Mathematical characterization of fractal river networks

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# 1 Abstract

River networks exhibit fractal characteristics. Quantifying river network complexity is crucial for unveiling the role of river fractals in driving riverine ecological dynamics, and researchers have used a metric of “branching probability” to do so. Previous studies showed that this metric reflects the fractal nature of river networks. However, a recent article by Carraro and Altermatt (2022) contradicted this classical observation and concluded that branching probability is “scale dependent.” I dispute this claim and argue that their major conclusion is derived merely from their misconception of scale invariance. Their analysis in the original article (Figure 3a) provided evidence that branching probability is scale invariant (i.e., branching probability exhibits a power-law scaling), although the authors erroneously interpreted this result as a sign of scale dependence. In this paper, I re-introduce the definition of scale invariance and show that branching probability meets this definition. This provided an opportunity to address the widespread confusion about the terms “scale invariance” and “scaling” in the ecological literature.

# 2 Maintext

Rivers form complex branching networks with fractal characteristics – “there are basins within basins within basins, all of them looking alike (Rinaldo et al. 2014).” The ecological implication of fractal structure has gained great interest over the past few decades (Grant et al. 2007; Carrara et al. 2012; Mari et al. 2014), and there have been concerted efforts to investigate the role of river fractals in controlling metapopulation (Yeakel et al. 2014; Mari et al. 2014; Terui et al. 2018) and metacommunity dynamics (Carrara et al. 2012; Anderson and Hayes 2018; Terui et al. 2021). The proper quantification of river network complexity is therefore crucial, and some researchers used a metric of “branching probability,” i.e., the probability of observing a joining tributary per unit river distance (Yeakel et al. 2014; Terui et al. 2018; Terui et al. 2021). It has been shown that this measure reflects the fractal nature of river networks because it exhibits the sign of scale invariance (Peckham and Gupta 1999; Rodríguez-Iturbe and Rinaldo 2001; Terui et al. 2018). However, a recent article by Carraro and Altermatt (2022) countered this classical observation and concluded that branching probability is “scale dependent.” They seriously criticized past research (Yeakel et al. 2014; Anderson and Hayes 2018; Terui et al. 2018; Terui et al. 2021) for the use of branching probability as a measure of river fractals, provocatively concluding “*an alleged property of such random networks (branching probability) is a scale dependent quantity that does not reflect any recognized metric of rivers’ fractal character…*” in the Abstract. Here, I would like to communicate my concern that their major conclusion is merely a misconception of the term “scale invariance.” Indeed, the reanalysis of their data proved that branching probability is scale invariant. In what follows, I re-introduce the definition of scale invariance and show that branching probability meets this definition.

# 3 Premise: the definition of scale invariance

The concept of scale invariance is inherently confusing and is often misunderstood among ecologists. To clarify this concept, let me use the most famous example of a scale invariant object – the complex structure of the coastline (Mandelbrot 1967). In Mandelbrot’s pioneering work, he measured the coastline length as multiples of a ruler with unit length (the “observation scale”). He observed that the length diverged to reach infinity as , because finer but similar zig-zag patterns emerged over and over as he used a shorter ruler (Figure 1); that is, there was no specific scale that uniquely characterizes the coastline length (the “lack of characteristic scale”; Table 1). However, when the coastline length was expressed as a function of scale (), the function obeyed a simple rule that reflects the fractal nature of the observed object: the power law.

where is the scaling constant and is the scaling exponent. The scale invariance of this function can be easily proved by multiplying an arbitrary scaling factor :

The above equation is interpreted as a sign of scale invariance because the multiplicative extension/shrink of scale by factor results in the same shape of the original object but with a *different scale* (Proekt et al. 2012) (i.e., the structural property or function is preserved). Hence, the observed object (= function ) is said to be scale invariant.

Although a power law is the most famous example of scale invariance, any function that suffices the following equation is said to be scale invariant (Rodríguez-Iturbe and Rinaldo 2001; Proekt et al. 2012):

A general property of this function is “scaling” (Table 1), in which the dimensional physical quantity of the object (e.g., length) changes predictably across scale (Rodríguez-Iturbe and Rinaldo 2001).

