Synthesizing Arbitrary Quantum States

in a Superconducting Resonator [1]

Lamia Ateshian Heidi Miller the first

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fulfillment of the requirements

for a certain degree of

understanding of Quoptics

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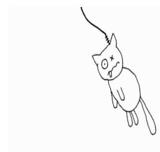

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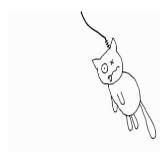

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Abstract

~~We explore how to take advantage of nonlinear optical effects~~ ...~~Superconducting electronic devices can operate at high speeds and low power switching, partly due to cryogenic operations~~, Fock states are attractive for quantum computing due to individual addressability, easy manipulation, and superior scalability [1]. Fock states always have a specified number of excitations. They contrast with coherent states, which are still quantum, but closest to classical theory since they have a [Poisson] probability distribution. Coherent states are easier to generate than Fock states since they require a classical driving force. Hofheinz and friends were able to experimentally generate an arbitrary superposition of Fock states by creating a particular qubit state and then having the qubit interact with the resonator for a controlled time [2]. To evaluate their results, they analyzed the resonator state using Wigner tomography and compared this to the theoretical prediction.

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To Do:

correct the derivation for the pulse

write in a table for the times

put in lamia's figures

write some stuff about the Wigner function.

Chapter 1

Approach the target from behind

Hofheinz generated states by exchanging excitations from the cavity into the qubit one by one. This is done by first placing the qubit in the excited state. The qubit is then immediately in resonance with the cavity until it places one excitation into the qubit.

not sure if this is correct!!!!!!!!!

This process is repeated until the target state has been hit. Hofheinz and friends showed that the time for which they took the qubit out of resonance with the cavity did not affect the coupling or the states in the qubit. In addition, they did a theoretical analysis on the decay of the system by applying the Linbald master equation, but decay proved to be negligible, so the master equation was not necessary to solve for the dynamics of the system. Lastly, the qubit can be assumed to be a two level system since the frequency spacing between the qubit levels is sufficiently different at higher excitations.

Experimentally, this setup involves a qubit with three Josephson Junctions (JJs) coupled to a cavity. A flux bias was applied to the qubit in order to tune the coupling between the cavity and qubit. The three JJ superconducting quantum interference device (SQUID) read out the qubit state. This set-up is shown in Fig. (1.1).

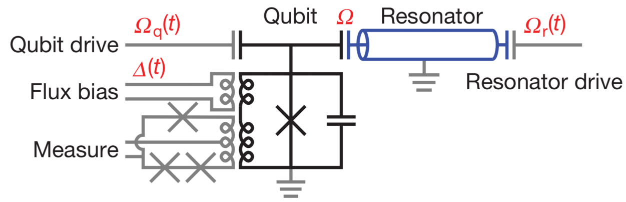

Figure 1.1: Hofheinz's Experimental Setup

The theoretical approach to model the dynamics can be described as follows:

1. Start at the target state and place the cavity and the qubit on resonance with each other so that the detuning is zero. Let them interact until the cavity transfers the photon excitation into the qubit.

2. Rotate the state around the z-axis of the Bloch sphere so that the phase changes allow the coupling between the cavity and qubit to be real.

3. Put the cavity and the qubit off resonance and apply a pi-pulse so that the qubit goes from the excited state to the ground state. Note that during this time, the state is also rotating around the z-axis.

4. Repeat steps 1-3 until all the excitations have been eliminated and the state $|g\rangle\,|0\rangle$ has been reached.

5. Time reverse everything.

Chapter 2

One step at a time

This chapter is divided into eight steps outlining the process by which each excitation is transferred to the qubit and subsequently the qubit is transferred to the ground state.

After the excitation goes out of the qubit, where does it go?!?!?

