Mathematical characterization of fractal river networks

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1 Maintext

Rivers form complex branching networks, and the ecological implication of river network complexity has gained a great interest over the past few decades[1–3]. To this end, there have been concerted efforts to construct virtual river networks, aiming to provide theoretical insights into how river network structure controls riverine ecological dynamics. Carraro and Altermatt[4] made a great contribution to this research area by comparing the scaling properties of virtual river networks produced by three different simulation methods - balanced binary trees (BBTs)[5], random branching networks (RBNs)[6,7], and optimal channel networks (OCNs)[8]. The first two methods have two parameters that control river network size (order or the number of nodes) and complexity (branching probability). In the meantime, OCNs are spanning trees that suffice a local minimum of a function describing total energy expenditure across the network. Carraro and Altermatt[4] performed extensive analysis and concluded that: (1) branching "probability" is better termed as branching "ratio" for real river networks because it is not a probability; (2) branching ratio is a scale-dependent quantity as the value changes across spatial resolutions at which river networks are extracted (expressed as the threshold catchment area A_T that initiates channels or pixel size l); (3) OCNs best represent the scaling properties of river networks, such as Horton's law and the exceedance probability of catchment area; (4) OCNs most accurately predicted the metapopulation stability and capacity. I agree that OCNs are capable of reproducing the scaling properties of real river networks. However, the supporting ground for the rest of the conclusions is either seriously flawed or inconclusive due to the improper use of branching probability, scale invariance, dimensions, and units.

First, branching probability p is not identical to branching ratio p_r in real river networks; in fact, p is certainly a probability by definition. In previous studies, branching probability p has been defined as a cumulative probability distribution of link length L [km]. For example, Terui et al.[6,7] fitted an exponential distribution to link length as $L_j \sim \text{Exp}(\theta)$ (θ is the rate parameter and j refers to an individual link) and estimated branching probability as $p = 1 - \exp(-\theta L)$ (L was set to be 1 [km] in Terui et al.). Thus, p [km^{-1}] in this example is the "probability" of including a confluence (or an upstream terminal) per unit river km. Here, the rate parameter θ is the inverse of average link length in a network, which is the definition of branching ratio $p_r = \frac{N_L}{N}$ (N_L the number of links in a network [dimensionless]; N the total river length [km]). To show this equality ($\theta = p_r$), I delineated 50 river networks analyzed in Carraro and Altermatt[4] using MERIT Hydro[9], with which θ and p_r are independently estimated as described above. MERIT Hydro has a constant pixel size of 3 arc-second (~90 m at the equator) across the globe. I used $A_T = 1$ [km^2] to capture sufficient details of real river networks. The relationship between p_r and θ fell exactly on a 1:1 line (Figure 1A), thus $\theta = p_r$. By definition, branching probability is a monotonic increasing function of p_r with a possible range of $0 - 1$ (Figure 1B); yet, they are not identical. Therefore, the author's claim "*…it is in fact improper to refer to a "probability" when analyzing the properties of a realized river network.*" is incorrect. Note that, in the following paragraphs, I use p_r for my analysis to be consistent with the original article. Since branching probability p is a simple transformation of p_r, the arguments are translatable between the two.