Conversely, a “scale dependent” object possesses contrasting properties (Table 1). A square exemplifies a scale dependent object. Assuming km on a side, the perimeter ( km) remains constant at the divisors of (); otherwise, the measured perimeter becomes shorter than the true length if we disregard rulers that do not fit the square’s side (, where denotes the integer part of the division; see Figure 1). Therefore, the square has characteristic scales that uniquely identify its physical quantity. This property is called “non-scaling,” which contrasts with the property of a scale invariant object (Rodríguez-Iturbe and Rinaldo 2001). Although I used a square for the sake of simplicity, a more general definition of non-scaling is . Thus, “scale dependence” and “scale invariance” are contrasting observations.

Key terminology

| Object type | Property |
| --- | --- |
| Scale invariant | Scaling, the lack of characteristic scale |
| Scale dependent | Non-scaling, the existence of characteristic scale |

![Figure 3.1: Graphical comparison of scale invariant and scale dependent objects. Black lines indicate an example of a scale invariant object (coastline), while gray lines show an example of a scale dependent object (square). For the square, the solid gray lines indicate the measured length as multiples of the integer ruler length x, with red points showing measurements at the divisors of 500 (the “characteristic scale”). The dotted lines denote the true perimeter of the square. The left (a) and right (b) panels are ordinary and log-log plots, respectively.](data:application/pdf;base64,)

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# 4 Power-law scaling of branching probability

Here, I reanalyze the data in Carraro and Altermatt (2022) to assess the scale invariance of branching probability. By definition, branching probability is said to be scale invariant if it follows a power law of observation scale . Although the observation scale can take a variety of forms, a common metric in river geomorphology is the threshold value that defines the minimum watershed area of the river channel. Since measures stream size, we extract a subset of wider river channels as increases (Figure 2A and 2B); in other words, the resolution of the river network becomes coarser. I extracted river networks of watersheds that were used in Carraro and Altermatt with MERIT Hydro (Yamazaki et al. 2019) (pixel size: ).

I used values of [km] ( with an equal interval at a scale, but confined to for small rivers), at which I estimated branching probability. Branching probability [] is the probability of observing a joining tributary per unit river distance. This quantity can be estimated from the probability distribution of link length (i.e., the length of the river channel from one confluence to another, or one confluence to the outlet/upstream terminal). Typically, the link length follows an exponential distribution, and branching probability can be calculated as a cumulative distribution function of the exponential distribution (Terui et al. 2018; Terui et al. 2021):

where is the rate parameter of an exponential distribution, which approximates the inverse of mean link length [unit: ] (, is the number of links). To evaluate the power law of branching probability (), I fitted the following log-log linear model with robust regression:

where is the branching probability ( represents a data point), is the watershed-specific scaling constant ( refers to a watershed), the scaling exponent, and the error term that is properly weighted by Huber’s function. I used robust regression analysis to account for outliers. Outliers were typical for coarse river networks with fewer links (i.e., large values) because the estimation accuracy of the mean link length depends on the number of links . The analysis provided strong support for the power-law scaling of branching probability (Figure 2C) with the estimated scaling exponent of .

![Figure 4.1: (a and b) Examples of extracted river network (Eel River, US). In panel (a), a fine-resolution river network was extracted with A_T = 12.7~\text{km}^2. In panel (b), a coarse river network was extracted with A_T = 233.6~\text{km}^2. (c) Log-log plot substantiates the power-law scaling of branching probability p along the axis of observation scale A_T. Colors indicate rivers highlighted in the original article (Carraro and Altermatt 2022). Individual data points are shown in dots, whose transparency is proportional to the number of links N_L (i.e., the sample size). Lines are predicted values from the robust regression model.](data:application/pdf;base64,)

Figure 4.1: (a and b) Examples of extracted river network (Eel River, US). In panel (a), a fine-resolution river network was extracted with . In panel (b), a coarse river network was extracted with . (c) Log-log plot substantiates the power-law scaling of branching probability along the axis of observation scale . Colors indicate rivers highlighted in the original article (Carraro and Altermatt 2022). Individual data points are shown in dots, whose transparency is proportional to the number of links (i.e., the sample size). Lines are predicted values from the robust regression model.