A sufficiently complex target state, eq. (2.1) is shown as an example in order to demonstrate the essential concepts:

$$|g\rangle \otimes \left(\frac{1}{\sqrt{2}} |1\rangle + \frac{i}{\sqrt{2}} |3\rangle \right) \tag{2.1}$$

We use the following constants:

$$\Delta_{off} = 2\pi \cdot (-463)\,\mathrm{MHz}$$

$$\Omega_q = 1000\,\mathrm{MHz}$$

Step 8: Swap

The Hamiltonian for this is:

$$H = \frac{\Omega}{2}\left(\sigma^+ a + a^\dagger \sigma^-\right) \tag{2.2}$$

Our initial state is:

$$c_{g_1}(0) = \frac{1}{\sqrt{2}} \tag{2.3}$$

$$c_{e_2}(0) = 0 \tag{2.4}$$

$$c_{g_3}(0) = \frac{i}{\sqrt{2}} \tag{2.5}$$

$$c_{g_n}(t) = c_{g_n}(0)\cos\left(\frac{\Omega\sqrt{n}}{2}t\right) + ic_{e_{n-1}}(0)\sin\left(\frac{\Omega\sqrt{n}}{2}t\right) \tag{2.6}$$

$$c_{e_{n-1}}(t) = c_{e_{n-1}}(0)\cos\left(\frac{\Omega\sqrt{n}}{2}t\right) + ic_{g_n}(0)\sin\left(\frac{\Omega\sqrt{n}}{2}t\right) \tag{2.7}$$

Eq. (2.6) and (2.7) are derived in the Appendix. Using eqs. (2.6), (2.7), (2.3), (2.4), and (2.5), we obtain the following expressions for c_{e2} and c_{g3} as functions of time:

$$c_{g_3}(t) = \frac{i}{\sqrt{2}}\cos\left(\frac{\Omega\sqrt{3}}{2}t\right) \tag{2.8}$$

$$c_{e_2}(t) = -\frac{1}{\sqrt{2}}\sin\left(\frac{\Omega\sqrt{3}}{2}t\right) \tag{2.9}$$

Since we want the excitation from the cavity to go into the qubit, we set $c_{g_3}(t) = 0$ and obtain:

$$\frac{\Omega\sqrt{3}}{2}t_8 = \frac{\pi}{2} \tag{2.10}$$

$$\rightarrow t_8 = \frac{\pi}{\Omega\sqrt{3}} \tag{2.11}$$

$$c_{e_2}(t_8) = -\frac{1}{\sqrt{2}}\sin\left(\frac{\Omega\sqrt{3}}{2}\frac{\pi}{\Omega\sqrt{3}}\right) = -\frac{1}{\sqrt{2}} \tag{2.12}$$

t_1 is the interaction time for the cavity to exchange excitations with the qubit from $n = 3$ total excitations to $n = 2$ total excitations.

For $n = 2$ excitations,

$$c_{g_2}(0) = 0 \tag{2.13}$$

$$c_{e_0}(0) = 0 \tag{2.14}$$

Therefore, there is no change in the second excitation. In other words, for two excitations, there is no interaction between the qubit and the cavity.

For $n = 1$ excitations,

$$c_{g_1}(0) = \frac{1}{\sqrt{2}} \tag{2.15}$$

$$c_{e_0}(0) = 0 \tag{2.16}$$

Using eq. (2.6) and eq. (2.7),

$$c_{g_1}(t) = \frac{1}{\sqrt{2}}\cos\left(\frac{\Omega}{2}t\right) \tag{2.17}$$

$$c_{e_0}(t) = \frac{i}{\sqrt{2}}\sin\left(\frac{\Omega}{2}t\right) \tag{2.18}$$

For time $t = t_8$,

$$c_{g_1}(t_8) = \frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{2\sqrt{3}}\right) \tag{2.19}$$

$$c_{e_0}(t_8) = \frac{i}{\sqrt{2}} \sin\left(\frac{\pi}{2\sqrt{3}}\right) \tag{2.20}$$

For $n = 0$ excitations, there cannot be any change in the state of the system because there are no excitations to be shifted around.

Therefore, after time $t = t_8$, we obtain the following state:

$$|\psi(t_8)\rangle = \frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{2\sqrt{3}}\right) |g,1\rangle + \frac{i}{\sqrt{2}} \sin\left(\frac{\pi}{2\sqrt{3}}\right) |e,0\rangle - \frac{1}{\sqrt{2}} |e,2\rangle \tag{2.21}$$

$$= 0.43571 |g,1\rangle + i0.556 |e,0\rangle - 0.7071 |e,2\rangle \tag{2.22}$$

Step 7: Pulse

The derivation for the ordering of this pulse and subsequent rotation (in Step 6) is shown in the appendix.

