared to the arithmetic mean. To calculate the median, you arrange the data in ascending order and select the middle value if the dataset has an odd number of values. If the dataset has an even number of values, you take the average of the two middle values.

Table 1: Common measures of central tendency.  $N$  refers to the number of data points.

Measure	Formula
Arithmetic mean $\mu$	$\frac{\sum_i^N x_i}{N}$
Geometric mean $\mu_{ge}$	$(\prod_i^N x_i)^{\frac{1}{N}}$
Median $\mu_{med}$	$\begin{cases} x_{(\frac{N+1}{2})^{th}} & \text{if } N \text{ is odd number} \\ \frac{1}{2}[x_{(\frac{N}{2})^{th}} + x_{(\frac{N}{2}+1)^{th}}] & \text{if } N \text{ is even number} \end{cases}$

### 1.1.2 R Exercise

To learn more about these measures, let's create vectors  $\mathbf{x} = \{15.9, 15.1, 21.9, 13.3, 24.4\}$  and  $\mathbf{y} = \{15.9, 15.1, 21.9, 53.3, 24.4\}$  –  $\mathbf{y}$  is identical to  $\mathbf{x}$  but contains one outlier value. How does this make difference? To construct vectors in R, we use `c()`, a function that stands for “construct.” Below is the script:

```
# construct vectors x and y
x <- c(15.9, 15.1, 21.9, 13.3, 24.4)
y <- c(15.9, 15.1, 21.9, 53.3, 24.4)
```

Confirm you constructed them correctly:

```
x
## [1] 15.9 15.1 21.9 13.3 24.4
y
## [1] 15.9 15.1 21.9 53.3 24.4
```

Cool! Now we can calculate summary statistics of x and y.

### Arithmetic mean

While R has a function for arithmetic `mean()`, let's try to calculate the value from scratch:

```
# for vector x
n_x <- length(x) # the number of elements in x = the number of data points
sum_x <- sum(x) # summation for x
mu_x <- sum_x / n_x # arithmetic mean
print(mu_x) # print calculated value

## [1] 18.12

# for vector y; we can calculate directly too
mu_y <- sum(y) / length(y)
print(mu_y) # print calculated value

## [1] 26.12
```

Compare with outputs from `mean()`:

```
print(mean(x))

## [1] 18.12
print(mean(y))

## [1] 26.12
```

### Geometric Mean

Unfortunately, there is no build-in function for geometric mean  $\mu_{ge}$  in R (as far as I know; there are packages though). But, we can calculate the value from scratch again:

```
# for vector x
prod_x <- prod(x) # product of vector x; x1 * x2 * x3...
n_x <- length(x)
mug_x <- prod_x^(1 / n_x) # ^ means power
print(mug_x)

## [1] 17.63648

# for vector y
mug_y <- prod(y)^(1 / length(y))
print(mug_y)

## [1] 23.28022
```

## Median

Lastly, let's do the same for median:

```
# for vector x
x <- sort(x) # sort x from small to large
index <- (length(x) + 1) / 2 # (N + 1)/2 th index as length(x) is an odd number
med_x <- x[index]
print(med_x)
```

```
## [1] 15.9
# for vector y
y <- sort(y) # sort y from small to large
med_y <- y[(length(y) + 1) / 2]
print(med_y)
```

```
## [1] 21.9
```

Compare with outputs from `median()`

```
print(median(x))
```

```
## [1] 15.9
```

```
print(median(y))
```

```
## [1] 21.9
```

## 1.2 Variation

### 1.2.1 Measures of Variation

Variation measures (Table 2) provide information about the spread of data points.

- **Variance.** Variance is a statistical measure that quantifies the spread or dispersion of a dataset. It provides a numerical value that indicates how far individual data points in a dataset deviate from the mean or average value. In other words, variance measures the average squared difference between each data point and the mean. **Standard deviation (SD) is the square root of variance.**
- **Inter-Quantile Range.** The interquartile range (IQR) is a statistical measure that provides information about the spread or dispersion of a dataset, specifically focusing on the middle 50% of the data. It is a robust measure of variability that is less affected by outliers compared to variance.
- **Median Absolute Deviation.** Median Absolute Deviation (MAD) is similar to variance, but provides a robust estimation of variability that is less affected by outliers compared to variance. MAD is defined as the median of the absolute deviations from the data's median.
- **Coefficient of Variation.** The coefficient of variation (CV) is a statistical measure that expresses the relative variability of a dataset in relation to its mean. It is particularly useful when comparing the variability of different datasets that have different scales or units of measurement.
- **MAD/Median.** MAD/Median is a statistical measure used to assess the relative variability of a dataset without assuming any specific distribution or parametric model. CV is sensitive to outliers because of its reliance on the arithmetic mean. However, MAD/Median is robust to this issue.

Table 2: Common measures of variation.  $N$  refers to the number of data points.

Measure	Formula
Variance $\sigma^2$ (standard deviation $\sigma$ )	$\frac{\sum_i^N (x_i - \mu)^2}{N}$
Inter-Quantile Range IQR ( $x_l$ and $x_h$ are $l^{th}$ and $h^{th}$ percentiles, respectively)	$ x_l - x_h $
Median Absolute Deviation (MAD)	Median( $ x_i - \mu_{med} $ )
Coefficient of Variation (CV)	$\frac{\sigma}{\mu}$
MAD/Median	$\frac{MAD}{\mu_{med}}$

### 1.2.2 R Exercise

#### Variance, SD, and CV

Let's try variance, SD, and CV:

```
# for x
sqd_x <- (x - mean(x))^2 # squared deviance
sum_sqd_x <- sum(sqd_x)

var_x <- sum_sqd_x / length(x)
sd_x <- sqrt(var_x) # sqrt(): square root
cv_x <- sd_x / mean(x)

print(var_x)

## [1] 18.2016
print(sd_x)

## [1] 4.266333
print(cv_x)

## [1] 0.2354489

# for y
var_y <- sum((y - mean(y))^2) / length(y)
sd_y <- sqrt(var_y)
cv_y <- sd_y / mean(y)

print(var_y)

## [1] 197.0816
print(sd_y)

## [1] 14.03858
print(cv_y)

## [1] 0.5374646
```

#### IQR, MAD, and MAD/Median

Let's try IQR, MAD, and MAD/Median. IQR can be defined for given percentiles. Here, let me use 25 and 75 percentiles as  $x_l$  and  $x_h$ .

```

# for x
## IQR
x_l <- quantile(x, 0.25) # quantile(): return quantile values, 25 percentile
x_h <- quantile(x, 0.75) # quantile(): return quantile values, 75 percentile
iqr_x <- abs(x_l - x_h) # abs(): absolute value

## MAD
ad_x <- abs(x - mean(x))
mad_x <- median(ad_x)

## MAD/median
mad_m_x <- mad_x / median(x)

print(iqr_x)

## 25%
## 6.8
print(mad_x)

## [1] 3.78
print(mad_m_x)

## [1] 0.2377358

# for y
## IQR
y_q <- quantile(y, c(0.25, 0.75)) # return as a vector
iqr_y <- abs(y_q[1] - y_q[2]) # y_q[1] = 25 percentile; y_q[2] = 75 percentile

## MAD
mad_y <- median(abs(y - mean(y)))

## MAD/median
mad_m_y <- mad_y / median(y)

print(iqr_y)

## 25%
## 8.5
print(mad_y)

## [1] 10.22
print(mad_m_y)

## [1] 0.4666667

```

## 1.3 Laboratory

### 1.3.1 Comparing Central Tendency Measures

What are the differences of the three measures of central tendency? To investigate this further, let's perform the following exercise.

1. Create a new vector  $z$  with length 1000 as `exp(rnorm(n = 1000, mean = 0, sd = 0.1))`, and calculate the arithmetic mean, geometric mean, and median.

2. Draw a histogram of `z` using functions `tibble()`, `ggplot()`, and `geom_histogram()`.
3. Draw vertical lines of arithmetic mean, geometric mean, and median on the histogram with different colors using a function `geom_vline()` .
4. Compare the values of the central tendency measures.
5. Create a new vector `z_rev` as `-z + max(z) + 0.1`, and repeat step 1 – 4.

### 1.3.2 Comparing Variation Measures

Why do we have absolute (variance, SD, MAD, IQR) and relative measures (CV, MAD/Median) of variation? To understand this, suppose we have 100 measurements of fish weight in unit “gram.” (`w` in the following script)

```
w <- rnorm(100, mean = 10, sd = 1)
head(w) # show first 10 elements in w

## [1] 9.237455 9.171337 10.834474 9.032348 9.971185 10.232525
```

Using this data, perform the following exercise:

1. Convert the unit of `w` to “milligram” and create a new vector `m`.
2. Calculate SD and MAD for `w` and `m`.
3. Calculate CV and MAD/Median for `w` and `m`.

## 2 Sampling

*“Why do I need statistics in the first place?”* This was the initial question that arose when I entered the field of ecology. Initially, I assumed it would be a straightforward query with an immediate response. However, I soon realized that it is a profound question with a complex answer. In short, “we need statistics because we often possess only partial information about what we seek to understand.” Now, let’s explore the more elaborate explanation below.

### 2.1 The Unknown: Garden Plant Example

Consider a scenario where we are conducting a study on plant height in a garden. In this garden, there exists a thousand of individual plants, making it impractical for a single researcher to measure all of them. Instead, due to resource limitations, a sample of 10 plants was selected to *calculate* the average height and the extent of variation among these plant individuals:

```
## # A tibble: 10 x 4
##   ...1 plant_id height unit
##   <dbl>    <dbl>  <dbl> <chr>
## 1     1        1    16.9 cm
## 2     2        2    20.9 cm
## 3     3        3    15.8 cm
## 4     4        4     28   cm
## 5     5        5    21.6 cm
## 6     6        6    15.9 cm
## 7     7        7    22.4 cm
## 8     8        8    23.7 cm
## 9     9        9    22.9 cm
## 10    10       10   18.5 cm
```

Cool. Let’s use this data set to learn about the pitfall behind this. Create a vector of plant height `h` and put it in a `tibble()` to analyze it:

```

h <- c(16.9, 20.9, 15.8, 28, 21.6, 15.9, 22.4, 23.7, 22.9, 18.5)

df_h1 <- tibble(plant_id = 1:10, # a vector from 1 to 10 by 1
                height = h, # height
                unit = "cm") # unit

```

This format (`tibble()`) is better than a raw vector of height because it allows more flexible analysis. Let's add columns of `mu_height` and `var_height` using `mutate()`, a function that adds new column(s) to an existing `tibble()` (or `data.frame()`):

```

# nrow() returns the number of rows
# while piping, "." refers to the dataframe inherited
# i.e., nrow(.) counts the number of rows in df_h1
df_h1 <- df_h1 %>%
  mutate(mu_height = mean(height),
        var_height = sum((height - mu_height)^2) / nrow(.))

```

Awesome, we were able to get the average height and the variance! – however, how confident are you? We obtained plant height only from 10...out of 1000. Are they different if we measure another set of 10 plant individuals? Let's see:

```

## # A tibble: 10 x 4
##   ...1 plant_id height unit
##   <dbl>     <dbl> <dbl> <chr>
## 1    1      11     11   27.6 cm
## 2    2      12     12   21.9 cm
## 3    3      13     13   16.9 cm
## 4    4      14     14    8.9  cm
## 5    5      15     15   25.6 cm
## 6    6      16     16   19.8 cm
## 7    7      17     17   19.9 cm
## 8    8      18     18   24.7 cm
## 9    9      19     19   24.1 cm
## 10   10     20     20   23   cm

```

Create another `tibble()`:

```

h <- c(27.6, 21.9, 16.9, 8.9, 25.6, 19.8, 19.9, 24.7, 24.1, 23)

df_h2 <- tibble(plant_id = 11:20, # a vector from 11 to 20 by 1
                height = h,
                unit = "cm") %>%
  mutate(mu_height = mean(height),
        var_height = sum((height - mu_height)^2) / nrow(.))

print(df_h2)

## # A tibble: 10 x 5
##   plant_id height unit  mu_height var_height
##       <int>  <dbl> <chr>    <dbl>      <dbl>
## 1      11    27.6 cm     21.2      25.8
## 2      12    21.9 cm     21.2      25.8
## 3      13    16.9 cm     21.2      25.8
## 4      14     8.9  cm    21.2      25.8
## 5      15    25.6 cm     21.2      25.8
## 6      16    19.8 cm     21.2      25.8

```

```

## 7      17   19.9 cm      21.2      25.8
## 8      18   24.7 cm      21.2      25.8
## 9      19   24.1 cm      21.2      25.8
## 10     20   23   cm      21.2      25.8

```

Wow, that's totally different.

## 2.2 Linking Part to the Whole

The exercise highlights an important takeaway: what we can determine from the above data is the average and variance of the sample, **which may not perfectly represent the characteristics of the entire garden.**

In the field of biological research, it is often impractical or impossible to sample the entire population, so we must rely on estimating the unknowns (in this case, the *mean* and *variance*) from the available samples. This is where statistics comes into play, offering a tool to infer information about the entire population based on partial information obtained from the samples.

The unknowns we are interested in, which the population mean and variance in this example, are referred to as “**parameters**.” These parameters cannot be directly measured but can be estimated from samples through statistical inference.

Provided that certain assumptions are met, the *sample mean* is the unbiased point estimate of the *population mean*. The “**unbiased**” means that the sample means – if we repeat the sampling process – are centered around the population mean. In the meantime, the sample variance – if we use the formula in Chapter 1 – is “**biased**.” It tends to be smaller than the population variance.

Let’s explore this concept further through simple simulations. Suppose we have data on a thousand plant individuals, although this scenario may be unrealistic in practice. However, by conducting these simulations, we can examine how different sample means and variances can deviate from the true values.

Download the data here containing height measurements of thousand individuals, and place this file under `data_raw/` in your project directory. You can load this `csv` file in R as follows:

```

# load csv data on R
df_h0 <- read_csv("data_raw/data_plant_height.csv")

# show the first 10 rows
print(df_h0)

## # A tibble: 1,000 x 4
##       ...1 plant_id height unit
##   <dbl>    <dbl>  <dbl> <chr>
## 1     1        1     16.9 cm
## 2     2        2     20.9 cm
## 3     3        3     15.8 cm
## 4     4        4     28   cm
## 5     5        5     21.6 cm
## 6     6        6     15.9 cm
## 7     7        7     22.4 cm
## 8     8        8     23.7 cm
## 9     9        9     22.9 cm
## 10   10       10    18.5 cm
## # i 990 more rows

```

Using this synthetic dataset (I generated through a random value generator), we can calculate the true mean and variance (reference values). It is important to note that in this case, we use the term “calculate” for the

mean and variance because they represent the parameters of the entire population, which are known to us in this scenario.

```
mu <- mean(df_h0$height)
sigma2 <- sum((df_h0$height - mu)^2) / nrow(df_h0)

print(mu)

## [1] 19.9426

print(sigma2)

## [1] 26.74083
```

We can simulate the sampling of 10 plant individuals by randomly selecting 10 rows from df\_h0:

```
df_i <- df_h0 %>%
  sample_n(size = 10) # size specifies the number of rows to be selected randomly

print(df_i)

## # A tibble: 10 x 4
##       plant_id height unit
##   <dbl>     <dbl> <dbl> <chr>
## 1     308      308  22.8 cm
## 2     461      461  17.7 cm
## 3     602      602  27.5 cm
## 4     181      181  13.8 cm
## 5     488      488  14.8 cm
## 6     565      565  17   cm
## 7     302      302  14.8 cm
## 8      96       96  22.8 cm
## 9     853      853  14.8 cm
## 10    987      987  15.9 cm
```

Since `sample_n()` selects rows randomly, you will (very likely) get different set of 10 individuals/rows every single time. Below is another set of 10 rows (notice that `df_i` is overwritten with the new data set):

```
df_i <- df_h0 %>%
  sample_n(size = 10)

print(df_i)

## # A tibble: 10 x 4
##       plant_id height unit
##   <dbl>     <dbl> <dbl> <chr>
## 1     962      962  19.9 cm
## 2     618      618  11   cm
## 3     614      614  13   cm
## 4     735      735  22.1 cm
## 5     215      215  24.9 cm
## 6      32       32  19.5 cm
## 7     230      230  21.3 cm
## 8     367      367  23.4 cm
## 9     270      270  15.3 cm
## 10    524      524  21   cm
```

Let's obtain 100 sets of 10 plant individuals (randomly selected) and *estimate* the mean and variance in each.

While we can perform random sampling one by one, this is cumbersome – at least, I do not want to do it. Instead, we can leverage a technique of `for` loop:

```
# for reproducibility
set.seed(3)

mu_i <- var_i <- NULL # create empty objects

# repeat the work in {} from i = 1 to i = 100
for (i in 1:100) {

  df_i <- df_h0 %>%
    sample_n(size = 10) # random samples of 10 individuals

  # save mean for sample set i
  mu_i[i] <- mean(df_i$height)

  # save variance for sample set i
  var_i[i] <- sum((df_i$height - mean(df_i$height))^2) / nrow(df_i)

}
```

Take a look at `mu_i` and `var_i`:

```
print(mu_i)

## [1] 19.97 17.86 22.55 22.03 17.00 23.28 21.33 21.23 18.55 19.29 22.14 19.84
## [13] 22.43 22.13 21.13 19.40 19.39 20.43 19.12 20.66 21.01 19.48 21.73 19.63
## [25] 21.03 20.60 21.11 20.42 18.76 23.70 20.31 22.22 21.34 20.70 20.96 20.03
## [37] 21.77 19.19 19.87 21.38 19.64 23.31 19.89 19.21 19.68 19.54 17.54 19.05
## [49] 18.91 20.57 18.33 18.07 19.48 17.70 20.24 17.74 20.45 16.48 18.93 17.60
## [61] 17.23 20.75 18.06 20.06 20.80 21.72 19.02 25.08 18.90 20.69 23.28 20.87
## [73] 18.65 19.74 21.47 17.95 16.98 18.30 19.77 17.25 19.60 21.27 19.28 20.42
## [85] 19.60 18.41 20.15 21.24 19.70 21.56 20.75 19.54 17.54 18.52 19.85 18.40
## [97] 20.39 17.07 17.84 20.66

print(var_i)

## [1] 14.1961 21.6884 28.2685 21.6341 6.5700 24.5996 23.6941 24.8681 21.6485
## [10] 36.3689 17.8844 31.5784 21.0521 34.2341 20.6281 10.4420 20.0469 31.2781
## [19] 32.0316 23.0964 9.8789 31.8916 28.3061 19.6861 8.2641 23.3640 26.8769
## [28] 48.9416 27.7644 42.4220 38.4989 12.7076 15.9004 8.6740 11.7284 26.2061
## [37] 21.7461 25.9269 28.0421 6.2616 16.5584 27.7489 17.3609 23.1349 38.6936
## [46] 25.0984 13.5864 25.2625 7.7149 17.5341 11.4681 38.1561 12.6376 42.5480
## [55] 31.9224 13.9824 14.9725 25.8296 33.6781 8.2440 31.4741 29.7805 26.7324
## [64] 47.3704 26.8660 24.0536 29.5076 23.7376 14.8000 51.8449 13.3536 18.5581
## [73] 18.9185 12.4544 36.2201 16.3225 17.9236 42.8760 18.5421 22.0205 34.1680
## [82] 28.8841 26.6976 22.7356 35.2900 19.2229 34.3905 17.3784 23.1880 42.4264
## [91] 18.5365 18.5424 21.5204 13.1276 17.7185 21.2940 27.1649 35.3141 43.5624
## [100] 55.1484
```

In each element of `mu_i` and `var_i`, we have saved estimated mean ( $\hat{\mu}$ ; reads mu hat) and variance ( $\hat{\sigma}^2$ ) of 10 plant height measures for dataset `i`. By drawing a histogram of these values, we can examine the distributions of mean and variance estimates. I use R package `patchwork` to make a better figure:

```
#install.packages("patchwork") # install only once
library(patchwork)
```

```

df_sample <- tibble(mu_hat = mu_i, var_hat = var_i)

# histogram for mean
g_mu <- df_sample %>%
  ggplot(aes(x = mu_hat)) +
  geom_histogram() +
  geom_vline(xintercept = mu)

# histogram for variance
g_var <- df_sample %>%
  ggplot(aes(x = var_hat)) +
  geom_histogram() +
  geom_vline(xintercept = sigma2)

# layout vertically
# possible only if "patchwork" is loaded
g_mu / g_var

```

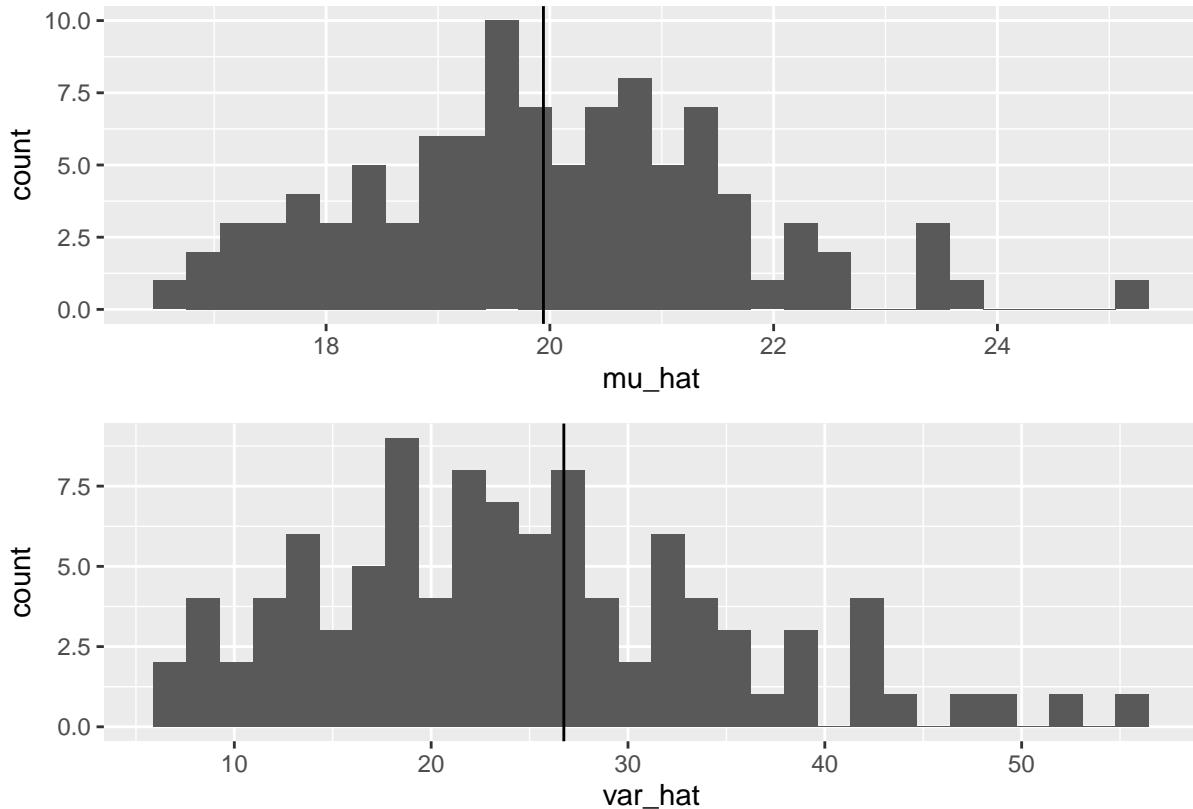


Figure 1: Sample mean and biased variance.

While sample means are indeed symmetrically distributed around the true mean, sample variances tend to be biased and skewed to the right, often underestimating the true variance.

The bias in estimating the variance arises due to inferring the parameter from a small number of samples. However, there is good news: an unbiased estimator of variance exists. The formula for the unbiased

estimator of variance is as follows:

$$\frac{\sum_i^N (x_i - \mu)^2}{N - 1}$$

The correction in the denominator ( $N$  replaced with  $N - 1$ ) compensates for the bias, providing an estimate of the true variance without systematic underestimation (although this seems a simple correction, a deep math underlies the derivation of  $N - 1$ ). This is the default formula in `var()` in R, a function used to estimate *unbiased* variance (and *unbiased* SD `sd()`). Comparison reveals how this works:

```
# for reproducibility
set.seed(3)

# redo simulations ----
mu_i <- var_i <- var_ub_i <- NULL # create empty objects

# repeat the work in {} from i = 1 to i = 100
for (i in 1:100) {

  df_i <- df_h0 %>%
    sample_n(size = 10) # random samples of 10 individuals

  # save mean for sample set i
  mu_i[i] <- mean(df_i$height)

  # save variance for sample set i
  var_i[i] <- sum((df_i$height - mean(df_i$height))^2) / nrow(df_i)

  var_ub_i[i] <- var(df_i$height)
}
```

Add the histogram of unbiased variance:

```
# draw histograms ----
df_sample <- tibble(mu_hat = mu_i,
                     var_hat = var_i,
                     var_ub_hat = var_ub_i)

# histogram for mu
g_mu <- df_sample %>%
  ggplot(aes(x = mu_hat)) +
  geom_histogram() +
  geom_vline(xintercept = mu)

# histogram for variance
# scale_x_continuous() adjusts scale in x-axis
g_var <- df_sample %>%
  ggplot(aes(x = var_hat)) +
  geom_histogram() +
  geom_vline(xintercept = sigma2) +
  scale_x_continuous(limits= c(min(c(var_i, var_ub_i)),
                               max(c(var_i, var_ub_i)))))

# histogram for unbiased variance
g_var_ub <- df_sample %>%
```

```

ggplot(aes(x = var_ub_hat)) +
  geom_histogram() +
  geom_vline(xintercept = sigma2) +
  scale_x_continuous(limits= c(min(c(var_i, var_ub_i)),
                               max(c(var_i, var_ub_i))))

```

`g_mu / g_var / g_var_ub`

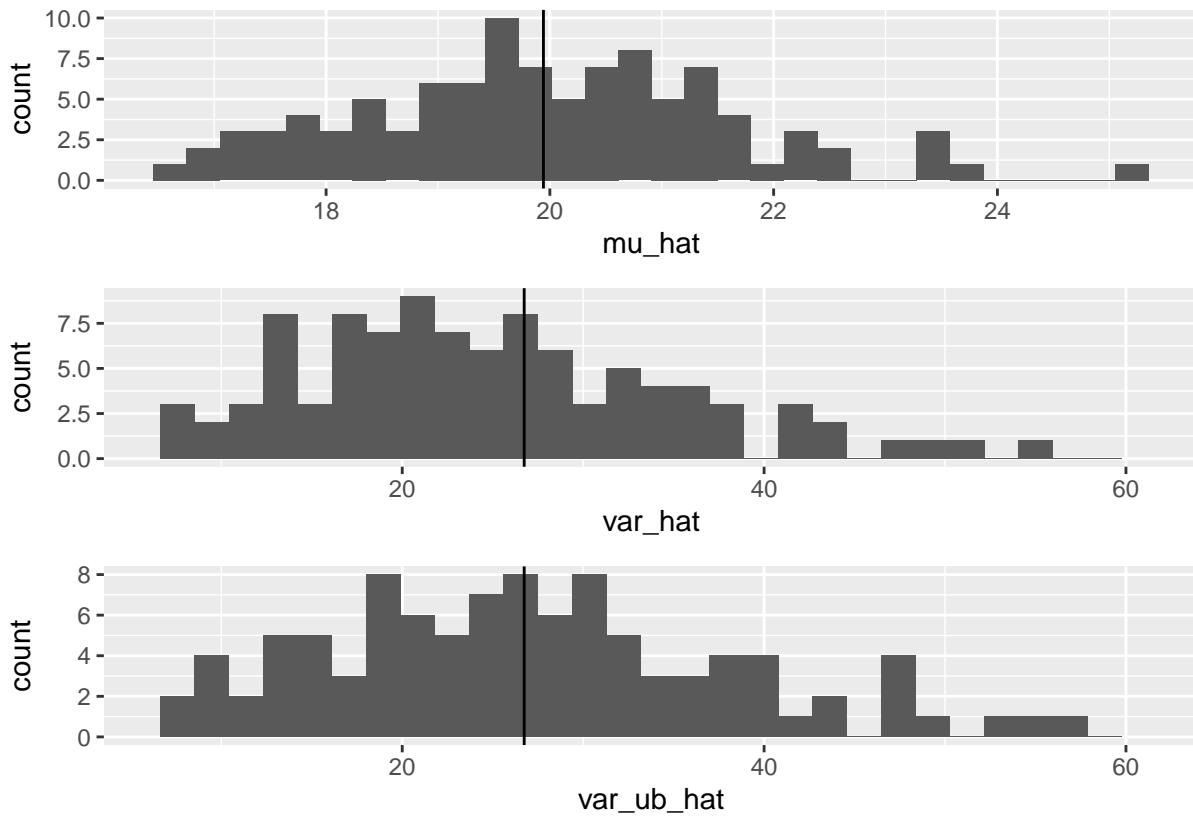


Figure 2: Comparison of biased and unbiased variances.

In summary, samples can only provide information about a part of the whole population. The complete picture of the entire population is often unknown, and **we rely on estimating key parameters from the available samples**. This concept applies to a wide range of parametric analyses in statistics, where we use sample data to make inferences about the population parameters.

Recognizing the limitations and uncertainties associated with working with samples is essential for proper statistical analysis and interpretation of results in various fields of study.

### 2.3 Laboratory

1. We used 10 plants to estimate sample means and variances. Obtain 100 sub-datasets with 50 and 100 measures each, and draw histograms of sample means and unbiased variances (use `var()`).
2. Sample means and unbiased variances are unbiased if samples are randomly selected. What happens if samples are non-random? Suppose the investigator was unable to find plants less than 10 cm in height – the following code excludes those less than 10 cm in height:

```
df_h10 <- df_h0 %>%
  filter(height >= 10)
```

Repeat step 1 with `df_h10` instead of `df_h0` and compare the results.

## 3 Probabilistic View

Chapter 2 emphasized the concept of sampling and introduced the crucial aspect of statistics: randomness. Although the mean represents the central tendency of the data, it does not encompass all the data points. Inevitable deviations occur, and we need a way to express this “randomness.” The concept of probability distributions aids in fully understanding the stochastic nature of observed data.

### 3.1 Probability Distribution of a Continuous Variable

#### 3.1.1 Probability Density Function

Let me use the plant height example in Chapter 2:

```
# load csv data on R
df_h0 <- read_csv("data_raw/data_plant_height.csv")
```

While Chapter 2 primarily focused on exploring mean and variance, let’s broaden our perspective to gain a comprehensive understanding of the entire distribution of height.

```
df_h0 %>%
  ggplot(aes(x = height)) +
  geom_histogram(binwidth = 1) + # specify binwidth
  geom_vline(aes(xintercept = mean(height))) # draw vertical line at the mean
```

This distribution comprises a thousand height measurements. However, there are certain patterns to consider. The data is centered around the mean and exhibits a symmetrical distribution. This characteristic implies that the distribution can be approximated by a simple formula that relies on only a few *key parameters*.

In statistics, the symmetrical bell-shaped form is commonly approximated by a Normal distribution, often denoted as “*variable  $x$  is assumed to follow a Normal distribution.*” We express this using the following mathematical expression:

$$x \sim \text{Normal}(\mu, \sigma^2)$$

Unlike an “equation,” we do not use the “=” sign because variable  $x$  is not equivalent to the Normal distribution. Rather, it represents a “stochastic” relationship that signifies the probability of  $x$  taking on a specific range of values.

A Normal distribution is characterized by two parameters: the mean and the variance. Once these parameters are determined, the formula that defines the Normal distribution will yield the probability density for a given range of values. This formula  $f(x)$  is known as the **probability density function (PDF)** and is displayed below:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

#### 3.1.2 PDF to frequency distribution

In R, the function `dnorm()` can be used to calculate the probability density for a given value. The first argument, `x`, should be a vector of values for which you want to calculate the probability density. The

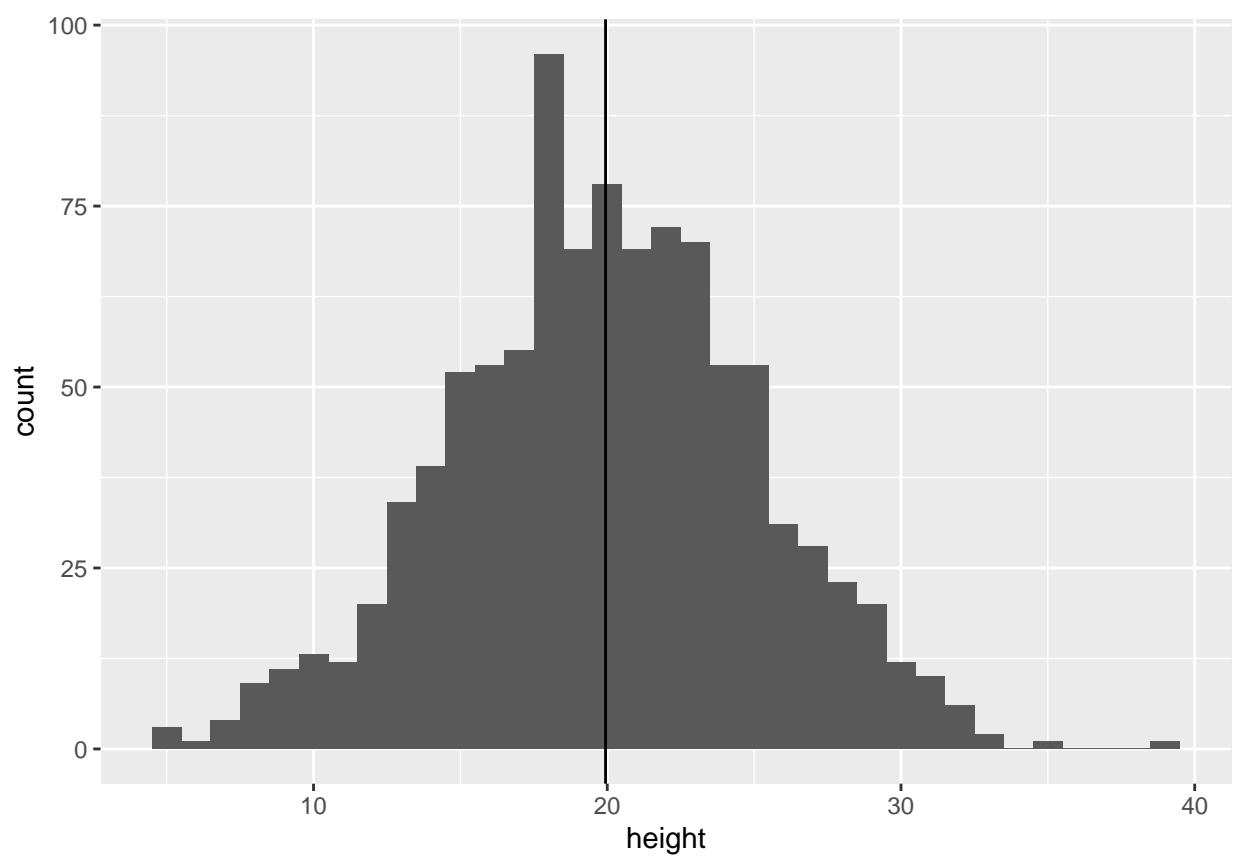


Figure 3: Distribution of plant height

second argument, `mean`, and the third argument, `sd`, correspond to the mean and standard deviation of the distribution, respectively. Note that we must provide `mean` and `sd` to calculate the probability density.

To encompass the entire range of observed heights, we can use `min(x)` and `max(x)` as the lower and upper limits, respectively.

```
# vector of x values
# seq() generate min to max values with specified numbers of elements or interval
# the following produce 100 elements
x <- seq(min(df_h0$height), max(df_h0$height), length = 100)

# calculate probability density
mu <- mean(df_h0$height)
sigma <- sd(df_h0$height)
pd <- dnorm(x, mean = mu, sd = sigma)

# figure
tibble(y = pd, x = x) %>% # data frame
  ggplot(aes(x = x, y = y)) +
  geom_line() + # draw lines
  labs(y = "Probability density") # re-label
```

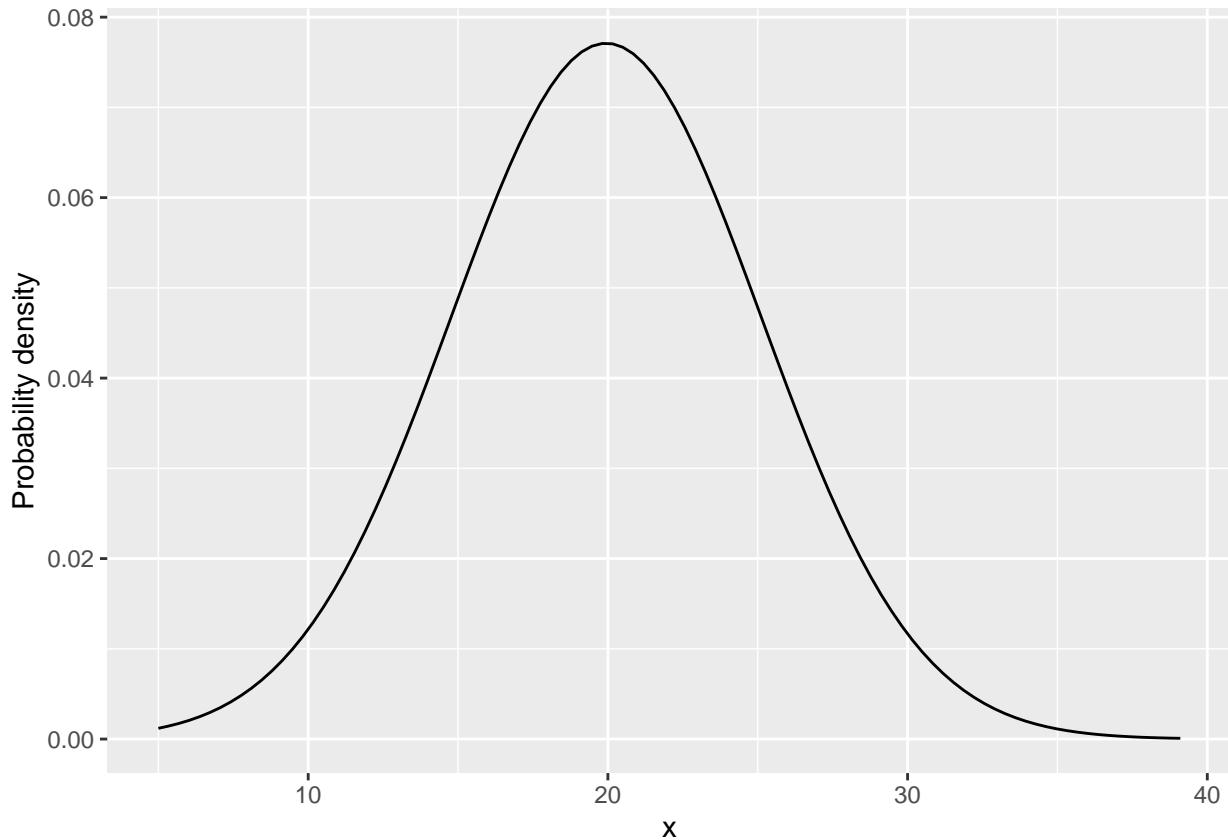


Figure 4: Probability density function of a Normal distribution

The shape of the curve appears quite similar to what we observed; however, the scale of the y-axis is different. This is because the y-axis represents “probability density.” To convert it into actual “probability,” we need to calculate the area under the curve. In R, we can utilize the `pnorm()` function for this purpose. It calculates

the probability of a variable being less than the specified value, which is provided in the first argument q.

```
# probability of x < 10
p10 <- pnorm(q = 10, mean = mu, sd = sigma)
print(p10)
```

```
## [1] 0.02731905
```

```
# probability of x < 20
p20 <- pnorm(q = 20, mean = mu, sd = sigma)
print(p20)
```

```
## [1] 0.5044426
```

```
# probability of 10 < x < 20
p20_10 <- p20 - p10
print(p20_10)
```

```
## [1] 0.4771069
```

To make the estimates comparable to the frequency data, you can calculate the probability for each 1 cm bin. The expected frequency can be obtained by multiplying the probability by the sample size, which in this case is 1000. This allows you to estimate the number of observations you would expect in each 1 cm bin based on the calculated probabilities.

```
x_min <- floor(min(df_h0$height)) # floor takes the integer part of the value
x_max <- ceiling(max(df_h0$height)) # ceiling takes the next closest integer
bin <- seq(x_min, x_max, by = 1) # each bin has 1cm

p <- NULL # empty object for probability
for (i in 1:(length(bin) - 1)) {
  p[i] <- pnorm(bin[i+1], mean = mu, sd = sigma) - pnorm(bin[i], mean = mu, sd = sigma)
}

# data frame for probability
# bin: last element [-length(bin)] was removed to match length
# expected frequency in each bin is "prob times sample size"
df_prob <- tibble(p, bin = bin[-length(bin)]) %>%
  mutate(freq = p * nrow(df_h0))
```

Overlay:

```
df_h0 %>%
  ggplot(aes(x = height)) +
  geom_histogram(binwidth = 1) +
  geom_point(data = df_prob,
             aes(y = freq,
                 x = bin),
             color = "salmon") +
  geom_line(data = df_prob,
            aes(y = freq,
                x = bin),
            color = "salmon")
```

It is remarkable that the probability distribution, characterized by just two parameters (mean and variance), can effectively reproduce the overall shape observed in the original data set of 1000 data points. This is great news because it means that once we determine these key parameters, we can infer the general pattern of the data.

