Analytical solution:

$C_-(x)$ can be integrated directly.

$$\frac{C_-'(\xi)}{C_-(\xi)} = + \psi'(\xi)$$

$$\Rightarrow \quad \log\left(\frac{C_-(\xi)}{C_-(0)}\right) = \left(\psi(\xi) - \cancel{\psi(0)}^{\,0}\right)$$

$$\Rightarrow \quad C_-(\xi) = C_-(0)\, e^{\psi(\xi)}$$

$$C_-(\xi) = C_b\, e^{\psi(\xi)}$$

- for $C_+(x)$ we have

$$C_+'(\xi) + C_+(\xi)\psi'(\xi) = \tilde{r}$$

Let

$$\mu(\xi) = e^{\int_0^{\xi} d\xi' \psi'(\xi')}$$

$$= e^{\psi(\xi) - \cancel{\psi(0)}^{\,0}}$$

Note that

$$\frac{d\mu(\xi)}{d\xi} = \psi'(\xi)\, e^{\psi(\xi)}$$

Thus,

$$e^{\psi(\xi)} C_+(\xi) + C_+(\xi)\psi'(\xi)\, e^{\psi(\xi)} = \tilde{r}\, e^{\psi(\xi)}$$

Solving for the potential

To solve this equation, we propose an expantion in terms of $\tilde{r}$.

$$C_-(\xi) = \sum_{n=0}^{\infty} \tilde{r}^{(n)} C_-^{(n)}(\xi)$$

$$C_+(\xi) = \sum_{n=0}^{\infty} \tilde{r}^{(n)} C_+^{(n)}(\xi).$$

$$\phi(\xi) = \sum_{n=0}^{\infty} \tilde{r}^{(n)} \phi^{(n)}(\xi).$$

we get up to ~~order~~ $n=1$ in the expantion:

$$C_-^{(0)}(\xi) + \tilde{r}\, C_-^{(1)}(\xi) = C_b e^{\psi^{(0)}(\xi) + \tilde{r}\psi^{(1)}(\xi)}$$

$$C_+^{(0)}(\xi) + \tilde{r}\, C_+^{(1)}(\xi) = C_b e^{-\left(\psi^{(0)}(\xi) + \tilde{r}\psi^{(1)}(\xi)\right)} \times \left(1 + \frac{\tilde{r}}{C_b}\int_0^{\xi} d\xi' \, e^{\psi^{(0)}(\xi') + \tilde{r}\psi^{(1)}(\xi')}\right)$$

$$\psi^{(0)}(\xi) + \tilde{r}\psi^{(1)}(\xi) =$$

ie zero order system:

$$\boxed{\begin{aligned} C_-^{(0)}(\xi) &= C_b e^{\psi^{(0)}(\xi)} \\ C_+^{(0)}(\xi) &= C_b e^{-\psi^{(0)}(\xi)} \\ \psi''^{(0)}(\xi) &= -\left[C_+^{(0)}(\xi) - C_-^{(0)}(\xi)\right] \end{aligned}} \qquad (1)$$

First order system:

$$\boxed{\begin{aligned} C_-^{(1)}(\xi) &= 0 \\ C_+^{(1)}(\xi) &= e^{-\psi^{(0)}(\xi)} \int_0^{\xi} d\xi' \, e^{\psi^{(0)}(\xi')} \\ \psi''^{(1)}(\xi) &= -\left(C_+^{(1)}(\xi) - C_-^{(1)}(\xi)\right) \end{aligned}} \qquad (2)$$

(1): already solved. $\psi^{(0)}(\xi) =$

$\psi =$

(2): $$\psi''^{(1)}(\xi) = -e^{-\psi^{(0)}(\xi)} \int_0^{\xi} d\xi' \, e^{\psi^{(0)}(\xi')}$$

$$\psi^{(0)''}(\zeta) = C_b \sinh(\psi^{(0)}(\zeta))$$

$$\int \frac{1}{2}\frac{d}{d\psi^{(0)}}\left[\left(\frac{d\psi^{(0)}}{d\zeta}\right)^2\right] d\zeta = \int d\zeta \, \frac{d^2\psi^{(0)}}{d\zeta}.$$

$$\Rightarrow \cancel{\psi^{(0)}(\zeta) = 2\log\left(1+\tanh(\psi_0/4)\,e^{-K\zeta}\right)}$$

$$\boxed{\psi^{(0)}(\zeta) = 2\log\left(\frac{1+\tanh(\psi_0/4)e^{-\zeta}}{1-\tanh(\psi_0/4)e^{-\zeta}}\right)}$$