First, rotation.

$$e^{i\Delta_{off}\sigma^+\sigma^- t} |g\rangle = 1 |g\rangle$$

$$e^{i\Delta_{off}\sigma^+\sigma^- t} |e\rangle = e^{i\Delta_{off}t} |e\rangle$$

The state becomes:

$$|\psi\rangle = 0.43571 |g,1\rangle + i0.556 e^{i\Delta_{off}t_7} |e,0\rangle - 0.7071 e^{i\Delta_{off}t_7} |e,2\rangle$$

Pulse:

$$|e,1\rangle \rightarrow i\,|g,1\rangle$$

$$\Omega_q t_7 = \pi$$

State:

$$|\psi\rangle = i0.43571\,|e,1\rangle - 0.556e^{i\Delta_{off}t_7}\,|g,0\rangle - i0.7071e^{i\Delta_{off}t_7}\,|g,2\rangle$$

Step 6: Rotation

$$|\psi\rangle = i0.43571e^{i\Delta_{off}t_6}\,|e,1\rangle - 0.556e^{i\Delta_{off}t_7}\,|g,0\rangle - i0.7071e^{i\Delta_{off}t_7}\,|g,2\rangle$$

Step 5: Swap

$$c_{g_2}(t) = -i0.7071\cos\left(\frac{\Omega\sqrt{n}}{2}t_5\right)e^{i\Delta_{off}t_7} - 0.43571\sin\left(\frac{\Omega\sqrt{n}}{2}t_5\right)e^{i\Delta_{off}t_6}$$

set $c_{g_2}(t_5) = 0$

$$0.7071\cos\left(\frac{\Omega\sqrt{n}}{2}t_5\right)e^{i\Delta_{off}t_7} = i0.43571\sin\left(\frac{\Omega\sqrt{n}}{2}t_5\right)e^{i\Delta_{off}t_6}$$

$$-\frac{i0.7071}{0.43571}e^{i\Delta_{off}(t_7-t_6)} = \tan\left(\frac{\Omega\sqrt{n}}{2}t_5\right)$$

Set Ω to be real and positive:

$$-ie^{i\Delta_{off}(t_7-t_6)} = 1$$

$$e^{i\Delta_{off}(t_7-t_6)} = i = e^{i\frac{\pi}{2}}$$

$$\Delta_{off}(t_7 - t_6) = \frac{\pi}{2}$$

$$t_6 = t_7 - \frac{\pi}{2\Delta_{off}}$$

Solving for t_5:

$$\tan\left(\frac{\Omega\sqrt{n}}{2}t_5\right) = \frac{0.7071}{0.43571}$$

$$\Omega t_5 = 1.44$$

More n=2 states. Note that c_{g_0} stays constant because there is no state for it to exchange excitations in.

$$c_{e_1}(t_5) = i0.43571\cos\left(\frac{\Omega\sqrt{2}}{2}t_5\right)e^{i\Delta_{off}t_6} + 0.7071\sin\left(\frac{\Omega\sqrt{n}}{2}t_5\right)e^{i\Delta_{off}t_7}$$

The final state is:

$$|\psi\rangle = (-0.7969429 - 0.2339386i)\,|e,1\rangle + (0.534368071079 + 0.156861063497i)\,|g,0\rangle$$

Step 4: Pulse

First, rotation.

$$e^{i\Delta_{off}\sigma^+\sigma^- t} |g\rangle = 1 |g\rangle$$
$$e^{i\Delta_{off}\sigma^+\sigma^- t} |e\rangle = e^{i\Delta_{off} t} |e\rangle$$

The state becomes:

$$|\psi\rangle = (-0.7969429 - 0.2339386i) e^{i\Delta_{off} t_4} |e, 1\rangle - (0.534368071079 + 0.156861063497i) |g, 0\rangle$$

Pulse:

$$|e, 2\rangle \to i |g, 2\rangle$$

$$\Omega_q t_4 = \pi$$

State:

$$|\psi\rangle = i (-0.7969429 - 0.2339386i) e^{i\Delta_{off} t_4} |g, 1\rangle - i (0.534368071079 + 0.156861063497i) |e, 0\rangle$$