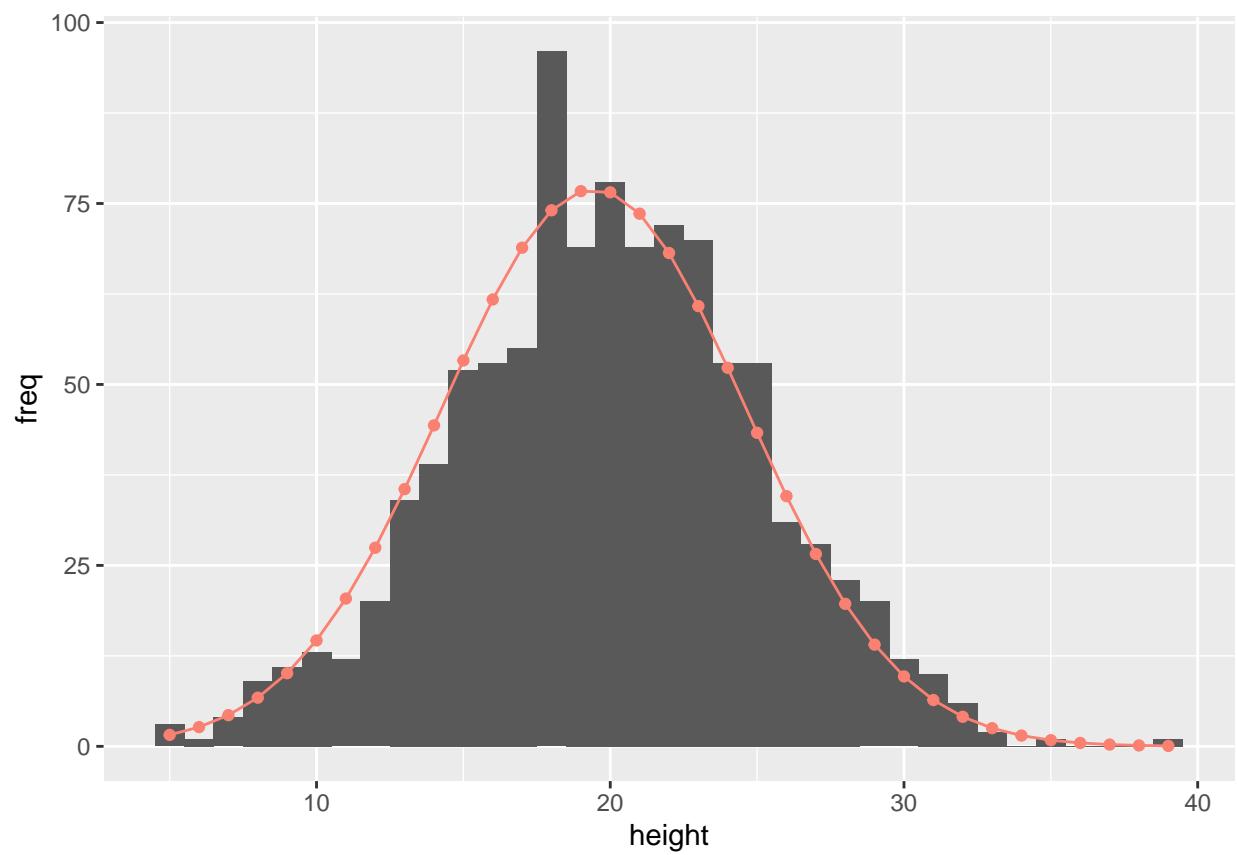


Figure 5: Histogram overlaid with predicted frequency (red dots and line)

## 3.2 Probability Distribution of a Discrete Variable

### 3.2.1 Probability Mass Function

Let's shift our perspective and examine the density of plants in the garden. For this analysis, we have counted the number of plant individuals within 30 plots, with each plot covering an area of one square meter (refer to Figure 6).

Download the data here and load it onto R:

```
df_count <- read_csv("data_raw/data_garden_count.csv")
print(df_count)
```

```
## # A tibble: 30 x 3
##       plot count nitrate
##   <dbl> <dbl>   <dbl>
## 1     1     1    14.6
## 2     2     4    32.6
## 3     3     6    29.3
## 4     4     4    30.4
## 5     5     7    34.1
## 6     6     2    26.6
## 7     7     2    30.2
## 8     8     2    28.7
## 9     9     3    32.5
## 10   10    2    26
## # i 20 more rows
```

Make a histogram:

```
df_count %>%
  ggplot(aes(x = count)) +
  geom_histogram(binwidth = 0.5, # define binwidth
                 center = 0) # relative position of each bin
```

There are several significant differences compared to the plant height example when considering the density of plants in the garden:

1. The possible values are discrete (count data) since we are counting the number of plant individuals in each plot.
2. The possible values are always positive, as counts cannot be negative.
3. The distribution appears to be non-symmetric around the sample mean.

Given these characteristics, a Normal distribution may not be an appropriate choice for representing such a variable. Instead, it would be more suitable to use a Poisson distribution to characterize the observed distribution of the discrete variable.

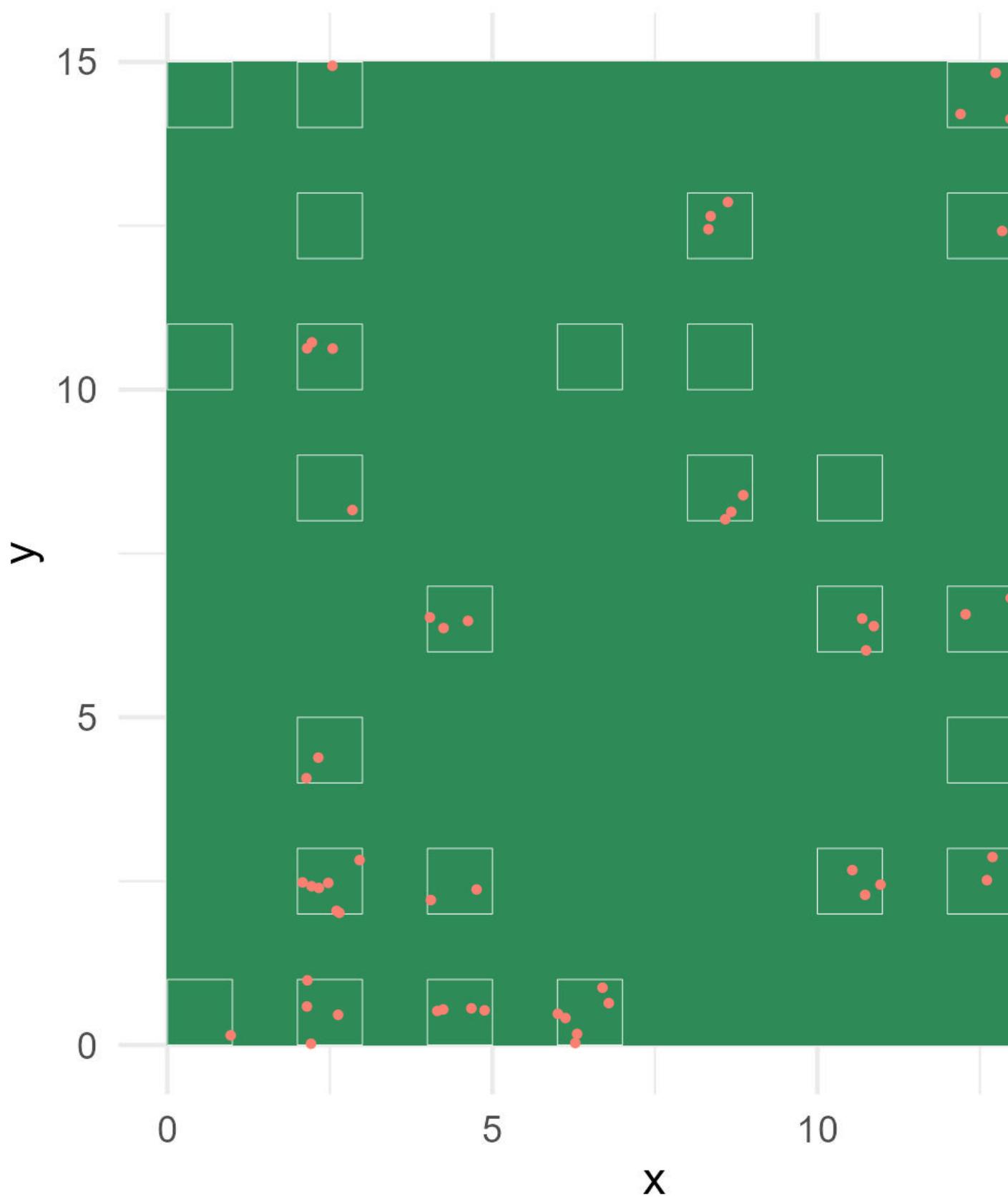
$$x \sim \text{Poisson}(\lambda)$$

In a Poisson distribution, the mean parameter ( $\lambda$ ) serves as the sole parameter<sup>1</sup>. Probability distributions that describe discrete variables are expressed using a **probability mass function** (PMF):

$$g(x) = \frac{\lambda^x \exp(-\lambda)}{x!}$$

---

<sup>1</sup>In a Poisson distribution, mean = variance



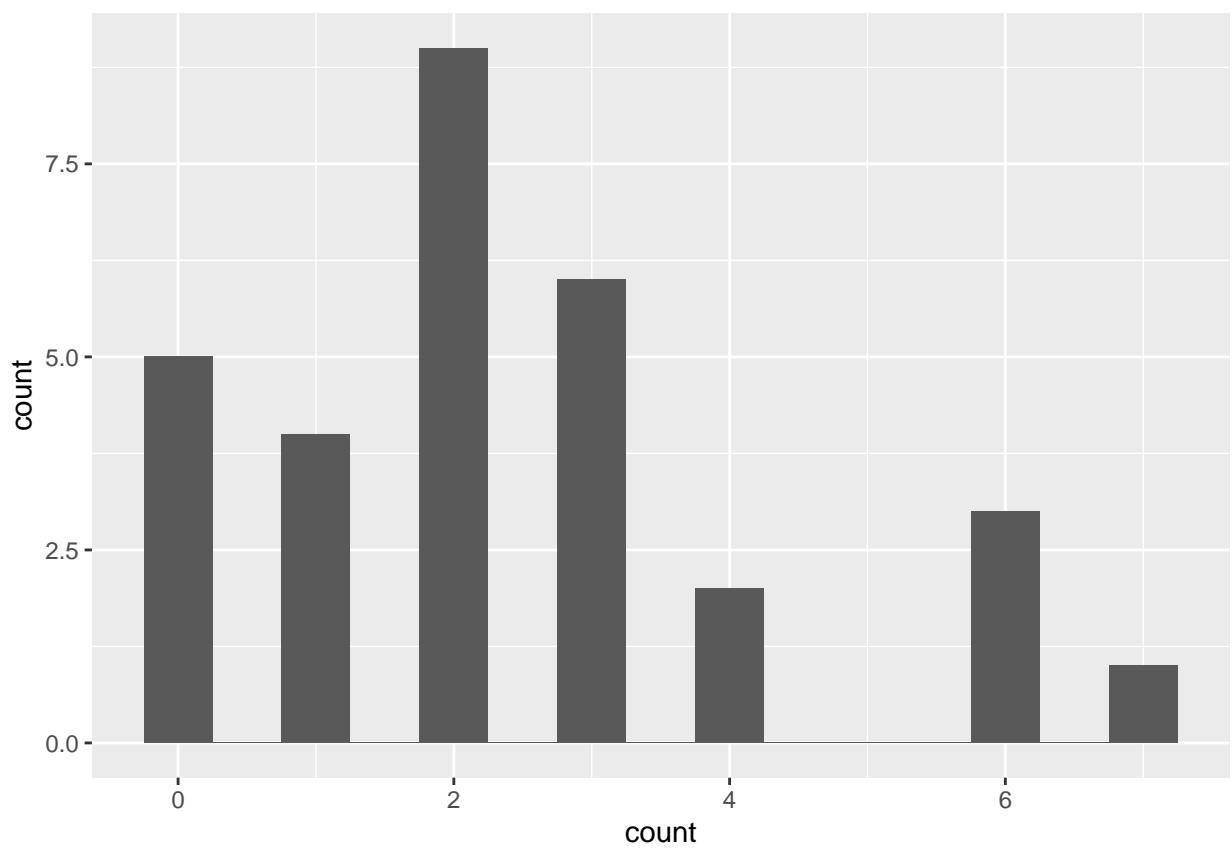


Figure 7: Histogram of plant individuals per plot

Unlike a probability density function (PDF), a probability mass function (PMF) represents the probability of a discrete variable  $x$  taking a specific value. For instance, the probability of  $x = 2$ , denoted as  $\Pr(x = 2)$ , can be calculated as follows:

$$\Pr(x = 2) = g(2) = \frac{\lambda^2 \exp(-\lambda)}{2!}$$

In a generalized form, we write:

$$\Pr(x = k) = g(k) = \frac{\lambda^k \exp(-\lambda)}{k!}$$

### 3.2.2 PMF to frequency distribution

In R, you can use the `dpois()` function to visualize a Poisson distribution. This function allows you to plot the probability mass function (PMF) of the Poisson distribution and gain insights into the distribution of the discrete variable.

```
# vector of x values
# create a vector of 0 to 10 with an interval one
# must be integer of > 0
x <- seq(0, 10, by = 1)

# calculate probability mass
lambda_hat <- mean(df_count$count)
pm <- dpois(x, lambda = lambda_hat)

# figure
tibble(y = pm, x = x) %>% # data frame
  ggplot(aes(x = x, y = y)) +
  geom_line(linetype = "dashed") + # draw dashed lines
  geom_point() + # draw points
  labs(y = "Probability",
       x = "Count") # re-label
```

To convert the y-axis from probability to frequency, multiply the probabilities by the sample size (total number of observations) to obtain the expected frequency. As in the plant height example, you can plot the expected frequency on the histogram, providing a visual representation of the distribution of the discrete variable.

```
df_prob <- tibble(x = x, y = pm) %>%
  mutate(freq = y * nrow(df_count)) # prob x sample size

df_count %>%
  ggplot(aes(x = count)) +
  geom_histogram(binwidth = 0.5,
                 center = 0) +
  geom_line(data = df_prob,
            aes(x = x,
                y = freq),
            linetype = "dashed") +
  geom_point(data = df_prob,
            aes(x = x,
                y = freq))
```

Cool, the Poisson distribution does an excellent job of characterizing the distribution of plant counts.

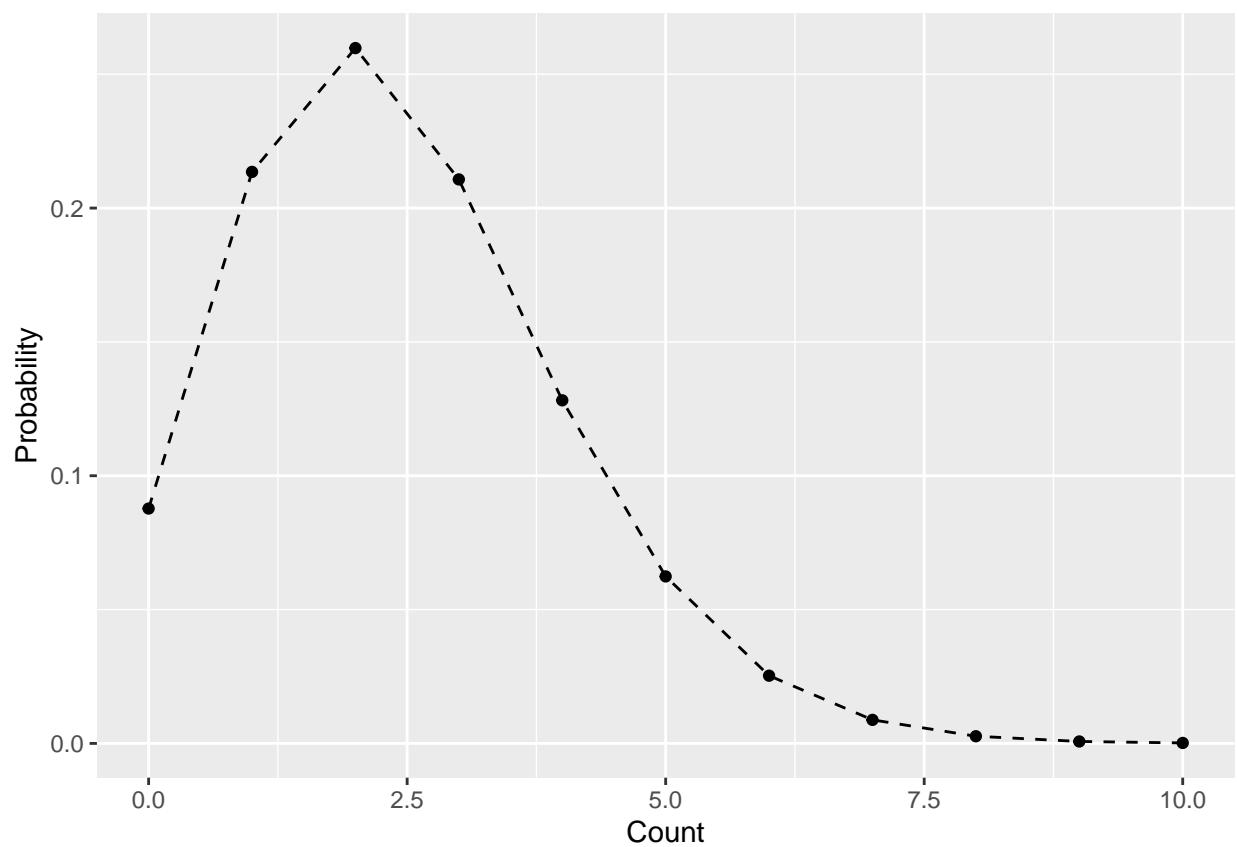


Figure 8: Example of a Poisson distribution

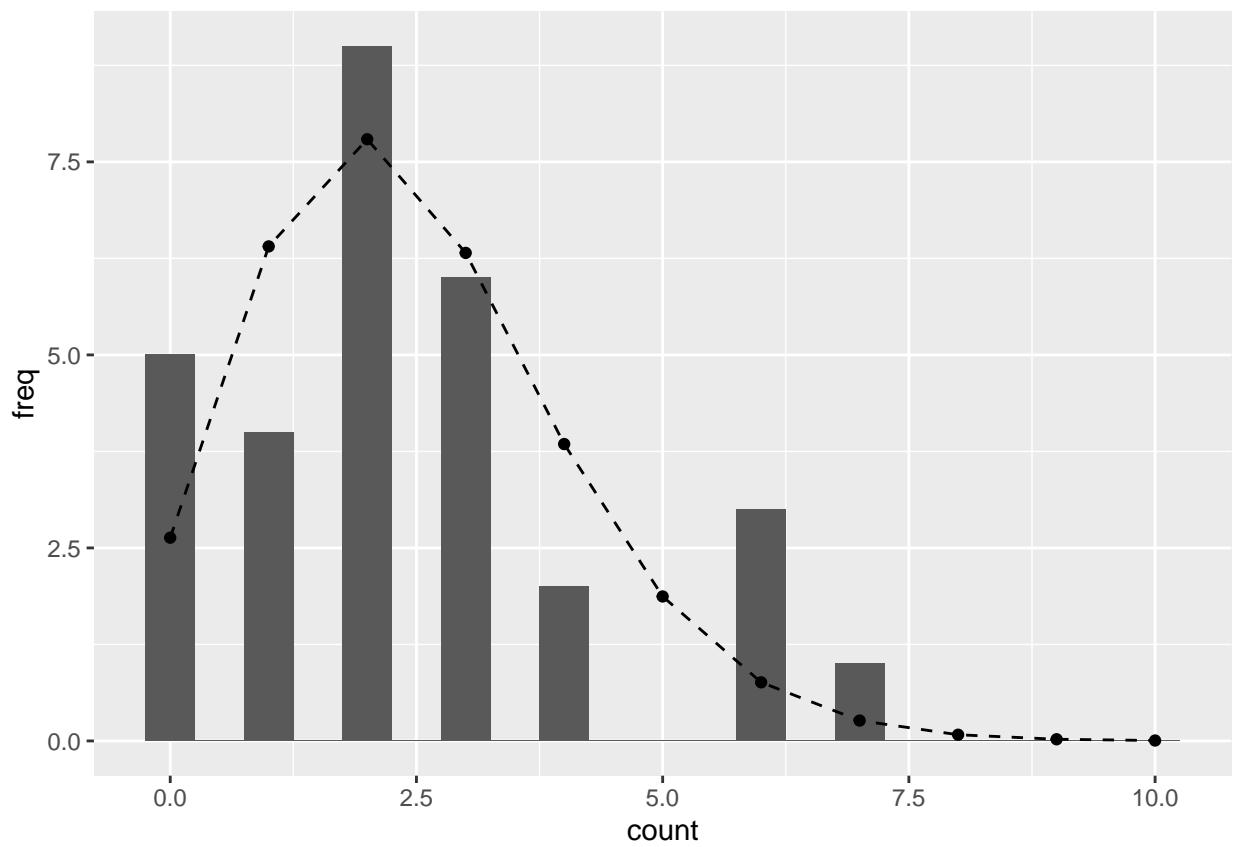


Figure 9: Observed histogram of the discrete variable overlaid with the Poisson expectation.

### 3.3 Why Probability Distributions?

As we have seen in the above examples, probability distributions provide a framework for describing the likelihood of different outcomes or events. By utilizing probability distributions, we can:

1. Make predictions and estimate probabilities of specific events or ranges of values.
2. Determine the most likely values or outcomes based on available information.
3. Quantify and analyze the uncertainty associated with random variables or processes.

Probability distributions serve as fundamental tools in statistics, enabling us to model, analyze, and make informed decisions in the face of uncertainty.

## 3.4 Laboratory

### 3.4.1 Normal Distribution

The function `rnorm()` produces a random variable that follows a Normal distribution with a specified mean and SD. Using this function,

1. Generate a variable with 50 observations.
2. Create a figure similar to Figure 5

### 3.4.2 Poisson Distribution

The function `rpois()` produces a random variable that follows a Poisson distribution with a specified mean. Using this function,

1. Generate a variable with 1000 observations.
2. Create a figure similar to Figure 9

## 4 Two-Group Comparison

Data consists of “samples” extracted from the population of interest, but it’s important to acknowledge that these samples are not flawless replicas of the entire population. Given this imperfect information, how can we effectively investigate and discern the distinction between two populations? In this Chapter, I will introduce one of the most basic statistical tests – “t-test.” The t-test takes the following steps:

1. Define a **test statistic** to signify the difference between groups (t-statistic).
2. Define a probability distribution of t-statistic under the **null hypothesis**, i.e., a scenario that we assume no difference between groups.
3. Estimate the probability of yielding greater or less than the observed test statistic.

### 4.1 Explore Data Structure

Suppose that we study fish populations of the same species in two lakes (Lake **a** and **b**). These lakes are in stark contrast of productivity (Lake **b** looks more productive), and our interest is the difference in *mean* body size between the two lakes. We obtained 50 data points of fish length from each lake (download here). Save under `data_raw/` and use `read_csv()` to read data:

```
library(tidyverse) # call add-in packages everytime you open new R session
df_f1 <- read_csv("data_raw/data_fish_length.csv")
print(df_f1)
```

```
## # A tibble: 100 x 3
##   lake   length unit
##   <chr>   <dbl> <chr>
## 1 a       10.8 cm
## 2 a       13.6 cm
```

```

## 3 a      10.1 cm
## 4 a      18.6 cm
## 5 a      14.2 cm
## 6 a      10.1 cm
## 7 a      14.7 cm
## 8 a      15.6 cm
## 9 a      15    cm
## 10 a     11.9 cm
## # i 90 more rows

```

In this data frame, fish length data from lake **a** and **b** are recorded. Confirm this with `unique()` or `distinct()` function:

```
# unique returns unique values as a vector
unique(df_f1$lake)
```

```
## [1] "a" "b"
# distinct returns unique values as a tibble
distinct(df_f1, lake)
```

```

## # A tibble: 2 x 1
##   lake
##   <chr>
## 1 a
## 2 b

```

Visualization provides a powerful tool for summarizing data effectively. By plotting individual data points overlaid with mean values and error bars, we can observe the distribution and patterns within the data, allowing us to identify trends, variations, and potential outliers:

```
# group mean and sd
df_f1_mu <- df_f1 %>%
  group_by(lake) %>% # group operation
  summarize(mu_1 = mean(length), # summarize by mean()
            sd_1 = sd(length)) # summarize with sd()

# plot
# geom_jitter() plot data points with scatter
# geom_segment() draw lines
# geom_point() draw points
df_f1 %>%
  ggplot(aes(x = lake,
             y = length)) +
  geom_jitter(width = 0.1, # scatter width
              height = 0, # scatter height (no scatter with zero)
              alpha = 0.25) +
  geom_segment(data = df_f1_mu, # switch data frame
               aes(x = lake,
                   xend = lake,
                   y = mu_1 - sd_1,
                   yend = mu_1 + sd_1)) +
  geom_point(data = df_f1_mu, # switch data frame
             aes(x = lake,
                 y = mu_1),
             size = 3) +
  labs(x = "Lake", # x label
```

```
y = "Fish body length") # y label
```

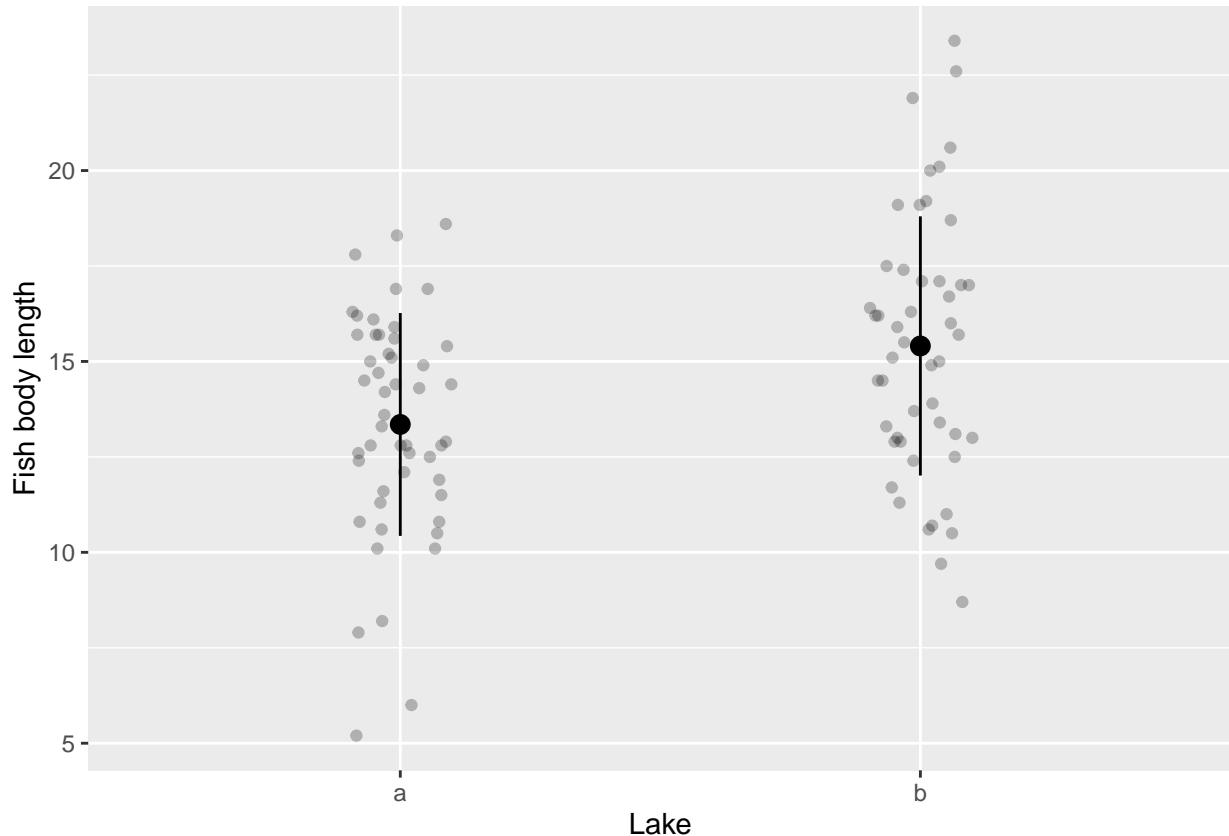


Figure 10: Example of mean and SD plot

Hmm, it is possible that there is a noticeable disparity in body size between Lake a and Lake b. However, how can we ascertain and provide evidence for this claim?

## 4.2 Test Statistic

### 4.2.1 t-statistic

Since our focus is on examining the difference in means, it is logical to estimate the disparity between the sample means in each lake. Let me denote the sample means as  $\hat{\mu}_a$  and  $\hat{\mu}_b$  for Lake a and Lake b, respectively. We can estimate the difference between the means using the following approach:

```
# take another look at df_fl_mu
print(df_fl_mu)

## # A tibble: 2 x 3
##   lake   mu_l   sd_l
##   <chr> <dbl> <dbl>
## 1 a     13.4  2.92
## 2 b     15.4  3.39

# pull mu_l from tibble as vector
v_mu <- df_fl_mu %>%
  pull(mu_l)
```

```

# lake a
print(v_mu[1])

## [1] 13.35

# lake b
print(v_mu[2])

## [1] 15.406

# difference
v_mu[1] - v_mu[2]

## [1] -2.056

```

The average fish body size in Lake b is approximately 2 cm larger than that in Lake a. However, it is crucial to recognize that we are still lacking vital information, specifically, the uncertainty associated with this difference. As discussed in Chapter 2, we must acknowledge that sample means are not flawless representations of the entire population.

Fortunately, we can utilize sample variances to address such uncertainties. The **t-statistic** is a common indicator of the difference of means that takes into consideration the uncertainty associated with sample means.

$$t = \frac{\hat{\mu}_a - \hat{\mu}_b}{\sqrt{\hat{\sigma}_p^2 \left( \frac{1}{N_a} + \frac{1}{N_b} \right)}}$$

where  $N_a$  and  $N_b$  are sample sizes in Lake a and b (i.e.,  $N_a = N_b = 50$  in this specific example), and  $\hat{\sigma}_p^2$  is the weighted mean of sample variances:

$$\hat{\sigma}_p^2 = \frac{N_a - 1}{N_a + N_b - 2} \hat{\sigma}_a^2 + \frac{N_b - 1}{N_a + N_b - 2} \hat{\sigma}_b^2$$

We can calculate this value in R manually:

```

# group mean, variance, and sample size
df_t <- df_f1 %>%
  group_by(lake) %>% # group operation
  summarize(mu_1 = mean(length), # summarize by mean()
            var_1 = var(length), # summarize with sd()
            n = n()) # count number of rows per group

print(df_t)

## # A tibble: 2 x 4
##   lake   mu_1   var_1     n
##   <chr> <dbl> <dbl> <int>
## 1 a      13.4  8.52    50
## 2 b      15.4  11.5    50

# pull values as a vector
v_mu <- pull(df_t, mu_1)
v_var <- pull(df_t, var_1)
v_n <- pull(df_t, n)

var_p <- ((v_n[1] - 1)/(sum(v_n) - 2)) * v_var[1] +

```

```
((v_n[2] - 1)/(sum(v_n) - 2)) * v_var[2]

t_value <- (v_mu[1] - v_mu[2]) / sqrt(var_p * ((1 / v_n[1]) + (1 / v_n[2])))

print(t_value)

## [1] -3.247322
```

The difference in sample means were -2.06; therefore, t-statistic further emphasizes the distinction between the study lakes. This occurrence can be attributed to the t-statistic formula.

In the denominator, we observe the inclusion of  $\hat{\sigma}_p^2$  (estimated pooled variance), as well as the inverses of sample sizes  $N_a$  and  $N_b$ . Consequently, as  $\hat{\sigma}_p^2$  decreases and/or the sample sizes increase, the t-statistic increases. This is a reasonable outcome since a decrease in variance and/or an increase in sample size enhance the certainty of the mean difference.

#### 4.2.2 Null Hypothesis

The observed t-statistic serves a dual purpose in accounting for both the disparity of means and the accompanying uncertainty. However, the significance of this t-statistic remains unclear.

To address this, the concept of the **Null Hypothesis** is utilized to substantiate the observed t-statistic. The Null Hypothesis, no difference between groups or  $\mu_a = \mu_b$ , allows us to draw the probability distribution of the test-statistic. Once we know the probability distribution, we can *estimate the probability of observing the given t-statistic by random chance* if there were no difference in mean body size between the lakes.

Let's begin by assuming that there is no difference in the mean body size of fish populations between the lakes, meaning the true difference in means is zero ( $\mu_a - \mu_b = 0$ ; without hats in this formula!). Under this assumption, we can define the probability distribution of t-statistics, which represents how t-statistics are distributed across a range of possible values.

This probability distribution is known as the Student's t-distribution and is characterized by three parameters: the mean, the variance, and the degrees of freedom (d.f.). Considering that we are evaluating the difference in body size as the test statistic, the mean (of the difference) is assumed to be zero. The variance is estimated using  $\hat{\sigma}_p^2$ , and the degrees of freedom are determined by  $N_a + N_b - 2$  (where d.f. can be considered as a measure related to the sample size<sup>2</sup>).

R has a function to draw the Student's t-distribution; let's try that out:

```
# produce 500 values from -5 to 5 with equal interval
x <- seq(-5, 5, length = 500)

# probability density of t-statistics with df = sum(v_n) - 2
y <- dt(x, df = sum(v_n) - 2)

# draw figure
tibble(x, y) %>%
  ggplot(aes(x = x,
             y = y)) +
  geom_line() +
  labs(y = "Probability density",
       x = "t-statistic")
```

The probability density in a distribution (Figure 11) determines the likelihood of observing a particular t-statistic. Higher probability density indicates a greater likelihood of the t-statistic occurring. Compare the observed t-statistic against this null distribution:

---

<sup>2</sup>Discussing the concept of degrees of freedom in depth would require a more comprehensive explanation, so I will refer to it here as a quantity relevant to the sample size.

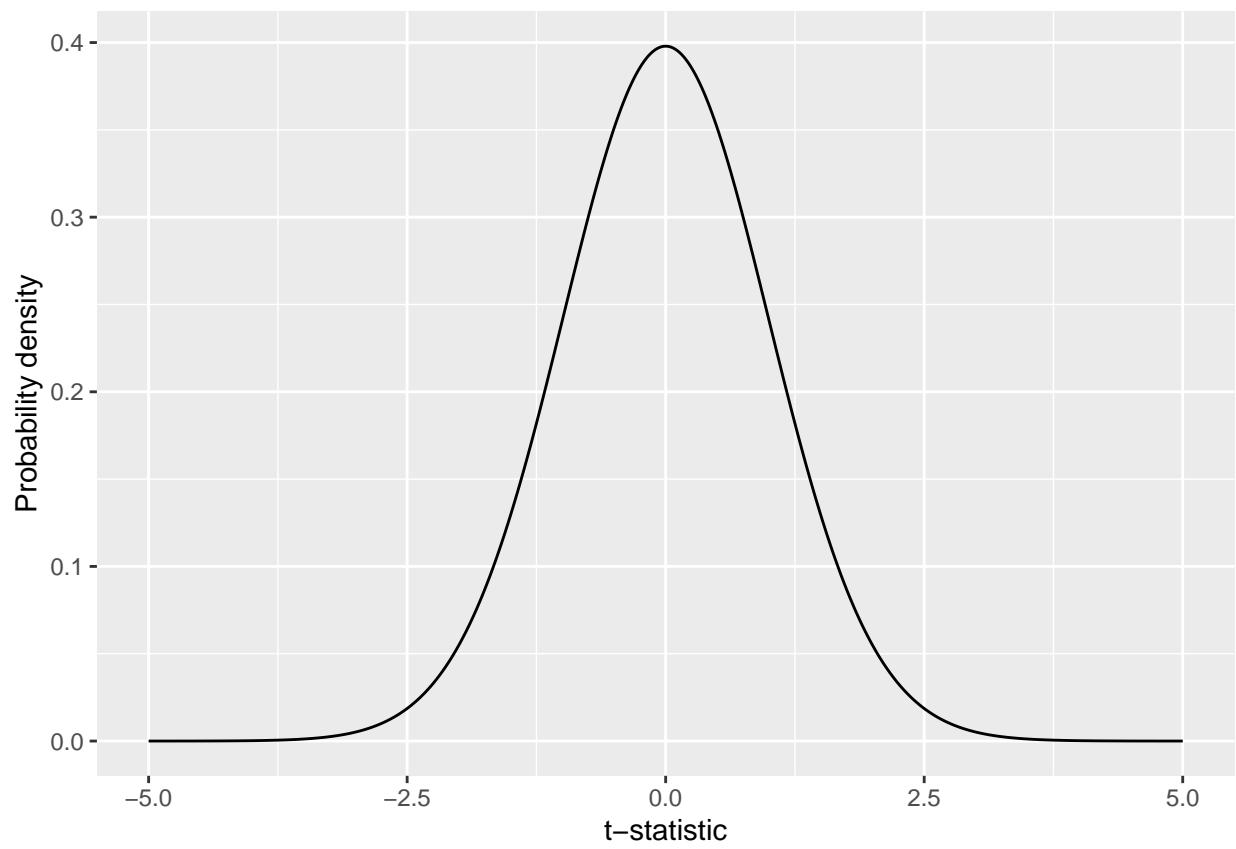


Figure 11: Distribution of t-statistics under null hypothesis. The probability density in a distribution determines the likelihood of observing a particular t-statistic. Higher probability density indicates a greater likelihood of the t-statistic occurring.

```

# draw entire range
tibble(x, y) %>%
  ggplot(aes(x =
             y = y)) +
  geom_line() +
  geom_vline(xintercept = t_value,
             color = "salmon") + # t_value is the observed t_value
  geom_vline(xintercept = abs(t_value),
             color = "salmon") + # t_value is the observed t_value
  labs(y = "Probability density",
       x = "t-statistic")

```

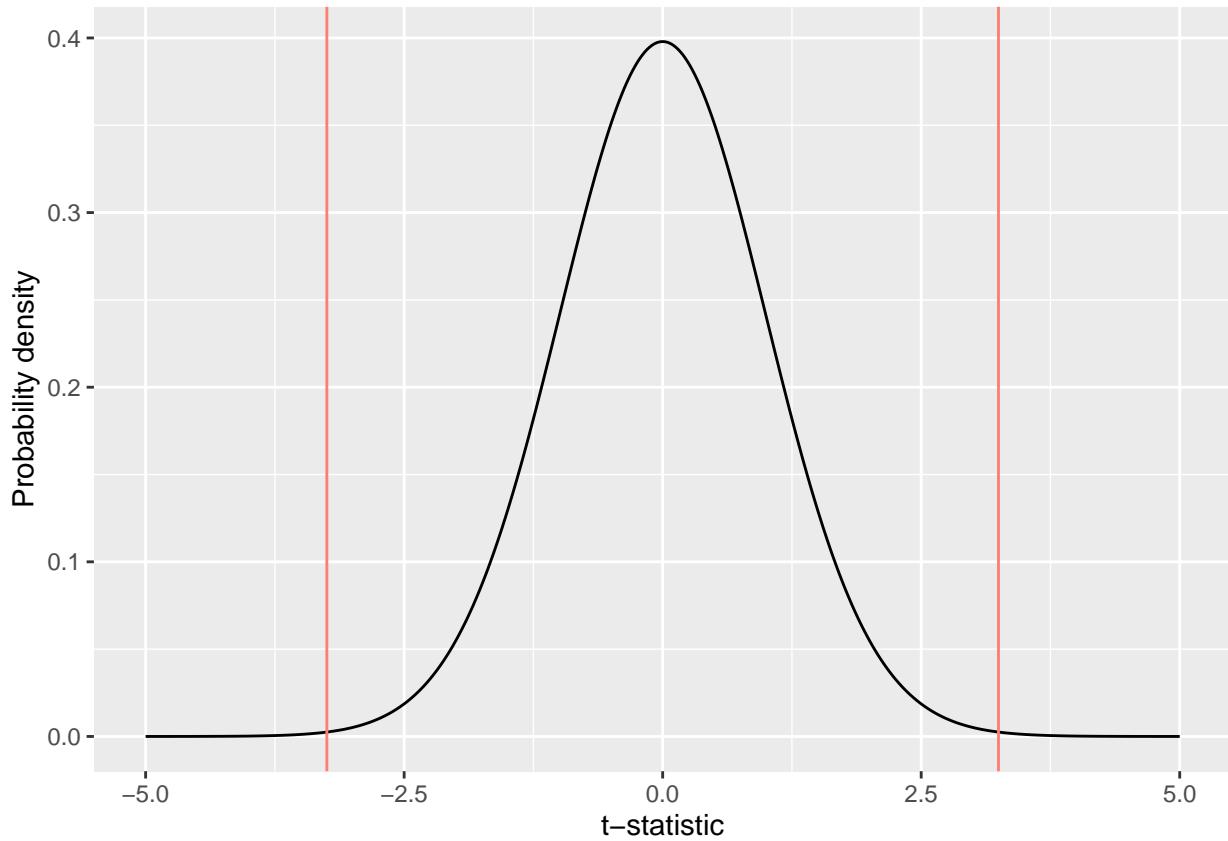


Figure 12: Observed t-statistic in comparison to the null distribution

In a probability distribution, the area under the curve corresponds to the probability. Notably, **the area under the curve that falls below or above the observed t-statistic (indicated by vertical red lines) is very small (Figure 12)**, meaning that the observed difference in body size is very unlikely to occur under the null hypothesis (no difference in true means).

The function `pt()` allows us to calculate the area under the curve:

```

# calculate area under the curve from -infinity to t_value
pr_below <- pt(q = t_value, df = sum(v_n) - 2)

# calculate area under the curve from abs(t_value) to infinity
pr_above <- 1 - pt(q = abs(t_value), df = sum(v_n) - 2)

```

The **p-value**, i.e., the probability of observing t-statistics less than (`pr_below`) or greater than (`pr_above`) the observed t-statistic under the null hypothesis, can be estimated as the sum of `pr_below` and `pr_above`.

```
# p_value
p_value <- pr_below + pr_above
print(p_value)
```

```
## [1] 0.001595529
```

### 4.2.3 Interpretation

The above exercise is called **t-test** – perhaps, the most well-known hypothesis testing. To explain the interpretation of the results, let me use the following notations.  $t_{obs}$ , the observed t-statistic;  $t_0$ , possible t-statistics under the null hypothesis;  $\Pr(\cdot)$ , the probability of “.” occurs – for example,  $\Pr(x > 2)$  means that the probability of  $x$  exceeding 2.

We first calculated the observed t-statistic  $t_{obs}$  using the data of fish body size, which was -3.25. Then, we calculated p-value, i.e.,  $\Pr(t_0 < t_{obs}) + \Pr(t_0 > t_{obs})$  under the null hypothesis of  $\mu_a = \mu_b$ . We found p-value was very low, meaning that the observed t-statistic is very unlikely to occur if  $\mu_a = \mu_b$  is the truth. Therefore, it is logical to conclude that the **Alternative Hypothesis**  $\mu_a \neq \mu_b$  is very likely.

