Mathematical characterization of fractal river networks

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# 1 Maintext

Rivers form complex branching networks, and the ecological implication of river network complexity has gained great interest over the past few decades1–3. To this end, there have been concerted efforts to construct virtual river networks to provide theoretical insights into how river network structure controls riverine ecological dynamics. Carraro and Altermatt4 compared the scaling properties of virtual river networks produced by three different simulation methods – balanced binary trees (BBTs)5, random branching networks (RBNs)6,7, and optimal channel networks (OCNs)8. The first two methods have two parameters that control river network size (order or the number of nodes) and complexity (branching probability). In the meantime, OCNs are spanning trees that suffice a local minimum of a function describing total energy expenditure across the network. Carraro and Altermatt4 performed extensive analysis and concluded that branching ratio is a scale-dependent quantity because the value and its rank change depending on spatial resolutions at which river networks are extracted (expressed as the threshold catchment area that initiates channels or pixel size ). However, the supporting ground for this conclusion is seriously flawed due to the improper use of scale invariance and units.

First, the term “scale invariance” was falsely used in their article. To explain this, let me consider an object whose structural property is assessed under observation scale . For example, Mandelbrot9 studied the length of a coastline as multiples of a ruler with a unit length (the “observation scale”). In this practice, the observed object is expressed as a function of scale (), and the function is said to be scale invariant if it suffices the following equation10,11:

where is an arbitrary constant and is the scaling exponent. The above equation is interpreted as the proof of “scale invariance” because the multiplicative extension/shrink of observation scale by factor results in the same shape of the original object but with a *different scale*11 (i.e., the structural property is retained).

The power-law function is the most famous example that suffices this definition of scale invariance10,12:

where is the scaling constant. The scale invariance can be easily proved by multiplying the scaling factor :

Carraro and Altermatt4 provided evidence that branching ratio follows a power law along the axes of observation scale and pixel size (i.e., length on a side) using OCNs (Equation 1 in the original article). Also, the authors visually showed that the relationship between and follows a power-law using the data from 50 real river networks (Figure 3a in the original article). Despite this, the authors claimed that “*Here we show that an alleged property of such random networks (branching probability) is a scale-dependent quantity that does not reflect any recognized metric of rivers’ fractal character*…” (Abstract) This interpretation is the opposite as it has been defined in the literature of scale-invariance10–13, including the author’s previous publication14. Instead, the conclusion derived from their analysis should have been “*branching ratio is a scale invariant feature that reflects rivers’ fractal character.”*

Second, units are inconsistent in their analysis of real river networks. The authors employed the number of pixels as a unit of measurement in their analysis. However, they used different pixel sizes (length on a side) among watersheds (103 m to 1268 m), meaning that the same number of pixels translates into very different lengths and areas. For example, in Figure 3 in the original article, the observation scale ranges pixels. This pixel range translates into 0.2 – 5.3 km for the Toss river (with smallest pixel size) and 32.2 – 803.9 km for the Stikine river (with largest pixel size). Further, the branching ratio is also affected by this variation in pixel size as its unit is [pixel length]. Once the pixel length is converted to a unit of km, the branching ratio represents the number of links per 0.1 km in Toss, whereas it represents the number of links per 1.3 km in Stikine. Hence, the authors compared incomparable values.

To explore the consequences of inconsistent units, I re-analyzed their data with MERIT Hydro15 that has a constant pixel size. I extracted river networks with 20 values of [km] ( with an equal interval at a scale, but confined to for small rivers), at which I estimated branching ratio as [km]. is the inverse of the mean link length. Thus, its estimation accuracy is affected by the number of links (i.e., the sample size). To statistically substantiate the power-law of , one must account for this statistical uncertainty. Therefore, I fitted the following log-linear models with robust regression:

where is the error term that is properly weighted by Huber’s function. Robust regression analysis is appropriate because it is robust to outliers caused by the small sample size (typical for large values). The first model (M0) assumes the “universal” scaling with the single exponent across watersheds; i.e., the branching ratio at all the watersheds follows the same power law with the watershed-specific constant ( is watershed for a data point ). In contrast, the second model (M1) assumes the “localized” scaling with the watershed-specific exponent . I estimated the evidence ratio of the two models using the approximated Bayes Factor (BF)16, which is defined as . In this definition, a value of gives the support for M0 over M1; for example, if , the model M0 is twice as likely as the model M1.

The analysis provided decisive support for M0 with . Under M0, as evident from its model formula, the rank of the expected branching ratio never changed across scales (Figure 1.1; see regression lines). This result is inconsistent with the author’s statement “*by extracting different river networks at various scales (i.e., various* val*ues) and assessing the rivers’ rank in terms of* *, one observes that rivers that look more ”branching” (i.e., have higher* *) than others for a given* *value can become less ”branching” for a different* *value (Fig. 3).*” I also must note that I did not find any significant correlation between watershed area [km] and branching ratio [km] when extracted with a constant value of across watersheds (Spearman’s rank correlation = , p-value = 0.19), as opposed to the statement in the original article “…*if … all of them extracted from the same DEM (same* *and same* *in km), then the larger river network will appear more branching (i.e., have larger* *).*” My re-analysis revealed that the author’s conclusion merely reflects the statistical artifact of inconsistent units across watersheds.

![Figure 1.1: Log-log plot substantiates the power-law scaling of dimensional branching ratio (p_r; Panel A) along the axis of observation scale A_T. Once non-dimensionalized, the properly rescaled branching ratio (\bar{p_r}; Panel B) will converge to values unique to each river as A_T \rightarrow 0. Colors indicate rivers highlighted in the original article4. Individual data points are shown in dots, whose transparency is proportional to the number of links N_L (i.e., sample size). Lines are predicted values from the model M0 (i.e., the expected value of dimensional or rescaled branching ratio). See Figures S1 and S2 for individual river plots.](data:application/pdf;base64,)

Figure 1.1: Log-log plot substantiates the power-law scaling of dimensional branching ratio (; Panel A) along the axis of observation scale . Once non-dimensionalized, the properly rescaled branching ratio (; Panel B) will converge to values unique to each river as . Colors indicate rivers highlighted in the original article4. Individual data points are shown in dots, whose transparency is proportional to the number of links (i.e., sample size). Lines are predicted values from the model M0 (i.e., the expected value of dimensional or rescaled branching ratio). See Figures S1 and S2 for individual river plots.

Carraro and Altermatt4 offered an important perspective on the use of the different classes of virtual river networks. Therefore, a full re-interpretation/re-analysis is warranted to provide useful insights into how we assess river network structure.

## 1.1 Data availability

Codes and data are available at <https://github.com/aterui/public-proj_fractal-river>.

## 1.2 Competing interest

None declared.

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