Step 3: Rotation

$$|\psi\rangle = i (-0.7969429 - 0.2339386i) e^{i\Delta_{off} t_4} |g, 1\rangle - i (0.534368071079 + 0.156861063497i) e^{i\Delta_{off} t_3} |e, 0\rangle$$

Step 2: Swap

$$c_{g_1}(t_2) = i\left(-0.7969429 - 0.2339386i\right) e^{i\Delta_{off}t_4} \cos\left(\frac{\Omega\sqrt{n}}{2}t_2\right) - \left(0.534368071079 + 0.156861063497i\right) e^{i\Delta_{off}t_3} \sin\left(\frac{\Omega\sqrt{n}}{2}t_2\right)$$

$$c_{e_0}(t_2) = -i\left(0.534368071079 + 0.156861063497i\right) e^{i\Delta_{off}t_3} \cos\left(\frac{\Omega\sqrt{n}}{2}t_2\right) - \left(-0.7969429 - 0.2339386i\right) e^{i\Delta_{off}t_4} \sin\left(\frac{\Omega\sqrt{n}}{2}t_2\right)$$

set $c_{g1}(t_2) = 0$

$$i\frac{\left(-0.7969429 - 0.2339386i\right)}{\left(0.534368071079 + 0.156861063497i\right)} e^{i\Delta_{off}t_4} \cos\left(\frac{\Omega\sqrt{n}}{2}t_2\right) = e^{i\Delta_{off}t_3} \sin\left(\frac{\Omega\sqrt{n}}{2}t_2\right)$$

$$\left(-1.49137460586i\right) e^{i\Delta_{off}(t_4-t_3)} = \tan\left(\frac{\Omega\sqrt{n}}{2}t_2\right)$$

Set Ω to be real and positive:

$$-ie^{i\Delta_{off}(t_4-t_3)} = 1$$

$$\Delta_{off}\left(t_4 - t_3\right) = \frac{\pi}{2}$$

$$t_3 = t_4 - \frac{\pi}{2\Delta_{off}}$$

$$t_3 = 0.00368$$

Solving for Ωt_2:

$$\tan\left(\frac{\Omega\sqrt{n}}{2}t_2\right) = 1.49137460586$$

$$\Omega t_2 = 1.9602$$

The final state is:

$$|\psi\rangle = (-0.31944698577 - 0.20522836891i)\,|e,0\rangle$$

Step 1: Pulse

First, rotation.

$$e^{i\Delta_{off}\sigma^+\sigma^- t}\,|g\rangle = 1\,|g\rangle$$
$$e^{i\Delta_{off}\sigma^+\sigma^- t}\,|e\rangle = e^{i\Delta_{off}t}\,|e\rangle$$

The state becomes:

$$|\psi\rangle = (-0.31944698577 - 0.20522836891i)\,e^{i\Delta_{off}t_1}\,|e,0\rangle$$

Pulse:

$$|e,0\rangle \rightarrow i\,|g,0\rangle$$

$$\Omega_q t_1 = \pi$$

$$t_1 = 0.00314$$

State:

$$|\psi\rangle = i\,(-0.31944698577 - 0.20522836891i)\,e^{i\Delta_{off} t_1}\,|g,0\rangle$$

$$|\psi\rangle = (-0.286895130017 + 0.248709156436j)\,|g,0\rangle$$

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Appendix

Qubit-Cavity dynamics

The qubit-cavity system dynamics for the two-level qubit can be derived as follows:

Hamiltonian:

$$\hat{H} = \hbar\omega_0 \hat{N} + \frac{\Omega}{2}\hbar\left(\sigma^+ a + a^\dagger \sigma^-\right) \tag{2.23}$$

Schrödinger Equation:

$$i\hbar\frac{d}{dt}\left|\tilde{\psi}(t)\right\rangle = \hat{H}\left|\tilde{\psi}(t)\right\rangle \tag{2.24}$$

Assume the following solution for the two-level system:

$$|\psi(t)\rangle = c_{e,n-1}(t)\,|e, n-1\rangle + c_{g,n}(t)\,|g, n\rangle \tag{2.25}$$