The same logic is used in many statistical analyses – define the null hypothesis, and estimate the probability of observing a given value under the null hypothesis. It is tricky, but I find this genius: we can substantiate the observation(s) objectively that are otherwise a subjective claim!

However, it is crucial to recognize that, even if we found the observed t-statistic that results in very high p-value (i.e., very common under the null hypothesis), finding such a result does NOT support the null hypothesis – we just can't reject it.

## 4.3 t-test in R

### 4.3.1 t-test with equal variance

In R, this t-test can be performed with `t.test()`. Let me examine if the function gives us identical results:

```
x <- df_f1 %>%
  filter(lake == "a") %>% # subset lake a
  pull(length)

y <- df_f1 %>%
  filter(lake == "b") %>% # subset lake b
  pull(length)

t.test(x, y, var.equal = TRUE)

##
##  Two Sample t-test
##
## data: x and y
## t = -3.2473, df = 98, p-value = 0.001596
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
## -3.3124399 -0.7995601
## sample estimates:
## mean of x mean of y
## 13.350    15.406
```

The reported t-statistic  $t = -3.2473$ , the p-value  $p\text{-value} = 0.001596$ , and the degrees of freedom all

agreed (`df = 98`) with what we have estimated manually. Importantly, this t-test assumes relative similarity of variance between groups.

### 4.3.2 t-test with unequal variance

The relative similarity of variance between groups could be an unrealistic assumption. Luckily, there is a variant of t-test “Welch’s t-test,” in which we assume unequal variance between groups. The implementation is easy – set `var.equal = FALSE` in `t.test()`:

```
t.test(x, y, var.equal = FALSE)
```

```
##  
## Welch Two Sample t-test  
##  
## data: x and y  
## t = -3.2473, df = 95.846, p-value = 0.001606  
## alternative hypothesis: true difference in means is not equal to 0  
## 95 percent confidence interval:  
## -3.3127929 -0.7992071  
## sample estimates:  
## mean of x mean of y  
## 13.350 15.406
```

In Welch’s t-test, t-statistics are defined differently to account for unequal variance between groups<sup>3</sup>:

$$t = \frac{\hat{\mu}_a - \hat{\mu}_b}{\sqrt{\left(\frac{\hat{\sigma}_a^2}{N_a} + \frac{\hat{\sigma}_b^2}{N_b}\right)}}$$

This t-statistic is known to follow the Student’s t-distribution with the degrees of freedom:

$$d.f. = \frac{\frac{\hat{\sigma}_a^2}{N_a} + \frac{\hat{\sigma}_b^2}{N_b}}{\frac{(\hat{\sigma}_a^2/N_a)^2}{N_a-1} + \frac{(\hat{\sigma}_b^2/N_b)^2}{N_b-1}}$$

Therefore, the reported values of t-statistic and d.f. are different from what we estimated. The Welch’s t-test covers the cases for equal variances; therefore, by default, we use the Welch’s test.

## 4.4 Laboratory

## 5 Multiple-Group Comparison

The t-test is used to compare two groups, but when there are more than two groups, ANOVA (Analysis of Variance) is employed. ANOVA allows for simultaneous comparison of means among multiple groups to determine if there are statistically significant differences. ANOVA uses the following steps:

1. Partition the total variability into **between-group** and **within-group** components.
2. Define a **test statistic** as a ratio of between-group variability to within-group variability (F statistic).
3. Define a probability distribution of F-statistic under the null hypothesis.
4. Estimate the probability of yielding greater the observed test statistic.

If appropriate, post-hoc tests can be conducted to identify specific differing groups.

---

<sup>3</sup>When  $\hat{\sigma}_a^2 \approx \hat{\sigma}_b^2$ , the t-statistic for Welch’s t-test is reduced to the original t-statistic.

## 5.1 Partition the Variability

The first step is to determine what aspect to examine. In the case of a t-test, we focus on the difference in sample means between groups. One might initially consider conducting t-tests for all possible combinations. However, this approach leads to a problem known as the multiple comparisons problem. Hence, this is not a viable option. Therefore, we need to explore the data from a different perspective.

To facilitate learning, let's once again utilize the lake fish data, but this time we have three lakes in our analysis:

```
df_anova <- read_csv("data_raw/data_fish_length_anova.csv")
distinct(df_anova, lake)
```

```
## # A tibble: 3 x 1
##   lake
##   <chr>
## 1 a
## 2 b
## 3 c
```

Visualization can be helpful in understanding how the data is distributed. While mean  $\pm$  SD is perfectly fine here, let me use a violin plot to show a different way of visualization.

```
# geom_violin() - function for violin plots
# geom_jitter() - jittered points

df_anova %>%
  ggplot(aes(x = lake,
             y = length)) +
  geom_violin(draw_quantiles = 0.5, # draw median horizontal line
              alpha = 0.2) + # transparency
  geom_jitter(alpha = 0.2) # transparency
```

It appears that Lake b exhibits a larger average body size compared to the other lakes. One approach to quantifying this difference between groups is by examining the ratio of between-group variability to within-group variability. **If we observe a greater between-group variability relative to within-group variability, it suggests that the differences among the groups are substantial.** In other words, much of the observed variation is explained by the group structure (lake).

Let me denote between-group and within-group variability as  $S_b$  and  $S_w$ , respectively, which are defined as follows:

$$S_b = \sum_g \sum_i (\hat{\mu}_{g(i)} - \hat{\mu})^2$$

$$S_w = \sum_g \sum_i (x_i - \hat{\mu}_{g(i)})^2$$

The double summation  $\sum_g \sum_i$  might scare you, but no worries. We can decompose this equation into two steps.

### 5.1.1 Between-group variability

Let me first consider  $S_b = \sum_g \sum_i (\hat{\mu}_{g(i)} - \hat{\mu})^2$ . In this equation,  $\hat{\mu}$  is the overall mean of the fish length and  $\hat{\mu}_{g(i)}$  is the group-mean in a given lake ( $g = \{a, b, c\}$ ). Let's perform this estimation in R:

```
# estimate overall mean
mu <- mean(df_anova$length)
```

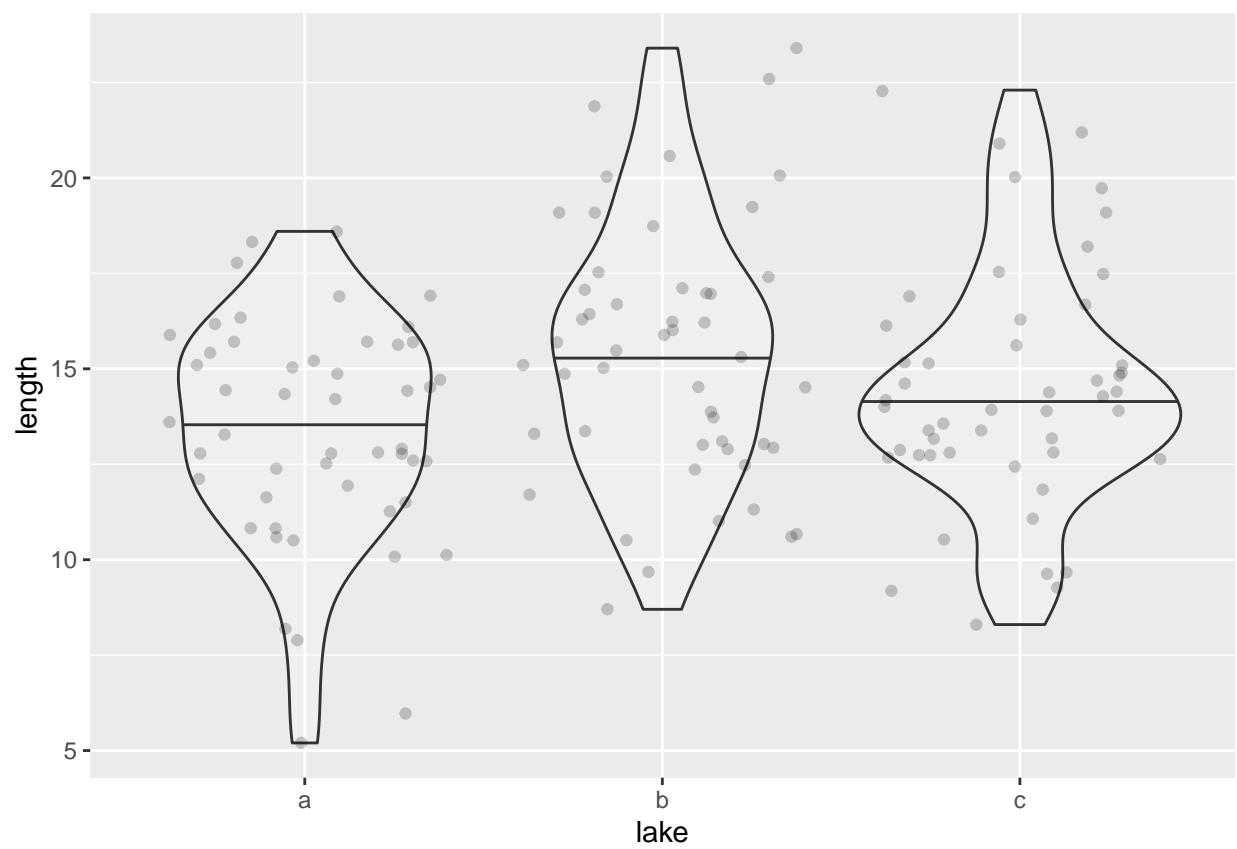


Figure 13: Violin plot for fish length in three lakes.

```

# estimate group means and sample size each
df_g <- df_anova %>%
  group_by(lake) %>%
  summarize(mu_g = mean(length), # mean for each group
            dev_g = (mu_g - mu)^2, # squared deviation for each group
            n = n()) # sample size for each group

print(df_g)

## # A tibble: 3 x 4
##   lake   mu_g   dev_g     n
##   <chr> <dbl>   <dbl> <int>
## 1 a      13.4   1.12     50
## 2 b      15.4   0.997    50
## 3 c      14.5   0.00344   50

```

In the column `dev_g`, we estimated  $(\hat{\mu}_{g(i)} - \hat{\mu})^2$ . We must sum over  $i$  (fish individual) to get the variability in each lake ( $\sum_i (\hat{\mu}_{g(i)} - \hat{\mu})^2$ ). Since  $\hat{\mu}_{g(i)}$  is constant in each lake, we can simply multiply `dev_g` with sample size `n` in each lake:

```

df_g <- df_g %>%
  mutate(ss = dev_g * n)

print(df_g)

## # A tibble: 3 x 5
##   lake   mu_g   dev_g     n     ss
##   <chr> <dbl>   <dbl> <int>  <dbl>
## 1 a      13.4   1.12     50  55.9
## 2 b      15.4   0.997    50  49.9
## 3 c      14.5   0.00344   50  0.172

```

Sum over  $g$  (lake) to get  $S_b$ .

```

s_b <- sum(df_g$ss)
print(s_b)

## [1] 105.9365

```

### 5.1.2 Within-group variability

We can follow the same steps to estimate the within-group variability  $S_w = \sum_g \sum_i (x_i - \hat{\mu}_{g(i)})^2$ . Let's estimate  $(x_i - \hat{\mu}_{g(i)})^2$  first:

```

df_i <- df_anova %>%
  group_by(lake) %>%
  mutate(mu_g = mean(length)) %>% # use mutate() to retain individual rows
  ungroup() %>%
  mutate(dev_i = (length - mu_g)^2) # deviation from group mean for each fish

```

You can take a look at group-level data with the following code; the column `mu_g` contains group-specific means of fish length, and `dev_i` contains  $(x_{g(i)} - \hat{\mu}_{g(i)})^2$ :

```

# filter() & slice(): show first 3 rows each group
print(df_i %>% filter(lake == "a") %>% slice(1:3))

print(df_i %>% filter(lake == "b") %>% slice(1:3))

```

```
print(df_i %>% filter(lake == "c") %>% slice(1:3))
```

Sum over  $i$  in each lake  $g$  ( $\sum_i (x_{g(i)} - \hat{\mu}_{g(i)})^2$ ):

```
df_i_g <- df_i %>%
  group_by(lake) %>%
  summarize(ss = sum(dev_i))

print(df_i_g)
```

```
## # A tibble: 3 x 2
##   lake     ss
##   <chr> <dbl>
## 1 a      417.
## 2 b      565.
## 3 c      486.
```

Then sum over  $g$  to get  $S_w$ :

```
s_w <- sum(df_i_g$ss)
print(s_w)

## [1] 1467.645
```

### 5.1.3 Variability to Variance

I referred to  $S_b$  and  $S_w$  as “variability,” which essentially represents the summation of squared deviations. To convert them into variances, we can divide them by appropriate numbers. In Chapter 2, I mentioned that the denominator for variance is the sample size minus one. The same principle applies here, but with caution.

For the between-group variability, denoted as  $S_b$ , the realized sample size is the number of groups,  $N_g$ , which in this case is three (representing the number of lakes). Therefore, we divide by three minus one to obtain an unbiased estimate of the between-group variance, denoted as  $\hat{\sigma}_b^2$ :

$$\hat{\sigma}_b^2 = \frac{S_b}{N_g - 1}$$

```
# n_distinct() count the number of unique elements
n_g <- n_distinct(df_anova$lake)
s2_b <- s_b / (n_g - 1)
print(s2_b)

## [1] 52.96827
```

Meanwhile, we need to be careful when estimating the within-group variance. Since the within-group variance is measured at the individual level, the number of data used is equal to the number of fish individuals. Yet, we subtract the number of groups – while the rationale behind this is beyond the scope, we are essentially accounting for the fact that some of the degrees of freedom are “used up” in estimating the group means. As such, we estimate the within-group variance  $\hat{\sigma}_w^2$  as follows:

$$\hat{\sigma}_w^2 = \frac{S_w}{N - N_g}$$

```
s2_w <- s_w / (nrow(df_anova) - n_g)
print(s2_w)
```

```
## [1] 9.983982
```

## 5.2 Test Statistic

### 5.2.1 F-statistic

In ANOVA, we use F-statistic – the ratio of between-group variability to within-group variability. The above exercise was essentially performed to yield this test statistic:

$$F = \frac{\text{between-group variance}}{\text{within-group variance}} = \frac{\hat{\sigma}_b^2}{\hat{\sigma}_w^2}$$

```
f_value <- s2_b / s2_w
print(f_value)
```

```
## [1] 5.305325
```

The F-statistic in our data was calculated to be 5.31. This indicates that the between-group variance is approximately five times higher than the within-group variance. While this difference appears significant, it is important to determine whether it is statistically substantial. To make such a claim, we can use the Null Hypothesis as a reference.

### 5.2.2 Null Hypothesis

The F-statistic follows an F-distribution when there is no difference in means among the groups. Therefore, the null hypothesis we are considering in our example is that the means of all groups are equal, represented as  $\mu_a = \mu_b = \mu_c$ . It's worth noting that the alternative hypotheses can take different forms, such as  $\mu_a \neq \mu_b = \mu_c$ ,  $\mu_a = \mu_b \neq \mu_c$ , or  $\mu_a \neq \mu_b \neq \mu_c$ . ANOVA, however, is unable to distinguish between these alternative hypotheses.

The degrees of freedom in an F-distribution are determined by two parameters:  $N_g - 1$  and  $N - N_g$ . To visualize the distribution, you can utilize the `df()` function. Similar to the t-test, we can plot the F-distribution and draw a vertical line to represent the observed F-statistic. This approach allows us to assess the position of the observed F-statistic within the distribution and determine the associated p-value.

```
x <- seq(0, 10, by = 0.1)
y <- df(x = x, df1 = n_g - 1, df2 = nrow(df_anova) - n_g)

tibble(x = x, y = y) %>%
  ggplot(aes(x = x,
             y = y)) +
  geom_line() + # F distribution
  geom_vline(xintercept = f_value,
             color = "salmon") # observed F-statistic
```

Unlike t-statistics, F-statistics can take only positive values (because F-statistics are the ratio of positive values). The p-value here is  $\Pr(F_0 > F)$ , where  $F_0$  is the possible F-statistics under the null hypothesis. Let's estimate this probability using `pf()`:

```
# pf() estimate the probability of less than q
# Pr(F0 > F) is 1 - Pr(F0 < F)
p_value <- 1 - pf(q = f_value, df1 = n_g - 1, df2 = nrow(df_anova) - n_g)
print(p_value)

## [1] 0.00596054
```

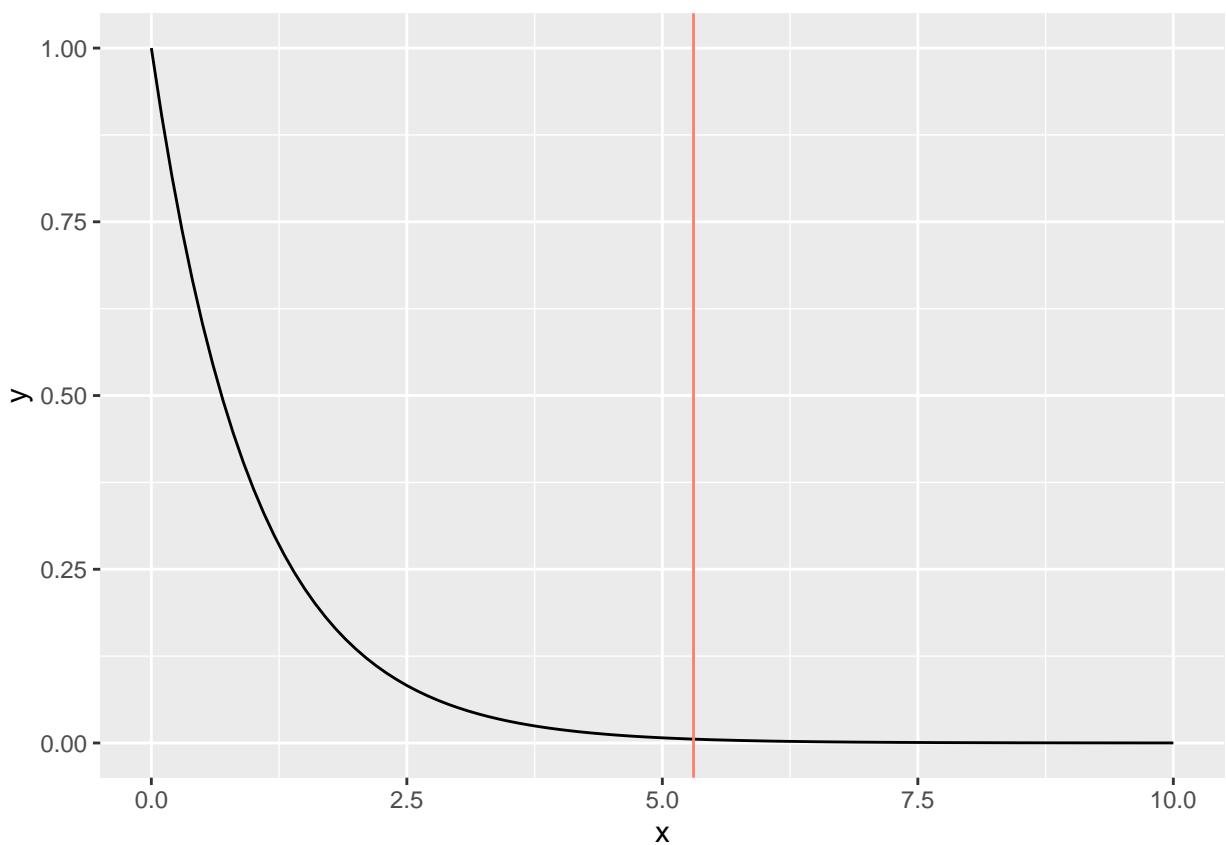


Figure 14: F-distribution. The vertical red line denotes the observed F-statistic.

### 5.3 ANOVA in R

Like the t-test, R provides functions to perform ANOVA easily. In this case, we will utilize the `aov()` function. The first argument of this function is the “formula,” which is used to describe the structure of the model. In our scenario, we aim to explain the fish body length by grouping them according to lakes. In the formula expression, we represent this relationship as `length ~ lake`, using the tilde symbol (`~`) to indicate the stochastic nature of the relationship.

```
# first argument is formula
# second argument is data frame for reference
# do not forget specify data = XXX! aov() refer to columns in the data frame
m <- aov(formula = length ~ lake,
          data = df_anova)

print(m)

## Call:
##   aov(formula = length ~ lake, data = df_anova)
##
## Terms:
##           lake Residuals
## Sum of Squares 105.9365 1467.6454
## Deg. of Freedom      2        147
##
## Residual standard error: 3.159744
## Estimated effects may be unbalanced
```

The function returns `Sum of Squares` and `Deg. of Freedom` – these value matches what we have calculated. To get deeper insights, wrap the object with `summary()`:

```
summary(m)

##           Df Sum Sq Mean Sq F value    Pr(>F)
## lake         2 105.9   52.97   5.305 0.00596 ***
## Residuals   147 1467.6    9.98
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The F-statistic and p-value are identical to our manual estimate.

### 5.4 Post-hoc Tests

When conducting an ANOVA and finding a significant difference among group means, a post-hoc test is often performed to determine which specific groups differ significantly from each other. Post-hoc tests help to identify pairwise comparisons that contribute to the observed overall difference.

There are several post-hoc tests available for ANOVA, including:

1. Tukey’s Honestly Significant Difference (HSD): This test compares all possible pairs of group means and provides adjusted p-values to control for the family-wise error rate. It is a commonly used and reliable post-hoc test.
2. Bonferroni correction: This method adjusts the significance level for each individual comparison to control the overall Type I error rate. The adjusted p-value is obtained by dividing the desired significance level (e.g., 0.05) by the number of comparisons.
3. Scheffe’s test: This test controls the family-wise error rate by considering all possible pairwise comparisons. It is more conservative than Tukey’s HSD, but it is suitable for cases where specific comparisons are of particular interest.

- Dunnett's test: This test is useful when comparing multiple groups against a control group. It controls the overall error rate by conducting multiple t-tests between the control group and each of the other groups.

The choice of post-hoc test depends on the specific research question, assumptions, and desired control of the Type I error rate. It is important to select an appropriate test and interpret the results accordingly to draw valid conclusions about group differences in an ANOVA analysis.

## 5.5 Laboratory

# 6 Regression

Our previous examples focused on cases that involved a distinct group structure. However, it is not always the case that such a structure exists. Instead, we might examine the relationship between continuous variables. In this chapter, I will introduce a technique called linear regression.

## 6.1 Explore Data Structure

For this exercise, we will use algae biomass data from streams. Download the data here and locate it in `data_raw/`.

```
library(tidyverse)
df_algae <- read_csv("data_raw/data_algae.csv")
print(df_algae)

## # A tibble: 50 x 4
##   biomass unit_biomass conductivity unit_cond
##   <dbl>    <chr>        <dbl>    <chr>
## 1 19.8 mg_per_m2     30.2 ms
## 2 24.4 mg_per_m2     40.4 ms
## 3 27.4 mg_per_m2     59.4 ms
## 4 48.2 mg_per_m2     91.3 ms
## 5 19.2 mg_per_m2     24.2 ms
## 6 57.0 mg_per_m2     90.4 ms
## 7 51.9 mg_per_m2     94.7 ms
## 8 40.8 mg_per_m2     67.8 ms
## 9 37.1 mg_per_m2     64.8 ms
## 10 3.55 mg_per_m2    10.9 ms
## # i 40 more rows
```

This dataset comprises data collected from 50 sampling sites. The variable `biomass` represents the standing biomass, indicating the dry mass of algae at the time of collection. On the other hand, `conductivity` serves as a proxy for water quality, with higher values usually indicating a higher nutrient content in the water. Let me now proceed to draw a scatter plot that will help visualize the relationship between these two variables.

```
df_algae %>%
  ggplot(aes(x = conductivity,
             y = biomass)) +
  geom_point()
```

It seems that there is a noticeable positive correlation where increased conductivity results in higher algal biomass. Nevertheless, how can we accurately represent this relationship with a line?

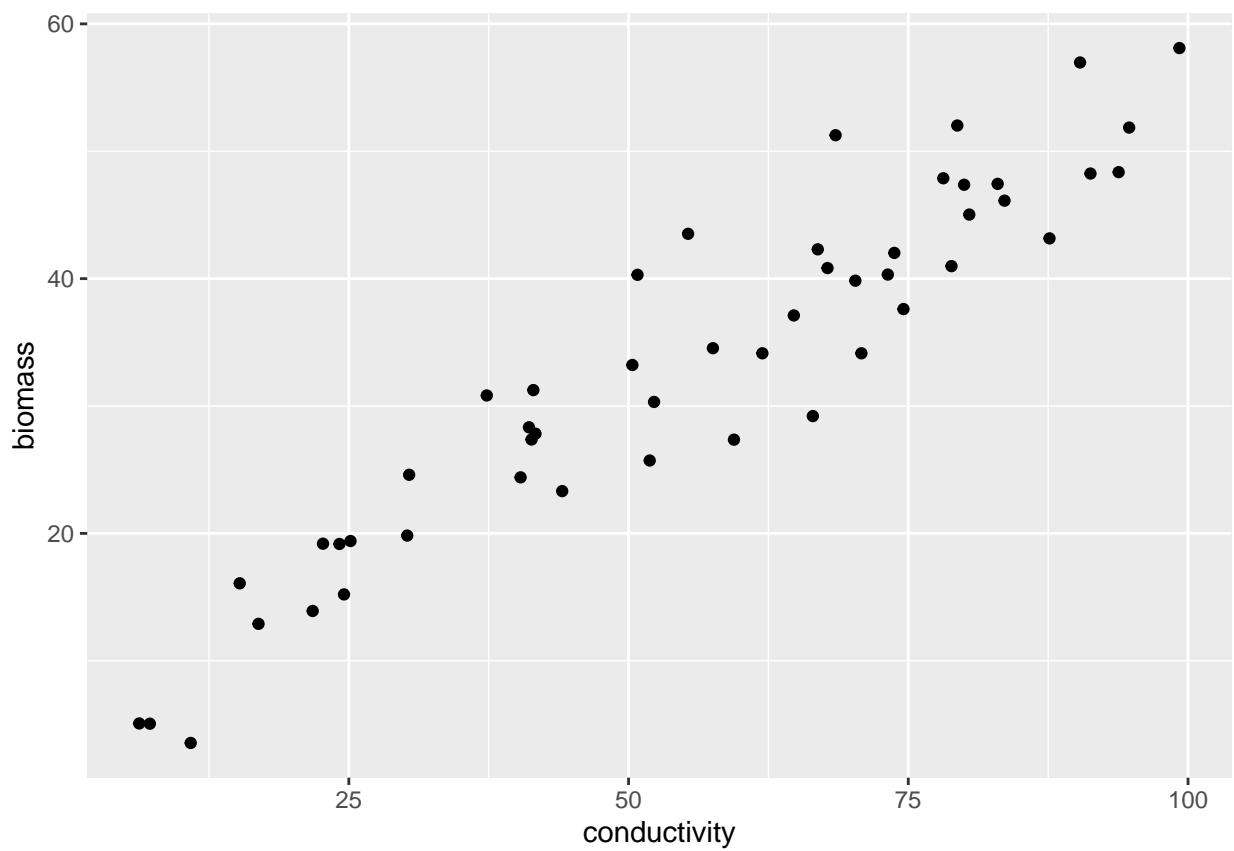


Figure 15: The relationship between algae biomass and conductivity.

## 6.2 Drawing the “fittest” line

### 6.2.1 Linear formula

The first step is to represent the relationship as a formula; let me ASSUME that the relationship can be described by the following linear formula:

$$y_i = \alpha + \beta x_i$$

In this formula, the algal biomass, denoted as  $y_i$ , at site  $i$ , is expressed as a function of conductivity, represented by  $x_i$ . The variable being explained ( $y_i$ ) is referred to as a **response (or dependent) variable**, while the variable used to predict the response variable ( $x_i$ ) is referred to as an **explanatory (or independent) variable**. However, there are two additional constants, namely  $\alpha$  and  $\beta$ .  $\alpha$  is commonly known as the **intercept**, while  $\beta$  is referred to as the **slope or coefficient**. The intercept represents the value of  $y$  when  $x$  is equal to zero, while the slope indicates the change in  $y$  associated with a unit increase in  $x$  (refer to Figure 16).

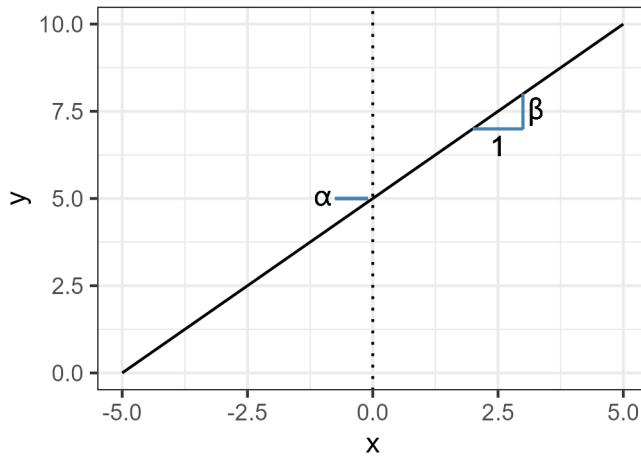


Figure 16: Intercept and slope.

However, this formula, or “model,” is incomplete. We must add the error term  $\varepsilon_i$  (**residual**) to consider the uncertainty associated with the observation process.

$$y_i = \alpha + \beta x_i + \varepsilon_i \sim \text{Normal}(0, \sigma^2)$$

Notice that  $y_i$ ,  $x_i$ , and  $\varepsilon_i$  have subscripts, while  $\alpha$  and  $\beta$  do not. This means that  $y_i$ ,  $x_i$ , and  $\varepsilon_i$  vary by data point  $i$ , and  $\alpha$  and  $\beta$  are constants. Although  $\alpha + \beta x_i$  cannot reproduce  $y_i$  perfectly, we “fill” the gaps with the error term  $\varepsilon_i$ , which is assumed to follow a Normal distribution.

In R, finding the best  $\alpha$  and  $\beta$  – i.e., the parameters in this model – is very easy. Function `lm()` does everything for you. Let me apply this function to the example data set:

```
# lm() takes a formula as the first argument
# don't forget to supply your data
m <- lm(biomass ~ conductivity,
         data = df_algae)

summary(m)

##
```

```

## Call:
## lm(formula = biomass ~ conductivity, data = df_algae)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -9.5578 -2.8729 -0.7307  2.5479 11.4700
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 5.29647   1.57178   3.37  0.00149 **
## conductivity 0.50355   0.02568  19.61 < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 4.649 on 48 degrees of freedom
## Multiple R-squared:  0.889, Adjusted R-squared:  0.8867
## F-statistic: 384.5 on 1 and 48 DF, p-value: < 2.2e-16

```

In `Coefficients:`, it says `(Intercept)` is 5.3 and `conductivity` is 0.5. These values corresponds to  $\alpha$  and  $\beta$ . Meanwhile, `Residual standard error:` indicates the SD of the error term  $\sigma$ . Thus, substituting these values into the formula yields:

$$y_i = 5.30 + 0.50x_i + \varepsilon_i \sim \text{Normal}(0, 4.6^2)$$

We can draw this “fittest” line on the figure:

```

# coef() extracts estimated coefficients
# e.g., coef(m)[1] is (Intercept)

alpha <- coef(m)[1]
beta <- coef(m)[2]

df_algae %>%
  ggplot(aes(x = conductivity,
             y = biomass)) +
  geom_point() +
  geom_abline(intercept = alpha,
              slope = beta) # draw the line

```

However, how did `lm()` find  $\alpha$  and  $\beta$ ?

### 6.2.2 Minimizing the errors

Notice that the error term is the difference between  $y_i$  and  $\alpha + \beta x_i$ :

$$\varepsilon_i = y_i - (\alpha + \beta x_i)$$

In other words, this term represents the portion of the observation  $y_i$  that cannot be explained by the conductivity  $x_i$ . Therefore, a logical approach would be to find values for  $\alpha$  and  $\beta$  that minimize this unexplained portion across all data points  $i$ .

**Least squares methods** are a widely employed statistical approach to achieve this objective. These methods aim to minimize the sum of squared errors, denoted as  $\sum_i \varepsilon_i^2$ , across all data points. As the deviation from the expected value of  $\alpha + \beta x_i$  increases, this quantity also increases. Consequently, determining parameter values ( $\alpha$  and  $\beta$ ) that minimize this sum of squared errors yields the best-fitting formula (see Section 6.2.3 for further details).

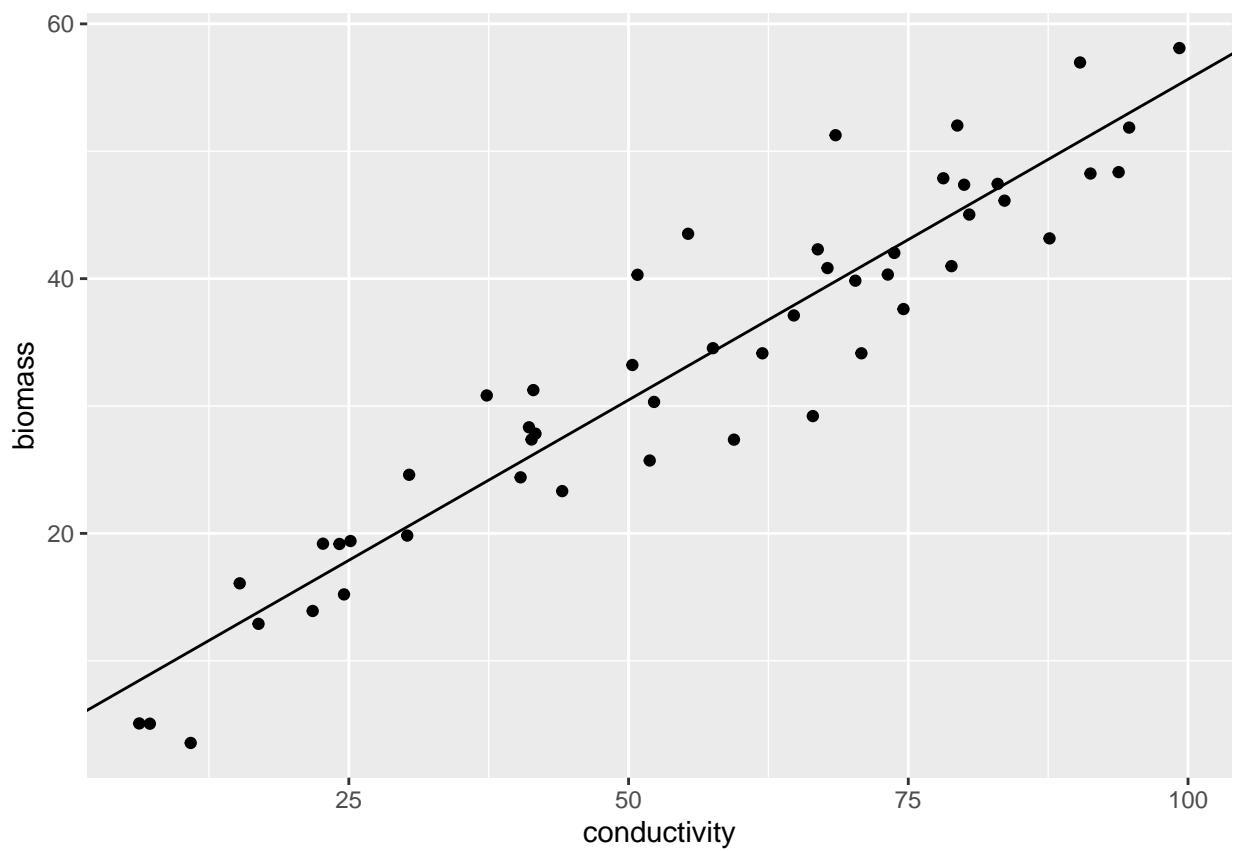


Figure 17: Drawing the fitted model prediction.

### 6.2.3 Least Squares

Introducing a matrix representation is indeed a helpful way to explain the least square method. Here, I represent the vector of observed values  $y_i$  as  $Y$  ( $Y = \{y_1, y_2, \dots, y_n\}^T$ ), the vector of parameters as  $\Theta$  ( $\Theta = \{\alpha, \beta\}^T$ ), and the matrix composed of 1s and  $x_i$  values as  $X$ . The matrix  $X$  can be written as:

$$X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$$

Using matrix notation, we can express the error vector  $\epsilon$  as:

$$\epsilon = Y - X\Theta$$

The column of 1s is required to represent the intercept;  $X\Theta$  reads:

$$X\Theta = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + \beta x_1 \\ \alpha + \beta x_2 \\ \alpha + \beta x_3 \\ \vdots \\ \alpha + \beta x_n \end{pmatrix}$$

Minimizing the squared magnitude of  $\epsilon$  (solve the partial derivative of  $\|\epsilon\|^2$  about  $\Theta$ ) leads to the solution (for detailed derivation, refer to Wikipedia):

$$\hat{\Theta} = (X^T X)^{-1} X^T Y$$

Let me see if this reproduces the result of `lm()`:

```
# create matrix X
v_x <- df_algae %>% pull(conductivity)
X <- cbind(1, v_x)

# create a vector of y
Y <- df_algae %>% pull(biomass)

# %*%: matrix multiplication
# t(): transpose a matrix
# solve(): computes the inverse matrix
theta <- solve(t(X) %*% X) %*% t(X) %*% Y
print(theta)
```

```
## [1]
##      5.2964689
## v_x 0.5035548
```

`lm()` output for a reference:

```
m <- lm(biomass ~ conductivity,
         data = df_algae)

coef(m)
```

```

## (Intercept) conductivity
##      5.2964689     0.5035548

```

#### 6.2.4 Standard Errors and t-values

Ensuring the reliability of the estimated point estimates for  $\alpha$  and  $\beta$  is crucial. In the `summary(m)` output, two statistical quantities, namely the `Std. Error` and `t-value`, play a significant role in assessing the uncertainty associated with these estimates.

The standard error (`Std. Error`; “SE”) represents the estimated standard deviation of the parameter estimates ( $\hat{\theta}$  is either  $\hat{\alpha}$  or  $\hat{\beta}$ ). A smaller value of the standard error indicates a higher degree of statistical reliability in the estimate. On the other hand, the `t-value` is a concept we have covered in Chapter 4. This value is defined as:

$$t = \frac{\hat{\theta} - \theta_0}{\text{SE}(\hat{\theta})}$$

The t-statistic is akin to the t-test; however, in the context of regression analysis, we do not have another group to compare with (i.e.,  $\theta_0$ ). Typically, in regression analysis, we use zero as the reference ( $\theta_0 = 0$ ). Therefore, higher t-values in `lm()` indicate a greater deviation from zero. Consequently, the **Null Hypothesis in regression analysis is  $\beta = 0$** . This hypothesis is sensible in our specific example since we are interested in quantifying the effect of conductivity. If  $\beta = 0$ , it implies that conductivity has no effect on algal biomass. Since  $\theta_0 = 0$ , the following code reproduces the reported t-values:

```

# extract coefficients
theta <- coef(m)

# extract standard errors
se <- sqrt(diag(vcov(m)))

# t-value
t_value <- theta / se
print(t_value)

## (Intercept) conductivity
##      3.369732     19.609446

```

After defining the Null Hypothesis and t-values, we can compute the probability of observing the estimated parameters under the Null Hypothesis. Similar to the t-test, `lm()` utilizes a Student’s t-distribution for this purpose. However, the difference lies in how we estimate the degrees of freedom. In our case, we have 50 data points but need to subtract the number of parameters, which are  $\alpha$  and  $\beta$ . Therefore, the degrees of freedom would be  $50 - 2 = 48$ . This value is employed when calculating the p-value:

```

# for intercept
# (1 - pt(t_value[1], df = 48)) calculates pr(t > t_value[1])
# pt(-t_value[1], df = 48) calculates pr(t < -t_value[1])
p_alpha <- (1 - pt(t_value[1], df = 48)) + pt(-t_value[1], df = 48)

# for slope
p_beta <- (1 - pt(t_value[2], df = 48)) + pt(-t_value[2], df = 48)

print(p_alpha)

## (Intercept)
## 0.001492023

```

```

print(p_beta)

## conductivity
## 7.377061e-25

```

## 6.3 Unexplained Variation

### 6.3.1 Retrieve Errors

The function `lm()` provides estimates for  $\alpha$  and  $\beta$ , but it does not directly provide the values of  $\varepsilon_i$ . To gain a deeper understanding of the statistical model, it is necessary to examine the residuals  $\varepsilon_i$ . By using the `resid()` function, you can obtain the values of  $\varepsilon_i$ . This allows you to explore and extract insights from the developed statistical model.

```

# eps - stands for epsilon
eps <- resid(m)
head(eps)

```

```

##           1          2          3          4          5          6
## -0.6838937 -1.2149034 -7.8576927 -3.0109473  1.7076481  6.1773586

```

Each element retains the order of data point in the original data frame, so `eps[1]` ( $\varepsilon_1$ ) should be identical to  $y_1 - (\alpha + \beta x_1)$ . Let me confirm:

```

# pull vector data
v_x <- df_algae %>% pull(conductivity)
v_y <- df_algae %>% pull(biomass)

# theta[1] = alpha
# theta[2] = beta
error <- v_y - (theta[1] + theta[2] * v_x)

# cbind() combines vectors column-wise
# head() retrieves the first 6 rows
head(cbind(eps, error))

```

```

##      eps      error
## 1 -0.6838937 -0.6838937
## 2 -1.2149034 -1.2149034
## 3 -7.8576927 -7.8576927
## 4 -3.0109473 -3.0109473
## 5  1.7076481  1.7076481
## 6  6.1773586  6.1773586

```

To ensure that the two columns are indeed identical, a visual check may be prone to overlooking small differences. To perform a more precise comparison, you can use the `all()` function to check if the data aligns (here, evaluated at the five decimal points using `round()`). By comparing the two columns in this manner, you can confirm whether they are identical or not.