Second, the term "scale invariance" was falsely used in their article. To explain this, let y^* and y be the perimeter length of a perfect circle and the length of a coastline, respectively, which are measured as multiples
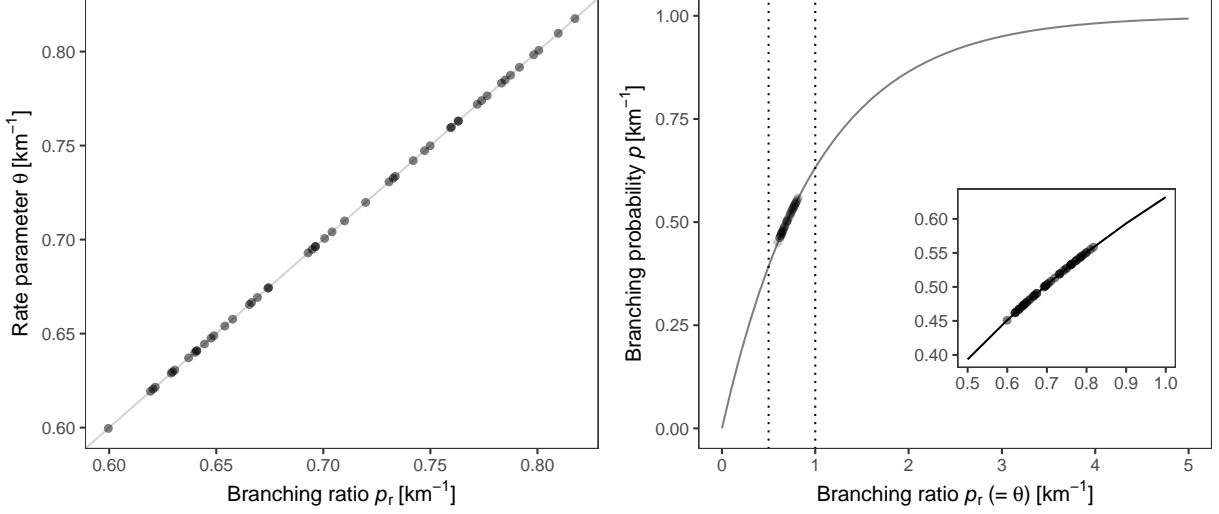
Figure 1: (A) Equality between rate parameter θ and branching ratio p_r. Dots indicate estimated values at 50 rivers analyzed in Carraro and Altermatt[4]. The gray line denotes a 1:1 relationship. (B) Relationship between branching probability ($p = 1 - \exp(\theta)$) and ratio ($p_r = \frac{N_L}{T}$). Dots are estimated values of p and p_r at the 50 rivers. Note that p and p_r were estimated independently; nevertheless, the data points fell exactly on the theoretical relationship between p and θ, confirming the equality between θ and p_r. Inlet zooms into the observed range of p_r, which is denoted by the vertical broken lines.

of a ruler with a unit length x (the "observation scale") – a classical comparison between "scale-dependent" and "scale-invariant" objects used in Mandelbrot[10]. The approximated length measured by the ruler is $q(x)x$, where $q(x)$ is the number of dividing steps with a given ruler length x. In the case of perimeter length y^* (scale-dependent), one will obtain the exact value as $x \to 0$ because a sufficiently short ruler can characterize the smooth curve of a perfect circle (the existence of characteristic scale). Clearly, $\lim_{x\to 0} q(x)x = 2\pi r$ (r is the radius) for the perimeter of a perfect circle. In contrast, the length of a complex coastline y (scale-invariant) diverges to reach infinity as $x \to 0$ because a shorter ruler will capture similar, but finer-scale complexity that is ignored by a longer ruler infinitely (the lack of characteristic scale; see Figure 1 in Mandelbrot[10] for the graphical comparison between the two types of objects). This property, i.e., the part is a reduced-scale image of the whole, is referred to as "statistical self-similarity" that underpins the mathematical characteristics of scale-invariance in the coastline length y. Mathematically, a scale-invariant structure can be epitomized by the power-law function[11,12]:

$$y = f(x) = cx^z$$

where c is the scaling constant and z is the scaling exponent. The scale invariance can be easily proved by multiplying the scaling factor λ:

$$y' = f(\lambda x) = c\lambda^z x^z = \lambda^z f(x) = \lambda^z y$$

Thus, the observed object $y = f(x)$ is said to be scale-invariant because the multiplicative extension/shrink of observation scale x results in the same shape of the original object y but with a different scale[13].

Carraro and Altermatt[4] provided some evidence that branching ratio p_r follows a power law along the axes of observation scale A_T and pixel size l (i.e., length on a side) using OCNs (Equation 1 in the original article):

$$\mathrm{E}(p_r) = 1.531 A_T^{-0.523} A^{-0.032}$$

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where A is the entire watershed area expressed in the number of pixels; thus, $A = A_{\text{true}}l^{-2}$ (A_{true} is the watershed area measured in the unit of l^2, such as km^2). Also, the authors "visually" showed that the relationship between p_r and A_T follows a power-law using the data from 50 real river networks (Figure 3a in the original article). Provided that this relationship statistically holds true (see the next section for this issue), branching ratio has a property of scale-invariance. Despite this, the authors claimed that "*Here we show that an alleged property of such random networks (branching probability) is a scale-dependent quantity that does not reflect any recognized metric of rivers' fractal character...*" (Abstract) This interpretation is the opposite as it has been defined in the literature of scale-invariance[11–14], including the author's previous publication[15]. Therefore, the sentences that build upon this interpretation are inappropriate and misleading (**Table 1**). Importantly, many of these sentences are the concluding sentence of a paragraph. As such, the progression of supporting sentences in these paragraphs is no longer logical, unfortunately.