Sub in these coefficients

$$
i\dot{c}_{e,n-1}(t)\left|e,n-1\right\rangle + i\dot{c}_{g,n}(t)\left|g,n\right\rangle = \begin{cases} & \omega_0 n c_{e,n-1}(t)\left|e,n-1\right\rangle + \omega_0 n c_{g,n}(t)\left|g,n\right\rangle \\ + & \frac{\Omega}{2} c_{e,n-1}(t) n \left|g,n\right\rangle + c_{g,n}(t)\sqrt{n}\left|e,n-1\right\rangle \end{cases} \tag{2.26}
$$

Separate equations for like terms:

$$
\left|e,n-1\right\rangle \qquad \dot{c}_{e,n-1}(t) = -i\omega_0 n c_{e,n-1}(t) - ig\sqrt{n}c_{g,n}(t) \tag{2.27}
$$

$$
\left|g,n\right\rangle \qquad \dot{c}_{g,n}(t) = -i\omega_0 n c_{g,n}(t) - ig\sqrt{n}c_{e,n-1}(t) \tag{2.28}
$$

Enter the rotating frame:

$$
\tilde{c} = c e^{i\omega_0 n t} \tag{2.29}
$$

$$
\dot{\tilde{c}}_{e,n-1}e^{-i\omega_0 n} - i\omega_0 n\tilde{c}_{e,n-1}e^{-i\omega_0 n} = -i\omega_0 n\tilde{c}_{e,n-1}(t)e^{-i\omega_0 n} - i\frac{\Omega}{2}\sqrt{n}\tilde{c}_{g,n}(t)e^{-i\omega_0 n} \tag{2.30}
$$

We get these equations:

$$
\dot{\tilde{c}}_{e,n-1} = -i\frac{\Omega}{2}\sqrt{n}\tilde{c}_{g,n}(t) \tag{2.31}
$$

$$
\dot{\tilde{c}}_{g,n} = -i\frac{\Omega}{2}\sqrt{n}\tilde{c}_{e,n-1}(t) \tag{2.32}
$$

From eq. (2.32):

$$
\dot{\tilde{c}}_{e,n-1} = \frac{i}{\frac{\Omega}{2}\sqrt{n}}\ddot{\tilde{c}}_{g,n} \tag{2.33}
$$

Sub into eq. (2.31)

$$\frac{i}{\frac{\Omega}{2}\sqrt{n}}\ddot{\tilde{c}}_{g,n}(t) = -i\frac{\Omega}{2}\sqrt{n}\tilde{c}_{g,n}(t) \tag{2.34}$$

Simplify

$$\ddot{\tilde{c}}_{g,n}(t) + \left(\frac{\Omega}{2}\right)^2 n\tilde{c}_{g,n}(t) = 0 \tag{2.35}$$

$$\tilde{c}_{g,n} = c_{g_n}(0)\cos\left(\frac{\Omega}{2}\sqrt{n}t\right) + ic_{e_{n-1}}(0)\sin\left(\frac{\Omega}{2}\sqrt{n}t\right) \tag{2.36}$$

$$c_{g_n}(t) = c_{g_n}(0)\cos\left(\frac{\Omega\sqrt{n}}{2}t\right) + ic_{e_{n-1}}(0)\sin\left(\frac{\Omega\sqrt{n}}{2}t\right) \tag{2.37}$$

$$c_{e_{n-1}}(t) = c_{e_{n-1}}(0)\cos\left(\frac{\Omega\sqrt{n}}{2}t\right) + ic_{g_n}(0)\sin\left(\frac{\Omega\sqrt{n}}{2}t\right) \tag{2.38}$$

Off-resonance pulse and rotation

$$\begin{aligned} H_{off} &= \Delta_{off}\sigma^+\sigma^- + \left(\frac{\Omega_q}{2}\sigma^+ e^{-i\Delta_{off}t} + \frac{\Omega_q^*}{2}\sigma^- e^{i\Delta_{off}t}\right) \\ i\frac{\partial\left|\psi(t)\right\rangle}{\partial t} &= H_{off}(t)\left|\psi\right\rangle \end{aligned}$$