```

# round values at 5 decimal
eps <- round(eps, 5)
error <- round(error, 5)

# eps == error return "TRUE" or "FALSE"
# all() returns TRUE if all elements meet eps == error
all(eps == error)

## [1] TRUE

```

### 6.3.2 Visualize Errors

To visualize the errors or residuals, which represent the distance from the best-fitted line to each data point, you can make use of the `geom_segment()` function:

```
# add error column
df_algae <- df_algae %>%
  mutate(eps = eps)

df_algae %>%
  ggplot(aes(x = conductivity,
             y = biomass)) +
  geom_point() +
  geom_abline(intercept = alpha,
              slope = beta) +
  geom_segment(aes(x = conductivity, # start-coord x
                  xend = conductivity, # end-coord x
                  y = biomass, # start-coord y
                  yend = biomass - eps), # end-coord y
               linetype = "dashed")
```

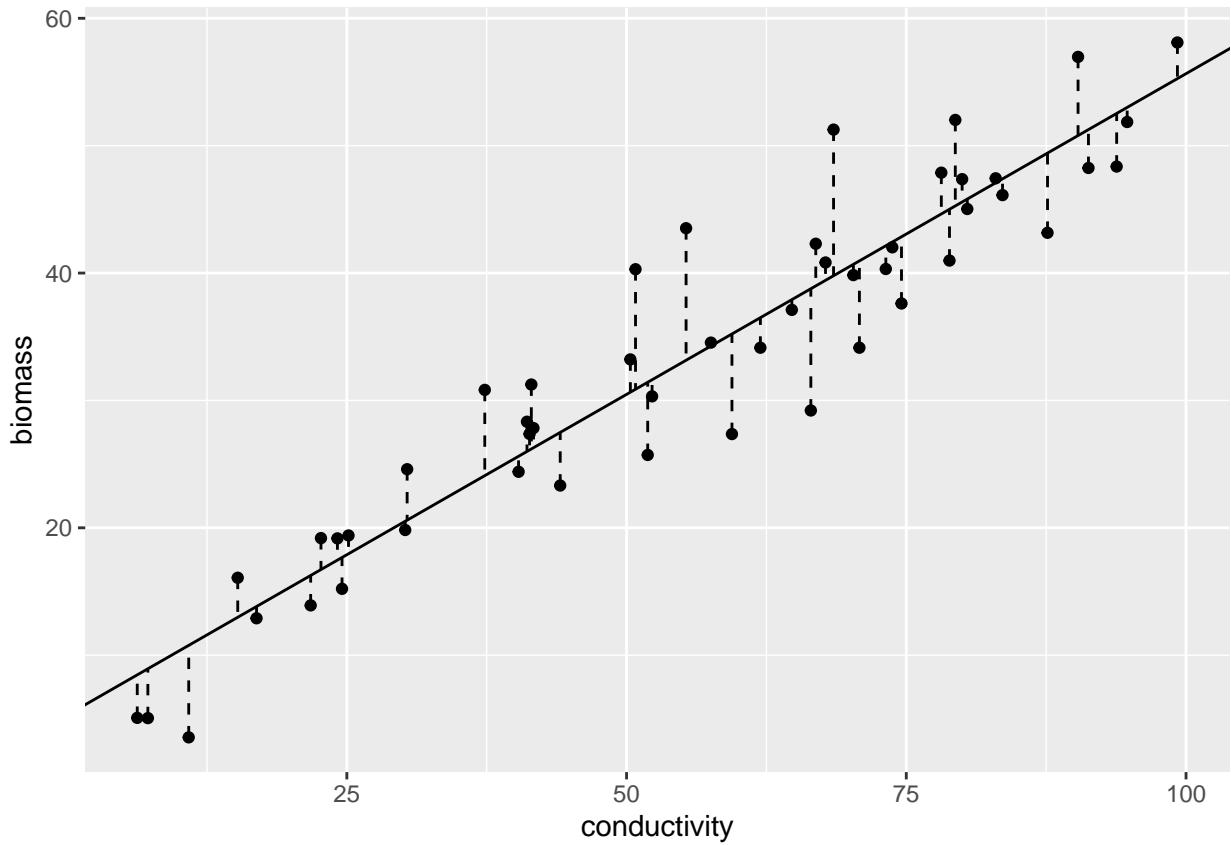


Figure 18: Vertical segments indicate errors.

### 6.3.3 Coefficient of Determination

One crucial motivation for developing a regression model, or any statistical model, is to assess the extent to which the explanatory variable ( $x_i$ ) explains the variation in the response variable ( $y_i$ ). By fitting a

regression model, we can quantify the amount of variability in the response variable that can be attributed to the explanatory variable. This assessment provides valuable insights into the strength and significance of the relationship between the variables under consideration.

The **coefficient of determination**, denoted as  $R^2$ , is a statistical measure that assesses the proportion of the variance in the response variable that can be explained by the explanatory variable(s) in a regression model. It provides an indication of how well the regression model fits the observed data. The formula for  $R^2$  is:

$$R^2 = 1 - \frac{SS}{SS_0}$$

where  $SS$  is the summed squares of residuals ( $\sum_i \varepsilon_i^2$ ), and  $SS_0$  is the summed squares of the response variable ( $SS_0 = \sum_i (y_i - \hat{\mu}_y)^2$ ). The term  $\frac{SS}{SS_0}$  represents the proportion of variability “unexplained” – therefore,  $1 - \frac{SS}{SS_0}$  gives the proportion of variability “explained.”  $R^2$  is a value between 0 and 1. An  $R^2$  value of 0 indicates that the explanatory variable(s) cannot explain any of the variability in the response variable, while an  $R^2$  value of 1 indicates that the explanatory variable(s) can fully explain the variability in the response variable.

While  $R^2$  is provided as a default output of `lm()`, let's confirm if the above equation reproduces the reported value:

```
# residual variance
ss <- sum(resid(m)^2)

# null variance
ss_0 <- sum((v_y - mean(v_y))^2)

# coefficient of determination
r2 <- 1 - ss / ss_0

print(r2)
```

```
## [1] 0.8890251
```

Compare it with `lm()` output:

```
summary(m)

##
## Call:
## lm(formula = biomass ~ conductivity, data = df_algae)
##
## Residuals:
##      Min    1Q   Median    3Q    Max 
## -9.5578 -2.8729 -0.7307  2.5479 11.4700 
## 
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) 5.29647   1.57178   3.37  0.00149 ***
## conductivity 0.50355   0.02568  19.61 < 2e-16 ***
## ---      
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 
## Residual standard error: 4.649 on 48 degrees of freedom
## Multiple R-squared:  0.889, Adjusted R-squared:  0.8867
```

```
## F-statistic: 384.5 on 1 and 48 DF,  p-value: < 2.2e-16
Multiple R-squared: corresponds to the calculated coefficient of determination.
```

## 6.4 Laboratory

# 7 Linear Model

We have extensively covered three important statistical analyses: the t-test, ANOVA, and regression analysis. While these methods may seem distinct, they all fall under the umbrella of the Linear Model framework<sup>4</sup>.

The Linear Model encompasses models that depict the connection between a response variable and one or more explanatory variables. It assumes that the error term follows a normal distribution. In this chapter, I will elucidate this framework and illustrate its relationship to the t-test, ANOVA, and regression analysis.

## 7.1 The Frame

The apparent distinctiveness between the t-test, ANOVA, and regression analysis arises because they are applied to different types of data:

- t-test: comparing differences between two groups.
- ANOVA: examining differences among more than two groups.
- Regression: exploring the relationship between a response variable and one or more explanatory variables.

Despite these differences, these analyses can be unified under a single formula. As discussed in the regression formula in Chapter 6, we have:

$$y_i = \alpha + \beta_1 x_{1,i} + \varepsilon_i \\ \varepsilon_i \sim \text{Normal}(0, \sigma^2)$$

where  $y_i$  represents the response variable (such as fish body length or plant height),  $x_{1,i}$  denotes the continuous explanatory variable,  $\alpha$  represents the intercept, and  $\beta_1$  corresponds to the slope. The equivalent model can be expressed differently as follows:

$$y_i \sim \text{Normal}(\mu_i, \sigma^2) \\ \mu_i = \alpha + \beta_1 x_{1,i}$$

This structure provides insight into the relationship between the t-test, ANOVA, and regression analysis. The fundamental purpose of regression analysis is to model how the mean  $\mu_i$  changes with increasing or decreasing values of  $x_{1,i}$ . Thus, the t-test, ANOVA, and regression all analyze the mean – the expected value of a Normal distribution. The primary distinction between the t-test, ANOVA, and regression lies in the nature of the explanatory variable, which can be either continuous or categorical (group).

### 7.1.1 Two-Group Case

To establish a connection among these approaches, one can employ “dummy indicator” variables to represent group variables. In the case of the t-test, where a categorical variable consists of two groups, typically denoted as **a** and **b** (although they can be represented by numbers as well), one can convert these categories into numerical values. For instance, assigning **a** to 0 and **b** to 1 enables the following conversion:

---

<sup>4</sup>Some textbooks refer to this framework as “General Linear Model.” However, to distinguish from Generalized Linear Model, I call this “Linear Model.”

$$\mathbf{x'}_2 = \begin{pmatrix} a \\ a \\ b \\ \vdots \\ b \end{pmatrix} \rightarrow \mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

The model can be written as:

$$y_i \sim \text{Normal}(\mu_i, \sigma^2)$$

$$\mu_i = \alpha + \beta_2 x_{2,i}$$

When incorporating this variable into the model, some interesting outcomes arise. Since  $x_{2,i} = 0$  when an observation belongs to group a, the mean of group a ( $\mu_i = \mu_a$ ) is determined as  $\mu_a = \alpha + \beta \times 0 = \alpha$ . Consequently, the intercept represents the mean value of the first group. On the other hand, if an observation is from group b, the mean of group b ( $\mu_i = \mu_b$ ) is given by  $\mu_b = \alpha + \beta \times 1 = \alpha + \beta$ . It's important to recall that  $\mu_a = \alpha$ . By substituting  $\alpha$  with  $\mu_a$ , the equation  $\beta = \mu_b - \mu_a$  is obtained, indicating that the slope represents the difference between the means of the two groups – the key statistic in the t-test.

Let me confirm this using the dataset in Chapter 4 (fish body length in two lakes):

```
library(tidyverse)
df_f1 <- read_csv("data_raw/data_fish_length.csv")
print(df_f1)
```

```
## # A tibble: 100 x 3
##   lake  length unit
##   <chr> <dbl> <chr>
## 1 a     10.8 cm
## 2 a     13.6 cm
## 3 a     10.1 cm
## 4 a     18.6 cm
## 5 a     14.2 cm
## 6 a     10.1 cm
## 7 a     14.7 cm
## 8 a     15.6 cm
## 9 a     15     cm
## 10 a    11.9 cm
## # ... i 90 more rows
```

In the `lm()` function, when you provide a “categorical” variable in character or factor form, it is automatically converted to a binary (0/1) variable internally. This means that you can include this variable in the formula as if you were conducting a standard regression analysis. The `lm()` function takes care of this conversion process, allowing you to seamlessly incorporate categorical variables into the `lm()` function.

```
# group means
v_mu <- df_f1 %>%
  group_by(lake) %>%
  summarize(mu = mean(length)) %>%
  pull(mu)

# mu_a: should be identical to intercept
v_mu[1]

## [1] 13.35
```

```

# mu_b - mu_a: should be identical to slope
v_mu[2] - v_mu[1]

## [1] 2.056

# in lm(), letters are automatically converted to 0/1 binary variable.
# alphabetically ordered (in this case, a = 0, b = 1)
m <- lm(length ~ lake,
         data = df_f1)

summary(m)

##
## Call:
## lm(formula = length ~ lake, data = df_f1)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -8.150 -2.156  0.022  2.008  7.994
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 13.3500    0.4477  29.819   <2e-16 ***
## lakeb       2.0560    0.6331   3.247   0.0016 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.166 on 98 degrees of freedom
## Multiple R-squared:  0.09715,   Adjusted R-squared:  0.08794
## F-statistic: 10.55 on 1 and 98 DF,  p-value: 0.001596

```

The estimated coefficient of Lake b (lakeb) is identical to the difference between group means. We can compare other statistics (t value and Pr(>|t|) with the output from t.test() as well:

```

lake_a <- df_f1 %>%
  filter(lake == "a") %>%
  pull(length)

lake_b <- df_f1 %>%
  filter(lake == "b") %>%
  pull(length)

t.test(x = lake_b, y = lake_a)

##
## Welch Two Sample t-test
##
## data: lake_b and lake_a
## t = 3.2473, df = 95.846, p-value = 0.001606
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  0.7992071 3.3127929
## sample estimates:
## mean of x mean of y
##      15.406      13.350

```

The t-statistic and p-value match the `lm()` output.

### 7.1.2 Multiple-Group Case

The same argument applies to ANOVA. In ANOVA, we deal with more than two groups in the explanatory variable. To handle this, we can convert the group variable into multiple dummy variables. For instance, if we have a variable  $\mathbf{x}'_2 = \{a, b, c\}$ , we can convert it to  $x_2 = \{0, 1, 0\}$  (where  $b \rightarrow 1$  and the others are 0) and  $x_3 = \{0, 0, 1\}$  (where  $c \rightarrow 1$  and the others are 0). Thus, the model formula would be:

$$y_i \sim \text{Normal}(\mu_i, \sigma^2)$$

$$\mu_i = \alpha + \beta_2 x_{2,i} + \beta_3 x_{3,i}$$

If you substitute these dummy variables, then:

$$\begin{aligned}\mu_a &= \alpha + \beta_2 \times 0 + \beta_3 \times 0 = \alpha && \text{both } x_{2,i} \text{ and } x_{3,i} \text{ are zero} \\ \mu_b &= \alpha + \beta_2 \times 1 + \beta_3 \times 0 = \alpha + \beta_2 && x_{2,i} = 1 \text{ but } x_{3,i} = 0 \\ \mu_c &= \alpha + \beta_2 \times 0 + \beta_3 \times 1 = \alpha + \beta_3 && x_{2,i} = 0 \text{ but } x_{3,i} = 1\end{aligned}$$

Therefore, the group **a** serves as the reference, and the  $\beta$ s represent the deviations from the reference group. Now, let me attempt the ANOVA dataset provided in Chapter 5:

```
df_anova <- read_csv("data_raw/data_fish_length_anova.csv")
print(df_anova)
```

```
## # A tibble: 150 x 3
##   lake  length unit
##   <chr> <dbl> <chr>
## 1 a     10.8  cm
## 2 a     13.6  cm
## 3 a     10.1  cm
## 4 a     18.6  cm
## 5 a     14.2  cm
## 6 a     10.1  cm
## 7 a     14.7  cm
## 8 a     15.6  cm
## 9 a     15    cm
## 10 a    11.9  cm
## # ... i 140 more rows
```

Again, the `lm()` function converts the categorical variable to the dummy variables internally. Compare the mean and differences with the estimated parameters:

```
# group means
v_mu <- df_f1 %>%
  group_by(lake) %>%
  summarize(mu = mean(length)) %>%
  pull(mu)

print(c(v_mu[1], # mu_a: should be identical to intercept
       v_mu[2] - v_mu[1], # mu_b - mu_a: should be identical to the slope for lakeb
       v_mu[3] - v_mu[1])) # mu_c - mu_a: should be identical to the slope for lakec

## [1] 13.350  2.056      NA
```

```

# lm() output
m <- lm(length ~ lake,
      data = df_anova)

summary(m)

##
## Call:
## lm(formula = length ~ lake, data = df_anova)
##
## Residuals:
##    Min     1Q Median     3Q    Max 
## -8.150 -1.862 -0.136  1.846  7.994 
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) 13.3500    0.4469  29.875 < 2e-16 ***
## lakeb       2.0560    0.6319   3.253  0.00141 **  
## lakec       1.1160    0.6319   1.766  0.07948 .    
## ---        
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.16 on 147 degrees of freedom
## Multiple R-squared:  0.06732, Adjusted R-squared:  0.05463 
## F-statistic: 5.305 on 2 and 147 DF,  p-value: 0.005961

```

Also, there is a report for F-statistic (5.305) and p-value (0.005961). Compare them with the `aov()` output:

```

m_aov <- aov(length ~ lake,
               data = df_anova)

summary(m_aov)

##           Df Sum Sq Mean Sq F value Pr(>F)    
## lake        2 105.9  52.97  5.305 0.00596 ** 
## Residuals  147 1467.6   9.98                
## ---        
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

The results are identical.

### 7.1.3 The Common Structure

The above examples show that the t-test, ANOVA, and regression has the same model structure:

$$y_i = (\text{deterministic component}) + (\text{stochastic component})$$

In the framework of the Linear Model, the deterministic component is expressed as  $\alpha + \sum_k \beta_k x_{k,i}$  and the stochastic component (random error) is expressed as a Normal distribution. This structure makes several assumptions. Here are the main assumptions:

1. Linearity: The relationship between the explanatory variables and the response variable is linear. This means that the effect of each explanatory variable on the response variable is additive and constant.
2. Independence: The observations are independent of each other. This assumption implies that the values of the response variable for one observation do not influence the values for other observations.

3. Homoscedasticity: The variance of the response variable is constant across all levels of the explanatory variables. In other words, the spread or dispersion of the response variable should be the same for all values of the explanatory variables.
4. Normality: The error term follows a normal distribution at each level of the explanatory variables. This assumption is important because many statistical tests and estimators used in the linear model are based on the assumption of normality.
5. No multicollinearity: The explanatory variables are not highly correlated with each other. Multicollinearity can lead to problems in estimating the coefficients accurately and can make interpretation difficult.

Violations of these assumptions can affect the validity and reliability of the results obtained from the Linear Model.

## 7.2 Combine Multiple Types of Variables

### 7.2.1 `iris` Example

The previous section clarified that the t-test, ANOVA, and regression have a common model structure; therefore, categorical and continuous variables can be included in the same model. Here, let me use the `iris` data (available in R by default) to show an example. This data set comprises continuous (Sepal.Length, Sepal.Width, Petal.Length, Petal.Width) and categorical variables (Species):

```
# convert the data format to tibble
iris <- as_tibble(iris)
print(iris)

## # A tibble: 150 x 5
##   Sepal.Length Sepal.Width Petal.Length Petal.Width Species
##       <dbl>      <dbl>      <dbl>      <dbl> <fct>
## 1         5.1        3.5        1.4       0.2  setosa
## 2         4.9        3.0        1.4       0.2  setosa
## 3         4.7        3.2        1.3       0.2  setosa
## 4         4.6        3.1        1.5       0.2  setosa
## 5         5.0        3.6        1.4       0.2  setosa
## 6         5.4        3.9        1.7       0.4  setosa
## 7         4.6        3.4        1.4       0.3  setosa
## 8         5.0        3.4        1.5       0.2  setosa
## 9         4.4        2.9        1.4       0.2  setosa
## 10        4.9       3.1        1.5       0.1  setosa
## # i 140 more rows
distinct(iris, Species)

## # A tibble: 3 x 1
##   Species
##   <fct>
## 1 setosa
## 2 versicolor
## 3 virginica
```

Here, let me model Petal.Length as a function of Petal.Width and Species:

```
# develop iris model
m_iris <- lm(Petal.Length ~ Petal.Width + Species,
              data = iris)

summary(m_iris)
```

```

## 
## Call:
## lm(formula = Petal.Length ~ Petal.Width + Species, data = iris)
## 
## Residuals:
##       Min     1Q   Median     3Q    Max 
## -1.02977 -0.22241 -0.01514  0.18180  1.17449 
## 
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) 1.21140   0.06524 18.568 < 2e-16 ***
## Petal.Width  1.01871   0.15224  6.691 4.41e-10 ***
## Speciesversicolor 1.69779   0.18095  9.383 < 2e-16 ***
## Speciesvirginica  2.27669   0.28132  8.093 2.08e-13 ***
## --- 
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1 
## 
## Residual standard error: 0.3777 on 146 degrees of freedom
## Multiple R-squared:  0.9551, Adjusted R-squared:  0.9542 
## F-statistic: 1036 on 3 and 146 DF, p-value: < 2.2e-16

```

Please note that the model output does not include `setosa` because it was used as the reference group (`(Intercept)`).

If you want to obtain predicted values, you can use the `predict()` function. To make predictions, you need to provide a new data frame containing the values of explanatory variables for the prediction.

```

# create a data frame for prediction
# variable names must be identical to the original dataframe for analysis
n_rep <- 100
df_pred <- tibble(Petal.Width = rep(seq(min(iris$Petal.Width),
                                         max(iris$Petal.Width),
                                         length = n_rep),
                                         n_distinct(iris$Species)),
                    Species = rep(unique(iris$Species),
                                  each = n_rep))

# make prediction based on supplied values of explanatory variables
y_pred <- predict(m_iris,
                   newdata = df_pred)

df_pred <- df_pred %>%
  mutate(y_pred = y_pred)

print(df_pred)

## # A tibble: 300 x 3
##   Petal.Width Species y_pred
##       <dbl> <fct>    <dbl>
## 1        0.1 setosa     1.31
## 2        0.124 setosa    1.34
## 3        0.148 setosa    1.36
## 4        0.173 setosa    1.39
## 5        0.197 setosa    1.41

```

```

## 6      0.221 setosa   1.44
## 7      0.245 setosa   1.46
## 8      0.270 setosa   1.49
## 9      0.294 setosa   1.51
## 10     0.318 setosa   1.54
## # i 290 more rows

```

By plotting the predicted values against the observed data points, we can visually evaluate the accuracy of the model and assess its goodness of fit.

```

iris %>%
  ggplot(aes(x = Petal.Width,
             y = Petal.Length,
             color = Species)) +
  geom_point(alpha = 0.5) +
  geom_line(data = df_pred,
            aes(y = y_pred)) # redefine y values for lines; x and color are inherited from ggplot()

```

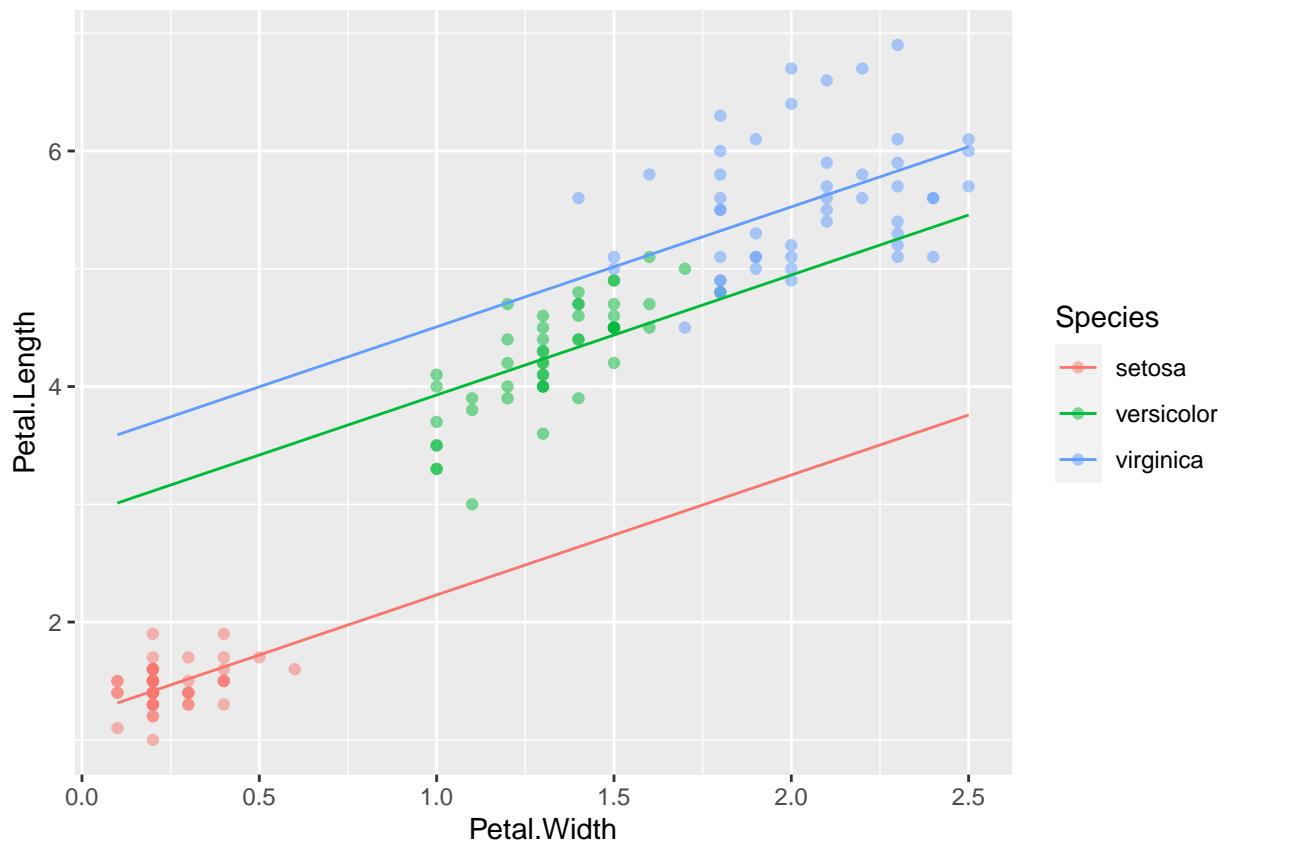


Figure 19: Example of model fitting to iris data. Points are observations, and lines are model predictions.

### 7.2.2 Variable Types

- Categorical Variables:** Qualitative or discrete characteristics that fall into specific categories or groups. They do not have a numerical value or order associated with them. Examples: gender (male/female), marital status (single/married/divorced), and color (red/green/blue).
- Ordinal Variables:** Categories or levels with a natural order or ranking. Examples: survey ratings

(e.g., Likert scale), education levels (e.g., high school, bachelor's, master's), or socioeconomic status (e.g., low, medium, high).

3. **Interval Variables:** Numerical values representing a continuous scale where the intervals between the values are meaningful and consistent. Unlike ordinal variables, interval variables have equally spaced intervals between adjacent values, allowing for meaningful arithmetic operations like addition and subtraction. However, interval variables do not contain a natural zero point; zero is arbitrary and merely a reference point. Examples: temperature
4. **Ratio Variables:** Numerical variable that shares the characteristics of interval variables but also has a meaningful and absolute zero point. Thus, this type of variables has the complete absence of the measured attribute/quantity. Examples: height (in centimeters), weight (in kilograms), time (in seconds), distance (in meters), and income (in dollars).

We cannot carry out meaningful arithmetic operations on categorical and ordinal variables; therefore, we should treat these variables as “character” or “factors” in R. This is because R recognizes that the intervals between different groups or levels of these variables are not meaningful in a numerical sense.

In contrast, interval and ratio variables should be considered “numerical” variables in R. These variables possess a meaningful numerical scale with equal intervals, and in the case of ratio variables, a true zero point.

## 7.3 Laboratory

# 8 Generalized Linear Model

One of the key assumptions in the Linear Model framework is Normality – that the error term follows a normal distribution (Chapter 7). However, this assumption frequently leads to situations where models predict unrealistic values for a response variable. In this chapter, I will introduce the Generalized Linear Model (GLM) framework, which enables more flexible modeling.

## 8.1 Count Data

### 8.1.1 Plant Density

Recall our data set of garden plant counts (Chapter 3):

```
df_count <- read_csv("data_raw/data_garden_count.csv")
print(df_count)

## # A tibble: 30 x 3
##   plot count nitrate
##   <dbl> <dbl>   <dbl>
## 1     1     1    14.6
## 2     2     4    32.6
## 3     3     6    29.3
## 4     4     4    30.4
## 5     5     7    34.1
## 6     6     2    26.6
## 7     7     2    30.2
## 8     8     2    28.7
## 9     9     3    32.5
## 10   10    2    26
## # i 20 more rows
```

The variable of plant count had the following characteristics:

1. The possible values are discrete (count data) since we are counting the number of plant individuals in each plot.

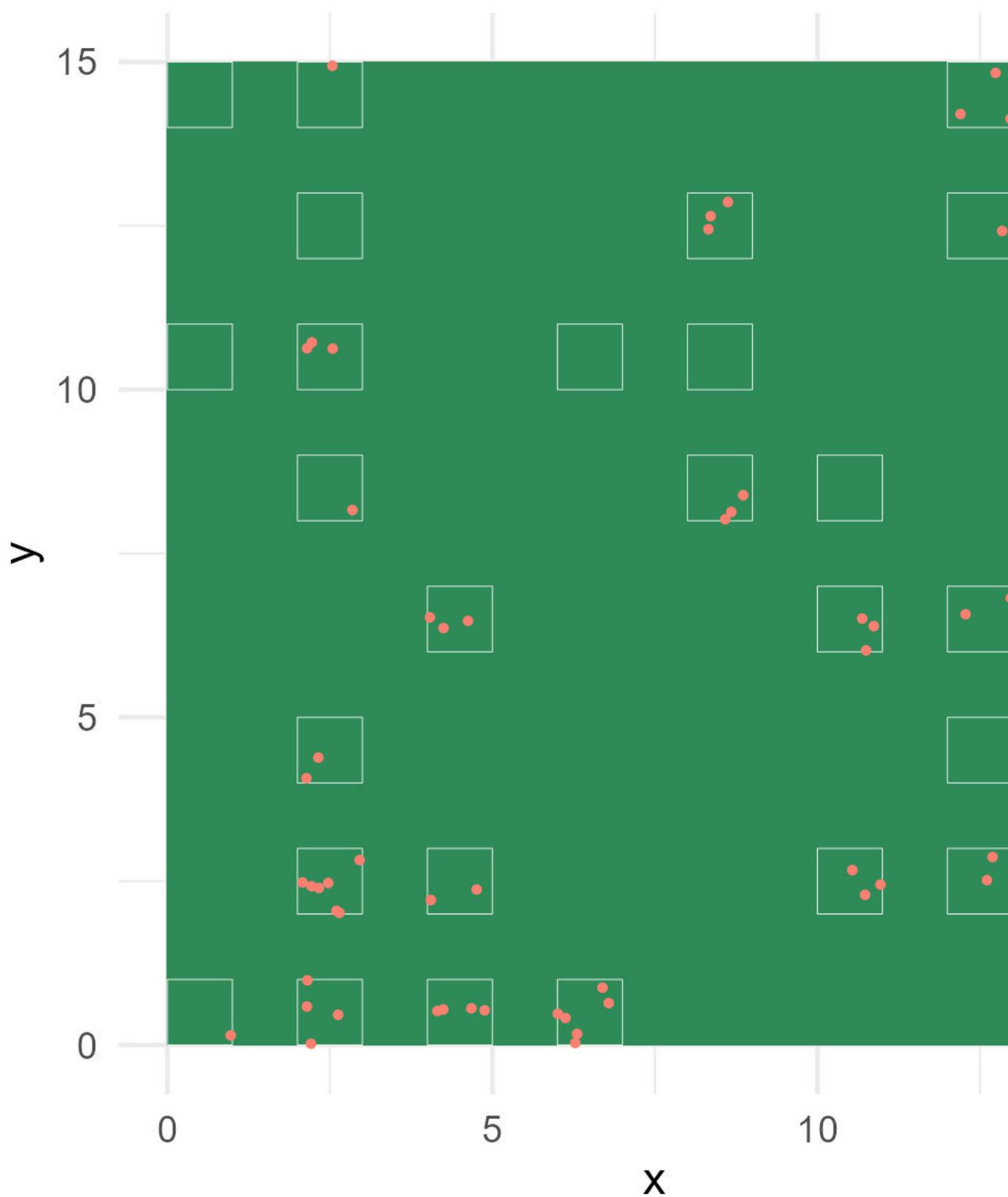


Figure 20: Garden view with sampling plots. White squares represent plots. Red dots indicate plant individuals counted.

2. The possible values are always positive, as counts cannot be negative.
3. The distribution appears to be non-symmetric around the sample mean.

What happens if we apply a simple linear model that assumes a Normal distribution? Let me model the plant count as a function of nitrate as follows:

$$y_i \sim \text{Normal}(\mu_i, \sigma^2)$$

$$\mu_i = \alpha + \beta \text{ nitrate}_i$$

```
m_normal <- lm(count ~ nitrate,
                 df_count)

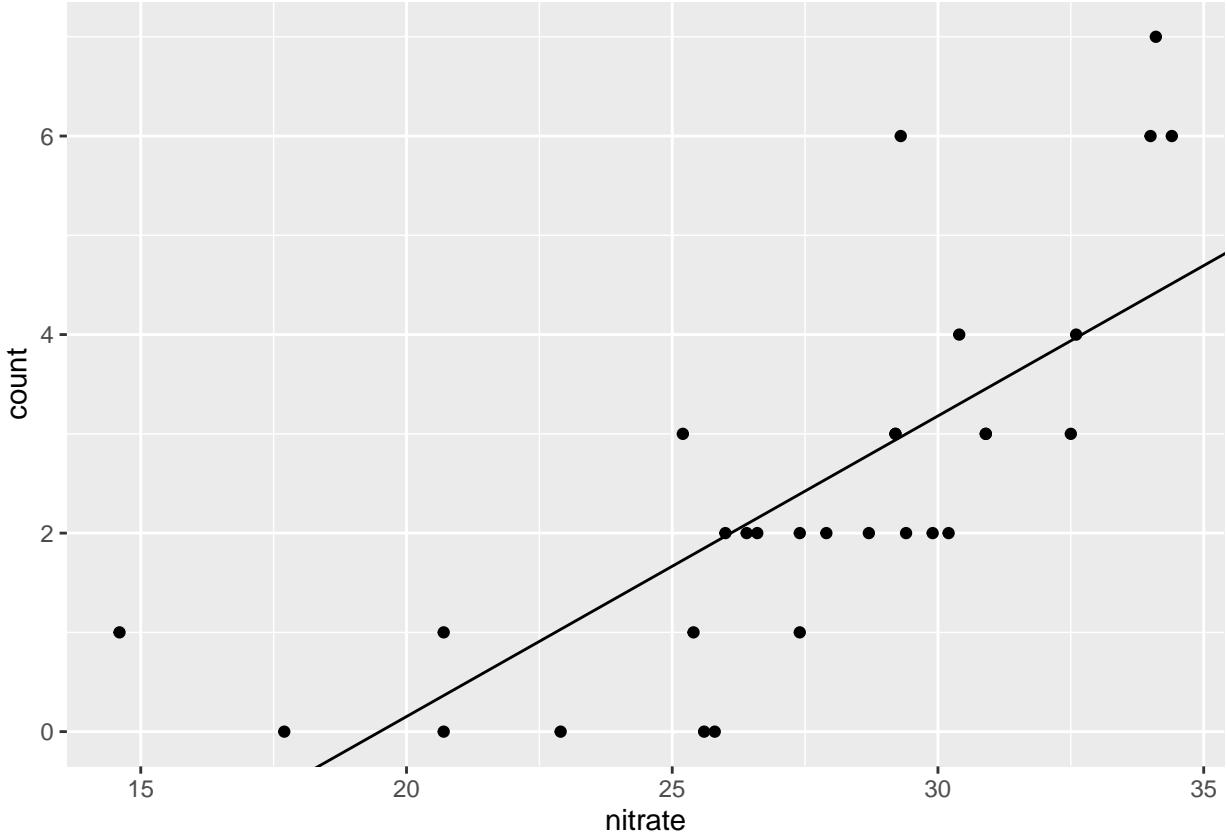
summary(m_normal)

##
## Call:
## lm(formula = count ~ nitrate, data = df_count)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.9084 -0.9000 -0.2572  0.6134  3.0316
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -5.90545   1.42516 -4.144 0.000285 ***
## nitrate      0.30286   0.05105  5.933 2.19e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.291 on 28 degrees of freedom
## Multiple R-squared:  0.557, Adjusted R-squared:  0.5411
## F-statistic: 35.2 on 1 and 28 DF,  p-value: 2.193e-06
```

This seems satisfactory, but an anomalous behavior occurs during the plotting process on the figure. Plot the data and the predicted values together.

```
# extract estimates
alpha <- coef(m_normal)[1] # intercept
beta <- coef(m_normal)[2] # slope

df_count %>%
  ggplot(aes(x = nitrate,
             y = count)) +
  geom_point() +
  geom_abline(intercept = alpha,
              slope = beta)
```



The intersection of the predicted line with the x-axis suggests the presence of negative predicted values for plant counts. However, it is logically implausible for plant counts to be negative.

This occurrence stems from the nature of the model employed.

1. In the Linear Model with a Normal error distribution, we allow the mean to encompass a wide range of values, including negative values.
2. Even when the mean predicted values do not intersect the x-axis, we consider the possibility of observations taking negative values, as dictated by the nature of a Normal distribution ( $\mu_i + \varepsilon_i$  can be negative even when  $\mu_i > 0$ ), which in reality can never occur.

### 8.1.2 Poisson Model

One possible approach is to consider the assumption that the error term follows a Poisson distribution. For the purpose of this discussion, let's assume that a model based on the Poisson distribution is appropriate. Unlike a Normal distribution, the Poisson distribution generates only non-negative discrete values, which makes it a suitable fit for the plant count variable. To incorporate this, we can make the following modifications to the model:

$$y_i \sim \text{Poisson}(\lambda_i)$$

$$\log \lambda_i = \alpha + \beta \text{ nitrate}_i$$

These changes are crucial:

1. The plant count, denoted as  $y_i$ , is now assumed to follow a Poisson distribution with parameter  $\lambda_i$ . This ensures that negative values of  $y_i$  are not possible (i.e.,  $\Pr(y_i < 0) = 0$ ).
2. The mean  $\lambda_i$  is log-transformed when expressing it as a function of nitrate. Given that the mean of a Poisson distribution cannot be negative, we must restrict the range of  $\lambda_i$  to positive values. The

log-transformation guarantees positivity of  $\lambda_i$  ( $\lambda_i = \exp(\alpha + \beta \text{ nitrate}_i)$ ) irrespective of the values of  $\alpha$ ,  $\beta$ , and  $\text{nitrate}_i$ .

The implementation of the Poisson model is quite simple. In the following example, we will use the function `glm()`:

```
m_pois <- glm(count ~ nitrate,
  data = df_count,
  family = "poisson")
```

The major difference from `lm()` is the argument `family = "poisson"`. This argument specifies the probability distribution to be used; in this example, I used a Poisson distribution to model count data. However, in looking at estimates...

```
summary(m_pois)

##
## Call:
## glm(formula = count ~ nitrate, family = "poisson", data = df_count)
##
## Deviance Residuals:
##      Min        1Q     Median        3Q       Max
## -1.6799   -0.5862   -0.1217    0.3250    1.8272
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept) -4.04899   1.05326 -3.844 0.000121 ***
## nitrate      0.17028   0.03467  4.911 9.06e-07 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for poisson family taken to be 1)
##
## Null deviance: 48.598 on 29 degrees of freedom
## Residual deviance: 19.073 on 28 degrees of freedom
## AIC: 93.858
##
## Number of Fisher Scoring iterations: 5
```

The estimated intercept is still negative...what's wrong? No sweat, we estimated these parameters in a log scale; this result translates into:

$$\lambda_i = \exp(-4.05 + 0.17 \text{ nitrate}_i)$$

Therefore, even when the intercept is estimated to be negative, the mean  $\lambda_i$  is guaranteed to be non-negative. Another important difference from `lm()` is that the test-statistic is no longer the t-statistic – instead, `glm()` reports the z-statistic. However, the z-statistic is very similar to the t-statistic, defined as  $z = \frac{\hat{\theta} - \theta_0}{\text{SE}(\hat{\theta})}$  ( $\hat{\theta} = \{\hat{\alpha}, \hat{\beta}\}$ ,  $\theta_0 = 0$ ). Therefore, you can reproduce the z-statistic by dividing the parameter estimates ( $\hat{\alpha}$  and  $\hat{\beta}$ ) by their standard errors:

```
# parameter estimates and their SEs
theta <- coef(m_pois)
se <- sqrt(diag(vcov(m_pois)))
z_value <- theta / se
```

```

print(z_value)

## (Intercept)      nitrate
## -3.844229    4.910935

```

We call this statistic “z-statistic” as it is known to follow the z-distribution (also known as the standardized Normal distribution  $\text{Normal}(0, 1)$ ), as opposed to the Student’s t-distribution. The reported p-value ( $\text{Pr}(>|z|)$ ) is estimated under this assumption, which can be seen as follows.

```

# estimate  $\text{Pr}(>|z|)$  using a standardized normal distribution
p_value <- (1 - pnorm(abs(z_value), mean = 0, sd = 1)) * 2
print(p_value)

```

```

## (Intercept)      nitrate
## 1.209323e-04 9.064303e-07

```

**Notice that these p-values are not identical to the output from the Normal model** – therefore, the choice of the probability distribution critically affects the results of statistical analysis, and often times, qualitatively (not in this particular example though).

Visualization provides the convincing difference between the Normal and Poisson models (Figure 21).

```

# make predictions
df_pred <- tibble(nitrate = seq(min(df_count$nitrate),
                                 max(df_count$nitrate),
                                 length = 100))

#  $y_{\text{pois}}$  is exponentiated because predict() returns values in log-scale
y_normal <- predict(m_normal, newdata = df_pred)
y_pois <- predict(m_pois, newdata = df_pred) %>% exp()

df_pred <- df_pred %>%
  mutate(y_normal,
        y_pois)

# figure
df_count %>%
  ggplot(aes(x = nitrate,
             y = count)) +
  geom_point() +
  geom_line(data = df_pred,
            aes(y = y_normal),
            linetype = "dashed") +
  geom_line(data = df_pred,
            aes(y = y_pois),
            color = "salmon")

```

Clearly, the Poisson model does a better job.

## 8.2 Proportional Data

### 8.2.1 Mussel Egg Fertilization

There is another type of data for which a Normal distribution is not suitable - proportional data. Let’s consider an example dataset where an investigation was conducted on the number of fertilized eggs out of 30 eggs examined for 100 female freshwater mussels. You can download the dataset here.