Third, dimensions and units are improperly treated in their analysis. A dimension is the power of an axis along which a physical quantity is measured, and a unit is a way to assign a number to a particular dimension to make it relative. For example, length is a dimension and meter is a unit of length. Throughout the article, the authors used the number of pixels to measure the total river length N, the total catchment area A, and the threshold catchment area A_T. There is no issue to use the number of pixels as a unit. However, a critical problem in their article is that they obscured the dimensions of pixels. For example, they made very unclear that the river length N has a dimension of pixel "length", while the total catchment area A and the threshold catchment area A_T have a dimension of pixel "area". In particular, the authors incorrectly defined p_r as a dimensionless quantity (Methods) despite its unit is [pixel length^{-1}]. Note that p_r has a non-discrete dimension of length due to the fractal nature of rivers (see the following paragraphs for details).

The improper use of units and dimensions caused serious problems in the analysis of "real" river networks; the authors used different pixel sizes (length on a side) among watersheds (103 m to 1268 m; Supplementary Table 2 in the original article), meaning that the same number of pixels translates into very different lengths and areas. For example, in Figure 3 in the original article, the observation scale A_T ranges $20 - 500$ pixels. This pixel range translates into $0.2 - 5.3$ km^2 for the Toss river (with smallest pixel size) and $32.2 - 803.9$ km^2 for the Stikine river (with largest pixel size). Further, the branching ratio p_r is also affected by this variation in pixel size as its unit is [pixel length^{-1}]. Once the pixel length is converted to a unit of km, the branching ratio represents the number of links per 0.1 km in Toss, whereas it represents the number of links per 1.3 km in Stikine. Hence, the authors compared values that are incomparable, and their results reported in Figure 3 are not trustable.

To explore the consequences of the improper use of pixels, I re-analyzed their data with MERIT Hydro[9]; as noted above, this layer has a constant pixel size, unlike the author's analysis. In addition, I use units of [km] (for line objects) and [km^2] (for aerial objects) throughout my analysis. I extracted river networks with 20 values of A_T [km^2] ($A_T = 1, ..., 1000$ with an equal interval at a $\log_{10}$ scale, but confined to $A_T < A$ for small rivers), at which I estimated branching ratio as $p_r = \frac{N_L}{N}$ [km^{-1}]. In the original article, the authors did not perform any statistical analysis despite the fact that p_r $(= \theta)$ is a statistical parameter characterizing a link length distribution (see above); thus, its estimation accuracy is affected by the number of links, i.e., the sample size. Instead, they "picked" specific values of p_r from only three observation scales to claim that the rank of p_r varies across scales (**Table 1**). To avoid such a subjective argument, and to statistically substantiate the power-law of p_r along the axis of A_T, I fitted the following log-linear models with robust regression (i represents an individual data point of p_r estimated in a given watershed at a given scale A_T).

$$\log_{10} p_{r,i} = \log_{10} c_{w(i)} + z \log_{10} A_{T,i} + \varepsilon_i \qquad \text{(M0)}$$
$$\log_{10} p_{r,i} = \log_{10} c_{w(i)} + z_{w(i)} \log_{10} A_{T,i} + \varepsilon_i \quad \text{(M1)}$$

where ε_i is the error term that is properly weighted by Huber's function. The first model (M0) assumes the "universal" scaling with the single exponent z across watersheds; i.e., the branching ratio at all the 50 watersheds follows the same power law with the watershed-specific constant $\log_{10} c_{w(i)}$ ($w(i)$ is watershed w for a data point i). In contrast, the second model (M1) assumes the "localized" scaling with the watershed-specific exponent $z_{w(i)}$. I estimated the evidence ratio of the two models using the approximated Bayes

Factor $(\text{BF})^{16}$, which is defined as $\text{BF} = \exp\left(\frac{\text{BIC(M1)}-\text{BIC(M0)}}{2}\right)$. In this definition, a value of $\text{BF} > 1$ gives the support for M0 over M1; for example, if $\text{BF} = 2$, the model M0 is twice as likely as the model M1.