Reframing the picture:

$$\begin{aligned} \left|\tilde{\psi}(t)\right\rangle &= e^{i\Delta_{off}\sigma^+\sigma^- t}\left|\psi(t)\right\rangle \\ \left|\psi(t)\right\rangle &= e^{-i\Delta_{off}\sigma^+\sigma^-}\left|\tilde{\psi}(t)\right\rangle \end{aligned}$$

$$
\begin{aligned}
\Delta_{off}\sigma^+\sigma^- e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|\tilde{\psi}(t)\right\rangle + ie^{-i\Delta_{off}\sigma^+\sigma^- t}\frac{\partial}{\partial t}\left|\tilde{\psi}(t)\right\rangle \\
= H_{off}e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|\tilde{\psi}(t)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
i\frac{\partial}{\partial t}\left|\tilde{\psi}(t)\right\rangle &= e^{i\Delta_{off}\sigma^+\sigma^- t}\left(H_{off} - \Delta_{off}\sigma^+\sigma^-\right)e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|\tilde{\psi}(t)\right\rangle \\
&= e^{i\Delta_{off}\sigma^+\sigma^- t}\left(\frac{\Omega_q}{2}\sigma^+ e^{-i\Delta_{off}t} + \frac{\Omega_q^*}{2}\sigma^- e^{i\Delta_{off}t}\right)e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|\tilde{\psi}(t)\right\rangle \\
&= \left[\frac{\Omega_q}{2}e^{i\Delta_{off}\sigma^+\sigma^- t}\sigma^+ e^{-i\Delta_{off}\sigma^+\sigma^- t}\cdot e^{-i\Delta_{off}t}\right. \\
&\quad \left. + \frac{\Omega_q^*}{2}e^{i\Delta_{off}\sigma^+\sigma^- t}\sigma^- e^{-i\Delta_{off}\sigma^+\sigma^- t}\cdot e^{i\Delta_{off}t}\right]\left|\tilde{\psi}(t)\right\rangle
\end{aligned} \tag{2.39}
$$

The middle of eq. (2.39) can be simplified.

Acting on the excited state:

$$
e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|e\right\rangle = e^{-i\Delta_{off}t}\left|e\right\rangle
$$

$$
\sigma^+\left|e\right\rangle = 0
$$

Acting on the ground state:

$$
e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|g\right\rangle = \left|g\right\rangle
$$

$$
\sigma^+\left|g\right\rangle = \left|e\right\rangle
$$

$$
e^{i\Delta_{off}\sigma^+\sigma^- t}\left|e\right\rangle = e^{i\Delta_{off}t}\left|e\right\rangle
$$

Which is the same for $\sigma^+ e^{i\Delta_{off}t}$. A similar argument can be made for using $\sigma^- e^{-i\Delta_{off}t}$:

Acting on the ground state:

$$e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|g\right\rangle = \left|g\right\rangle$$
$$\sigma^-\left|g\right\rangle = 0$$

Acting on the excited state:

$$e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|e\right\rangle = \left|e\right\rangle$$
$$\sigma^-\left|e\right\rangle = \left|g\right\rangle$$
$$e^{i\Delta_{off}\sigma^+\sigma^- t}\left|g\right\rangle = e^{i\Delta_{off}t}\left|g\right\rangle$$

Therefore, eq. (2.39) can be written as:

$$i\frac{\partial}{\partial t}\left|\tilde{\psi}(t)\right\rangle = \left(\frac{\Omega_q}{2}\sigma^+ + \frac{\Omega_q^*}{2}\sigma^-\right)\left|\tilde{\psi}(t)\right\rangle$$

Solving this expression:

$$\left|\tilde{\psi}(t)\right\rangle = \exp\left\{-i\left(\frac{\Omega_q}{2}\sigma^+ + \frac{\Omega_q^*}{2}\sigma^-\right)t\right\}\left|\tilde{\psi}(0)\right\rangle$$

Taking the picture out of the rotating frame:

$$\left|\psi(t)\right\rangle = e^{-i\Delta_{off}\sigma^+\sigma^- t}\exp\left\{-i\left(\frac{\Omega_q}{2}\sigma^+ + \frac{\Omega_q^*}{2}\sigma^-\right)t\right\}\left|\psi(0)\right\rangle$$

Pulse Derivation

Here, we show an example derivation for how the π-pulse is in fact a rotation over π radians.