$$\psi^{(1)}(\zeta) = -\exp\left\{2\log\left(\frac{1-\tanh(\psi_0/4)e^{-\zeta}}{1+\tanh(\psi_0/4)e^{-\zeta}}\right)\right\}$$

$$\times \int_0^{\zeta} d\zeta' \exp\left\{2\log\left(\frac{1+\alpha e^{-\zeta'}}{1-\alpha e^{-\zeta'}}\right)\right\}$$

$$= \left(\frac{1-\alpha e^{-\zeta}}{1+\alpha e^{-\zeta}}\right)\underbrace{\int_0^{\zeta} d\zeta \left(\frac{1+\alpha e^{-\zeta}}{1-\alpha e^{-\zeta'}}\right)^2}$$

$$\int_0^{\zeta} d\zeta \left(\coth\left(\frac{\zeta-\zeta_0}{2}\right)\right)^2 ; \quad e^{\zeta_0} = \alpha.$$

$$= \int_{-\zeta_0/2}^{(\zeta-\zeta_0)/2} d\eta \, \coth^2(\eta) = \int_{\zeta_0/2}^{(\zeta-\zeta_0)/2} d\eta \left(1+\operatorname{cosech}^2(\eta)\right)$$

$$\psi^{(1)'}(\zeta) = \left(\frac{1-\alpha e^{-\zeta}}{1+\alpha e^{-\zeta}}\right)\left(\frac{\zeta}{2} + \coth(-\zeta_0/2) - \coth\left(\frac{\zeta-\zeta_0}{2}\right)\right)$$

$$\psi^{(1)'}(\zeta) = \left(\frac{1-\alpha e^{-\zeta}}{1+\alpha e^{-\zeta}}\right)\left(\frac{\zeta}{2} - \coth\left(\frac{\zeta_0}{2}\right) - \coth\left(\frac{\zeta-\zeta_0}{2}\right)\right)$$

Tenemos,

$$\boxed{\begin{aligned}\psi^{(0)}(\zeta) &= 2\log\left(\frac{1+\tanh(\psi_0/4)\,e^{-\zeta}}{1-\tanh(\psi_0/4)\,e^{-\zeta}}\right)\\ \psi^{(1)'}(\zeta) &= \left(\frac{1-\alpha e^{-\zeta}}{1+\alpha e^{-\zeta}}\right)\left(\frac{\zeta}{2} - \coth\left(\frac{\zeta_0}{2}\right) - \coth\left(\frac{\zeta-\zeta_0}{2}\right)\right)\end{aligned}}$$

y

$$\boxed{\psi(\zeta) = \psi^{(0)}(\zeta) + \tilde{r}\,\psi^{(1)}(\zeta)}$$

Zero order solution:

$$C_s'(\xi) = -s C_s^{(0)}(\xi)\,\psi^{(0)\prime}(\xi)$$

$$\Rightarrow \quad \frac{C_s'(\xi)}{C_s(\xi)} = -s\,\psi^{(0)\prime}(\xi) \quad \Big| \int_\xi^\delta d\xi'$$

$$-\log\left(C_s(\xi)/C_s(\delta)\right) = -s\int_\xi^\delta d\xi'\,\psi'(\xi')$$

$$\Rightarrow \quad \log\left(C_s(\xi)/C_s(\delta)\right) = s\left(\psi(\delta) - \psi(\xi)\right)$$

$$\Rightarrow \quad C_s(\xi) = C_s(\delta)\,e^{-s(\psi(\xi)-\psi(\delta))}$$

Let $\psi(\delta) \to 0$.

$$\boxed{C_s(\xi) = C_s(\delta)\,e^{-s\psi(\xi)}}$$

$$\boxed{\begin{aligned} \psi^{(0)}(\xi) &= 2\log\left(\tanh^{-1}\left(\frac{\xi-\xi_0}{2}\right)\right) \\ \xi_0 &= \log\left(\tanh(\psi_0/4)\right) \end{aligned}}$$

Solution to zero order

$$C_+(\xi) = C_b e^{-\psi(\xi)}$$

$$C_+(\xi) = C_b e^{+\psi(\xi)}$$

$$\psi^{(0)}(\xi) = -2 \log \left| \tanh\left(\frac{\xi-\xi_0}{2}\right) \right|$$

First order system

$$C_-(\xi) = 0$$

$$C_+(\xi) = -e^{-\psi(\xi)} \left(\xi - 2\coth\left(\frac{\xi-\xi_0}{2}\right) \right)$$

$$\psi^{(1)''}(\xi) = -\left(\frac{1}{C_b}\right) C_+^{(1)}(\xi)$$