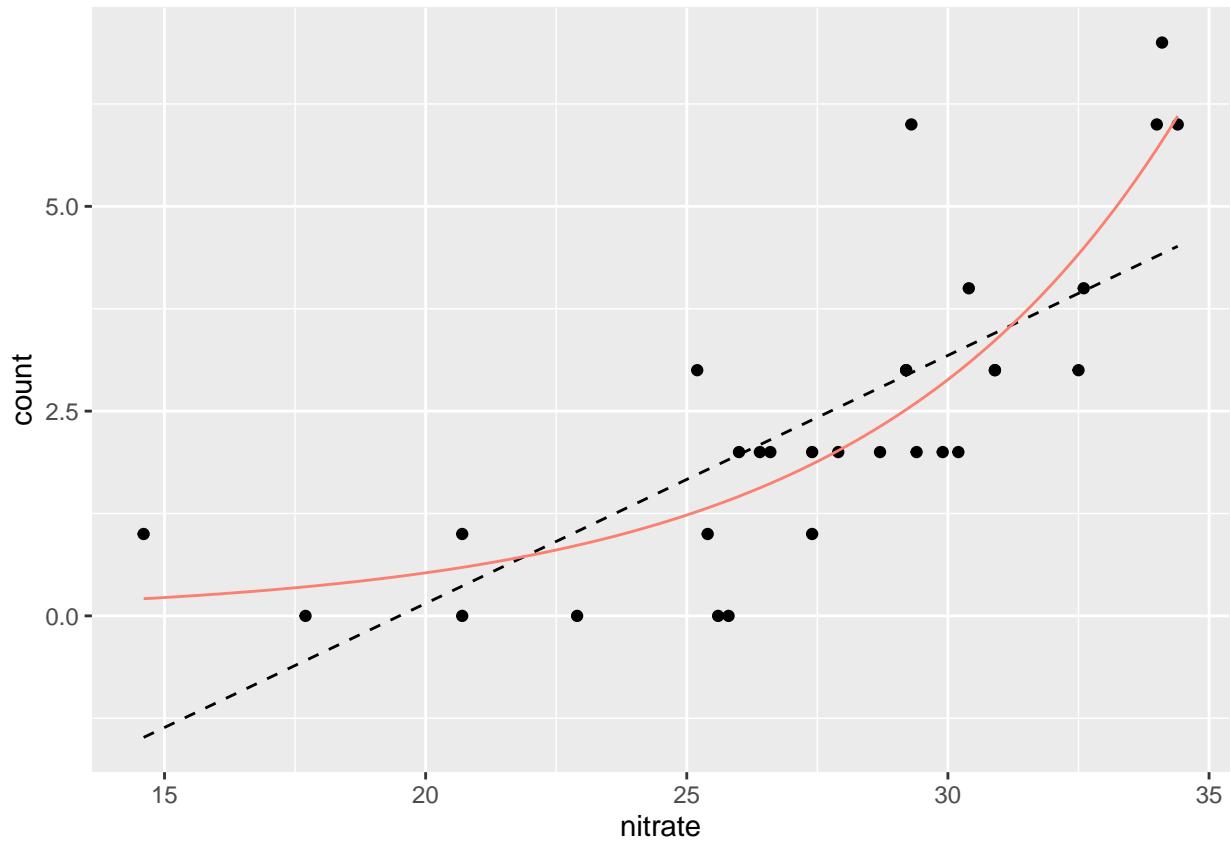


Figure 21: Comparison of the predicted values between Normal (broken, black) and Poisson models (solid, red).



Male freshwater mussels release sperm into the water, which is then drawn into female mussels downstream. Considering the nature of the fertilization process, female mussels are expected to have a higher probability of fertilization when they are surrounded by a greater number of males. To explore this relationship, the data can be loaded into R and visualized using a plot. By examining the data, we can observe the proportion of fertilized eggs based on the number of males. This visualization will provide insights into the impact of male presence on the fertilization process of female mussels.

```
df_mussel <- read_csv("data_raw/data_mussel.csv")
print(df_mussel)

## # A tibble: 100 x 4
##   ind_id n_fertilized n_examined density
##   <dbl>      <dbl>      <dbl>    <dbl>
## 1     1          0        30      3
## 2     2          1        30     14
## 3     3          2        30     14
## 4     4          6        30     19
## 5     5          2        30     17
## 6     6          0        30      8
## 7     7          4        30     19
## 8     8          0        30     11
## 9     9          0        30      7
## 10    10         0        30      7
## # i 90 more rows
```

This data frame contains variables such as `ind_id` (mussel individual ID), `n_fertilized` (number of fertilized eggs), `n_examined` (number of eggs examined), and `density` (number of mussels in a 1 m<sup>2</sup> quadrat). To visualize the relationship between the proportion of eggs fertilized and the density gradient, plot the proportion of fertilized eggs against the density gradient (Figure 22).

```
# calculate the proportion of fertilized eggs
df_mussel <- df_mussel %>%
  mutate(prop_fert = n_fertilized / n_examined)

# plot
df_mussel %>%
  ggplot(aes(x = density,
             y = prop_fert)) +
  geom_point() +
  labs(y = "Proportion of eggs fertilized",
```

```
x = "Mussel density")
```

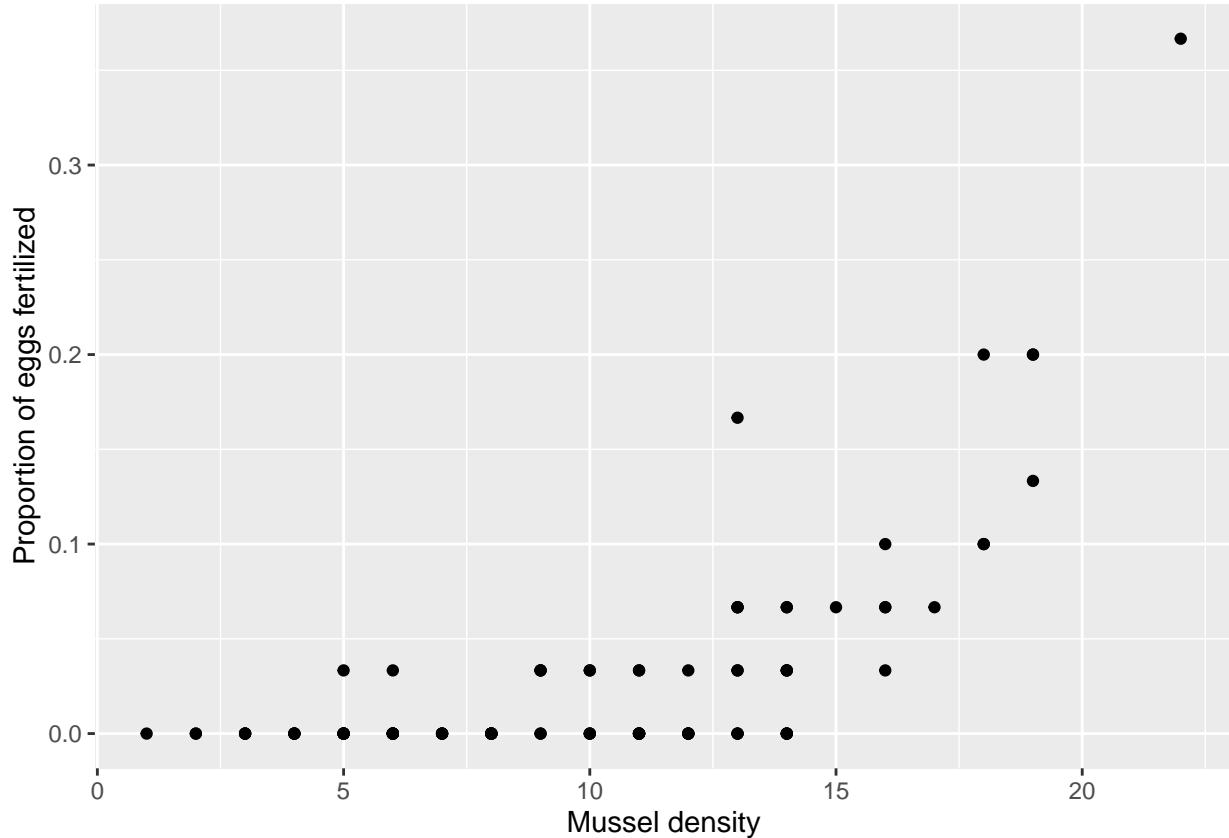


Figure 22: Relationship between the proportion of fertilized eggs and mussel density in the surround.

In this example, the response variable, which represents the number of eggs fertilized, possesses the following characteristics:

1. Discreteness: The variable takes on discrete values rather than continuous ones.
2. Upper limit: The number of eggs fertilized is bounded by an upper limit (in this case, 30).

The discrete nature of the variable might initially lead one to consider the Poisson distribution as a potential candidate. However, due to the presence of an upper limit where the number of fertilized eggs cannot exceed the number of eggs examined, the Poisson distribution is not a suitable option.

### 8.2.2 Binomial Model

A Binomial distribution is a natural and appropriate choice for modeling this variable. The Binomial distribution is characterized by two parameters: the number of trials ( $N$ ) and the probability of success ( $p$ ). Each trial has a certain probability of success, and the outcome of each trial is either assigned a value of 1 (indicating success) or 0 (indicating failure). In the given example, we can consider the “fertilization” as a successful outcome among the 30 trials. Consequently, the number of eggs fertilized ( $y_i$ ) for each individual mussel ( $i$ ) can be described as:

$$y_i \sim \text{Binomial}(N_i, p_i)$$

This formulation adequately captures the nature of the response variable, as the outcome of the Binomial distribution is constrained within the range of 0 to  $N_i$ , aligning with the upper limit imposed by the number of examined eggs.

Our prediction was that the fertilization probability  $p_i$  would increase with increasing mussel's density – how do we relate  $p_i$  to mussel density? The value of  $p_i$  must be constrained within a range of 0.0 – 1.0 because it is a “probability.” Given this, the log-transformation, which we used in the Poisson model, is not a suitable choice. Instead, we can use a logit-transformation to convert the linear formula to fall within a range of 0.0 – 1.0.

$$y_i \sim \text{Binomial}(N_i, p_i)$$

$$\log\left(\frac{p_i}{1 - p_i}\right) = \alpha + \beta \text{ density}_i$$

The logit transformation, represented as  $\log\left(\frac{p_i}{1-p_i}\right)$ , guarantees that the values of  $p_i$  are confined within the range of 0.0 to 1.0. This relationship is expressed by the following equation:

$$p_i = \frac{\exp(\alpha + \beta \text{ density}_i)}{1 + \exp(\alpha + \beta \text{ density}_i)}$$

A straightforward coding can verify this.

```
# x: produce 100 numbers from -100 to 100 (assume logit scale)
# y: convert with inverse-logit transformation (ordinary scale)
df_test <- tibble(logit_x = seq(-10, 10, length = 100),
                    x = exp(logit_x) / (1 + exp(logit_x)))

df_test %>%
  ggplot(aes(x = logit_x,
             y = x)) +
  geom_point() +
  geom_line() +
  labs(y = "x",
       x = "logit(x)")
```

The `glm()` function can be used to implement this modeling approach. However, certain modifications need to be made to the response variable in order to ensure compatibility with the binomial distribution.

```
m_binom <- glm(cbind(n_fertilized, n_examined - n_fertilized) ~ density,
                 data = df_mussel,
                 family = "binomial")
```

In contrast to the Normal or Poisson models, the response variable was encoded using the `cbind(n_fertilized, n_examined - n_fertilized)` function. The `cbind()` function combines the number of successes (fertilized) and failures (not fertilized) into a single matrix. This approach allows for modeling the data using the binomial distribution, taking into account both the successes and the total number of trials within each observation.

```
cbind(df_mussel$n_fertilized, df_mussel$n_examined - df_mussel$n_fertilized) %>%
  head()
```

```
##      [,1] [,2]
## [1,]    0   30
## [2,]    1   29
## [3,]    2   28
## [4,]    6   24
```

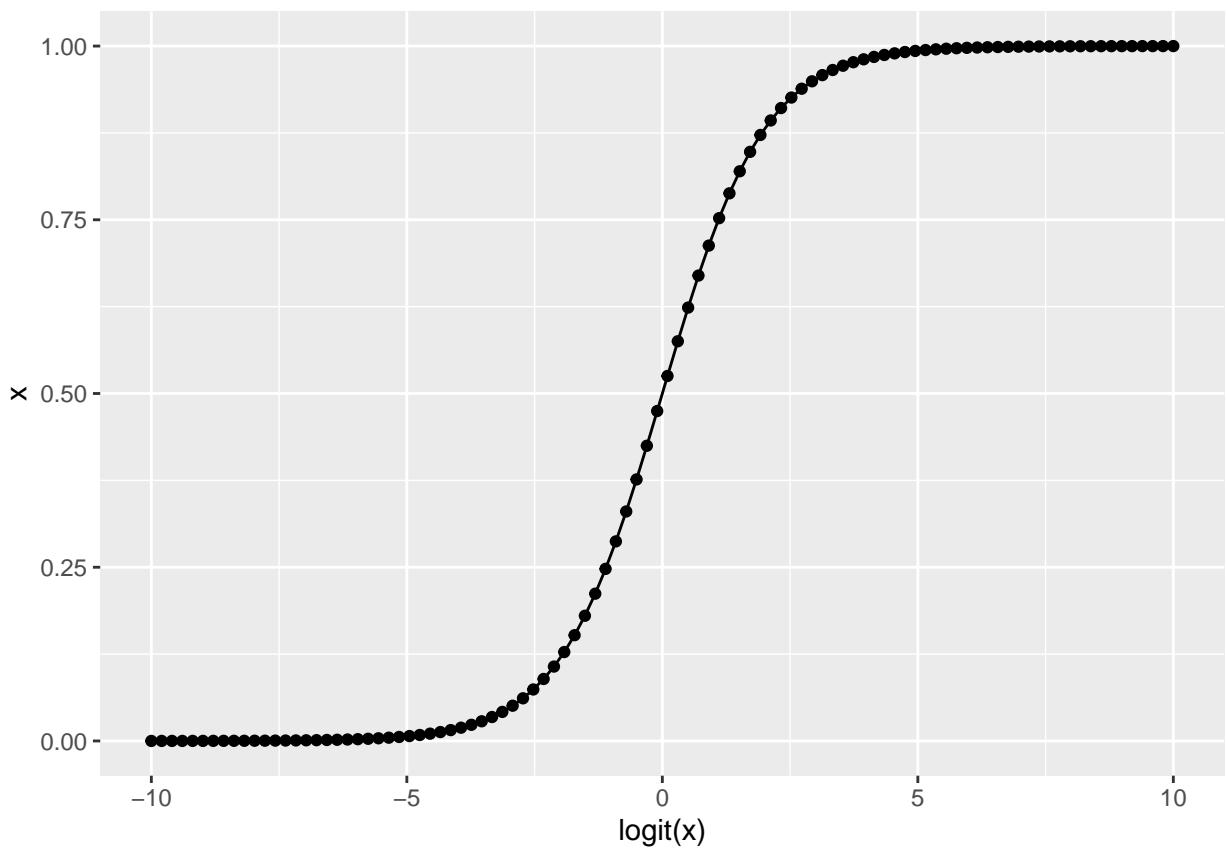


Figure 23: Logit-transformation ensures that the transformed variable fall in the range of zero to one.

```
## [5,]    2   28
## [6,]    0   30
```

Similar to the previous examples, the `summary()` function provides estimates of the model parameters. By applying the `summary()` function to the fitted model, you can obtain information such as coefficient estimates, standard errors, p-values, and other relevant statistics.

```
summary(m_binom)

##
## Call:
## glm(formula = cbind(n_fertilized, n_examined - n_fertilized) ~
##       density, family = "binomial", data = df_mussel)
##
## Deviance Residuals:
##      Min        1Q     Median        3Q       Max
## -1.4925  -0.6394  -0.3832  -0.0666   3.2751
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept) -8.06411   0.49944 -16.15 <2e-16 ***
## density      0.34209   0.03077  11.12 <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
## Null deviance: 233.752 on 99 degrees of freedom
## Residual deviance: 70.744 on 98 degrees of freedom
## AIC: 158.3
##
## Number of Fisher Scoring iterations: 6
```

This result translates into:

$$y_i \sim \text{Binomial}(N_i, p_i)$$

$$\log\left(\frac{p_i}{1-p_i}\right) = -8.06 + 0.34 \text{ density}_i$$

To make predictions, it is necessary to back-transform the predicted fertilization probability since it is estimated on a logit scale. The back-transformation will convert the predicted values back to the original probability scale (0.0 – 1.0).

```
# make prediction
df_pred <- tibble(density = seq(min(df_mussel$density),
                                 max(df_mussel$density),
                                 length = 100))

# y_binom is inv.logit-transformed because predict() returns values in logit-scale
y_binom <- predict(m_binom, newdata = df_pred) %>% boot::inv.logit()

df_pred <- df_pred %>%
  mutate(y_binom)
```

Draw on the figure (Figure 24).

```

df_mussel %>%
  ggplot(aes(x = density,
             y = prop_fert)) +
  geom_point() +
  geom_line(data = df_pred,
            aes(y = y_binom)) +
  labs(y = "Proportion of eggs fertilized",
       x = "Mussel density")

```

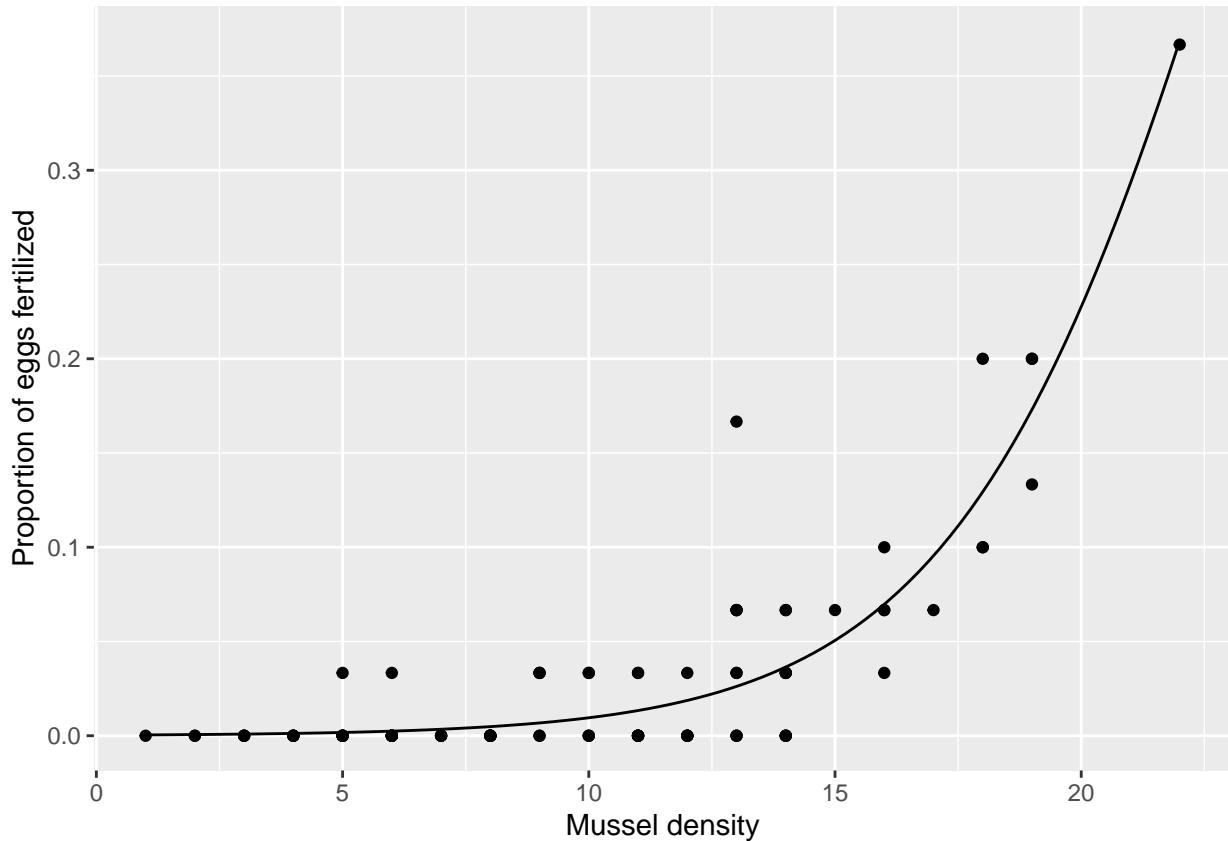


Figure 24: Prediction by the Binomial model.

Perfect.

### 8.3 The GLM Framework

These models, including those assuming a Normal distribution, fall into the category of the Generalized Linear Model (GLM) framework. The framework follows a common structure:

$$y_i \sim D(\Theta)$$

$$\text{Link}(f(\Theta)) = \alpha + \sum_k \beta_k x_{k,i}$$

In this structure,  $\Theta$  represents a parameter vector of the probability distribution, while  $D(\cdot)$  and  $\text{Link}(\cdot)$  denote the chosen probability distribution and its associated link function, respectively. For example, if we select a Poisson distribution as  $D(\cdot)$ , then the corresponding  $\text{Link}(\cdot)$  function is the natural logarithm (log)

Table 3: Common distributions used in GLM analysis

Node	Distribution	Range	R
1	Beta-binomial	\$0 \le y \le N\$	'glmmTMB::glmmTMB()'
2	Binomial	\$0 \le y \le N\$	'glm()'
3	Negative binomial	\$0 \le y\$	'glm.nb()'
4	Poisson	\$0 \le y\$	'glm()'
5	Beta	\$0 < y < 1\$	'glmmTMB::glmmTMB()'
6	Normal	\$-\infty < y < \infty\$	'lm()', 'glm()'

(see Section 8.1.2). The function  $f(\Theta)$  defines the modeled parameter of the probability distribution, which is related to the explanatory variables  $x$  and represented as  $g(x)$ . For instance, in Normal and Poisson models, we model the mean, while in the Binomial model, we model the success probability. In certain distributions,  $f(\Theta)$  may be a function of multiple parameters.

The GLM framework differs from the Linear Model framework by offering greater flexibility in the choice of probability distribution, achieved through the introduction of the Link function. This function allows us to constrain the modeled parameter within a specific range. As non-normal distributions are common in natural phenomena, this framework plays a critical role in modern statistical analysis.

However, how do we determine the appropriate probability distribution? While there is no single correct answer to this question, there are clear factors to consider—specifically, the characteristics of the response variable. The following criteria can be particularly helpful:

1. Is the variable discrete?
2. Is there an upper bound?
3. Is the sample variance far greater than the mean?

To assist you in selecting an appropriate probability distribution for modeling, I have provided a concise decision tree graph. Please note that the choices presented are frequently encountered in ecological modeling, rather than an exhaustive list of probability distributions (see Table 3). The full list of probability distributions can be found here (Wikipedia page)

## 8.4 Laboratory

```
source(here::here("code/library.R"))
df_count <- read_csv(here::here("data_raw/data_garden_count.csv"))
```

# 9 Likelihood

In Chapter 8, we introduced non-normal distributions to represent count and proportional data. The `glm()` function was used to obtain parameter estimates, but how does this function achieve that? **In reality, the least squares approach is only applicable to normal distributions.** Therefore, we require an alternative approach to estimate parameters within the GLM framework. In this section, I will introduce the concept of likelihood as a fundamental principle for parameter estimation in the GLM framework.

## 9.1 Finding the “More Likely”

In the least squares method, our goal is to minimize the sum of squared errors, ensuring that the fitted model has the smallest deviation from the observed data points. However, this approach is not applicable to non-normal variables. In such cases, we need to rely on the “probability” of observing a data point or numerical value.

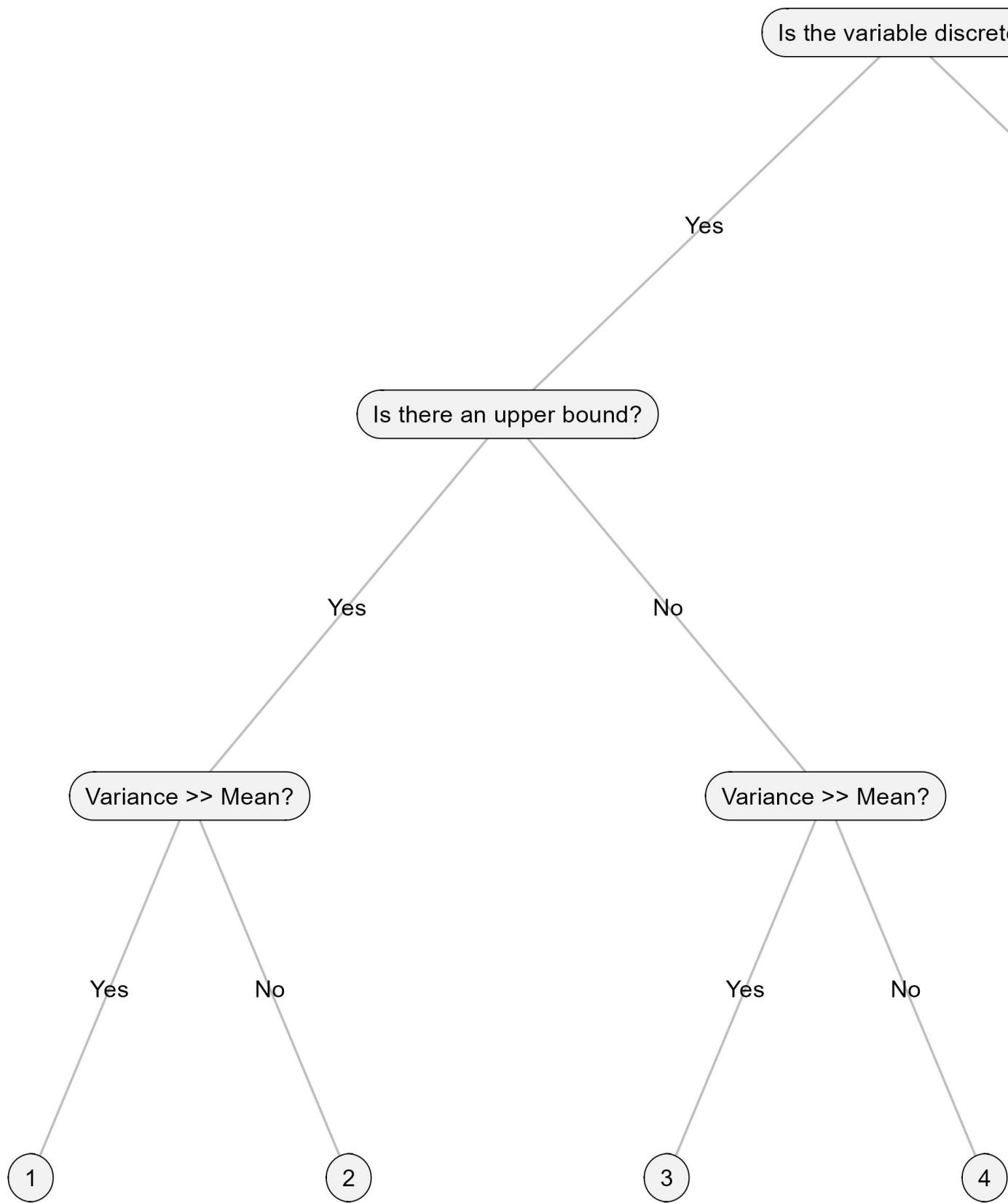


Figure 25: Tree chart on how to choose a probability distribution in GLM analysis.  
75

Let's consider an example where we have count data, denoted as  $\mathbf{y} = \{y_1, y_2, \dots, y_N\}$ . To simplify the matter, let's focus on a specific data point, say  $y_1 = 3$ . Assuming a Poisson distribution, we can calculate the probability of observing this particular value by using an arbitrary parameter value for the mean, denoted as  $\lambda$ . To refresh our memory, we can refer back to the Poisson distribution discussed in Chapter 3.

$$\Pr(y = k) = \frac{\lambda^k \exp(-\lambda)}{k!}$$

This equation represents the probability of the variable  $y$  taking a specific value  $k$ . In our current example, we are interested in the case where  $k = 3$ . By specifying a value for the mean parameter  $\lambda$ , we can compute the probability associated with this value. Let's try using  $\lambda = 3.5$  for this calculation:

$$\Pr(y_1 = 3) = \frac{3.5^3 \exp(-3.5)}{3!} \approx 0.215$$

We can calculate this with `dpois()` in R:

```
# dpois()
# the first argument is "k"
# the second argument is "lambda"
dpois(3, lambda = 3.5)

## [1] 0.2157855

# write the formula to confirm
(p <- (3.5^3 * exp(-3.5)) / factorial(3))

## [1] 0.2157855
```

If we assume a mean of 3.5 in the Poisson distribution, the probability of observing a value of 3 is approximately 0.215. We can experiment with different values of  $\lambda$  to see if there is a potentially “better” value. By exploring different values of  $\lambda$ , we can observe how the probability of observing a specific value changes. This allows us to assess which value of  $\lambda$  provides a better fit or captures the data more accurately. The following example explores  $\lambda = 0 - 10$  with a 0.1 interval.

```
# change lambda from 0 to 10 by 0.1
lambda <- seq(0, 10, by = 0.1)

# probability
pr <- dpois(3, lambda = lambda)

# create a data frame
df_pois <- tibble(y = 3,
                    lambda = lambda,
                    pr = pr)

print(df_pois)

## # A tibble: 101 x 3
##       y   lambda      pr
##   <dbl>   <dbl>    <dbl>
## 1     3     0     0
## 2     3     0.1  0.000151
## 3     3     0.2  0.00109
## 4     3     0.3  0.00333
## 5     3     0.4  0.00715
## 6     3     0.5  0.0126
```

```

## 7      3      0.6 0.0198
## 8      3      0.7 0.0284
## 9      3      0.8 0.0383
## 10     3      0.9 0.0494
## # i 91 more rows

```

Make a plot:

```

df_pois %>%
  ggplot(aes(x = lambda,
             y = pr)) +
  geom_point() +
  geom_line() +
  labs(x = "lambda",
       y = "Pr(k = 3)")

```

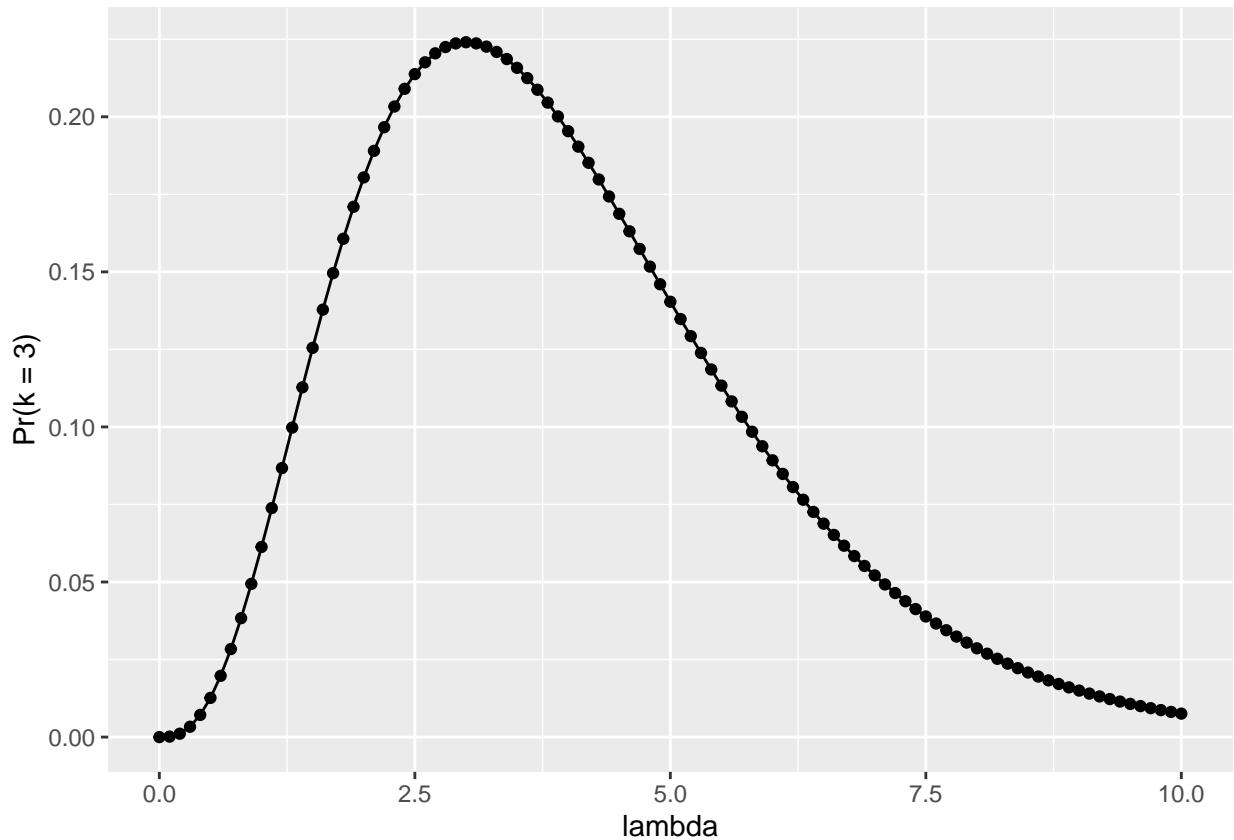


Figure 26: Relationship between the parameter  $\lambda$  and the probability of observing a value of three.

By examining the figure or data, it appears that the probability is highest around  $\lambda = 3$ . To confirm this observation, let's arrange the data frame `df_pois`.

```

# arrange() re-orders the dataframe based on the specified column in an ascending order
# desc() flips the order (descending)

df_pois %>%
  arrange(desc(pr)) %>%
  print()

```

```

## # A tibble: 101 x 3
##       y lambda   pr
##   <dbl> <dbl> <dbl>
## 1     3    3  0.224
## 2     3    3  0.224
## 3     3    3  0.224
## 4     3    3  0.223
## 5     3    3  0.222
## 6     3    3  0.221
## 7     3    3  0.220
## 8     3    3  0.219
## 9     3    3  0.218
## 10    3    3  0.216
## # i 91 more rows

```

That reasoning is sound because the probability of  $y_1$  being equal to 3 is maximized when  $\lambda = 3$ . This represents the simplest form of a **likelihood function** denoted as  $L(\lambda | y_1)$ , where  $(\cdot | y_1)$  indicates that  $y_1$  is fixed or given. To evaluate the probability of observing the fixed value of  $y_1$ , we vary the value of  $\lambda$ .

However, what if we have multiple values of  $y_i$ ? Let's consider the scenario where  $\mathbf{y} = \{3, 2, 5\}$ . In such cases, we must account for the probability of simultaneously observing these values. When the events are independent, the probability of observing multiple events can be expressed as the product of their individual probabilities. Consequently, the likelihood function takes the following form:

$$\begin{aligned}
L(\lambda | \mathbf{y}) &= \Pr(y_1 = 3) \times \Pr(y_2 = 2) \times \Pr(y_3 = 5) \\
&= \frac{\lambda^3 \exp(-\lambda)}{3!} \times \frac{\lambda^2 \exp(-\lambda)}{2!} \times \frac{\lambda^5 \exp(-\lambda)}{5!} \\
&= \prod_i^3 \frac{\lambda^{y_i} \exp(-\lambda)}{y_i!}
\end{aligned}$$

Implement this in R:

```

# try lambda = 3 for y = 3, 2, 5
pr <- dpois(c(3, 2, 5), lambda = 3)
print(pr)

## [1] 0.2240418 0.2240418 0.1008188
# probability of observing 3, 2, 5 simultaneously
# with lambda = 3
prod(pr)

## [1] 0.005060573

```

Similar to the previous example, let's search for a more suitable value of  $\lambda$  that better captures the observed data.

```

# lambda = 0 - 10 by 0.01
y <- c(3, 2, 5)
lambda <- seq(0, 10, by = 0.01)

# sapply repeats the task in FUN
# each element in "X" will be sequentially substituted in "z"
pr <- sapply(X = lambda,
              FUN = function(z) prod(dpois(y, lambda = z)))

```

```

# make a data frame and arrange by pr (likelihood)
df_pois <- tibble(lambda = lambda,
                  pr = pr)

df_pois %>%
  arrange(desc(pr)) %>%
  print()

## # A tibble: 1,001 x 2
##   lambda      pr
##   <dbl>    <dbl>
## 1 3.33 0.00534
## 2 3.34 0.00534
## 3 3.32 0.00534
## 4 3.35 0.00534
## 5 3.31 0.00534
## 6 3.36 0.00534
## 7 3.3  0.00534
## 8 3.37 0.00534
## 9 3.29 0.00533
## 10 3.38 0.00533
## # i 991 more rows

# visualize
df_pois %>%
  ggplot(aes(x = lambda,
             y = pr)) +
  geom_line() +
  labs(y = "Likelihood")

```

In this exercise with an interval of 0.01,  $\lambda \approx 3.33$  is identified as the most suitable value. Interestingly, the sample mean matches this value:

```
mean(c(3, 2, 5))
```

```
## [1] 3.333333
```

Why? See below.

## 9.2 Maximum Likelihood Method

### 9.2.1 Simple case

Finding the parameter value that maximizes the probability of observing a set of observed values is known as the **Maximum Likelihood Estimate (MLE)**. This estimation method is commonly employed in the GLM framework and various other statistical inference techniques. The MLE approach is applicable as long as the likelihood can be defined, making it a versatile method for many statistical analyses.

However, the procedure I employed in the previous section (Section 9.1) is not mathematically rigorous because the optimal value of  $\lambda$  is dependent on the resolution of the interval. As the interval becomes smaller, it is possible to discover an even better value of  $\lambda$  ad infinitum due to its continuous nature.

A more reliable approach to identifying the “peak” is by utilizing the first-order derivative. The first-order derivative represents the slope of the likelihood function at a given  $\lambda$  value. If the derivative equals zero, it indicates that we have reached the peak of the likelihood function. We can leverage this mathematical principle to determine the maximum likelihood estimate of  $\lambda$ . Since computing the derivative of a product is mathematically challenging, we can apply a logarithm transformation to the likelihood function. Recall that

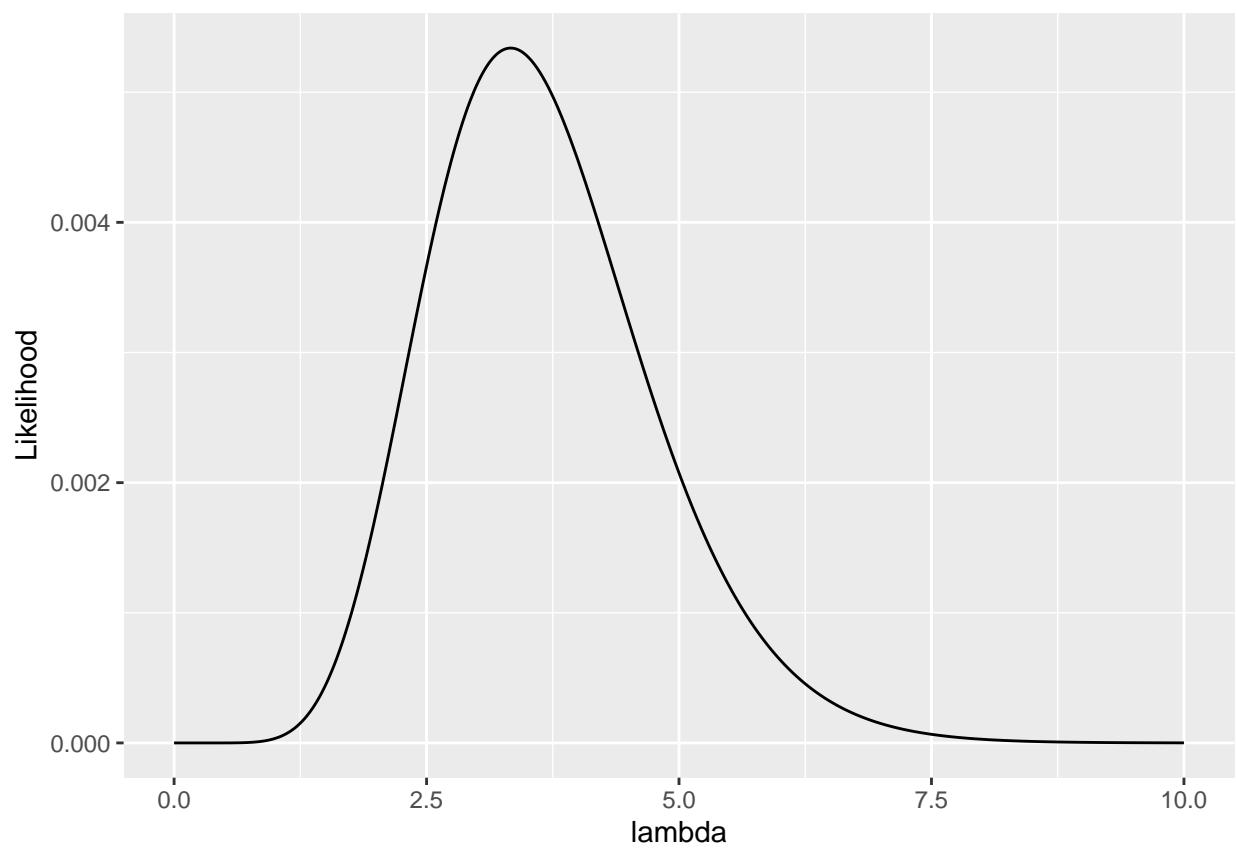


Figure 27: Likelihood of observing three count data points  $\mathbf{y}$  simultaneously.

$\log ab = \log a + \log b$ . By taking the logarithm, we can convert the product operation  $\prod$  into a summation operation  $\sum$ . Specifically:

$$\begin{aligned}\log L(\lambda|\mathbf{y}) &= \log \prod_i^N \frac{\lambda^{y_i} \exp(-\lambda)}{y_i!} \\ &= \log \frac{\lambda^{y_1} \exp(-\lambda)}{y_1!} + \log \frac{\lambda^{y_2} \exp(-\lambda)}{y_2!} + \dots + \log \frac{\lambda^{y_N} \exp(-\lambda)}{y_N!} \\ &= \sum_i^N \log \frac{\lambda^{y_i} \exp(-\lambda)}{y_i!} \\ &= \sum_i^N (y_i \log \lambda - \lambda - \log y_i!)\end{aligned}$$

Taking the first derivative of  $\log L(\lambda|\mathbf{y})$ :

$$\begin{aligned}\frac{\partial \log L(\lambda|\mathbf{y})}{\partial \lambda} &= \sum_i^N \left( \frac{y_i}{\lambda} - 1 \right) \\ &= \frac{\sum_i^N y_i}{\lambda} - \sum_i^N 1 \\ &= \frac{\sum_i^N y_i}{\lambda} - N\end{aligned}$$

To obtain the value of  $\lambda$  that maximizes the likelihood, we set the derivative equal to zero:

$$\begin{aligned}\frac{\partial \log L(\lambda|\mathbf{y})}{\partial \lambda} &= \frac{\sum_i^N y_i}{\lambda} - N = 0 \\ \lambda &= \frac{\sum_i^N y_i}{N}\end{aligned}$$

Interesting – the maximum likelihood estimate of  $\lambda$  is equal to the sample mean. This is THE reason why we could use the sample mean as the estimate of  $\lambda$  in Chapter 3.