Our state before the pulse is:

$$
\begin{aligned}
|\psi(t_2)\rangle_1 = &\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{2\sqrt{3}}\right)\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) + i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|g,1\rangle \\
&+ \frac{i}{\sqrt{2}}\sin\left(\frac{\pi}{2\sqrt{3}}\right)\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) - i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|e,0\rangle \\
&+ \frac{1}{\sqrt{2}}\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) - i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|e,2\rangle
\end{aligned} \tag{2.40}
$$

The Hamiltonian evolves in time as such:

$$e^{iHt} = e^{i\left(\frac{\Omega_q}{2}\sigma^+ + \frac{\Omega_q^*}{2}\sigma^-\right)t} \tag{2.41}$$

$$= e^{\frac{i}{2}\left((\mathrm{Re}\Omega_q + i\mathrm{Im}\Omega_q)\sigma^+ + (\mathrm{Re}\Omega_q - i\mathrm{Im}\Omega_q)\sigma^-\right)} \tag{2.42}$$

$$= e^{\frac{i}{2}\left(\mathrm{Re}\Omega_q\left(\sigma^+ + \sigma^-\right) + i\mathrm{Im}\Omega_q\left(\sigma^+ - \sigma^-\right)\right)} \tag{2.43}$$

$$= e^{\frac{i}{2}(\mathrm{Re}\Omega_q\sigma^x - \mathrm{Im}\Omega_q\sigma^y)} \tag{2.44}$$

$$= e^{i\frac{|\Omega_q|}{2}\left[\frac{\mathrm{Re}\Omega_q}{|\Omega_q|}\sigma^x - \frac{\mathrm{Im}\Omega_q}{|\Omega_q|}\sigma^y\right]t} \tag{2.45}$$

$$= e^{i\frac{|\Omega_q|}{2}\hat{n}\cdot\vec{\sigma}t} \tag{2.46}$$

where:

$$\hat{n} = \frac{\mathrm{Re}\Omega_q}{|\Omega_q|}\hat{x} - \frac{\mathrm{Im}\Omega_q}{|\Omega_q|}\hat{y} \tag{2.47}$$

Expanding eq. (2.46):

$$\mathbb{I}\cos\left(\frac{|\Omega_q|}{2}t\right) + i\left[\frac{\mathrm{Re}\Omega_q}{|\Omega_q|}\sigma^x - \frac{\mathrm{Im}\Omega_q}{|\Omega_q|}\sigma^y\right]\sin\left(\frac{|\Omega_q|}{2}t\right) \tag{2.48}$$

Applying eq. (2.48) to the state (2.40), term by term:

$$\begin{aligned}
|\psi(t_2)\rangle = &\cos\left(\frac{|\Omega_q|}{2}t_2\right)\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{2\sqrt{3}}\right)\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) + i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|g,1\rangle \\
&+ i\frac{\mathrm{Re}\Omega_q}{|\Omega_q|}\sin\left(\frac{|\Omega_q|}{2}t_2\right)\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{2\sqrt{3}}\right)\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) + i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|e,1\rangle \\
&+ \frac{\mathrm{Im}\Omega_q}{|\Omega_q|}\sin\left(\frac{|\Omega_q|}{2}t_2\right)\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{2\sqrt{3}}\right)\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) + i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|e,1\rangle \\
&\\
&+ \cos\left(\frac{|\Omega_q|}{2}t_2\right)\frac{i}{\sqrt{2}}\sin\left(\frac{\pi}{2\sqrt{3}}\right)\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) - i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|e,0\rangle \\
&+ i\frac{\mathrm{Re}\Omega_q}{|\Omega_q|}\sin\left(\frac{|\Omega_q|}{2}t_2\right)\frac{i}{\sqrt{2}}\sin\left(\frac{\pi}{2\sqrt{3}}\right)\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) - i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|g,0\rangle \\
&- \frac{\mathrm{Im}\Omega_q}{|\Omega_q|}\sin\left(\frac{|\Omega_q|}{2}t_2\right)\frac{i}{\sqrt{2}}\sin\left(\frac{\pi}{2\sqrt{3}}\right)\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) - i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|g,0\rangle \\
&\\
&+ \cos\left(\frac{|\Omega_q|}{2}t_2\right)\frac{1}{\sqrt{2}}\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) - i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|e,2\rangle \\
&+ i\frac{\mathrm{Re}\Omega_q}{|\Omega_q|}\sin\left(\frac{|\Omega_q|}{2}t_2\right)\frac{1}{\sqrt{2}}\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) - i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|g,2\rangle \\
&- \frac{\mathrm{Im}\Omega_q}{|\Omega_q|}\sin\left(\frac{|\Omega_q|}{2}t_2\right)\frac{1}{\sqrt{2}}\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right) - i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|g,2\rangle
\end{aligned} \tag{2.49}$$

