1 **Reply to "Mathematical characterization of fractal river networks"**

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7 Optimal Channel Networks (OCNs) are virtual constructs that reproduce topology and scaling of real

8 rivers[1,2,3]. Commonly used in hydrology, they also proved useful for ecological studies[4,5]. However, also

9 because their computation has been less accessible until recently[6], other approximations, such as random

10 networks, have been used in ecological studies. In a recent study[7], we compared random networks with

11 OCNs and real rivers, both in terms of their geomorphological character and of ecologically-relevant

12 metrics. We found that OCNs accurately match properties of real rivers, while this is generally not true for

13 random networks. We also resolved a common misunderstanding in the use of random river models in

14 ecology: the branching probability with which these are derived is not an inherent property of rivers, but

15 depends on the observational scale. In a Matters Arising article[8], Terui disputed our conclusions, and

16 criticized two aspects of our work: concerning the definition of scale invariance, and on the use of

17 dimensions and units. We respectfully but strongly disagree with both claims. Here, we show that both are

18 incorrect and inappropriate, and neither of them is related to the core finding of our work, namely that

19 OCNs are best at capturing ecologically-relevant features of river networks.

20 **Definition of scale invariance**

21 Firstly, Terui claimed that scale invariance is falsely defined in our study[7], and specifically argued that

22 "*branching probability has a property of scale invariance*"[8] because we found that branching ratio p_r scales

23 as a power law of threshold area A_T. First of all, we note that we did not provide any formal definition of

24 scale invariance in our study[7], as this was not required to support our findings. Notably, scale invariance is

25   defined as a property of a function $y = f(x)$ (that is, the relationship between y and x), not of the

26   dependent variable y itself. Quoting Proekt et al.[9] (also referred to by ref.[8]): "*a function $f(x)$ is said to be*

27   *scale-invariant, if on multiplying the argument of the function by some constant […] the same shape is*

28   *retained but with a different scale*". Therefore, the claim on branching probability being scale-invariant is

29   misleading. If $p_r \sim A_T^z$, one can say that the relationship between p_r and A_T is scale-invariant, not that p_r is

30   scale-invariant per se. To make an ecologically relevant example, if this were not the case, one would

31   conclude that the number of species is scale-invariant, because species-area relationships are power laws.

32   Yet, the number of species does depend on the scale (habitat area) at which it is observed; it is the

33   relationship between area and species number that is scale-invariant.

34   More importantly, the interpretation of the definition of scale invariance is immaterial for the limitation we

35   highlighted in our study[7]: given a river network, it is not possible to univocally characterize its p_r, because it

36   depends on the scale at which the network is observed (that is, the depth/width/discharge above which a

37   river is defined, subsumed by A_T in our study[7]), just as one cannot assign an intrinsic length to a coastline

38   because it depends on the length of the "ruler" used to measure it. Even if one could univocally determine

39   the position of channel heads (and hence the number of links, that is the numerator of p_r), p_r would still

40   depend on its denominator, that is the number of network nodes. We[7] used the pixel size as the extent of a

41   network node, because this is the smallest dimension at which one can measure the features of a river

42   network. We[7] also argued that the dimension of a node could be defined from an ecological perspective as

43   the extent of a local population. In this case, different species (with different local population ranges)

44   inhabiting the same real river network would perceive the network as partitioned into a different number

45   of nodes, hence leading to different p_r values.

46   **On dimensions and units**

47   Secondly, Terui[8] claimed an improper treatment of dimensions by our use[7] of pixels as units to measure

48   lengths and areas, which would allegedly lead to issues in the analysis of real river networks.

49   First, the distinction between pixel lengths and pixel areas is immaterial and does not affect our analyses in

50   any way. Indeed, the numerator of p_r is dimensionless, while the denominator is measured in pixel lengths,

51   and p_r is hence in principle measured in pixel lengths^{-1}. However, we treated p_r as de facto dimensionless,

52   as we did not attribute any specific dimension to the pixel size. This allowed a comparison between real

53   river networks spanning different catchment areas in km^2 (see following paragraph). To this end, we also

54   observe that previous works[10−13] using the concept of branching probability p expressed p as a

55   dimensionless parameter, which is self-evident for a probability. The distinction between branching

56   probability p and branching ratio p_r that we operated in our study is here irrelevant, because also p

57   depends on the length that is attributed to a node.