### 9.2.2 General case

In practice, obtaining the MLE analytically can be highly complex and often infeasible. Consequently, the GLM framework relies on numerical search algorithms to approximate the MLE estimates. Within GLMs, it is common to establish relationships between the mean  $\lambda$  and explanatory variables in order to investigate their influences. For instance:

$$\begin{aligned}y_i &\sim \text{Poisson}(\lambda_i) \\ \log(\lambda_i) &= \alpha + \sum_k \beta_k x_k\end{aligned}$$

where  $x_k$  represents  $k$ th explanatory variable. This translates into:

$$\log L(\alpha, \beta_k | \mathbf{y}) = \sum_i \left[ y_i (\alpha + \sum_k \beta_k x_k) - \exp(\alpha + \sum_k \beta_k x_k) - \log y_i! \right]$$

and solving the  $k+1$  partial derivatives:

$$\frac{\partial \log L(\alpha, \beta_k | \mathbf{y})}{\partial \alpha} = 0$$

$$\frac{\partial \log L(\alpha, \beta_k | \mathbf{y})}{\partial \beta_1} = 0$$

...

$$\frac{\partial \log L(\alpha, \beta_k | \mathbf{y})}{\partial \beta_k} = 0$$

Thus, the function `glm()` does a lot for us! The log likelihood of the fitted model can be extracted with `logLik()`:

```
# load garden plant data
# df_count <- read_csv("data_raw/data_garden_count.csv")

m_pois <- glm(count ~ nitrate,
               data = df_count,
               family = "poisson")

logLik(m_pois)

## 'log Lik.' -44.92912 (df=2)
```

### 9.3 Laboratory

## 10 Appendix: Project Management

### 10.1 R Project

Throughout this book, I will use the ‘R Project’ as the core workspace unit. It serves as a centralized location where all relevant materials, such as R scripts (.R) and data files, are consolidated. There are various ways to organize your project, but I prefer creating a single ‘R Project’ for a set of scripts and data that contribute to a specific publication (see example here). To set up an ‘R Project,’ you’ll need to have both *RStudio* and the base *R* software installed. While *R* can be used independently, I highly recommend using it in conjunction with *RStudio* due to the latter’s numerous features that facilitate data analysis. You can download *R* and *RStudio* from the following websites:

- R (you can choose any CRAN mirror for downloading)
- RStudio

Upon launching *RStudio*, you will encounter the interface depicted in Figure 28. Initially, the interface comprises three primary panels: the **Console**, **Environment**, and **Files**. The **Console** is where you write code and execute calculations, data manipulation, and analysis. The **Environment** panel lists the saved objects, while the **Files** panel displays files in a designated location on your computer.

After pasting the script into the console, you will notice that the variable `x` appears in the environment panel.

```
x <- c(1, 2)
```

`x` is an object that stores information. In this particular case, we have stored a sequence of 1 and 2 in the object `x`. Once you have stored information in `x`, you can access it by simply typing `x`.

```
x
```

```
## [1] 1 2
```

Excellent! While it is possible to work with your data using this approach, it is important to highlight that **it is generally regarded as a poor practice**. As your project evolves, you will accumulate a substantial

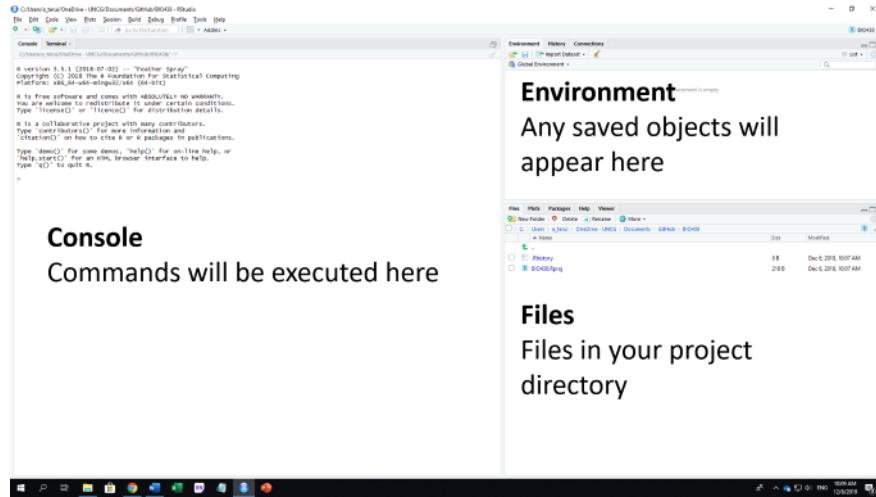


Figure 28: RStudio interface.

amount of material, such as writing more than 2000 lines of code for a single project. Consequently, it becomes essential to implement effective code management strategies. How do you presently handle code management?

### 10.1.1 Script Editor

It is highly recommended to manage your scripts in the **Editor** instead. The **Editor** is where you draft and fine-tune your code before executing it in the **Console**. To create space for the **Editor**, press **Ctrl + Shift + N**. A new panel will appear in the top left corner. Let's type the following script in the **Editor**. Please note that the key combination **Ctrl + Shift + N** assumes a Windows or Linux operating system. If you're using a Mac, you can use **Command + Shift + N** instead.

```
y <- c(3, 4)
```

Then, hit **Ctr + S** to save the **Editor** file. *RStudio* will prompt you to enter the file name of the **Editor**<sup>5</sup>.

### 10.1.2 File Name

It is also crucial to establish consistent naming **rules** for your files. As your project progresses, the number of files within each sub-directory may increase significantly. Without clear and consistent naming rules for your files, navigating through the project can become challenging, not only for yourself but also for others involved. To alleviate this issue, consider implementing the following recommendations for file naming:

- **NO SPACE.** Use underscore.
  - Do: `script_week1.R`
  - Don't: `script week1.R`
- **NO UPPERCASE.** Use lowercase for file names.
  - Do: `script_week1.R`
  - Don't: `Script_week1.R`
- **BE CONSISTENT.** Apply consistent naming rules within a project.
  - Do: R scripts for figures always start with a common prefix, e.g., `figure_XXX.R` `figure_YYY.R` (`XXX` and `YYY` specifies further details).
  - Don't: R scripts for figures start with random text, e.g., `XXX_fig.R` , `Figure_Y2.R` , `plotB.R`.

<sup>5</sup>In *R*, an editor file has an extension of `.R`.

### 10.1.3 Structure Your Project

If you fail to save or haphazardly store your code files on your computer, the risk of losing essential items becomes inevitable sooner or later. To mitigate this risk, I highly recommend gathering all the relevant materials within a single **R Project**. To create a new **R Project**, follow the procedure outlined below:

- a. Go to **File > New Project** on the top menu
- b. Select **New Directory**
- c. Select **New Project**

A new window will appear, prompting you to name a directory and select a location on your computer. To choose a location for the directory, click on the ‘Browse’ button. When organizing your project directories on your computer, I highly recommend creating a dedicated space. For instance, on my computer, I have a folder named `/github` where I store all my **R Project** directories.

The internal structure of an **R Project** is crucial for effective navigation and ensures clarity for both yourself and others when it is published. An **R Project** typically consists of various file types, such as `.R`, `.csv`, `.rds`, `.Rmd`, and others. Without an organized arrangement of these files, there is a high probability of encountering significant coding errors. Therefore, I place great importance on maintaining a well-structured project. In Table 4, I present my recommended subdirectory structure.

Table 4: Suggested internal structure of **R Project**

Name	Content
<code>README.md</code>	Markdown file explaining contents in the <b>R Project</b> . Can be derived from <code>README.Rmd</code> .
<code>/code</code>	Sub-directory for R scripts ( <code>.R</code> ).
<code>/data_raw</code>	Sub-directory for raw data before data manipulation ( <code>.csv</code> or other formats). Files in this sub-directory MUST NOT be modified unless there are changes to raw data entries.
<code>/data_fmt</code>	Sub-directory for formatted data ( <code>.csv</code> , <code>.rds</code> , or other formats).
<code>/output</code>	Sub-directory for result outputs ( <code>.csv</code> , <code>.rds</code> , or other formats). This may include statistical estimates from linear regression models etc.
<code>/rmd</code>	(Optional) Sub-directory for Rmarkdown files ( <code>.Rmd</code> ). Rmarkdown allows seamless integration of R scripts and text.

## 10.2 Robust coding

While it is not mandatory, I highly recommend using *RStudio* in conjunction with *Git* and *GitHub*. Coding is inherently prone to errors, and even the most skilled programmers make mistakes—without exception. However, the crucial difference between beginner and advanced programmers lies in their ability to develop robust coding practices accompanied by a self-error-detection system. *Git* plays a vital role in this process. Throughout this book, I will occasionally delve into the importance of *Git* and its usage.

# 11 Appendix: Git & GitHub

## 11.1 Git & GitHub

In this section, I will explain how to integrate Git and GitHub into R Studio. While R Studio is already an excellent tool, it becomes even more powerful when combined with Git and GitHub. Git is a free and open-source distributed version control system that tracks changes in your code as you work on your project, ensuring you are aware of any modifications made to your script (and other) files. Tracking changes is crucial to avoid unintended errors in your code and helps prevent the creation of redundant files. Although Git is primarily a local system, it has an online counterpart called GitHub.

To set up this system, you'll need to follow a few steps. The first step is to install Git on your computer:

- For **Windows**: Install Git from here. During the installation process, you'll be prompted about "Adjusting your PATH environment." Choose "Git from the command line and also from 3rd-party software" if it's not already selected.
- For **Mac**: Follow the instructions provided here.

Once Git is installed, open R Studio and navigate to **Create Project > New Directory > New Project**. If you see the checkbox labeled "Create a git repository," make sure to select it before creating your new project (refer to Figure 29). This action will display the Git pane on the upper right panel of R Studio, indicating that Git integration is enabled.

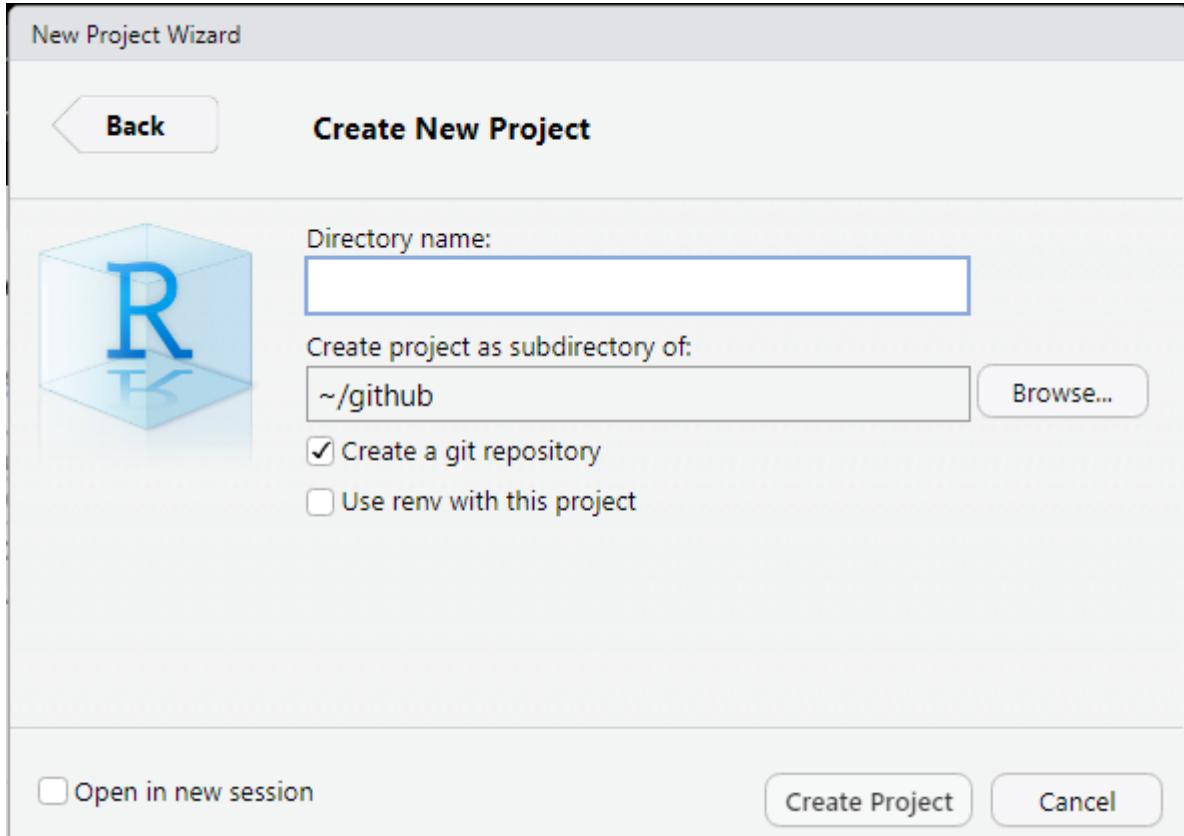


Figure 29: After installing Git, you should see 'Create a git repository'.

If you are unable to find the options mentioned above, follow these steps:

1. Click on **Tools** in the menu bar of R Studio.
2. Select **Terminal** and then choose **New Terminal**.
3. In the terminal window, type `where git` and press Enter. This command will display the location of the Git executable on your computer.
4. Next, go back to **Tools** in the menu bar and select **Global Options**.
5. In the options window, navigate to **Git/SVN**.
6. Look for the field labeled *Git executable* and specify the location of the Git executable that you obtained from the terminal.

After setting up Git, you can proceed to create an account on GitHub. It's a free platform, and when choosing a username, consider using lowercase letters and including your name to make it easier for others to find you.

While R Studio seamlessly integrates with Git and GitHub, using a Git client can provide additional visual aids. There are various options for Git clients (see choices here), but for this exercise, we will use GitHub Desktop. Install GitHub Desktop on your computer from here.

## 11.2 Commit & Push

### 11.2.1 Register Your Git repo

To open the **R Project** you've just created as a git repository, follow these steps:

1. Open R Studio.
2. Click on **File** in the menu bar.
3. Select **Open Project**.
4. Browse to the directory where you created the **R Project** and select the corresponding **.Rproj** file.
5. R Studio will open the project, and you will see the project name in the top-right corner of the window.

To create a sample **.R** file named **sample.R** within the project, you can use the shortcut **Ctrl + Shift + N** or follow these steps:

1. Click on **File** in the menu bar.
2. Select **New File**.
3. Choose **R Script**.
4. A new script editor will open, where you can write your R code.
5. Write your code in the editor and save it as **sample.R** by clicking on **File** and selecting **Save** or using the shortcut **Ctrl + S**.

Remember to save the file after writing your code.

```
## produce 100 random numbers that follows a normal distribution
x <- rnorm(100, mean = 0, sd = 1)

## estimate mean
mean(x)

## estimate SD
sd(x)
```

To open the GitHub Desktop app, locate the application on your computer and launch it. Once opened, you will see a GUI similar to the one depicted in Figure 30.

Click on the “Current Repository” button located at the top left corner of the GitHub Desktop app interface. Then, select “Add” followed by “Add existing repository” from the dropdown menu, as shown in Figure 31.

### 11.2.2 Commit

GitHub Desktop will prompt you to enter a local path to your Git repository. Browse and select the directory where your **R Project** is located. Once you have selected the directory, the local Git repository will appear in the list of repositories on the left side bar of GitHub Desktop, as shown in Figure 31.

Now that your repository is added to GitHub Desktop, you are ready to start committing your files to Git. Committing is the process of recording the changes made to your files in Git. To proceed with committing, click on your Git repository in GitHub Desktop. You will see the following options and information displayed:

- The list of changed files: This section shows the files that have been modified since the last commit.
- A summary of the changes: This section provides an overview of the changes made to the selected files.
- Commit message: This is where you can enter a descriptive message explaining the changes made in the commit.

By providing a meaningful commit message, you can keep track of the changes made to your files and easily understand the purpose of each commit.

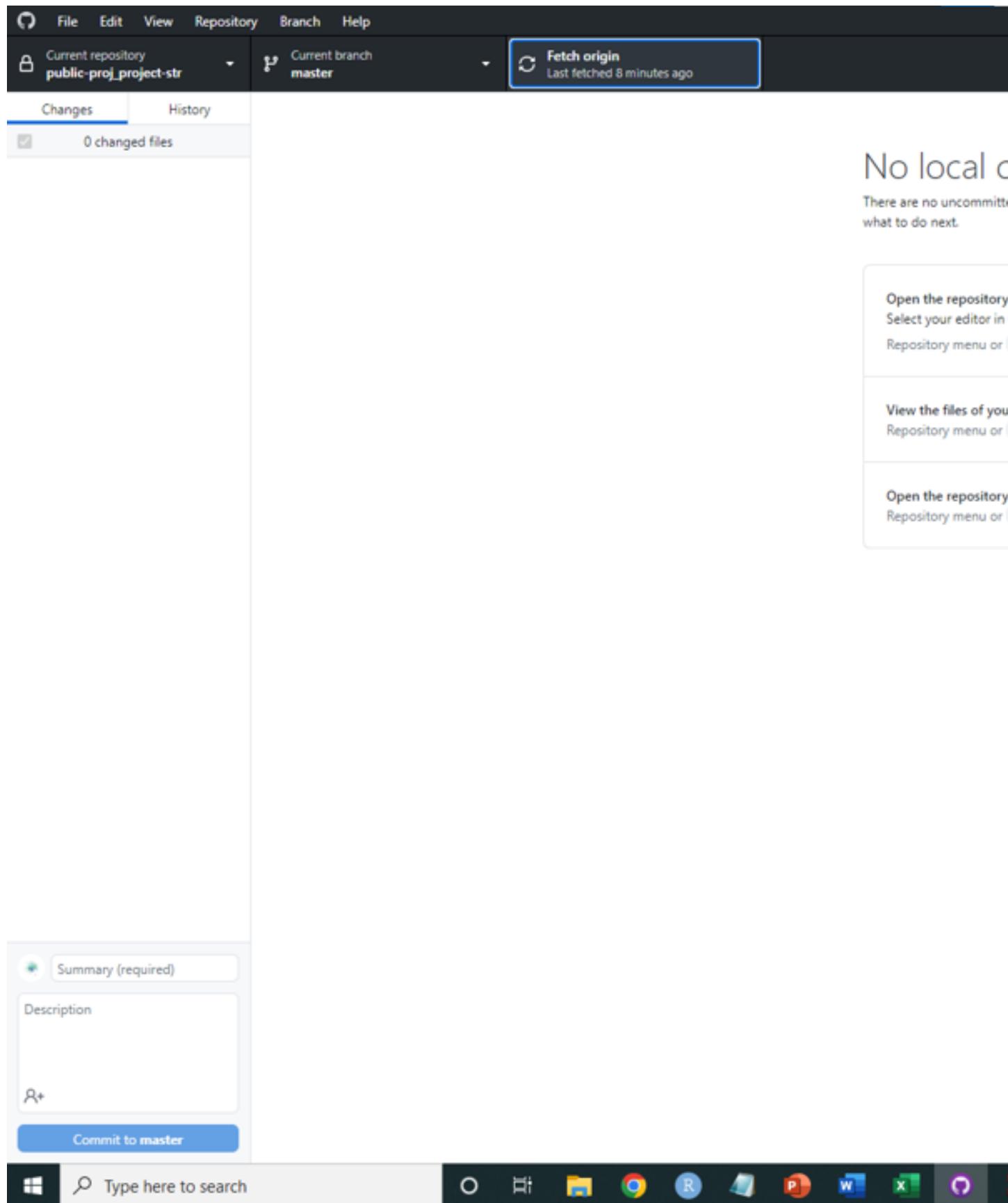


Figure 30: GUI for GitHub Desktop  
87

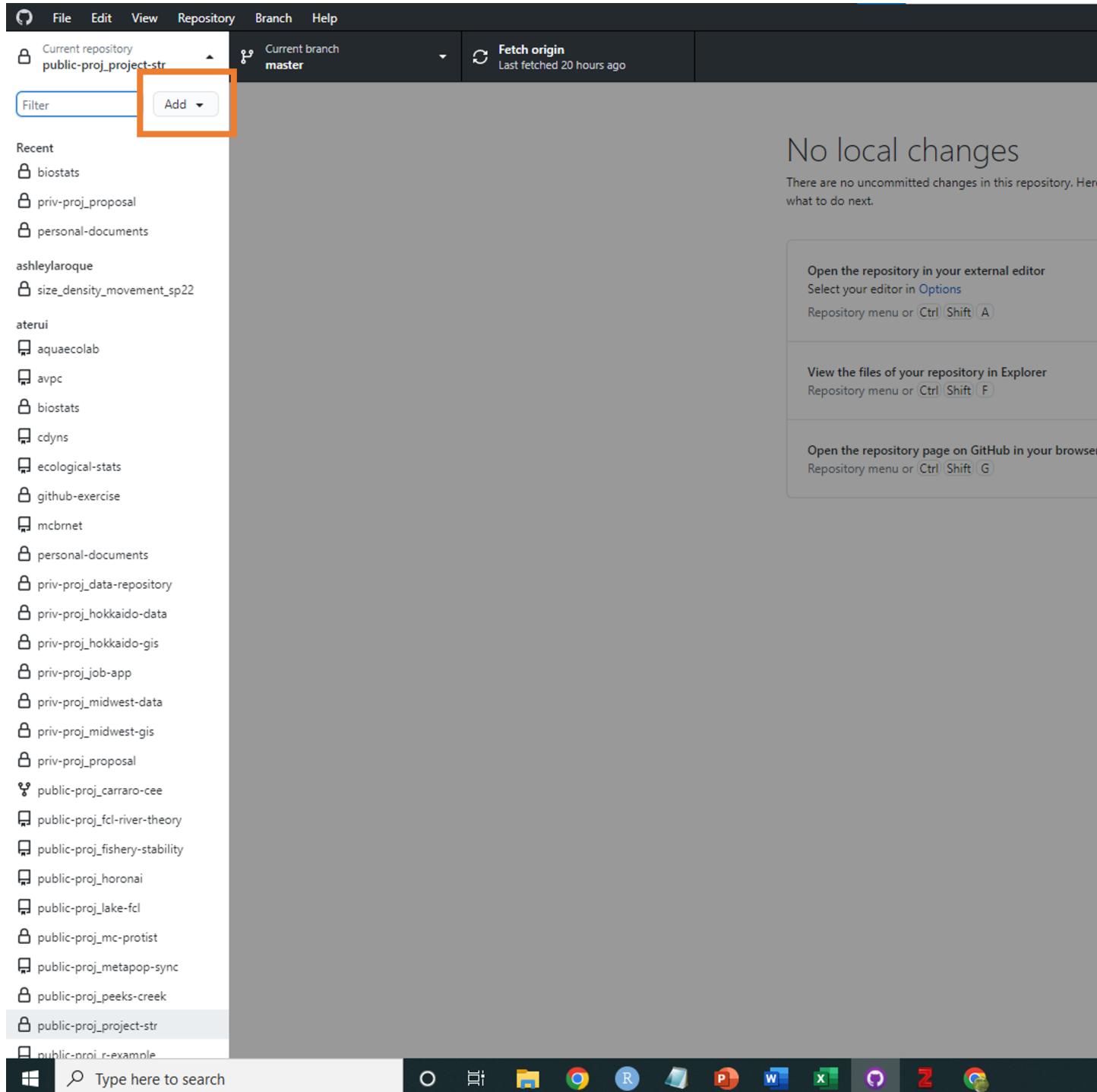


Figure 31: Add dropdown on the top left

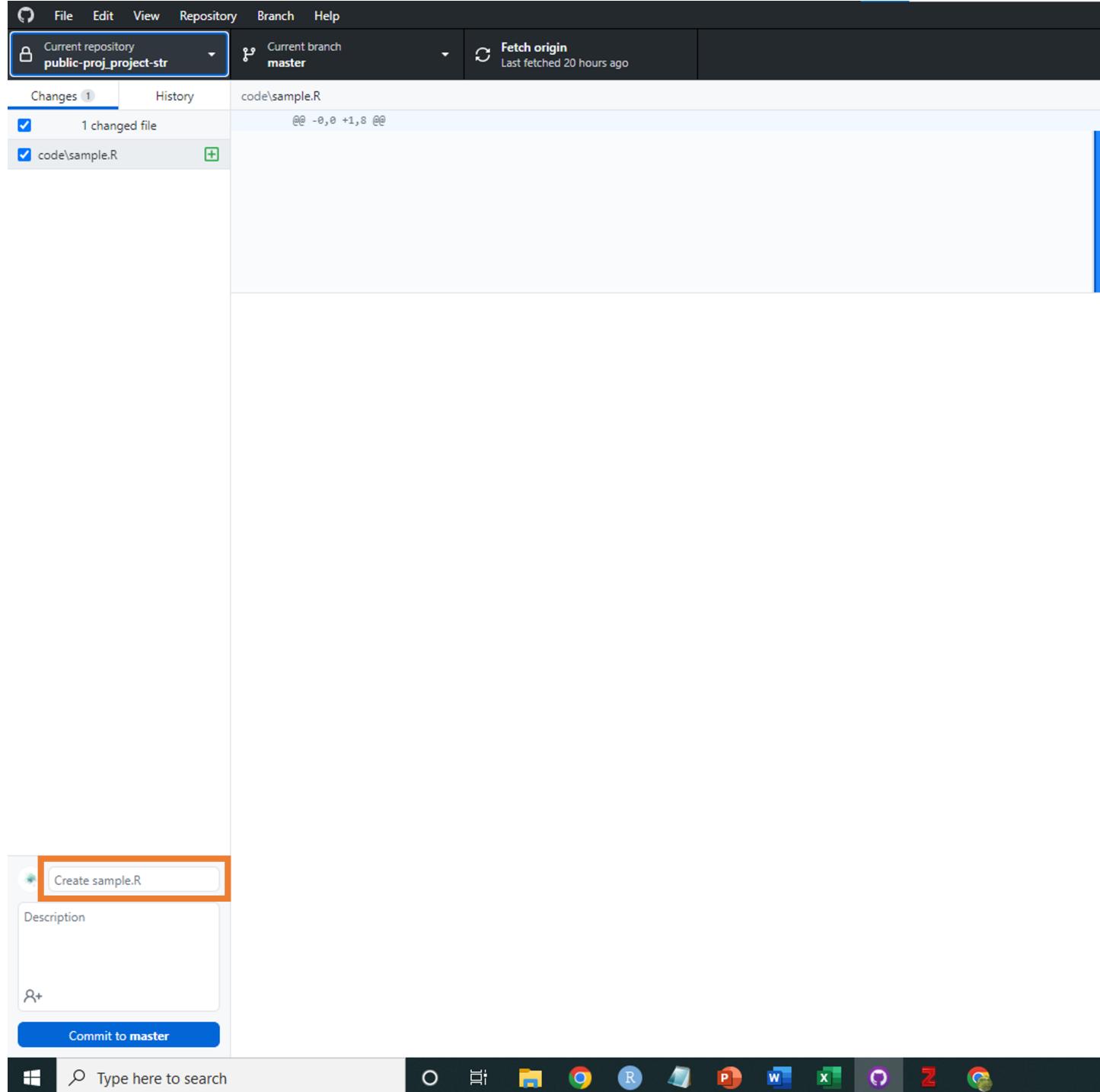


Figure 32: You will see ‘Create sample.R‘ or ‘summary (required)‘ on the bottom left

At this stage, your file (.R or any other file) is saved in your local directory but has not been recorded in Git as a change history. To commit the changes, follow these steps:

1. First, select the file(s) you want to commit. You can do this by checking the checkboxes next to the files listed in the “Changes” section of the GitHub Desktop app.
2. Once you have selected the files, go to the bottom left of the GitHub Desktop app window. You will see a small box that says **summary (required)** or **Create sample.R**.
3. In this box, enter a descriptive title that explains what you did in this commit. It is important to provide a meaningful summary, as you cannot proceed with the commit unless you enter this information. For example, for this exercise, you can enter **Initial commit** as the commit title. For subsequent commits, it is recommended to provide more informative commit messages that accurately describe the changes made. You can refer to online resources for recommendations on how to write effective commit titles and descriptions.
4. After entering the commit title, click on the button that says **Commit to master**. This action will record the changes made to the selected files in Git.

Now, the changes to the selected files have been successfully recorded in Git as a commit. It’s important to commit regularly and provide meaningful commit messages so that you can track changes and understand the evolution of your project when needed.

#### 11.2.3 Push

It’s important to note that your changes are currently recorded in your local computer’s Git repository but have not been published to your online GitHub repository. To send the local changes to the online GitHub repository, you need to use the “Push” operation through GitHub Desktop.

After you make a commit in GitHub Desktop, you will be prompted with a dialog box asking if you want to push the commit to an online repository, as shown in Figure 33. If this is your first push, there won’t be a corresponding repository on GitHub linked to your local repository. In this case, GitHub Desktop will ask if you want to publish it on GitHub. Please note that “publishing” here means making the repository available on GitHub, but it will remain private unless you explicitly choose to make it public.

If you are comfortable with the changes you made and want to send them to the online repository, click the “Push” button in GitHub Desktop. This action will push your commits to the remote repository on GitHub, effectively synchronizing your local repository with the online repository.

#### 11.2.4 Edit

We have created a file named **sample.R** as per the previous instructions. Now, let’s make a minor change to the content of the **sample.R** file. Open the **sample.R** file in your preferred text editor or in R Studio, and make a modification to the code. For example, you can update the code from:

```
## produce 100 random numbers that follows a normal distribution
x <- rnorm(100, mean = 0, sd = 1)

## estimate mean
mean(x)

## estimate SD
sd(x)
```

to:

```
## produce 100 random numbers that follows a normal distribution
x <- rnorm(100, mean = 0, sd = 1)
```

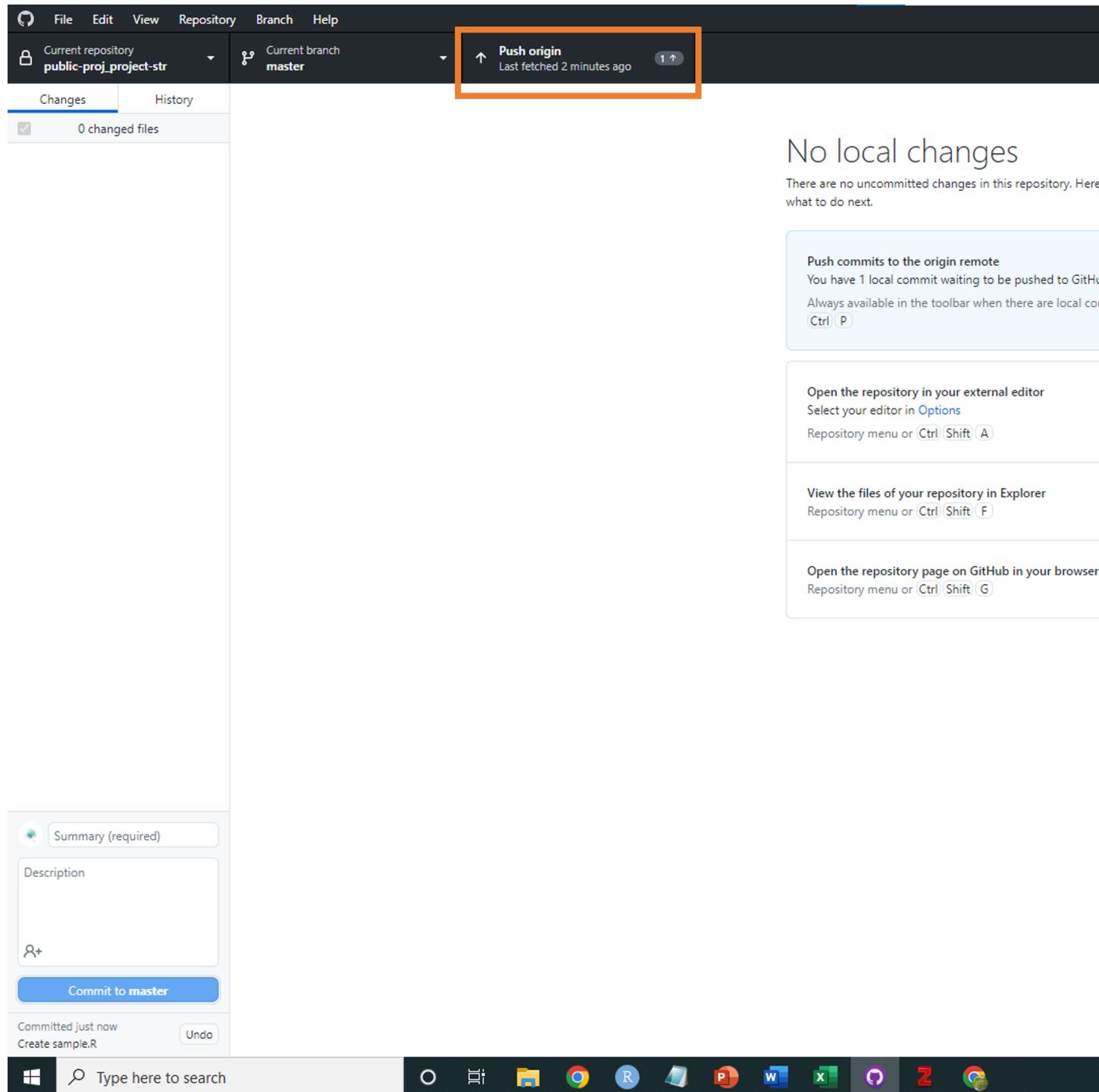


Figure 33: To Push your code, hit the highlighted menu button

```

## estimate mean
median(x)

## estimate SD
var(x)

```

Save the changes to the `sample.R` file. Now, let's go back to GitHub Desktop and see how we can handle this change.

After making the change to the `sample.R` file, open GitHub Desktop again. You will notice that GitHub Desktop automatically detects the difference between the new and old versions of the file. It highlights the specific parts of the script that have been edited, which can be extremely helpful when reviewing and comparing changes in your code.

This feature provided by GitHub Desktop makes it easier to track modifications and understand the specific updates made to your files. It helps streamline the coding process by providing a clear visual representation of the differences between different versions of your code.

## 12 Appendix: Data Structure

### 12.1 Overview

R has 6 basic **data types**.

- character: "aquatic", "ecology" (no order)
- factor: similar to character, but has *levels* (alphabetically ordered by default)
- numeric: 20.0 , 15.5
- integer: 3, 7
- logical: TRUE , FALSE
- complex: 1+2i (complex numbers with real and imaginary parts)

These elements form one of the following **data structures**.

- **vector**: a series of elements. A single data type is allowed in a single vector
- **matrix**: elements organized into rows and columns. A single data type is allowed in a single matrix
- **data frame**: looks similar to a matrix, but allows different data types in different columns

### 12.2 Vector

#### 12.2.1 Create Vector

Below are examples of atomic character vectors, numeric vectors, integer vectors, etc. There are many ways to create vector data. The following examples use `c()`, `:`, `seq()`, `rep()`:

```

#ex.1a manually create a vector using c()
x <- c(1,3,4,8)
x

```

```
## [1] 1 3 4 8
```

```
#ex.1b character
```

```
x <- c("a", "b", "c")
x
```

```
## [1] "a" "b" "c"
```

```
#ex.1c logical
```

```
x <- c(TRUE, FALSE, FALSE)
x
```

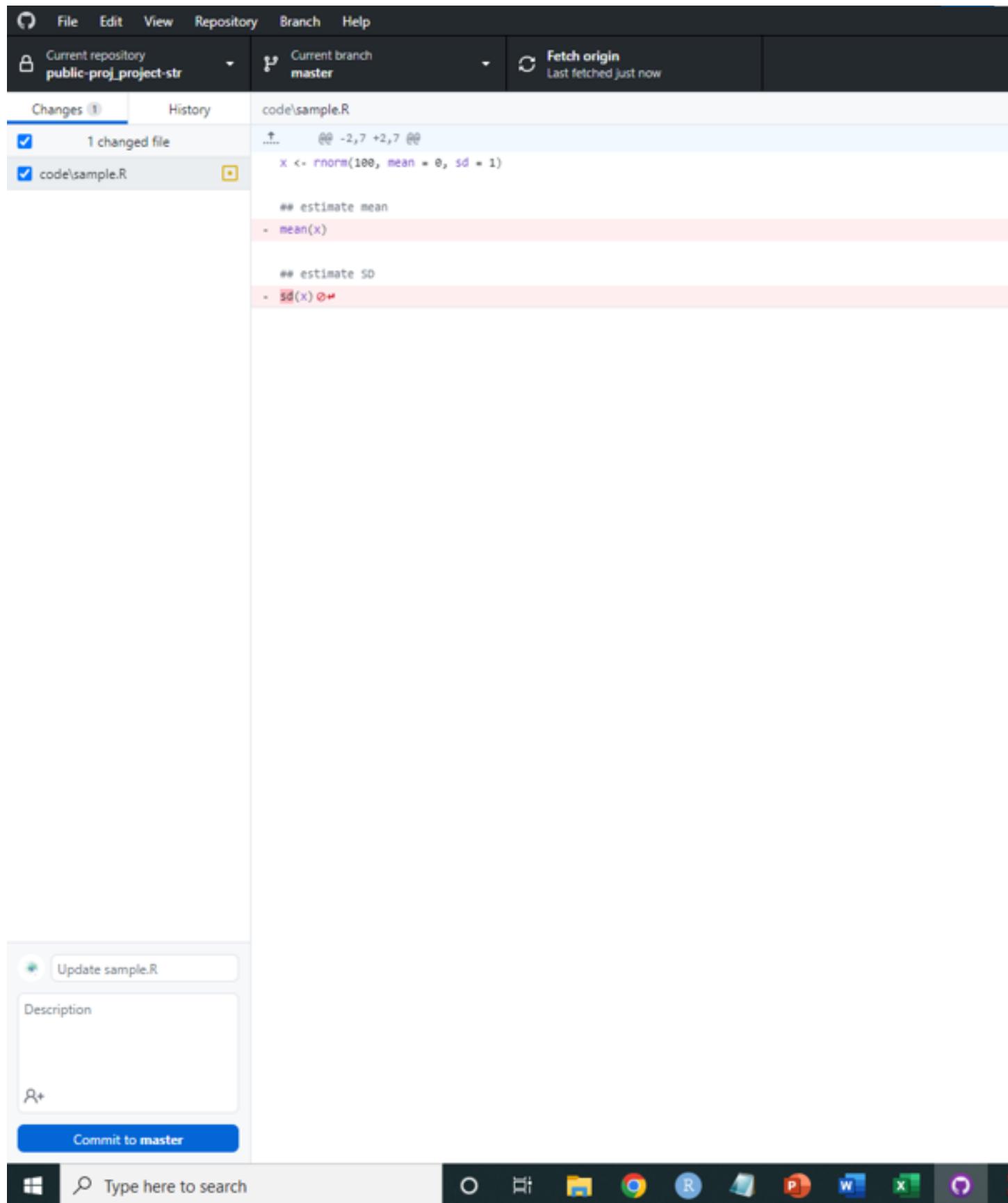


Figure 34: Git detects edits to your codes  
93

```

## [1] TRUE FALSE FALSE
#ex.2 sequence of numbers
x <- 1:5
x

## [1] 1 2 3 4 5
#ex.3a replicate same numbers or characters
x <- rep(2, 5) # replicate 2 five times
x

## [1] 2 2 2 2 2
#ex.3b replicate same numbers or characters
x <- rep("a", 5) # replicate "a" five times
x

## [1] "a" "a" "a" "a" "a"
#ex.4a use seq() function
x <- seq(1, 5, by = 1)
x

## [1] 1 2 3 4 5
#ex.4b use seq() function
x <- seq(1, 5, by = 0.1)
x

## [1] 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8
## [20] 2.9 3.0 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8 3.9 4.0 4.1 4.2 4.3 4.4 4.5 4.6 4.7
## [39] 4.8 4.9 5.0

```

## 12.2.2 Check Features

R provides many functions to examine features of vectors and other objects, for example:

- `class()` - return high-level data structure of the object
- `typeof()` - return low-level data structure of the object
- `attributes()` - metadata of the object
- `length()` - number of elements in the object
- `sum()` - sum of object's elements
- `mean()` - mean of object's elements

### Numeric Vector

```

x <- c(1.2, 3.1, 4.0, 8.2)
x

## [1] 1.2 3.1 4.0 8.2
class(x)

## [1] "numeric"
typeof(x)

## [1] "double"
length(x)

## [1] 4

```

```

sum(x)

## [1] 16.5
mean(x)

## [1] 4.125

Character Vector

y <- c("a", "b", "c")
class(y)

## [1] "character"

length(y)

## [1] 3

```

### 12.2.3 Access

#### Element ID

Use brackets [] when accessing specific elements in an object. For example, if you want to access element #2 in the vector x, you may specify as x[2]:

```

x <- c(2,2,3,2,5)
x[2] # access element #2

## [1] 2
x[c(2,4)] # access elements #2 and 4

## [1] 2 2
x[2:4] # access elements #2-4

## [1] 2 3 2

```

#### Equation

R provides many ways to access elements that suffice specific conditions. You can use mathematical symbols to specify what you need, for example:

- == equal
- > larger than
- >= equal & larger than
- < smaller than
- <= equal & smaller than
- which() a function that returns element # that suffices the specified condition

The following examples return a logical vector indicating whether each element in x suffices the specified condition:

```

# creating a vector
x <- c(2,2,3,2,5)

# ex.1a equal
x == 2

## [1] TRUE TRUE FALSE TRUE FALSE
# ex.1b larger than
x > 2

```

```
## [1] FALSE FALSE TRUE FALSE TRUE
```

You can access elements that suffice the specified condition using brackets, for example:

```
# ex.2a equal  
x[x == 2]
```

```
## [1] 2 2 2
```

```
# ex.2b larger than  
x[x > 2]
```

```
## [1] 3 5
```

Using `which()`, you can see which elements (i.e., #) matches what you need:

```
# ex.3a equal  
which(x == 2) # returns which elements are equal to 2
```

```
## [1] 1 2 4
```

```
# ex.3b larger than  
which(x > 2)
```

```
## [1] 3 5
```

## 12.3 Matrix

### 12.3.1 Create Matrix

Matrix is a set of elements (*single data type*) that are organized into rows and columns:

```
#ex.1 cbind: combine objects by column  
x <- cbind(c(1,2,3), c(4,5,6))  
x
```

```
##      [,1] [,2]  
## [1,]     1     4  
## [2,]     2     5  
## [3,]     3     6
```

```
#ex.2 rbind: combine objects by row  
x <- rbind(c(1,2,3), c(4,5,6))  
x
```

```
##      [,1] [,2] [,3]  
## [1,]     1     2     3  
## [2,]     4     5     6
```

```
#ex.3 matrix: specify elements and the number of rows (nrow) and columns (ncol)  
x <- matrix(1:9, nrow = 3, ncol = 3)  
x
```

```
##      [,1] [,2] [,3]  
## [1,]     1     4     7  
## [2,]     2     5     8  
## [3,]     3     6     9
```

### 12.3.2 Check Features

R provides many functions to examine features of matrix data, for example:

- `dim()` number of rows and columns

- `rowSums()` row sums
- `colSums()` column sums

### Integer Matrix

```
x <- matrix(1:9, nrow = 3, ncol = 3)
x

##      [,1] [,2] [,3]
## [1,]    1    4    7
## [2,]    2    5    8
## [3,]    3    6    9

class(x)

## [1] "matrix" "array"
typeof(x)

## [1] "integer"
dim(x)

## [1] 3 3
```

### Character Matrix

```
y <- matrix(c("a", "b", "c", "d", "e", "f"), nrow = 3, ncol = 2)
y

##      [,1] [,2]
## [1,] "a"   "d"
## [2,] "b"   "e"
## [3,] "c"   "f"

class(y)

## [1] "matrix" "array"
typeof(y)

## [1] "character"
dim(y)

## [1] 3 2
```

#### 12.3.3 Access

When accessing matrix elements, you need to pick row(s) and/or column(s), for example:

```
x <- matrix(1:9, nrow = 3, ncol = 3)
x

##      [,1] [,2] [,3]
## [1,]    1    4    7
## [2,]    2    5    8
## [3,]    3    6    9

x[2,3] # access an element in row #2 and column #3

## [1] 8
```

```

x[2,] # access elements in row #2

## [1] 2 5 8

x[c(2,3),] # access elements in rows #2 and 3

##      [,1] [,2] [,3]
## [1,]    2    5    8
## [2,]    3    6    9

x[,c(2,3)] # access elements in columns #2 and 3

##      [,1] [,2]
## [1,]    4    7
## [2,]    5    8
## [3,]    6    9

```

You can assess each element with mathematical expressions just like vectors:

```

x == 2 # equal

##      [,1] [,2] [,3]
## [1,] FALSE FALSE FALSE
## [2,] TRUE FALSE FALSE
## [3,] FALSE FALSE FALSE

x > 2 # larger than

##      [,1] [,2] [,3]
## [1,] FALSE TRUE TRUE
## [2,] FALSE TRUE TRUE
## [3,] TRUE TRUE TRUE

```

However, care must be taken when accessing elements, as it will be automatically converted to vector data:

```
x[x == 2] # equal
```

```

## [1] 2
x[x > 2] # larger than

```

```
## [1] 3 4 5 6 7 8 9
```

`which()` needs an additional argument to return both row and column #:

```
which(x == 2, arr.ind = TRUE)
```

```

##      row col
## [1,]    2    1
which(x > 2, arr.ind = TRUE)

##      row col
## [1,]    3    1
## [2,]    1    2
## [3,]    2    2
## [4,]    3    2
## [5,]    1    3
## [6,]    2    3
## [7,]    3    3

```

## 12.4 Data Frame

A data frame is a collection of elements organized into rows and columns, but it differs from a matrix in several ways.