The time t_2 must be set so that the $|e,2\rangle$ term is zero.

$$\frac{|\Omega_q|}{2}t_2 = \frac{\pi}{2} \tag{2.50}$$

$$\rightarrow t_2 = \frac{\pi}{|\Omega_q|} \tag{2.51}$$

$$+ i\frac{\mathrm{Re}\Omega_q}{|\Omega_q|}\sin\left(\frac{|\Omega_q|}{2}t_2\right)\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{2\sqrt{3}}\right)\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right)+i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|e,1\rangle$$

$$-\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{2\sqrt{3}}\right)\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right)-i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|g,0\rangle \tag{2.52}$$

$$+ i\frac{\mathrm{Re}\Omega_q}{|\Omega_q|}\sin\left(\frac{|\Omega_q|}{2}t_2\right)\frac{1}{\sqrt{2}}\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right)-i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|g,2\rangle$$

Since we can control the drive on the qubit, we can control the phase as well, so let's set that equal to zero. That means that this is true: $\frac{\mathrm{Im}\Omega_q}{|\Omega_q|} = 0$ and $\frac{\mathrm{Re}\Omega_q}{|\Omega_q|} = 1$. Therefore, eq. (2.49) becomes:

$$\begin{aligned}|\psi(t_2)\rangle = &\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{2\sqrt{3}}\right)\left[-i\cos\left(\frac{\Delta_{off}}{2}t_2\right)+\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|e,1\rangle\\ &+\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{2\sqrt{3}}\right)\left[\cos\left(\frac{\Delta_{off}}{2}t_2\right)-i\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|g,0\rangle\\ &-\frac{1}{\sqrt{2}}\left[i\cos\left(\frac{\Delta_{off}}{2}t_2\right)+\sin\left(\frac{\Delta_{off}}{2}t_2\right)\right]|g,2\rangle\end{aligned} \tag{2.53}$$

$$\begin{aligned}&= (0.43128 - 0.06199i)\,|e,1\rangle\\ &+ (-0.07924 + 0.55125i)\,|g,0\rangle\\ &+ (-0.69991 + 0.10061i)\,|g,2\rangle\end{aligned} \tag{2.54}$$

Spin Matrices

$$\sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.55}$$

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.56}$$

$$\sigma^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{2.57}$$

$$\sigma^+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{2.58}$$

$$\sigma^- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{2.59}$$

$$\sigma^x = \sigma^+ + \sigma^- \tag{2.60}$$

$$\sigma^y = i\left(\sigma^+ - \sigma^-\right) \tag{2.61}$$

$$\sigma^+\sigma^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{2.62}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.63}$$

$$= \frac{1}{2}\left(-\sigma^z + 1\right) \tag{2.64}$$

$\sigma^x \left|g\right\rangle$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |e\rangle \tag{2.65}$$

$\sigma^y |g\rangle$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i\,|e\rangle \tag{2.66}$$

$\sigma^z |g\rangle$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\,|g\rangle \tag{2.67}$$

$\sigma^+ |g\rangle$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \tag{2.68}$$

$\sigma^- |g\rangle$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |e\rangle \tag{2.69}$$

$\sigma^x |e\rangle$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |g\rangle \tag{2.70}$$

$\sigma^y |e\rangle$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = i \begin{bmatrix} 0 \\ 1 \end{bmatrix} = i |g\rangle \tag{2.71}$$

$\sigma^z |e\rangle$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |e\rangle \tag{2.72}$$

$\sigma^+ |e\rangle$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |g\rangle \tag{2.73}$$

$\sigma^- |e\rangle$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \tag{2.74}$$