58   Second, we respectfully disagree with the claim on the improper use of dimensions in our study. The

59   probability distribution of drainage areas is invariant under coarse graining[3], hence it is absolutely correct

60   to compare DEMs constituted by pixels of different sizes, which we did in our analysis. We specifically

61   included real river basins spanning a wide range of catchment areas, as this proves that the same scaling

62   features (e.g., relationship between p_r and A_T) emerge irrespective of the pixel size, i.e., irrespective of the

63   size of these catchments. In contrast, we respectfully note that Fig. 1 in ref.[8] does not invalidate our results

64   whatsoever: panel b is simply obtained by shifting the lines of panel a along the x-axis. However, values on

65   the y-axis are unchanged, hence our conclusion that different rivers rank differently in terms of p_r when

66   extracted at different scales remains unaltered. For instance, in Fig. 1b, river Magra has the highest p_r

67   value when extracted at the coarsest resolution, but does not rank among the most "branching" rivers

68   when extracted at the other two resolutions. If one extrapolated the trend lines of Fig. 1b and drew a

69   vertical line at, say, A_T=1 km^2, one would indeed observe a higher p_r value for the larger catchments (e.g.,

70   Klamath, Owyhee) than the smaller ones (e.g., Kleine Emme, Chisone). However, this comparison is

71   improper, as A_T=1 km^2 may be a very fine scale to extract river networks in large (in terms of km^2) basins,

72   but a very coarse scale for smaller catchments. A fair comparison with A_T expressed in km^2 would require

73   the use of basins of similar areas in km^2 and extracted from DEMs with same (or similar) pixel size. Fig. 1b in

74   ref.[8] indeed shows that even for such rivers (e.g. Piave, Tanaro, Magra – see data for these rivers in Table

75   S2 in ref.[7]), the p_r-rank is not maintained as A_T changes.

76   Also other claims, such as *"it is very likely that substantial errors are contained […] that underpin their main*

77   *conclusion that OCNs most accurately predicted the metapopulation stability and capacity"*[8] are neither

78   substantiated nor proven. Our analysis was designed to compare networks with similar numbers of pixels,

79   the only meaningful way to perform a fair comparison, as the number of pixels (and hence, of network

80   nodes) exerts a major control on the metapopulation metrics. In fact, we ensured that BBTs, RBNs, OCNs

81   and real rivers have the same parameters (number of nodes and branching probability, Fig. S8 in ref.[7])

82   when metapopulation properties are compared. It is indeed unclear how one should have "*tweaked virtual*

83   *networks*" to overcome this alleged "*fundamental problem*"[8], given that the only two parameters of

84   random networks are number of nodes and branching probability, and no parameter controlling for the

85   scale/dimension of these networks is provided.

86   We conclude that neither of Terui's critiques are justified, and we strongly oppose the statement that our

87   findings are *"merely an artefact"*. Both claims are not only incorrect, but actually inappropriate, as not

88   substantiated. We also respectfully disagree with alleged indirect critiques, such as implications that

89   random networks would be better than OCNs because these *"methods have only few parameters that*

90   *control river network size"*[8]. The choice of a method is not (only) about the number of parameters needed,

91   but rather about its accuracy and suitability. Importantly, we acknowledge that Terui agrees that OCNs are

92   capable of reproducing the scaling properties of real river networks, and concludes that a re-interpretation

93   of the importance of river network structure in ecology should be assessed. Actually, this is exactly what we

94   did in our study[7]: proving that OCNs can reproduce ecologically-relevant metrics of real river networks,

95   while random networks do not. Consequently, the use of random networks in ecological studies should be

96   abandoned in favor of OCNs.

97 **Data availability statement**

98 Data sharing not applicable to this article as no datasets were generated or analysed during the current

99 study.

100 **Author contributions**

101 L.C. and F.A. equally contributed to this work.

102 **Competing interests**

103 The authors declare no competing interests.

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109 **References**

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