The analysis provided decisive support for M0 with $\text{BF} \approx 3.00 \times 10^{20}$. This result remains valid even when I used a subset of watersheds with area sufficient to cover the entire range of observation scale A_T ($\text{BF} \approx 3.18 \times 10^{10}$ for 21 watersheds with $A > 5000$ km^2). Under M0, as evident from its model formula, the rank of the expected branching ratio $\text{E}(\log_{10} p_r)$ never changed across scales, maintaining the order of the watershed-specific constant $\log_{10} c_{w(i)}$ (Figure 2; see regression lines). This result is inconsistent with the author's statement "*by extracting different river networks at various scales (i.e., various A_T values) and assessing the rivers' rank in terms of p_r, one observes that rivers that look more "branching" (i.e., have higher p_r) than others for a given A_T value can become less "branching" for a different A_T value (Fig. 3).*" I also must note that I did not find any significant correlation between watershed area A [km^2] and branching ratio p_r [km^{-1}] when extracted with a constant value of $A_T = 1$ km^2 across watersheds (Spearman's rank correlation = -0.19; p-value = 0.19), as opposed to the statement in the original article "*...if different river networks spanning different catchment areas (say, in km^2) are compared, all of them extracted from the same DEM (same l and same A_T in km^2), then the larger river network will appear more branching (i.e., have larger p_r)^4.*" The lack of correlation between branching probability p and watershed area A has also been reported in two previous studies[6,7], one of which used 184 watersheds for the analysis[7]. This result is unsurprising at all because p_r is the number of links "divided" by the total river length N by definition. My re-analysis revealed that the author's statements merely reflect the lack of appropriate quantitative analysis and/or a statistical artifact of inconsistent units across watersheds.

It is worthwhile to note that, if scaling exponent z varies by watershed (albeit not the case for my analysis), the dimensional p_r is incomparable across watersheds because its dimension ($= 2z$) is different (just like length and area are not comparable). To understand this argument, one must recognize that, by writing $p_r = c_w A_T^z$, we equate the dimension in both sides of the equation. From this equation, the dimension of p_r is "estimated" to be $2z$ since the scaling exponent z applies to the unit of A_T as $p_r = c_w A_T^z$ [km^{2z}]. Therefore, to compare branching ratio across watersheds, we must non-dimensionalize p_r as $\bar{p}_r = \frac{p_r}{A^z} = c_w$ [-] (see Figure 2B for this exercise). Evidently, the expected value of dimensionless branching ratio $\text{E}(\bar{p}_r)$ [-] remains constant across scales (c_w) with no possibility of changes in the rank (Figure 2B). This technique of dimensional analysis has been widely used when comparing the self-similar structure of scale-invariant objects (see Figure 1 in Rinaldo et al.[14] and equation (2.2) in Rodriguez-Iturbe and Rinaldo[11] for examples). In fact, past empirical studies used $A_T = 1$ km^2 such that the estimated dimensional p_r approximates the rescaled $\bar{p}_r$ ($p_r = \bar{p}_r \approx c_w$ for $A_T = 1$ in a given unit, regardless of the value of z)[6,7]. Therefore, when z varies by watershed, the comparison of dimesional branching ratio p_r is meaningless (unless properly rescaled as in Terui et al.[6,7]). A valid implication from this re-analysis is that $\bar{p}_r$ fluctuates unpredictably at coarser observation scales (larger A_T relative to A; Figure 2B). This is simply because of a small sample size (i.e., the number of links) at coarser resolutions, which inflates the statistical uncertainty of parameter estimation for p_r. However, a solution is simple: use a sufficiently small observation scale (e.g., $A_T = 1$ km^2) to yield a large sample size N_L. The $\bar{p}_r$ will converge to the single value that represents the self-similar structure unique to each river network (Figure 2B). Therefore, there is no mathematical supporting ground for the author's statement "*...branching probability is a non-descriptive property of a river network, which by no means describes its inherent branching character...*"