- It allows for the inclusion of *multiple data types* in different columns.
- Each column in a data frame has a *name* associated with it.
- You can access columns in a data frame by their respective names using the `$` operator.

The data frame is the most commonly used data structure when manipulating ecological data. When loading a dataset from a spreadsheet (which we will discuss later), it is automatically recognized as a data frame. Let's consider an example:

### Creating a data frame

In the following example, the variables `x` and `y` are organized into a single data frame named `df0`. The variables are renamed as part of the process of creating the data frame.

```
# Create data frame
x <- c("Pristine", "Pristine", "Disturbed", "Disturbed", "Pristine") # Lake type
y <- c(1.2, 2.2, 10.9, 50.0, 3.0) # TSS: total suspended solids (mg/L)
df0 <- data.frame(LakeType = x, TSS = y) # x is named as "LakeType" while y is named as "TSS"
df0
```

```
##      LakeType   TSS
## 1    Pristine  1.2
## 2    Pristine  2.2
## 3 Disturbed 10.9
## 4 Disturbed 50.0
## 5    Pristine  3.0
```

### Call column names

```
colnames(df0) # call column names
```

```
## [1] "LakeType" "TSS"
```

### Access by columns

```
df0$LakeType # access LakeType
```

```
## [1] "Pristine"  "Pristine"  "Disturbed" "Disturbed" "Pristine"
```

```
df0$TSS # access TSS
```

```
## [1] 1.2 2.2 10.9 50.0 3.0
```

You can access elements like a matrix as well:

```
df0[,1] # access column #1
```

```
## [1] "Pristine"  "Pristine"  "Disturbed" "Disturbed" "Pristine"
```

```
df0[1,] # access row #1
```

```
##      LakeType   TSS
```

```
## 1    Pristine  1.2
```

```
df0[c(2,4),] # access row #2 and 4
```

```
##      LakeType   TSS
```

```
## 2    Pristine  2.2
```

```
## 4 Disturbed 50.0
```

## 12.5 Exercise

### 12.5.1 Vector

- a. Create three numeric vectors with length 3, 6 and 20, respectively. Each vector must be created using different functions in R.
- b. Create three character vectors with length 3, 6 and 20, respectively. Each vector must be created using different functions in R.
- c. Copy the following script to your R script and perform the following analysis:
  - Identify element IDs of `x` that are greater than 2.0
  - Identify element values of `x` that are greater than 2.0

```
set.seed(1)
x <- rnorm(100)
```

### 12.5.2 Matrix

- a. Create a numeric matrix with 4 rows and 4 columns. Each column must contain identical elements.
- b. Create a numeric matrix with 4 rows and 4 columns. Each row must contain identical elements.
- c. Create a character matrix with 4 rows and 4 columns. Each column must contain identical elements.
- d. Create a character matrix with 4 rows and 4 columns. Each row must contain identical elements.
- e. Copy the following script to your R script and perform the following analysis:
  - Identify element IDs of `x` that are greater than 2.0 (**specify row and column IDs**)
  - Identify element values of `x` that are greater than 2.0 and calculate the mean.

```
set.seed(1)
x <- matrix(rnorm(100), nrow = 10, ncol = 10)
```

### 12.5.3 Data Frame

- a. Create a data frame of 3 variables with 10 elements (name variables as `x`, `y` and `z`. `x` must be `character` while `y` and `z` must be `numeric`.
- b. Check the data structure (higher-level) of `x`, `y` and `z`
- c. Copy the following script to your R script and perform the following analysis:
  - Calculate the means of `temperature` and `abundance` for states `VA` and `NC` separately.

```
set.seed(1)
x <- rnorm(100, mean = 10, sd = 3)
y <- rpois(100, lambda = 10)
z <- rep(c("VA", "NC"), 50)
df0 <- data.frame(temperature = x, abundance = y, state = z)
```

## 13 Appendix: Tidyverse

### 13.1 Overview

R packages are collections of functions that extend the functionality of the R programming language. They provide a convenient way for users to access and utilize specialized tools for data analysis, visualization, statistical modeling, and more. Among them, `tidyverse` provides very useful functions for data manipulation and visualization.

`tidyverse`:: bundles R packages designed to make data manipulation, exploration, and visualization more efficient and intuitive. Developed by Hadley Wickham and other contributors, the Tidyverse packages share a common philosophy and syntax, emphasizing a consistent and tidy data format. The core packages, such

as `dplyr::`, `tidyverse::`, and `ggplot2::`, provide powerful tools for data wrangling, reshaping, and creating visualizations.

In this Chapter, we will use `iris` data, which is available by default in R. While the default data structure `iris` is prepared in `data.frame()`, I convert this data format to `tibble()` in this exercise.

```
library(tidyverse)
iris <- as_tibble(iris)
```

I cover the basics for those who are not familiar with `tidyverse::`, but there is plenty of documentation available for the `tidyverse` package.

- Tidyverse website
- `dplyr`
- `tidyverse`
- `ggplot2`
- from Data to Viz

## 13.2 Data Manipulation

### 13.2.1 Data Format

`dplyr::` and `tidyverse::` (and others) offer useful functions for data manipulation. Key manipulations include: row/column manipulations, group operation, reshape, and more.

### 13.2.2 Row Manipulation

`filter()`: select/remove rows

```
# single match ==
filter(iris, Species == "virginica")

# multiple match "%in%"
filter(iris, Species %in% c("virginica", "versicolor"))

# except !=
filter(iris, Species != "virginica")

# except multiple "!(x %in% c("a", "b"))"
filter(iris, !(Species %in% c("virginica", "versicolor")))

# greater than ">"
filter(iris, Sepal.Length > 5)

# equal & greater than ">="
filter(iris, Sepal.Length >= 5)

# less than "<"
filter(iris, Sepal.Length < 5)

# equal & less than "<="
filter(iris, Sepal.Length <= 5)
```

`arrange()` : arrange the order of rows

```
# arrange in an ascending order
arrange(iris, Sepal.Length)
```

```
# arrange in an descending order
arrange(iris, desc(Sepal.Length))
```

### 13.2.3 Column Manipulation

```
select(): select/remove column(s)

# select one column
select(iris, Sepal.Length)

# select multiple columns
select(iris, c(Sepal.Length, Sepal.Width))

# remove one column
select(iris, -Sepal.Length)

# remove multiple columns
select(iris, -c(Sepal.Length, Sepal.Width))

# select/remove multiple columns with a start rule
# starts_with("x")
select(iris, starts_with("Sepal"))
select(iris, -starts_with("Sepal"))

# select/remove multiple columns with an end rule
# ends_with("x")
select(iris, ends_with("Width"))
select(iris, -ends_with("Width"))
```

```
mutate(): add column(s)
```

```
# add a new column
x <- 1:150
mutate(iris, x = x)
```

```
## # A tibble: 150 x 6
##   Sepal.Length Sepal.Width Petal.Length Petal.Width Species     x
##       <dbl>      <dbl>      <dbl>      <dbl> <fct>    <int>
## 1         5.1        3.5        1.4        0.2 setosa     1
## 2         4.9        3.0        1.4        0.2 setosa     2
## 3         4.7        3.2        1.3        0.2 setosa     3
## 4         4.6        3.1        1.5        0.2 setosa     4
## 5         5.0        3.6        1.4        0.2 setosa     5
## 6         5.4        3.9        1.7        0.4 setosa     6
## 7         4.6        3.4        1.4        0.3 setosa     7
## 8         5.0        3.4        1.5        0.2 setosa     8
## 9         4.4        2.9        1.4        0.2 setosa     9
## 10        4.9        3.1        1.5        0.1 setosa    10
## # i 140 more rows
```

### 13.2.4 Piping

%>% (pipe) allows sequential operations of multiple functions (shortcut: Ctr + Shift + M). The pipe passes the object to the following function as the first argument.

```

# the following codes produce the same data frame
# apply functions separately
df_vir <- filter(iris, Species == "virginica")
df_vir_sl <- select(df_vir, Sepal.Length)
print(df_vir_sl)

## # A tibble: 50 x 1
##   Sepal.Length
##   <dbl>
## 1     6.3
## 2     5.8
## 3     7.1
## 4     6.3
## 5     6.5
## 6     7.6
## 7     4.9
## 8     7.3
## 9     6.7
## 10    7.2
## # i 40 more rows

# piping
iris %>%
  filter(Species == "virginica") %>%
  select(Sepal.Length)

## # A tibble: 50 x 1
##   Sepal.Length
##   <dbl>
## 1     6.3
## 2     5.8
## 3     7.1
## 4     6.3
## 5     6.5
## 6     7.6
## 7     4.9
## 8     7.3
## 9     6.7
## 10    7.2
## # i 40 more rows

```

### 13.2.5 Reshape

```

pivot_wider() : reshape a data frame to a wide format

iris_w <- iris %>%
  mutate(id = rep(1:50, 3)) %>% # add an ID column
  select(id, Sepal.Length, Species) %>%
  pivot_wider(id_cols = "id", # unique row ID based on
              values_from = "Sepal.Length", # values in each cell from
              names_from = "Species") # new column names from

print(iris_w)

## # A tibble: 50 x 4
##   id   Sepal.Length Species
##   <dbl>      <dbl> <fct>
## 1 1       6.3     setosa
## 2 1       5.8     versicolor
## 3 1       7.1     virginica
## 4 2       6.3     setosa
## 5 2       5.8     versicolor
## 6 2       6.5     virginica
## 7 3       7.6     setosa
## 8 3       4.9     versicolor
## 9 3       7.3     virginica
## 10 4       6.3    setosa
## 11 4       6.5    versicolor
## 12 4       6.7    virginica
## 13 5       7.6    setosa
## 14 5       7.2    versicolor
## 15 5       7.2    virginica
## 16 6       4.9    setosa
## 17 6       6.3    versicolor
## 18 6       6.5    virginica
## 19 7       7.3    setosa
## 20 7       6.7    versicolor
## 21 7       6.7    virginica
## 22 8       7.6    setosa
## 23 8       7.3    versicolor
## 24 8       7.3    virginica
## 25 9       4.9    setosa
## 26 9       6.3    versicolor
## 27 9       6.7    virginica
## 28 10      7.2   setosa
## 29 10      7.2   versicolor
## 30 10      7.2   virginica
## 31 11      6.3   setosa
## 32 11      5.8   versicolor
## 33 11      7.1   virginica
## 34 12      6.5   setosa
## 35 12      6.5   versicolor
## 36 12      6.5   virginica
## 37 13      7.6   setosa
## 38 13      4.9   versicolor
## 39 13      7.3   virginica
## 40 14      4.9   setosa
## 41 14      6.3   versicolor
## 42 14      6.5   virginica
## 43 15      7.2   setosa
## 44 15      7.2   versicolor
## 45 15      7.2   virginica
## 46 16      6.3   setosa
## 47 16      6.5   versicolor
## 48 16      6.5   virginica
## 49 17      7.3   setosa
## 50 17      6.7   versicolor
## 51 17      6.7   virginica

```

```

##      <int> <dbl>     <dbl>     <dbl>
## 1     1     5.1      7     6.3
## 2     2     4.9      6.4      5.8
## 3     3     4.7      6.9      7.1
## 4     4     4.6      5.5      6.3
## 5     5     5       6.5      6.5
## 6     6     5.4      5.7      7.6
## 7     7     4.6      6.3      4.9
## 8     8     5       4.9      7.3
## 9     9     4.4      6.6      6.7
## 10    10    4.9      5.2      7.2
## # i 40 more rows

pivot_longer(): reshape a data frame to a long format

iris_l <- iris_w %>%
  pivot_longer(cols = c("setosa",
                      "versicolor",
                      "virginica"), # columns with values to be reshaped
               names_to = "Species", # column IDs move to "Species"
               values_to = "Sepal.Length") # column values move to "Sepal.Length"

print(iris_l)

## # A tibble: 150 x 3
##       id Species   Sepal.Length
##   <int> <chr>        <dbl>
## 1     1 setosa        5.1
## 2     2 versicolor    7
## 3     3 virginica     6.3
## 4     2 setosa        4.9
## 5     2 versicolor    6.4
## 6     2 virginica     5.8
## 7     3 setosa        4.7
## 8     3 versicolor    6.9
## 9     3 virginica     7.1
## 10    4 setosa        4.6
## # i 140 more rows

```

### 13.2.6 Group Operation

```

group_by() & summarize(): group-by-group operation. summarize() does not retain individual rows.

# grouping by "Species", then take means "Sepal.Length" for each species
iris %>%
  group_by(Species) %>%
  summarize(mu_sl = mean(Sepal.Length))

## # A tibble: 3 x 2
##   Species   mu_sl
##   <fct>     <dbl>
## 1 setosa     5.01
## 2 versicolor 5.94
## 3 virginica  6.59

# grouping by "Species", then take means & SD "Sepal.Length" for each species
iris %>%

```

```

group_by(Species) %>%
  summarize(mu_sl = mean(Sepal.Length),
           sd_sl = sd(Sepal.Length))

## # A tibble: 3 x 3
##   Species     mu_sl    sd_sl
##   <fct>      <dbl>   <dbl>
## 1 setosa     5.01    0.352
## 2 versicolor 5.94    0.516
## 3 virginica  6.59    0.636

group_by() & mutate(): group-by-group operation. mutate() retains individual rows along with summary columns. Do not forget ungroup() to avoid errors in following operations.

# grouping by "Species", then take means "Sepal.Length" for each species
iris %>%
  group_by(Species) %>%
  mutate(mu_sl = mean(Sepal.Length)) %>%
  ungroup()

## # A tibble: 150 x 6
##   Sepal.Length Sepal.Width Petal.Length Petal.Width Species mu_sl
##       <dbl>        <dbl>        <dbl>        <dbl> <fct>   <dbl>
## 1         5.1        3.5        1.4        0.2 setosa   5.01
## 2         4.9        3.0        1.4        0.2 setosa   5.01
## 3         4.7        3.2        1.3        0.2 setosa   5.01
## 4         4.6        3.1        1.5        0.2 setosa   5.01
## 5         5.0        3.6        1.4        0.2 setosa   5.01
## 6         5.4        3.9        1.7        0.4 setosa   5.01
## 7         4.6        3.4        1.4        0.3 setosa   5.01
## 8         5.0        3.4        1.5        0.2 setosa   5.01
## 9         4.4        2.9        1.4        0.2 setosa   5.01
## 10        4.9        3.1        1.5        0.1 setosa   5.01
## # i 140 more rows

```

### 13.2.7 Join

`left_join()`: merge data frames based on column(s)

```

# matching by a single column
## left join by "Species": one to one
df1 <- tibble(Species = c("A", "B", "C"),
               x = c(1, 2, 3))

df2 <- tibble(Species = c("A", "B", "C"),
               y = c(4, 5, 6))

left_join(x = df1,
          y = df2,
          by = "Species")

```

```

## # A tibble: 3 x 3
##   Species     x     y
##   <chr>   <dbl> <dbl>
## 1 A         1     4
## 2 B         2     5

```

```

## 3 C          3      6
# matching by a single column
## left join by "Species": one to many
df3 <- tibble(Species = c("A", "A", "B", "C"),
              y = c(4, 5, 6, 7))

left_join(x = df1,
          y = df3,
          by = "Species")

## # A tibble: 4 x 3
##   Species     x     y
##   <chr>    <dbl> <dbl>
## 1 A            1     4
## 2 A            1     5
## 3 B            2     6
## 4 C            3     7

# matching by a single column
## left join by "Species": one to missing
df4 <- tibble(Species = c("A", "A", "C"),
              y = c(4, 5, 7))

left_join(x = df1,
          y = df4,
          by = "Species")

## # A tibble: 4 x 3
##   Species     x     y
##   <chr>    <dbl> <dbl>
## 1 A            1     4
## 2 A            1     5
## 3 B            2    NA
## 4 C            3     7

# matching by multiple columns
## one to one
df5 <- tibble(Species = c("A", "B", "C"),
              x = c(1, 2, 3),
              z = c("cool", "awesome", "magical"))

left_join(x = df1,
          y = df5,
          by = c("Species", "x"))

## # A tibble: 3 x 3
##   Species     x     z
##   <chr>    <dbl> <chr>
## 1 A            1 cool
## 2 B            2 awesome
## 3 C            3 magical

# matching by multiple columns
## one to many
df6 <- tibble(Species = c("A", "A", "B", "C"),
              x = c(1, 1, 2, 3),

```

```

      z = c("cool", "cool", "awesome", "magical"))

left_join(x = df1,
          y = df6,
          by = c("Species", "x"))

## # A tibble: 4 x 3
##   Species     x     z
##   <chr>    <dbl> <chr>
## 1 A            1 cool
## 2 A            1 cool
## 3 B            2 awesome
## 4 C            3 magical

# matching by multiple columns
## one to missing
df6 <- tibble(Species = c("A", "B", "C"),
              x = c(1, 2, 4),
              z = c("cool", "awesome", "magical"))

left_join(x = df1,
          y = df6,
          by = c("Species", "x"))

## # A tibble: 3 x 3
##   Species     x     z
##   <chr>    <dbl> <chr>
## 1 A            1 cool
## 2 B            2 awesome
## 3 C            3 <NA>

```

### 13.3 Visualization

`ggplot2::` offers a range of convenient functions for data visualization. The foundational function, `ggplot()`, provides the initial framework for adding supplementary layers using the `+` operator. In `ggplot()`, we define variables plotted on x- and y-axis through `aes()`. For example:

```

# without pipe
ggplot(data = iris,
        mapping = aes(x = Sepal.Length,
                      y = Sepal.Width)) +
  # additional layers...

# with pipe
iris %>%
  ggplot(mapping = aes(x = Sepal.Length,
                      y = Sepal.Width)) +
  # additional layers...

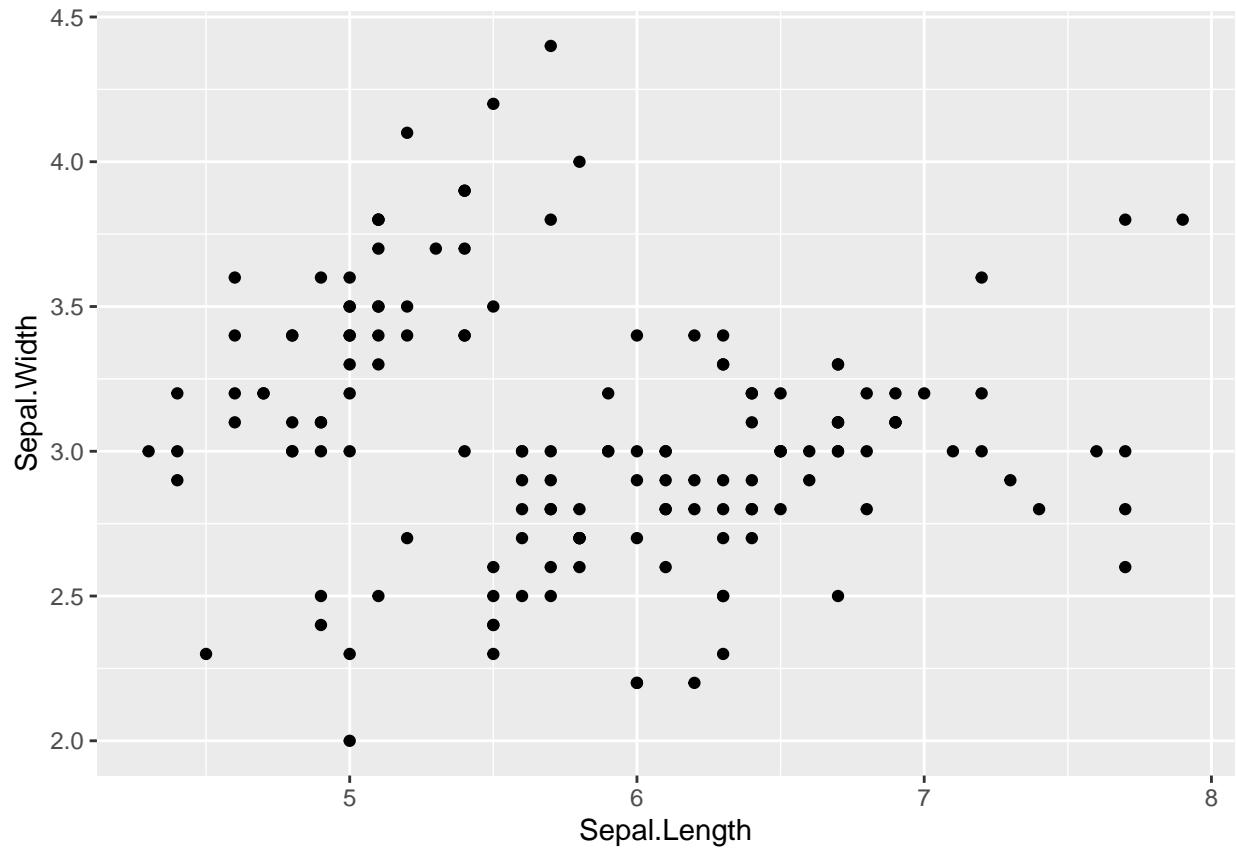
```

Please note that `aes()` refers to columns in the data frame. Variables names that do not exist in the data frame cannot be used.

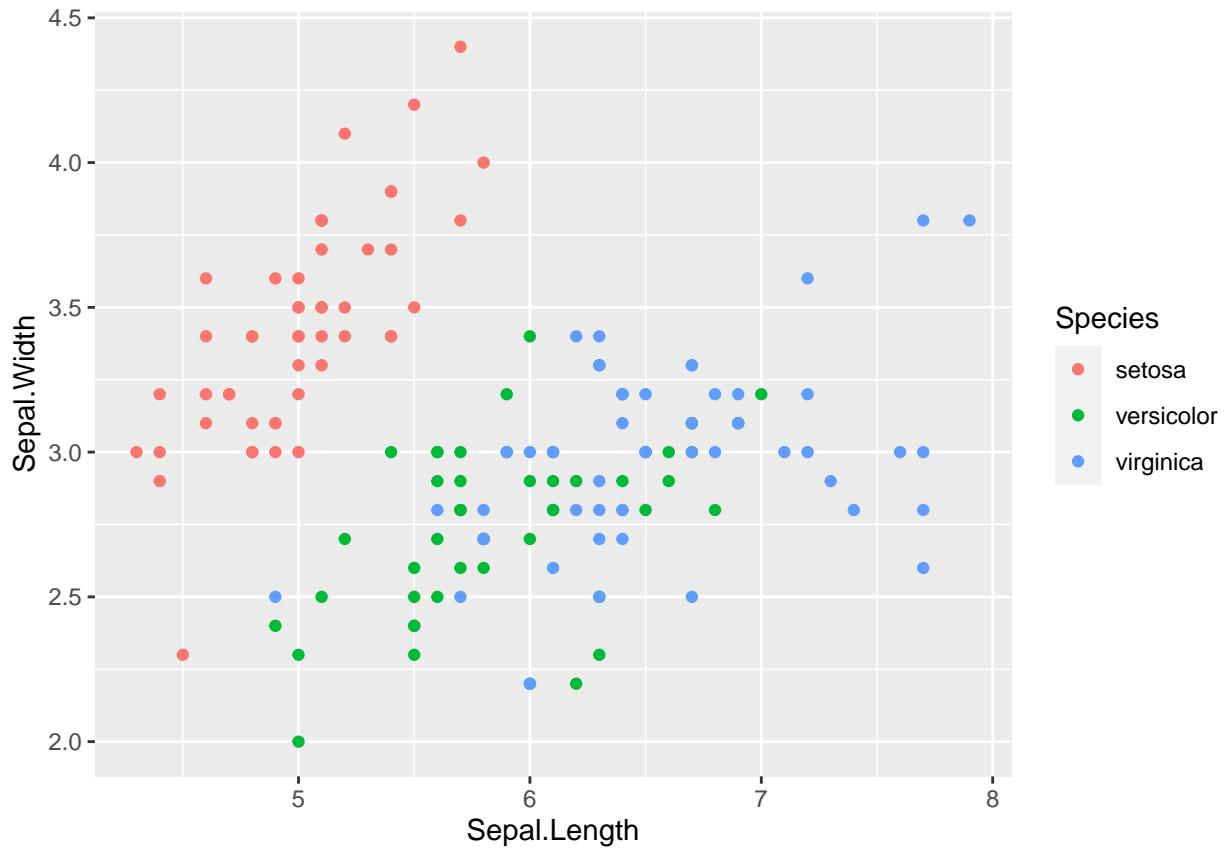
#### 13.3.1 Point

`geom_point()` : Add a point layer

```
# basic plot
iris %>%
  ggplot(aes(x = Sepal.Length,
             y = Sepal.Width)) +
  geom_point()
```



```
# change color by "Species" column
iris %>%
  ggplot(aes(x = Sepal.Length,
             y = Sepal.Width,
             color = Species)) +
  geom_point()
```

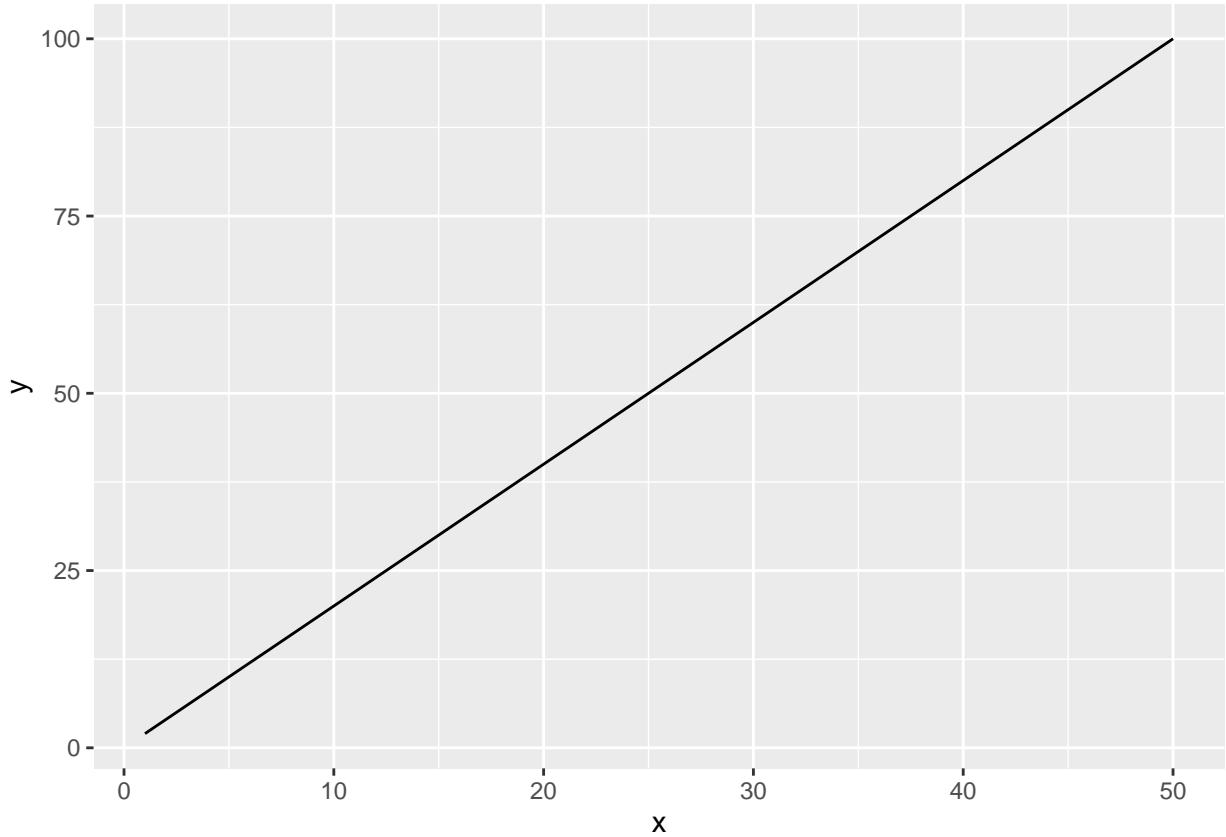


### 13.3.2 Line

`geom_line()` : Add a line layer

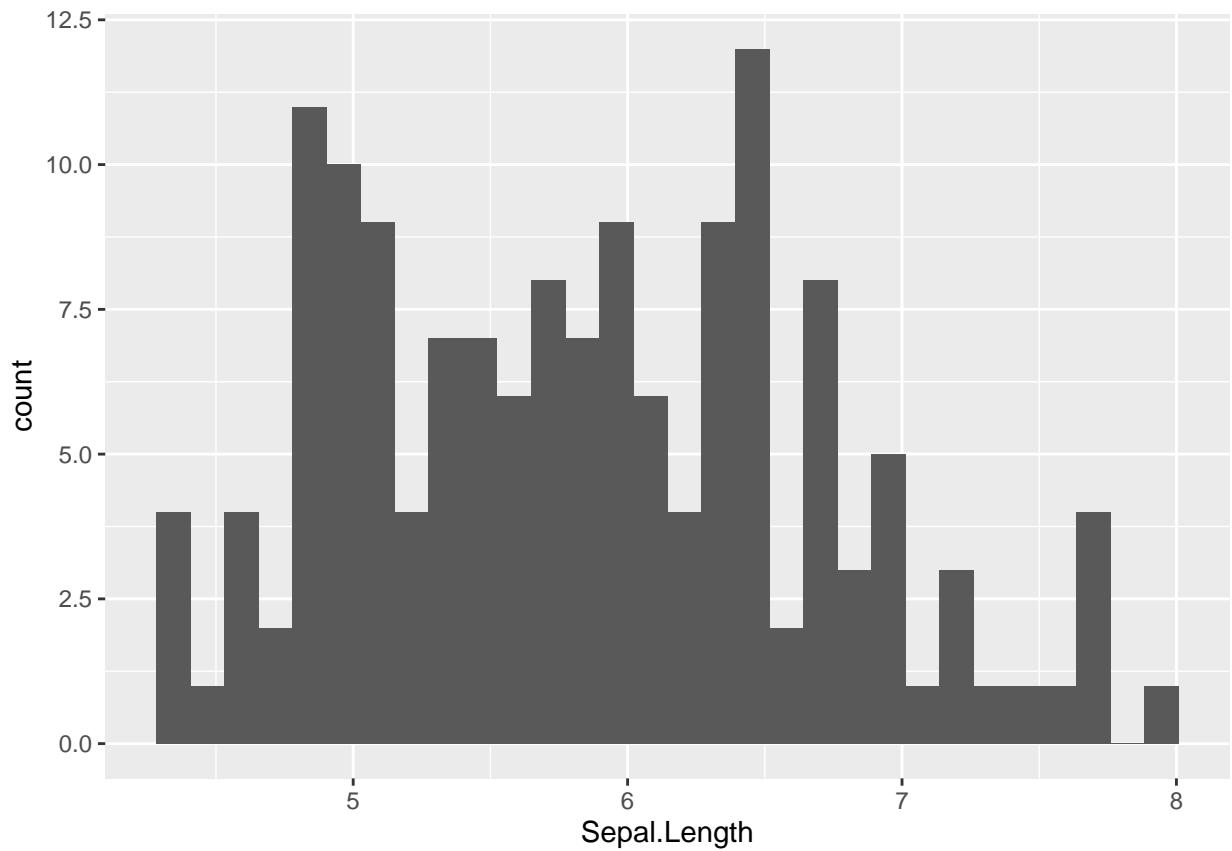
```
# sample data
df0 <- tibble(x = rep(1:50, 3),
               y = x * 2)

# basic plot
df0 %>%
  ggplot(aes(x =
             y)) +
  geom_line()
```

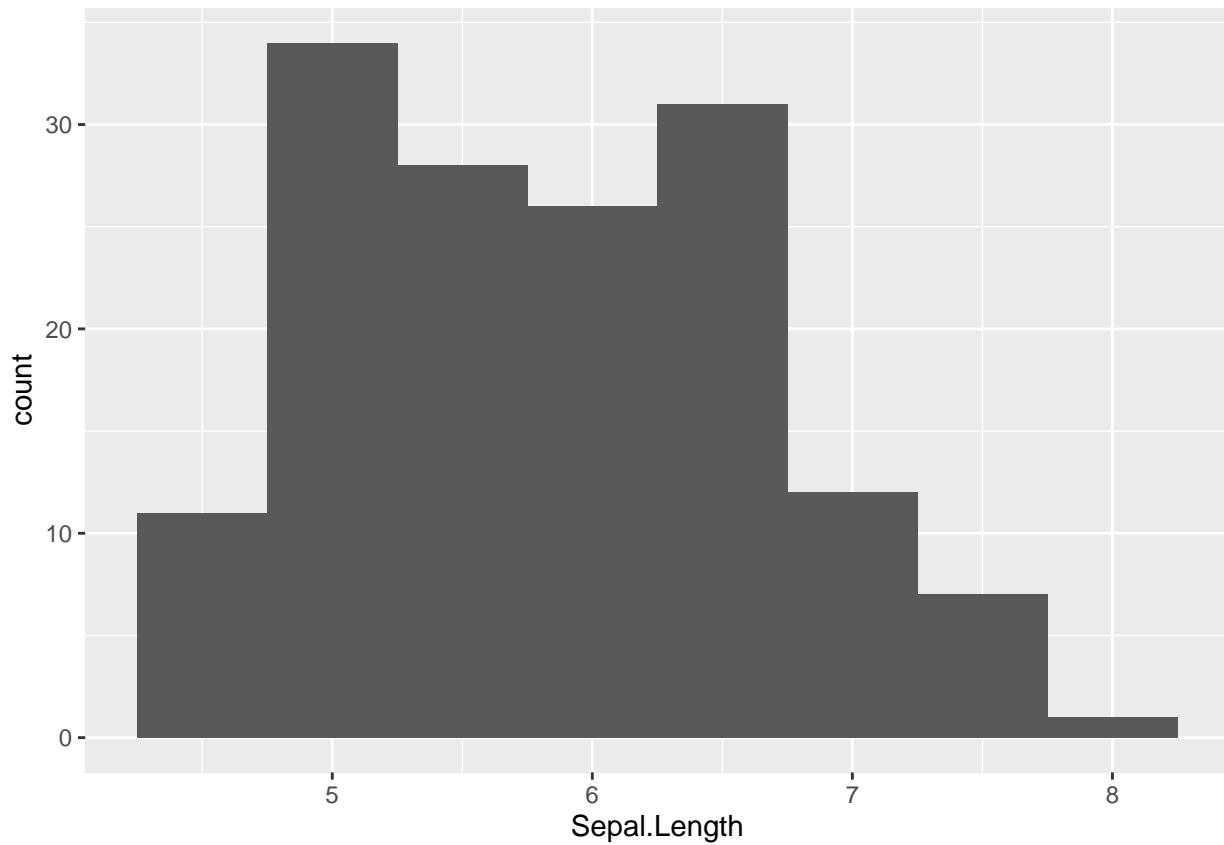


### 13.3.3 Histogram

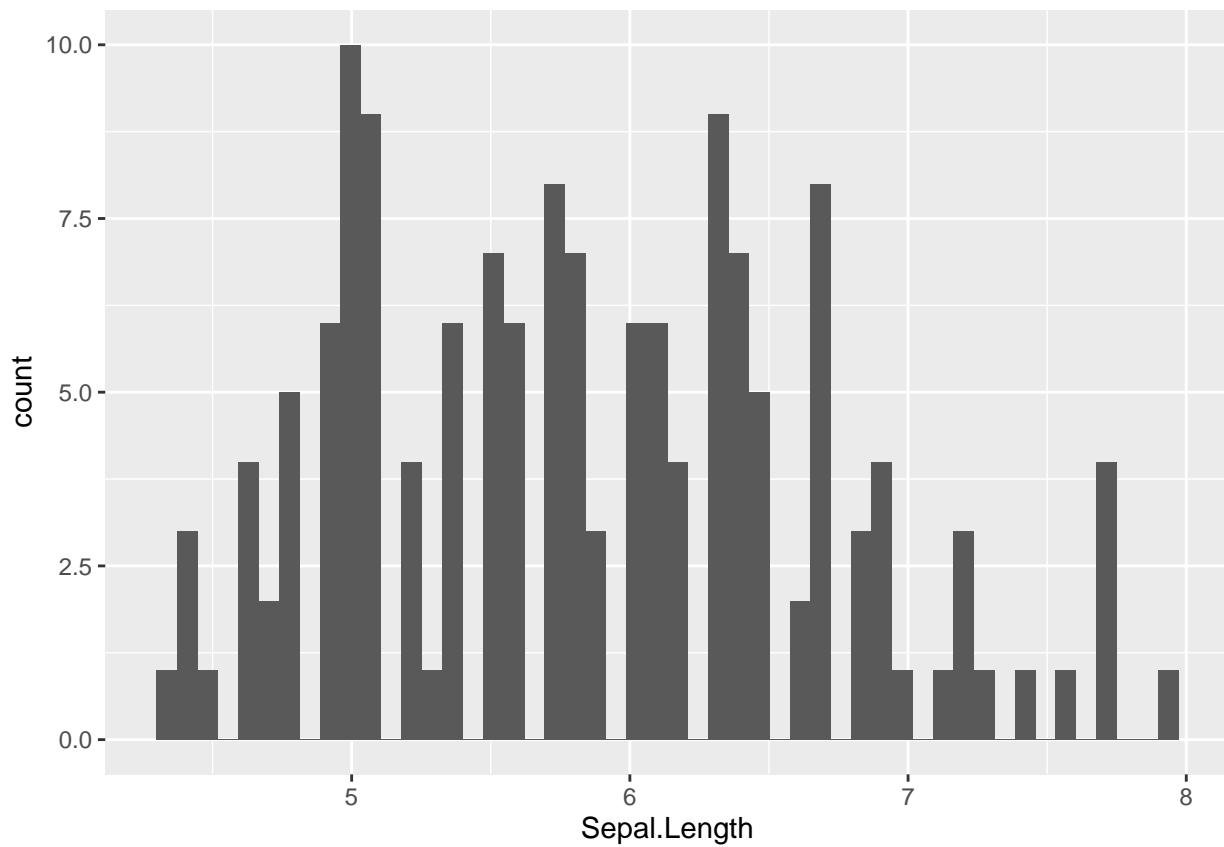
```
geom_histogram() : add a histogram layer  
# basic plot; bins = 30 by default  
iris %>%  
  ggplot(aes(x = Sepal.Length)) +  
  geom_histogram()
```



```
# change bin width
iris %>%
  ggplot(aes(x = Sepal.Length)) +
  geom_histogram(binwidth = 0.5)
```



