Synthesizing Arbitrary Quantum States in a Superconducting Resonator [1]

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fulfillment of the requirements

for a certain degree of

understanding of Quoptics

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To Law and Eberly for being utterly useless

Acknowledgments:

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Abstract

~~We explore how to take advantage of nonlinear optical effects~~ ...~~Superconducting electronic devices can operate at high speeds and low power switching, partly due to cryogenic operations~~, Fock states are attractive for quantum computing due to individual addressability, easy manipulation, and superior scalability [1]. Fock states always have a specified number of excitations. They contrast with coherent states, which are still quantum, but closest to classical theory since they have a [Poisson] probability distribution. Coherent states are easier to generate than Fock states since they require a classical driving force. Hofheinz and friends were able to experimentally generate an arbitrary superposition of Fock states by creating a particular qubit state and then having the qubit interact with the resonator for a controlled time [2]. To evaluate their results, they analyzed the resonator state using Wigner tomography and compared this to the theoretical prediction. We attempted to reproduce one of their states with theory. We tried to simulate this, but our simulations were off by a phase, so the end result did not match the predicted outcome. We also reproduced their Fock states using the Wigner function on Python's Qutip package.

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Chapter 1

Approach the target from behind

Hofheinz generated states by exchanging excitations from the cavity into the qubit one by one. This is done by first placing the qubit in the excited state. The qubit is then immediately in resonance with the cavity until it places one excitation into the cavity.

This process is repeated until the target state has been hit. Hofheinz and his gang of wily experimentalists showed that the time for which they took the qubit out of resonance with the cavity did not affect the coupling or the states in the qubit. In addition, they did a theoretical analysis on the decay of the system by applying the Linbald master equation, but decay proved to be negligible, so the master equation was not necessary to solve for the dynamics of the system. Lastly, the qubit can be assumed to be a two level system since the frequency spacing between the qubit levels is sufficiently different at higher excitations.

Experimentally, this setup involves a qubit with three Josephson Junctions (JJs) coupled to a cavity. A flux bias was applied to the qubit in order to tune the coupling between the cavity and qubit. The three JJ superconducting quantum interference device (SQUID) read out the qubit state. This set-up is shown in Fig. (1.1).

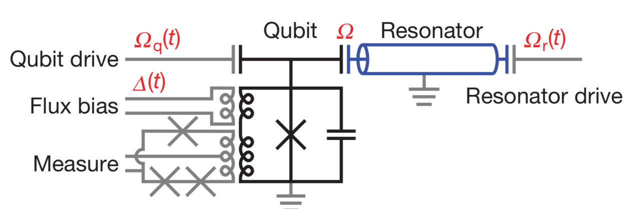

Figure 1.1: Hofheinz's Experimental Setup

The following is the Hamiltonian for this system, and Table. (1.1) shows the meaning of each term.

$$H = \Delta(t)\sigma^+\sigma^- + \left(\frac{\Omega}{2}\sigma^+ a + \frac{\Omega_q(t)}{2}\sigma^+ + \frac{\Omega_r(t)}{2}a^\dagger\right) + \text{h.c.} \tag{1.1}$$

term	meaning
$\Delta(t)$	$= \omega_q(t) - \omega_r$ (detuning)
$\sigma^+\sigma^-$	$\sigma^+\sigma^- = \frac{1}{2}\left(-\sigma^z + 1\right)$: rotation around z axis of Bloch sphere
$\frac{\Omega}{2}\left(\sigma^+ a + \sigma^- a^\dagger\right)$	JC interaction between cavity and qubit
Ω	Coupling strength between resonator and qubit
$\Omega_q(t)$	Drive on the qubit (complex)
$\Omega_r(t)$	Drive on the cavity. Experimentally tunable parameter (complex)

Table 1.1: Terms in the Hamiltonian

The theoretical approach to model the dynamics can be described as follows. This is shown schematically in Fig. (1.2).

1. Start at the target state and place the cavity and the qubit on resonance with each other so that the detuning is zero. Let them interact until the cavity transfers the photon excitation into the qubit.

2. Put the cavity and the qubit off resonance and apply a pi-pulse so that the qubit goes from the excited state to the ground state. Note that during this time, the state is also rotating around the z-axis.

3. Rotate the state around the z-axis of the Bloch sphere so that the phase changes allow the coupling between the cavity and qubit to be real.

4. Repeat steps 1-3 until all the excitations have been eliminated and the state $|g\rangle\,|0\rangle$ has been reached.

5. Time reverse everything.

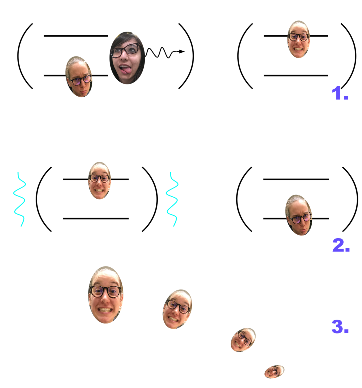

Figure 1.2: Schematic of the excitation transfer

Chapter 2

One step at a time

This chapter is divided into eight steps outlining the process by which each excitation is transferred to the qubit and subsequently the qubit is transferred to the ground state.