# 5 Practical issues

It is noteworthy that the similar power-law scaling was reported in Carraro and Altermatt (2022) using the inverse of mean link length (referred to as “branching ratio ” in the original article; see Figure 3a and Equation 1 in Carraro and Altermatt (2022)). Nevertheless, the authors erroneously interpreted this result as a sign of “scale dependence.” In addition, they used the term “scaling” as if it describes a property of scale dependence (page 3, right column). The misuse of the terms is not a simple issue of terminology. Based on their misconception, they expanded a discussion over two pages and concluded “*We therefore conclude that branching probability is a non-descriptive property of a river network, which by no means describes its inherent branching character, and depends on the observational scale*” (page 3). Hence, Carraro and Altermatt (2022) boldly used their misconception to invalidate past research that correctly referred to branching probability (or a cumulative distribution function of link length) as scale invariant (Peckham and Gupta 1999; Rodríguez-Iturbe and Rinaldo 2001; Moore et al. 2015; Terui et al. 2018; Terui et al. 2021).

Along this line, Carraro and Altermatt (2022) also argued that branching probability does not represent river’s branching character (page 3 to page 4, left column) because of the lack of characteristic scale (i.e., the unique value of branching probability does not exist for a given river network). This is clearly incorrect. We just need to perform a proper rescaling (or non-dimensionalization) to capture the structure of scale-invariant objects (Rodríguez-Iturbe and Rinaldo 2001). Such rescaling is possible once the scaling equation is established:

As evident from equation (6), we can obtain the unique value of (dimensionless) branching probability for a given river network after proper rescaling. This rescaling technique has been used for decades to characterize the physical/biological structure of scale invariant objects (Rodríguez-Iturbe and Rinaldo 2001; Brown et al. 2004; Rinaldo et al. 2014). In fact, previous studies used a unit scale of (; notice that when ) to approximate the rescaled branching probability (Terui et al. 2018; Terui et al. 2021). As such, Carraro and Altermatt (2022) provided no evidence that limits the use of branching probability as a measure of river network structure.

Overall, it is clear that the authors fundamentally misunderstood the mathematical definition of scale invariance. Inappropriate arguments arising from the misconception spanned from page 2 to 4; thus, correcting this misconception compromises the substantial portion of their article.

# 6 Concluding remarks: relevance to other ecological research

Scale invariance is not unique to rivers (Brown et al. 2002). It is observed in predator-prey interactions (Hatton et al. 2015), biological metabolism (Brown et al. 2004), food chain length (Sabo et al. 2010), and species-area relationships (Hubbell 2001), to name just a few. Scaling is a property of scale invariant observations, enabling us to make generalizations across scales (Levin 1992; Clark et al. 2021). This concept is central to ecological research.

However, in ecology, the terms are often misused or (implicitly) defined in stark contrast with those in fractal theory (Rodríguez-Iturbe and Rinaldo 2001; Proekt et al. 2012). Ecologists use “scale dependence” when ecological patterns change across scales (Levin 1992; Clark et al. 2021). This use of scale dependence *per se* is not an issue; however, a confusion emerges when it is paired with the term “scaling” to characterize scale dependence. Countless numbers of ecological research incorrectly use “scaling” as a property of “scale dependence” (Keil et al. 2018; Jarzyna and Jetz 2018; Chase et al. 2018; Gonzalez et al. 2020; Galiana et al. 2021; Nishizawa et al. 2022), although some ecologists follow the definition of scaling in fractal theory (Hubbell 2001; Brown et al. 2002, 2004). This confusion is so widespread that the diverged use of the same terms between ecology and mathematics is unavoidable.

Yet, such misconception (in a mathematical sense) must not be used as the ground for invalidating previous research that followed the mathematical definition of scale invariance. This is particularly true when the research concerns about fractals, as Carraro and Altermatt (2022) does. Otherwise, it introduces unnecessary debates that hinder the progress of this research field.

## 6.1 Data availability

Codes and data are available at <https://github.com/aterui/public-proj_fractal-river>.

## 6.2 Competing interest

None declared.

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