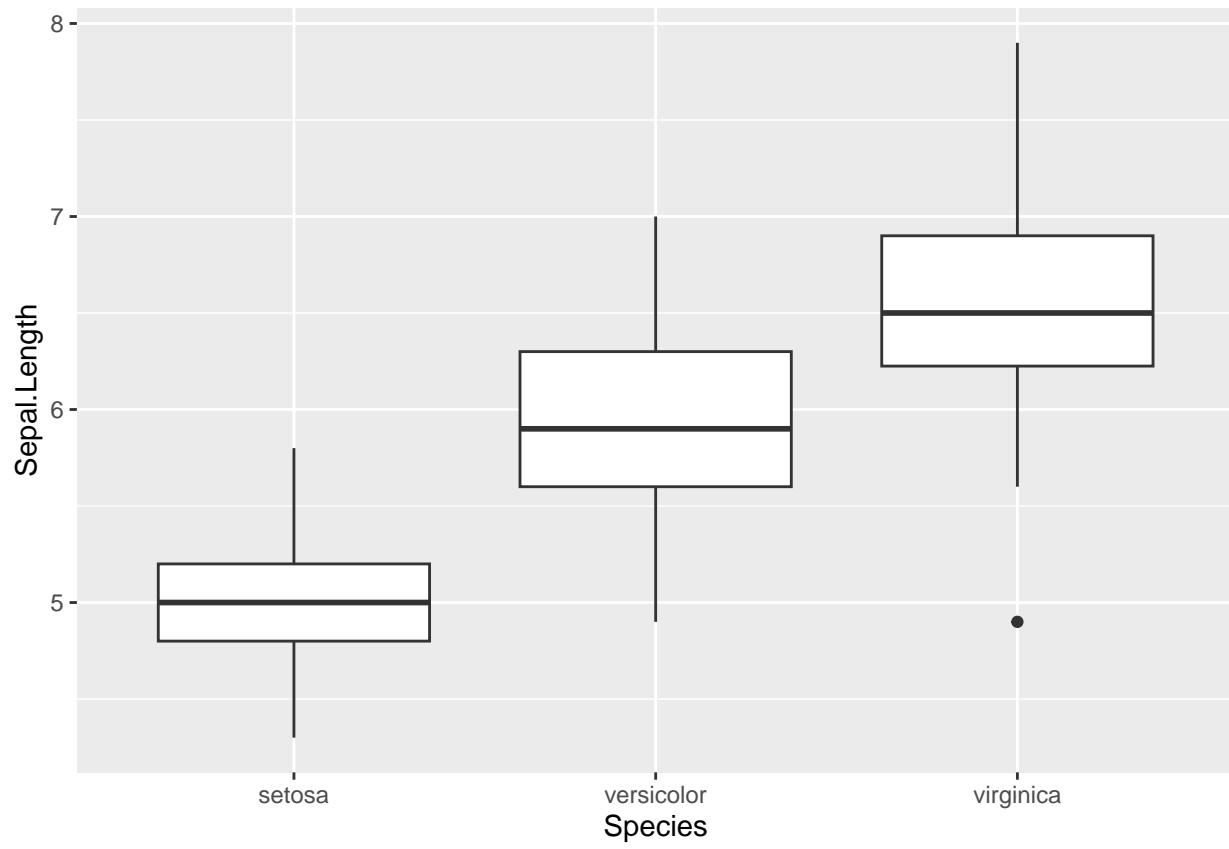
```
# change bin number
iris %>%
  ggplot(aes(x = Sepal.Length)) +
  geom_histogram(bins = 50)
```



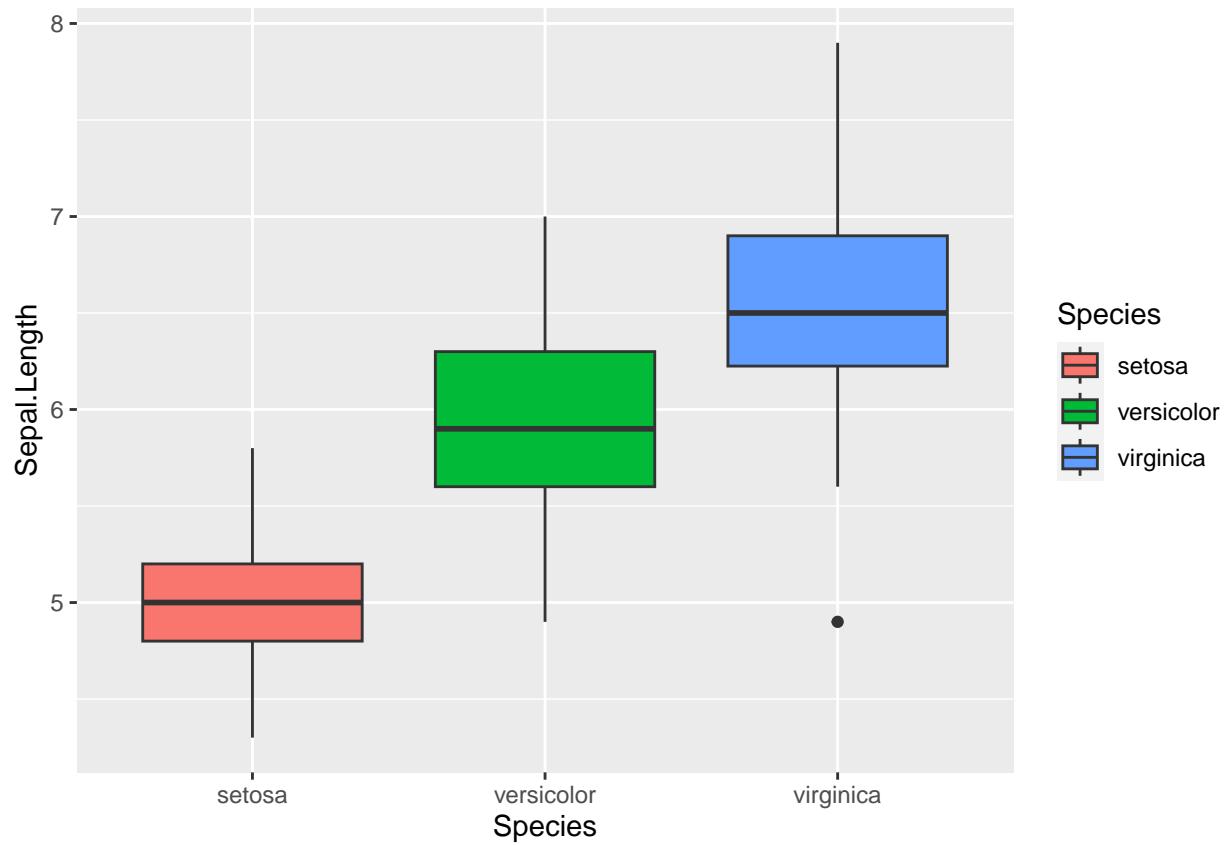
#### 13.3.4 Boxplot

`geom_boxplot()` : add a boxplot layer

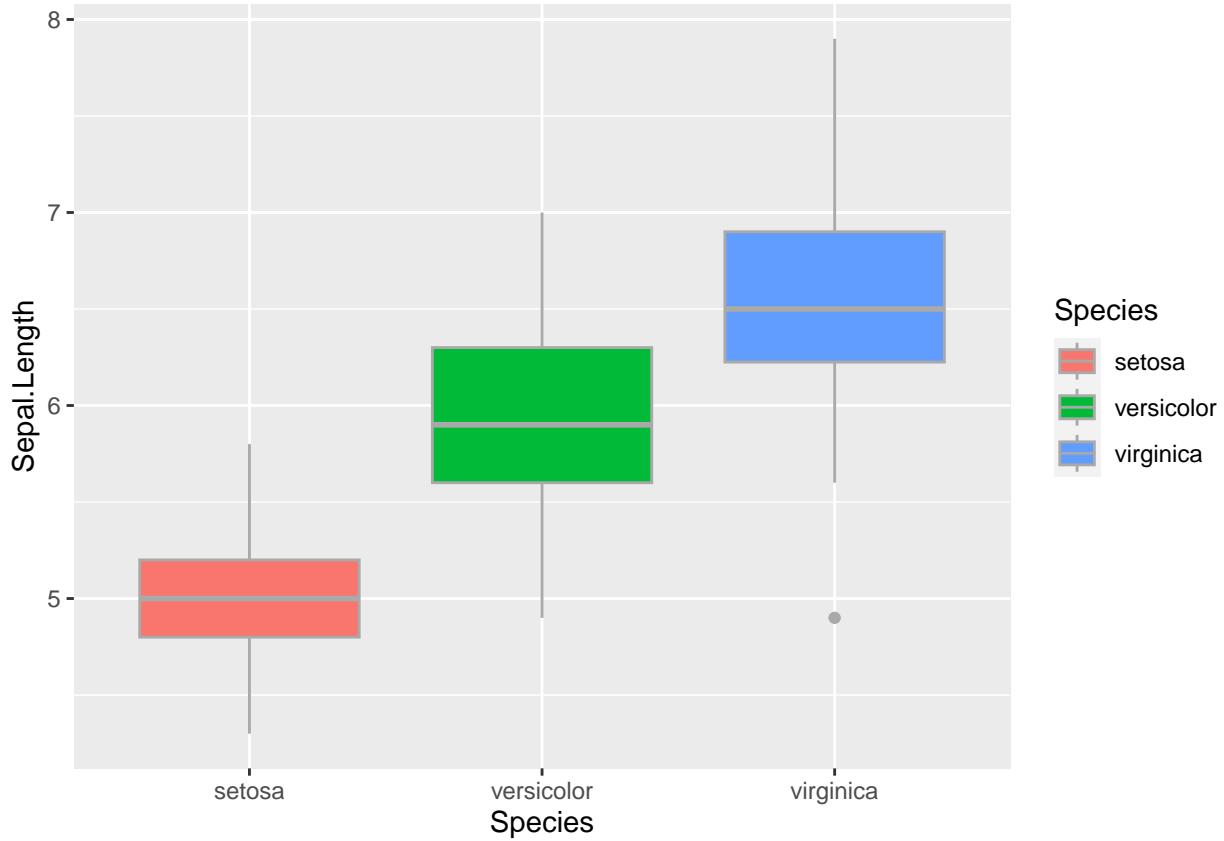
```
# basic plot
iris %>%
  ggplot(aes(x = Species,
             y = Sepal.Length)) +
  geom_boxplot()
```



```
# change fill by "Species"
iris %>%
  ggplot(aes(x = Species,
             y = Sepal.Length,
             fill = Species)) +
  geom_boxplot()
```



```
# change fill by "Species", but consistent color
iris %>%
  ggplot(aes(x = Species,
             y = Sepal.Length,
             fill = Species)) +
  geom_boxplot(color = "darkgrey")
```



## 14 Appendix: Base Plot

### 14.1 Overview

R offers a variety of functions that aid in visualizing data. The `graphics` package in R provides a set of functions for basic graphics (list of functions). To demonstrate the functionality of these `graphics` functions, I will utilize the built-in `iris` dataset in R.

```
head(iris)
```

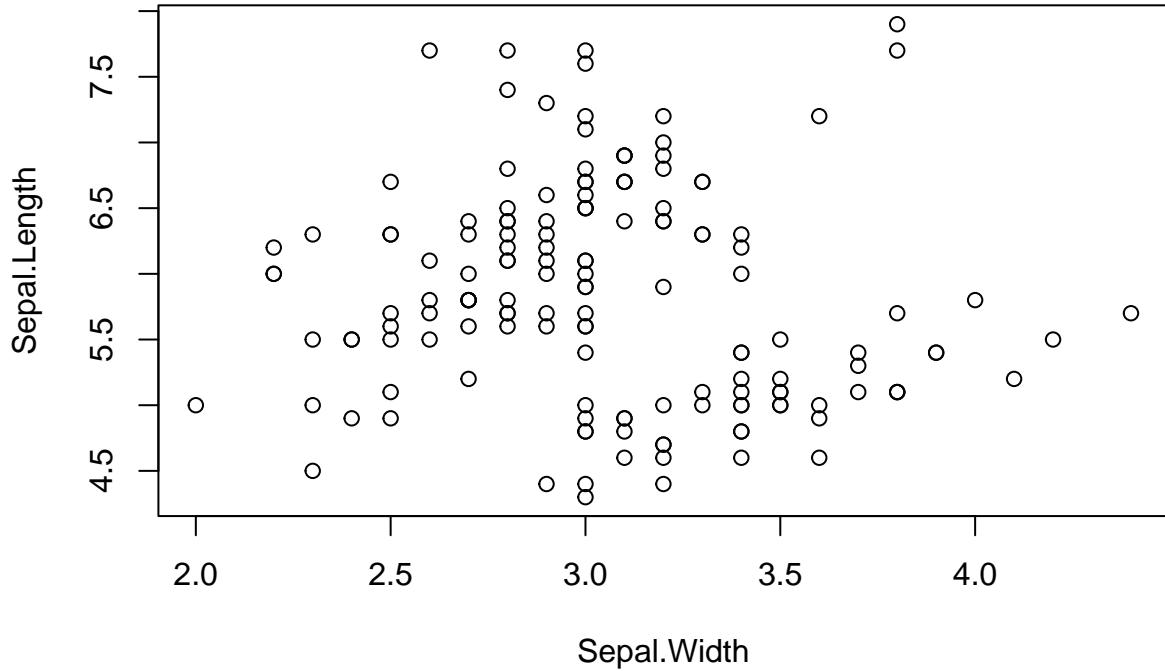
```
## # A tibble: 6 x 5
##   Sepal.Length Sepal.Width Petal.Length Petal.Width Species
##       <dbl>      <dbl>      <dbl>      <dbl>   <fct>
## 1       5.1      3.5       1.4      0.2  setosa
## 2       4.9      3.0       1.4      0.2  setosa
## 3       4.7      3.2       1.3      0.2  setosa
## 4       4.6      3.1       1.5      0.2  setosa
## 5       5.0      3.6       1.4      0.2  setosa
## 6       5.4      3.9       1.7      0.4  setosa
```

### 14.2 Plot

When creating a plot, you typically need to specify the **formula** to define the relationship between variables. For instance, if you wish to visualize the association between **x** and **y** (with **y** on the vertical axis and **x** on the horizontal axis), the formula would be **y ~ x** (where the left side of the formula represents the vertical axis). In the `iris` dataset, you have access to the following columns: `Sepal.Length`, `Sepal.Width`, `Petal.Length`,

`Petal.Width`, and `Species`. In the subsequent example, we will plot the relationship between `Sepal.Length` and `Sepal.Width`:

```
plot(Sepal.Length ~ Sepal.Width, data = iris)
```

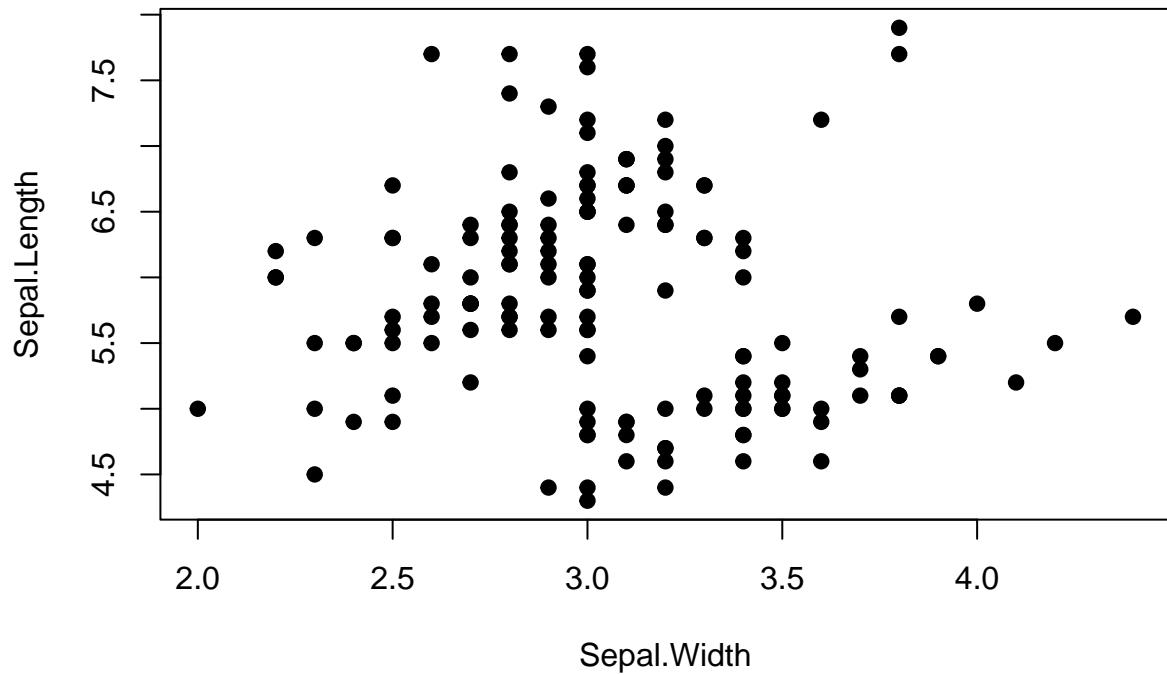


The `data =` argument informs the function about the dataset from which the variables (`Sepal.Length` and `Sepal.Width`) should be extracted.

#### 14.2.1 Symbol

`pch` argument. Choose from 1 to 25 (google `r plot pch` for details)

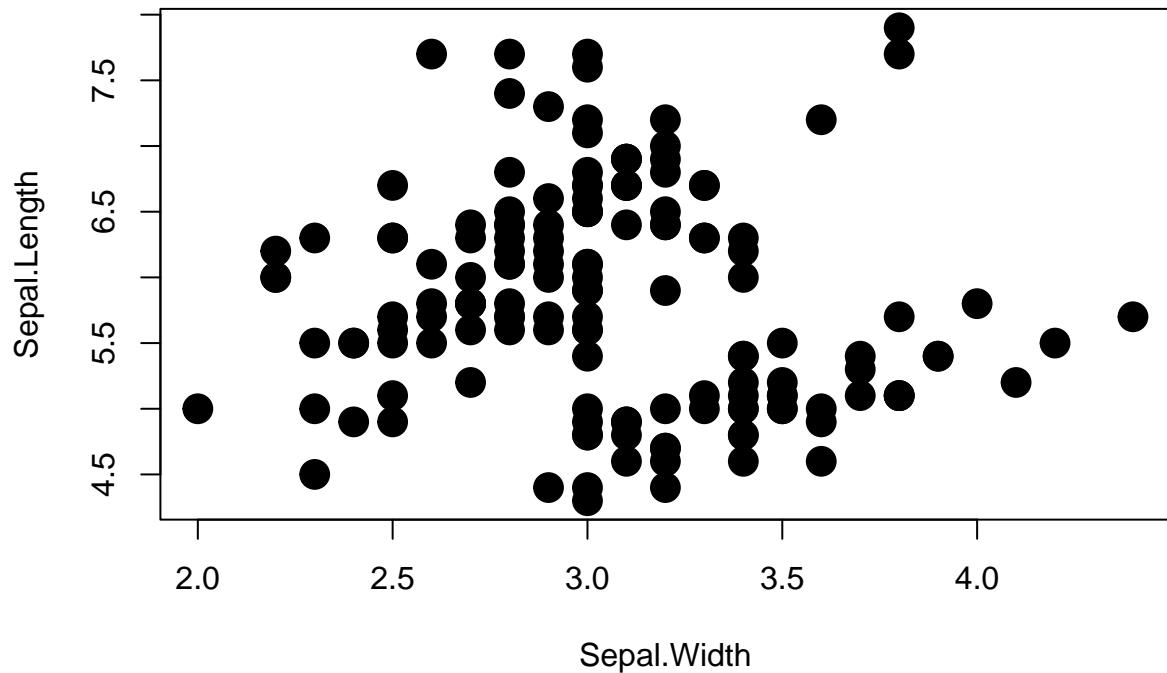
```
plot(Sepal.Length ~ Sepal.Width, data = iris,  
     pch = 19)
```



#### 14.2.2 Symbol size

`cex` argument. `cex = 1` is the default value. `cex = 2` is as twice large as default value.

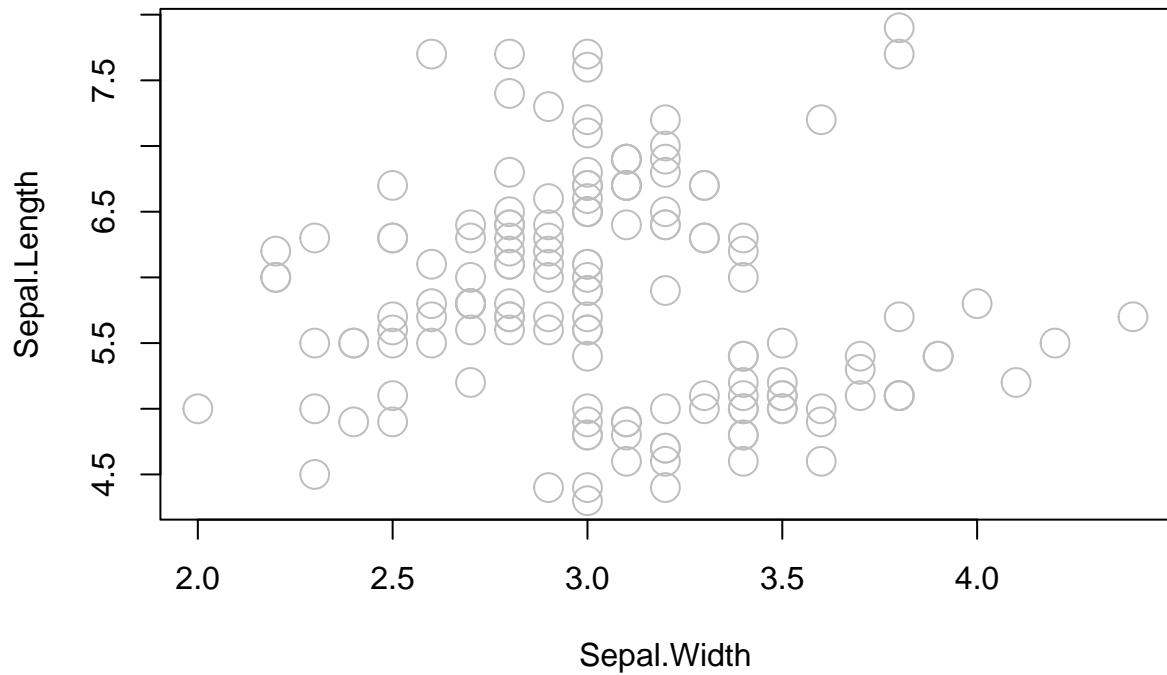
```
plot(Sepal.Length ~ Sepal.Width, data = iris,  
     pch = 19, cex = 2)
```



#### 14.2.3 Symbol color (border)

col argument (quote "color name" when specifying). Google r **color name** for color options.

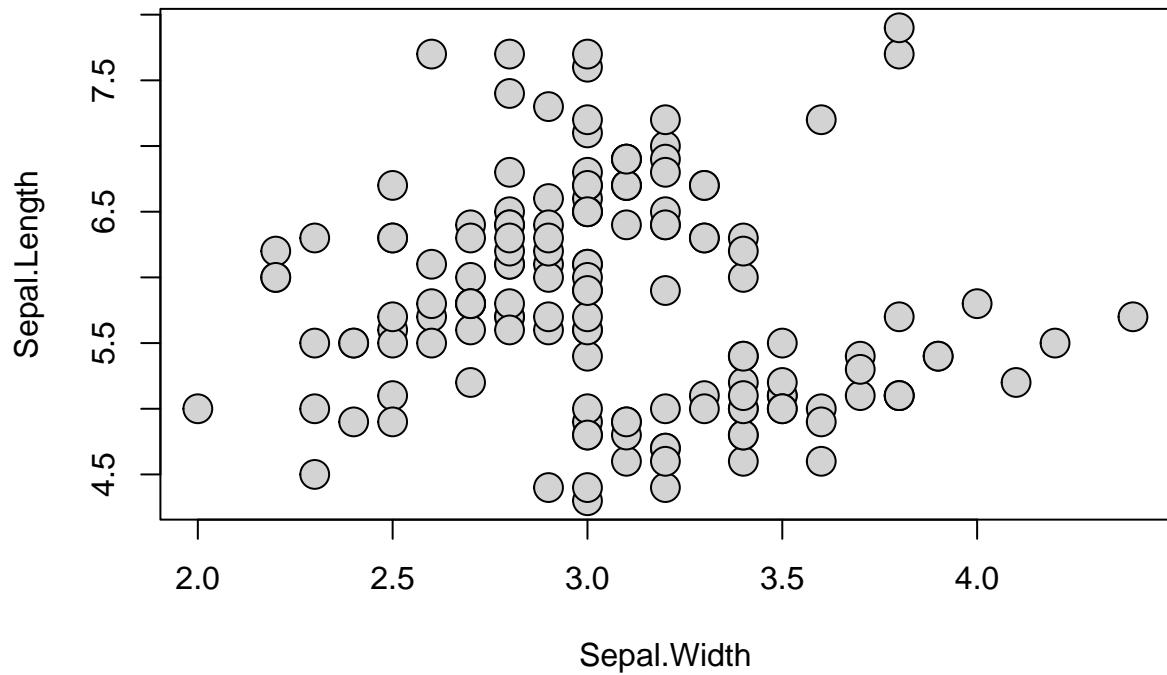
```
plot(Sepal.Length ~ Sepal.Width, data = iris,  
      pch = 21, cex = 2, col = "gray")
```



#### 14.2.4 Symbol color (fill)

`bg` argument (quote "color name" when specifying). Available for a subset of symbol options (some symbols have pre-defined filled color).

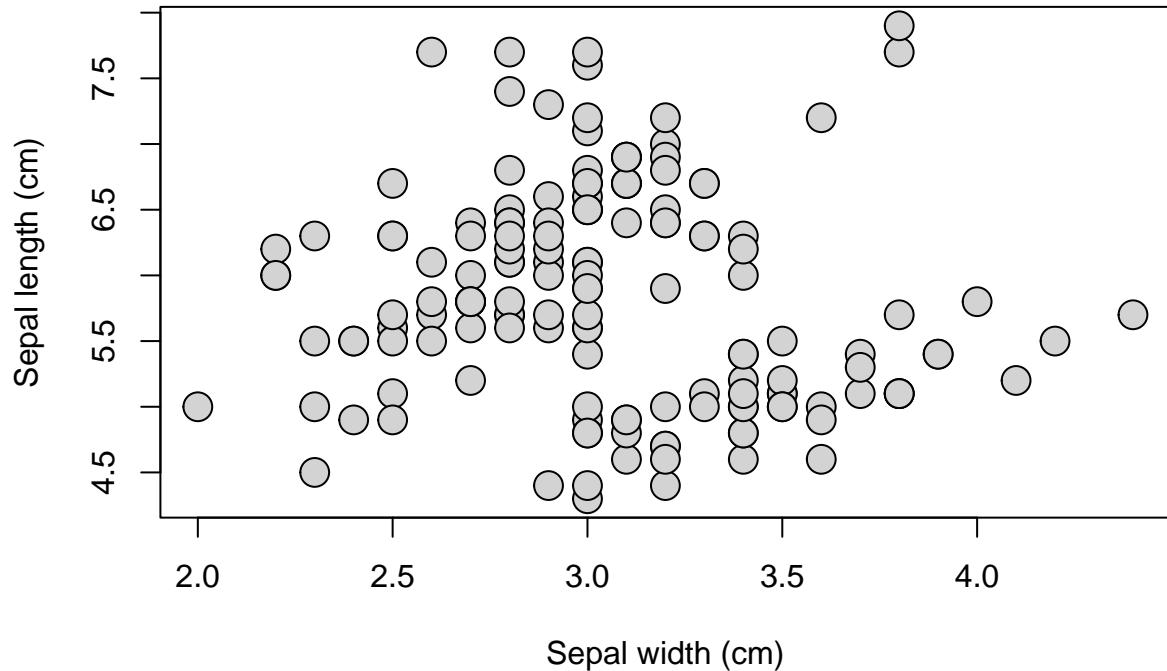
```
plot(Sepal.Length ~ Sepal.Width, data = iris,  
      pch = 21, cex = 2, bg = "lightgray")
```



#### 14.2.5 Label

ylab or xlab arguments. Provide "quoted text".

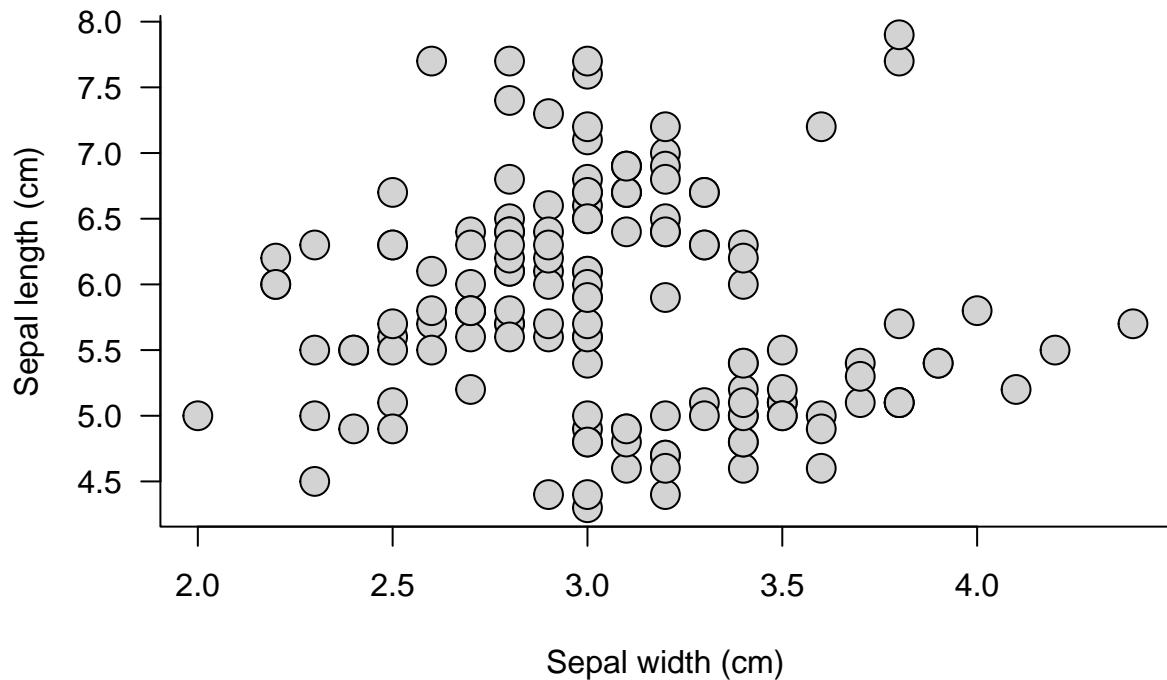
```
plot(Sepal.Length ~ Sepal.Width, data = iris,
      pch = 21, cex = 2, bg = "lightgray",
      xlab = "Sepal width (cm)", ylab = "Sepal length (cm)")
```



#### 14.2.6 Axis

Delete axes with `axes = F` and re-draw with `box()` and `axis()` functions.

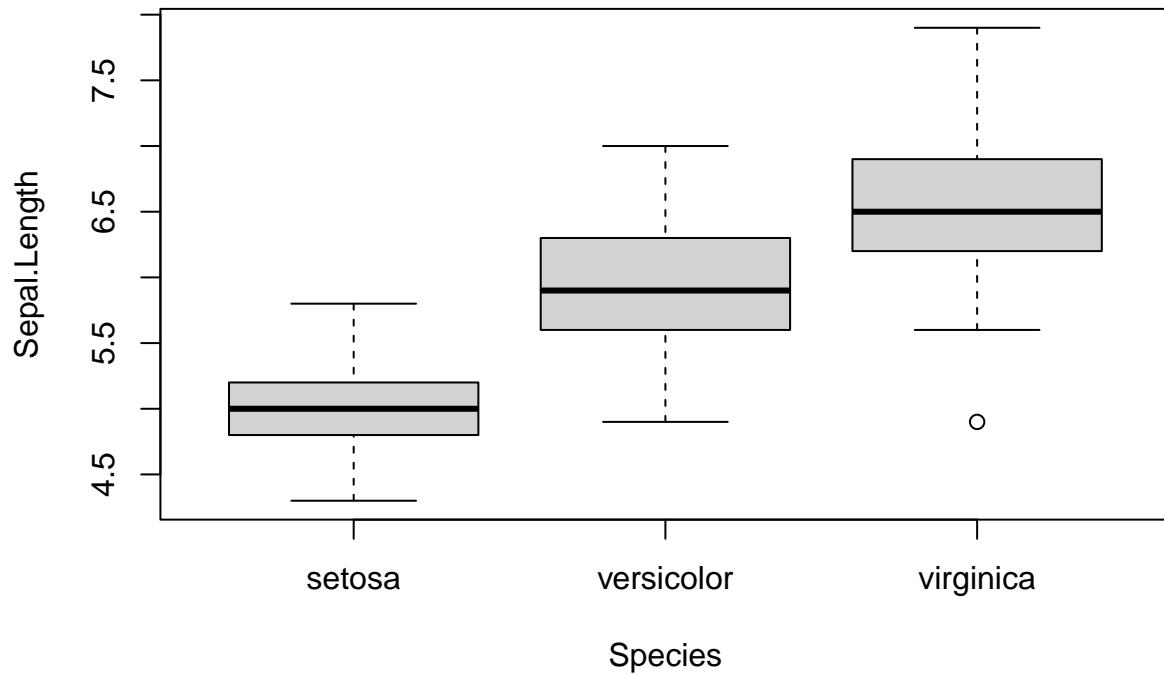
```
plot(Sepal.Length ~ Sepal.Width, data = iris,
  pch = 21, cex = 2, bg = "lightgray",
  xlab = "Sepal width (cm)", ylab = "Sepal length (cm)",
  axes = F)
box(bty = "1") # L-shaped border lines
axis(1) # 1: draw x-axis
axis(2, las = 2) # 2: draw y-axis, las = 2: make axis lables horizontal
```



### 14.3 Boxplot

`boxplot()` is used when the x-axis is factor-type data (by default, `plot()` will produce a boxplot when x-axis is a factor variable). In the `iris` dataset, the column `Species` is a factor variable. Compare `Sepal.Length` among species using `boxplot()`.

```
boxplot(Sepal.Length ~ Species, data = iris)
```

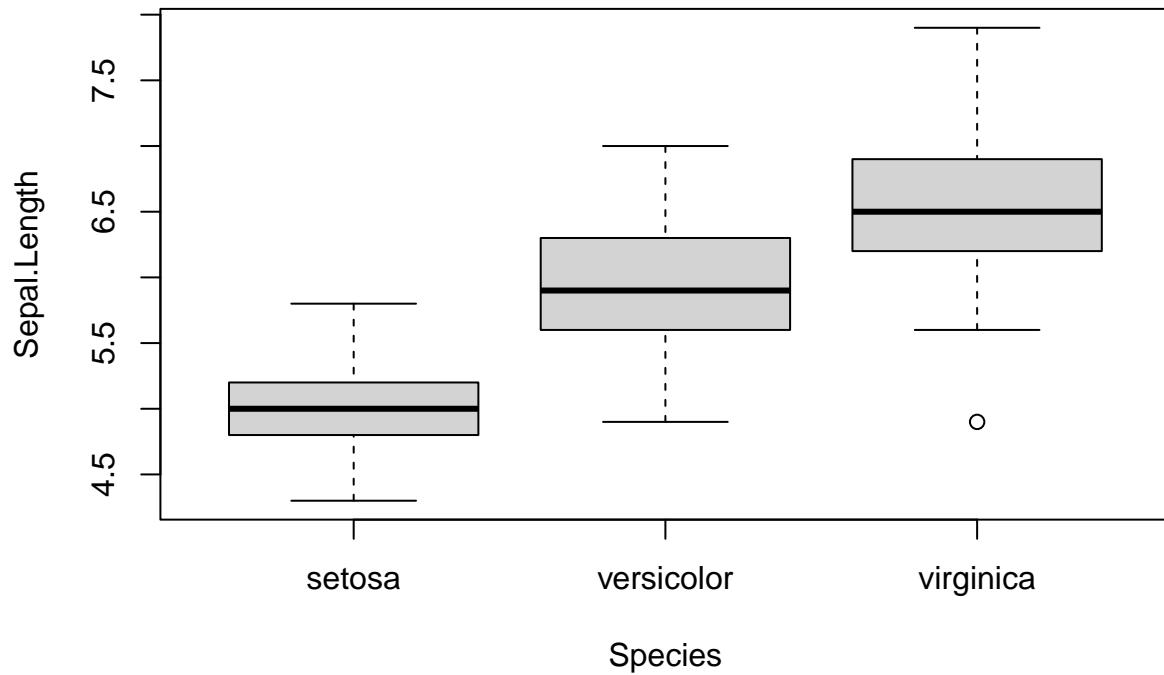


You can customize as in `plot()`, but slightly different.

#### 14.3.1 Box color

`col` argument.

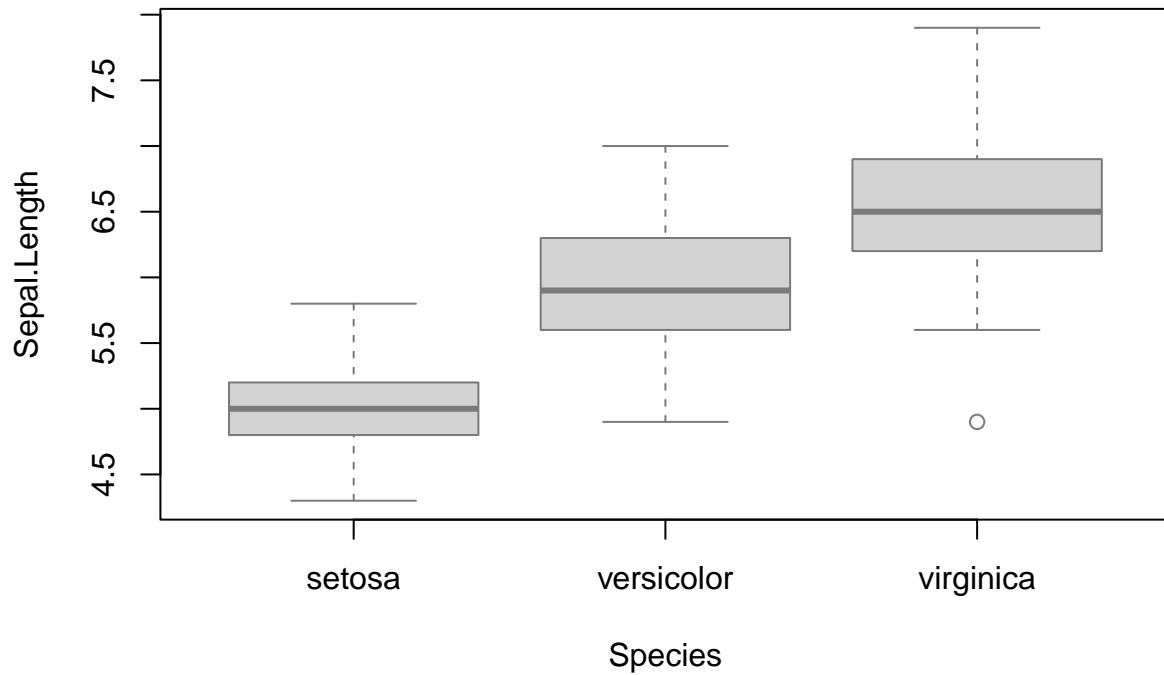
```
boxplot(Sepal.Length ~ Species, data = iris,  
        col = "lightgray")
```



#### 14.3.2 Border color

border argument.

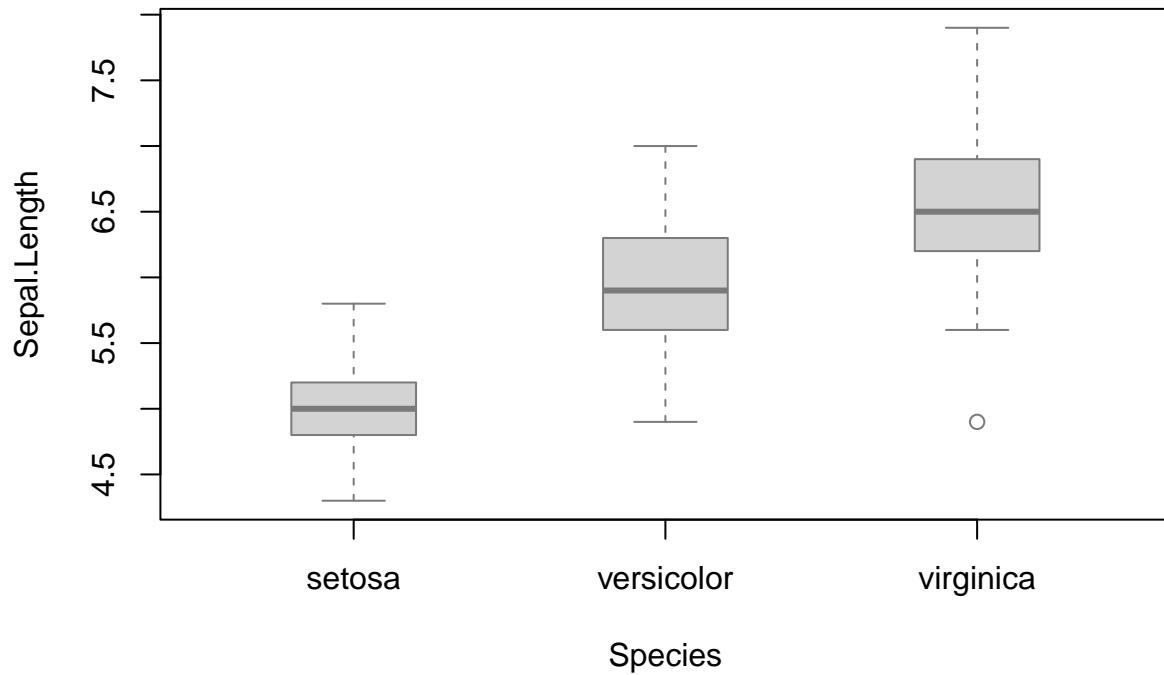
```
boxplot(Sepal.Length ~ Species, data = iris,
        col = "lightgray", border = "grey48")
```



#### 14.3.3 Box width

boxwex argument.

```
boxplot(Sepal.Length ~ Species, data = iris,
        col = "lightgray", border = "grey48",
        boxwex = 0.4 )
```



#### 14.3.4 Axis

Delete axes with `axes = F` and re-draw with `box()` and `axis()` functions.

```
boxplot(Sepal.Length ~ Species, data = iris,
        col = "lightgray", border = "grey48",
        boxwex = 0.4, ylab = "Sepal length (cm)",
        axes = F)
box(bty = "1")
axis(1, at = c(1, 2, 3), labels = c("Setosa", "Versicolor", "Virginica"))
axis(2, las = 2)
```

