# Reaction Diffusion system with ext electric field.

We are considering a system of electrolytes coming from a symetric salt.

To analyse the system, we first consider the continuity equation,

$$\boxed{\frac{\partial C_s}{\partial t} + \nabla \cdot \vec{N}_s = r}$$

where $C_s$ is the concentration, $\vec{N}_s$ the flux of particles and $r$ the reaction rate.

a typical model for the flux in such context is

$$\boxed{\vec{N}_s = \mathcal{D}\left(\nabla \cdot C_s + \frac{sz\mathcal{F}}{RT} C_+(x) \nabla \phi\right)}$$

~~which~~
where
- $\mathcal{D}$ : Diffusion coef.
- $s$ : sign of the charge of the electrolyte
- $z$ : valence of the electrolyte
- $\mathcal{F} := e N_A$ : Faraday's constant
- $R := K_B N_A$ : Gases constant.
- $T$ : temperature
- $\phi$ : potential

is consider as a border condition such that

$$\left[\frac{\partial C_s}{\partial t} + \nabla \cdot \vec{N}_s\right]\Big|_{x=\delta} = \begin{cases} \Gamma & \text{if } \delta > 0 \\ 0, & \delta < 0. \end{cases}$$

Steady State solution:

we first consider

$$\frac{\partial C_s}{\partial t} = 0$$

which yields

$$\nabla \cdot \vec{N}_s = 0$$

$$N_s = \text{Constant}.$$

we get the following two equations:

$$\boxed{\begin{aligned} &\frac{\partial C_+(x)}{\partial x} + \frac{zF}{RT} C_+(x) \nabla\phi = \Gamma \\ &\frac{\partial C_-(x)}{\partial x} - \frac{zF}{RT} C_-(x) \nabla\phi = 0 \end{aligned}}$$

and

$$\boxed{\nabla^2 \phi = -\frac{1}{\varepsilon} zF \left(C_+(x) - C_-(x)\right)}$$

as,

$$\phi''(x) = -\frac{z\tilde{F}}{RT} C_b \left( e^{-\frac{z\tilde{F}}{RT}\phi(x)} - e^{\frac{z\tilde{F}}{RT}\phi(x)} \right)$$

we define,

$$\frac{zF}{RT}\phi(x) = \psi$$

as the dimensional potential

$$\psi''(x) = -\frac{z^2\tilde{F}^2}{R^2T^2} C_b \left( e^{-\psi(x)} - e^{+\psi(x)} \right)$$

Let

$$\boxed{K^2 = \frac{2z^2 F}{R^2T^2} C_b}$$

$$\psi''(x) = -\frac{K^2}{2} \left( e^{-\psi(x)} - e^{\psi(x)} \right)$$

$$\boxed{\psi''(x) = -K^2 \sinh(\psi(x))}$$

. Define the Dimentionless potential as

$$\boxed{\psi = \frac{z\mathcal{F}}{RT}\phi}$$

~~Also, we must too rescale~~

From ~~the~~ poisson's equation :

$$\phi''(x) = -\frac{z\mathcal{F}}{RT}\left(C_+(x) - C_-(x)\right)$$

(*) No reaction case,

$$\frac{\partial C_s(x)}{\partial x} = -\frac{sz\mathcal{F}}{RT} C_s(x) \nabla\phi$$

$$\frac{dC_s(x)}{C_s(x)} = -\frac{sz\mathcal{F}}{RT}\phi'(x)\,dx$$

$$\Rightarrow \log\left(\frac{C_s(x)}{C_s(0)}\right) = -\frac{sz\mathcal{F}}{RT}\int_0^x dx\,\phi'(x)$$

$$\Rightarrow C_s(x) = C_s(0)\,e^{-\frac{sz\mathcal{F}}{RT}\left(\phi(x)-\phi(0)\right)}$$

Defining $\phi(0) = 0$, we get

$$C_s(x) = C_b\, e^{-\frac{sz\mathcal{F}}{RT}\phi(x)}$$

redefine the system in terms of these variables.