Lastly, the issue of dimensions/units is pertinent to their metapopulation simulations, as they used pixel size l as a unit of "population scale" and "dispersal distance". This makes their simulation results hardly interpretable. For example, they assumed a population scale of 103 m in Toss, while assuming 1268 m in Stikine. A reasonable interpretation of this simulation setup is that the authors simulated metapopulation dynamics of "different" species among watersheds. This is not trivial as the population scale defines the number of local populations within a metapopulation – one of the most influential parameter dictating metapopulation CV (see Equations (2) and (3) in the original article). If one assumes the same population scale across watersheds (i.e., the same species), then Stikine should be ~12 times greater in the number of local populations N_p compared to Toss. Nevertheless, the authors used a nearly-constant number of N_p (e.g., $\text{E}(N_p) \approx 1088$ for $A_T = 500$ [pixel area]) by changing the population scale for real rivers. Similarly, the average dispersal distance α was measured in the unit of pixel length ($\alpha = 100$ [pixel length] in Figure 6 in

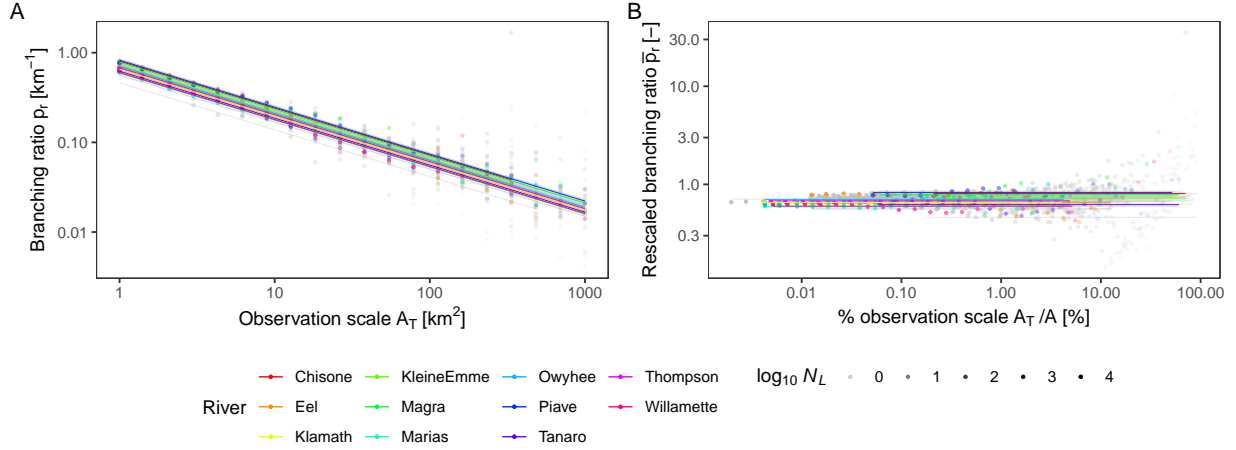Figure 2: Power law scaling of brnaching ratio p_r. Colors indicate rivers highlighted in the original article[4]. (A) Dimesional branching ratio p_r. Dots are individual data points. Lines are predicted values from the model M0 (i.e., the expected value of branching ratio). (B) Rescaled branching ratio $\bar{p}_r$. Dots are individual data points. Lines are predicted values from the model M0 after proper rescaling.

the original article), meaning that it varies from 10.3 km to 126.8 km among watersheds when converted to a unit of km. It is difficult to envision that such a huge variation in dispersal capability exists within the same species. Again, the parameter α has a decisive influence on metapopulation CV and capacity in the author's model (see Methods in the original article). Importantly, Figure 6 and Supplementary Figures $4-9$ of the original article "aggregated" these metapopulations of different species in real rivers and compared the overall summary statistics (e.g., percentiles) for BBTs, RBNs, OCNs, and real rivers. Here again, the authors compared values that are incomparable. A better alternative should have been to use different number of N_p among watersheds to assume metapopulations of the same species, then comparing metapopulation CV and capacity by watershed (as performed in Figure 3 of the original article). Although I do not know how this improvement will alter one of the main conclusions "*OCNs most accurately predicted the metapopulation stability and capacity*", it is clear that their simulation results are not ecologically interpretable in its current form.

Carraro and Altermatt[4] offered an important perspective to the use of the different classes of virtual river networks. Therefore, a full re-interpretation/re-analysis is warranted to provide valid insights into how we assess river network structure.

1.1 Data availability

Codes and data are available at https://github.com/aterui/public-proj_fractal-river.

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