$$\boxed{\begin{aligned} &\frac{\partial C_+(x)}{\partial x} + C_+(x)\nabla\psi(x) = r \\ &\frac{\partial C_-(x)}{\partial x} - C_-(x)\nabla\psi(x) = 0 \\ &\psi'' = -K_1^2 \sinh(\psi(x)) \end{aligned}}$$

we want to get the adimentional lengthscale

$$\xi = K_1 x \rightarrow x = \xi / K_1 .$$

$$\Rightarrow \frac{\partial C_s(\xi)}{\partial \xi} = \frac{\partial C_s}{\partial x}\frac{\partial x}{\partial \xi} = C_s'(x)\cdot\frac{1}{K_1}.$$

$$\frac{\partial \psi}{\partial \xi} = \frac{\partial \psi}{\partial x}\cdot\frac{\partial x}{\partial \xi} = \psi'(x)\frac{1}{K_1}$$

Thus,

$$C_s'(x) = K_1 C_s'(\xi).$$

$$\psi'(x) = K_1 \psi'(\xi).$$

Also

$$\frac{\partial^2}{\partial \xi^2}\psi = \frac{\partial}{\partial \xi}\left(\psi'(x)\frac{1}{K_1}\right) = \frac{1}{K_1}\frac{\partial \psi'}{\partial x}\cdot\frac{\partial x}{\partial \xi} = \frac{1}{K_1^2}\cdot\psi''(x)$$

$$\psi''(x) = K_1^2 \psi''(\xi)$$

fore,

$$C_s'(x) = \kappa\, C_s'(\xi).$$

$$\psi'(x) = \kappa\, \psi'(\xi).$$

$$\psi''(x) = \kappa^2\, \psi''(\xi).$$

Now the range of integration is:

$$x = (0, \delta) \longrightarrow \xi = (0, \kappa\delta)$$

where $\kappa\delta \sim 1$.

thus, we rewrite the equations in terms of these variables.

$$\kappa C_+'(\xi) + \kappa C_+(\xi)\, \psi'(\xi) = r$$

$$\kappa C_-'(\xi) - \kappa C_-(\xi)\, \psi'(\xi) = 0$$

$$\kappa^2 \psi''(\xi) = -\kappa^2 \sinh(\psi(\xi))$$

$$\Rightarrow \boxed{\begin{aligned} &C_+'(\xi) + C_+(\xi)\,\psi'(\xi) = \tilde{r} \\ &C_-'(\xi) - C_-(\xi)\,\psi'(\xi) = 0 \\ &\psi''(\xi) = \cancel{-\sinh(\psi(\xi))}^{-z} \end{aligned}} \qquad \tilde{r} = \frac{r}{\kappa}$$

$$\frac{d}{d\xi}\left(e^{\psi(\xi)} C_+(\xi)\right) = \tilde{r} e^{\psi(\xi)}$$

$$\int_0^{\xi} d\xi' \left(e^{\psi(\xi)} C_+(\xi)\right)' = \int_0^{\xi} d\xi' \tilde{r} e^{\psi(\xi)}$$

$$e^{\psi(\xi)} C_+(\xi) - e^{\psi(0)^{\to 0}} C_+(0)_{\to C_b} = \tilde{r} \int_0^{\xi} d\xi' e^{\psi(\xi')}$$

$$C_+(\xi) = e^{-\psi(\xi)}\left(C_b + \tilde{r} \int_0^{\xi} d\xi' e^{\psi(\xi')}\right)$$

thus we get :

$$C_-(\xi) = C_b e^{\psi(\xi)}$$

$$C_+(\xi) = C_b e^{-\psi(\xi)}\left(1 + \frac{\tilde{r}}{C_b}\int_0^{\xi} d\xi' e^{\psi(\xi')}\right)$$

$$K^2\psi''(\xi) = -\frac{(zF)^2}{\varepsilon RT}\left(C_+(\xi) - C_-(\xi)\right)$$

That is

$$\boxed{\begin{aligned} C_-(\xi) &= C_b e^{\psi(\xi)} \\ C_+(\xi) &= C_b e^{-\psi(\xi)}\left(1 + \frac{\tilde{r}}{C_b}\int_0^{\xi} d\xi' e^{\psi(\xi')}\right) \\ \psi''(\xi) &= -\left(C_+(\xi) - C_-(\xi)\right). \end{aligned}}$$