A sufficiently complex target state, eq. (2.1) is shown as an example in order to demonstrate the essential concepts:

$$|g\rangle \otimes \left(\frac{1}{\sqrt{2}} |1\rangle + \frac{i}{\sqrt{2}} |3\rangle \right) \tag{2.1}$$

We use the following constants:

$$\Delta_{off} = 2\pi \cdot (-463)\,\mathrm{MHz}$$

$$\Omega_q = 1000\,\mathrm{MHz}$$

$$\Omega = 2\pi \cdot 19\,\mathrm{MHz}$$

Step 8: Swap

The Hamiltonian for this is:

$$H = \frac{\Omega}{2} \left(\sigma^+ a + a^\dagger \sigma^- \right) \tag{2.2}$$

Our initial state is:

$$c_{g_1}(0) = \frac{1}{\sqrt{2}} \tag{2.3}$$

$$c_{e_2}(0) = 0 \tag{2.4}$$

$$c_{g_3}(0) = \frac{i}{\sqrt{2}} \tag{2.5}$$

$$c_{g_n}(t) = c_{g_n}(0)\cos\left(\frac{\Omega\sqrt{n}}{2}t\right) + ic_{e_{n-1}}(0)\sin\left(\frac{\Omega\sqrt{n}}{2}t\right) \tag{2.6}$$

$$c_{e_{n-1}}(t) = c_{e_{n-1}}(0)\cos\left(\frac{\Omega\sqrt{n}}{2}t\right) + ic_{g_n}(0)\sin\left(\frac{\Omega\sqrt{n}}{2}t\right) \tag{2.7}$$

Eq. (2.6) and (2.7) are derived in the Appendix. Using eqs. (2.6), (2.7), (2.3), (2.4), and (2.5), we obtain the following expressions for c_{e2} and c_{g3} as functions of time:

$$c_{g_3}(t) = \frac{i}{\sqrt{2}}\cos\left(\frac{\Omega\sqrt{3}}{2}t\right) \tag{2.8}$$

$$c_{e_2}(t) = -\frac{1}{\sqrt{2}}\sin\left(\frac{\Omega\sqrt{3}}{2}t\right) \tag{2.9}$$

Since we want the excitation from the cavity to go into the qubit, we set $c_{g_3}(t) = 0$ and obtain:

$$\frac{\Omega\sqrt{3}}{2}t_8 = \frac{\pi}{2} \tag{2.10}$$

$$\rightarrow t_8 = \frac{\pi}{\Omega\sqrt{3}} \tag{2.11}$$

$$c_{e_2}(t_8) = -\frac{1}{\sqrt{2}}\sin\left(\frac{\Omega\sqrt{3}}{2}\frac{\pi}{\Omega\sqrt{3}}\right) = -\frac{1}{\sqrt{2}} \tag{2.12}$$

t_1 is the interaction time for the cavity to exchange excitations with the qubit from $n = 3$ total excitations to $n = 2$ total excitations.

For $n = 2$ excitations,

$$c_{g_2}(0) = 0 \tag{2.13}$$

$$c_{e_0}(0) = 0 \tag{2.14}$$

Therefore, there is no change in the second excitation. In other words, for two excitations, there is no interaction between the qubit and the cavity.

For $n = 1$ excitations,

$$c_{g_1}(0) = \frac{1}{\sqrt{2}} \tag{2.15}$$

$$c_{e_0}(0) = 0 \tag{2.16}$$

Using eq. (2.6) and eq. (2.7),

$$c_{g_1}(t) = \frac{1}{\sqrt{2}} \cos\left(\frac{\Omega}{2} t\right) \tag{2.17}$$
$$c_{e_0}(t) = \frac{i}{\sqrt{2}} \sin\left(\frac{\Omega}{2} t\right) \tag{2.18}$$

For time $t = t_8$,

$$c_{g_1}(t_8) = \frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{2\sqrt{3}}\right) \tag{2.19}$$
$$c_{e_0}(t_8) = \frac{i}{\sqrt{2}} \sin\left(\frac{\pi}{2\sqrt{3}}\right) \tag{2.20}$$

For $n = 0$ excitations, there cannot be any change in the state of the system because there are no excitations to be shifted around.

Therefore, after time $t = t_8$, we obtain the following state:

$$|\psi(t_8)\rangle = \frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{2\sqrt{3}}\right) |g, 1\rangle + \frac{i}{\sqrt{2}} \sin\left(\frac{\pi}{2\sqrt{3}}\right) |e, 0\rangle - \frac{1}{\sqrt{2}} |e, 2\rangle \tag{2.21}$$
$$= 0.43571 |g, 1\rangle + i0.556 |e, 0\rangle - 0.7071 |e, 2\rangle \tag{2.22}$$

Step 7: Pulse

The derivation for the ordering of this pulse and subsequent rotation (in Step 6) is shown in the appendix.

First, rotation.

$$e^{i\Delta_{off}\sigma^+\sigma^- t} |g\rangle = 1 |g\rangle \tag{2.23}$$
$$e^{i\Delta_{off}\sigma^+\sigma^- t} |e\rangle = e^{i\Delta_{off} t} |e\rangle \tag{2.24}$$

The state becomes:

$$|\psi\rangle = 0.43571 |g, 1\rangle + i0.556 e^{i\Delta_{off} t_7} |e, 0\rangle - 0.7071 e^{i\Delta_{off} t_7} |e, 2\rangle \tag{2.25}$$

Pulse:

$$|e, 1\rangle \rightarrow i |g, 1\rangle \tag{2.26}$$

$$\Omega_q t_7 = \pi \tag{2.27}$$

State:

$$|\psi\rangle = i0.43571 |e, 1\rangle - 0.556 e^{i\Delta_{off} t_7} |g, 0\rangle - i0.7071 e^{i\Delta_{off} t_7} |g, 2\rangle \tag{2.28}$$

Step 6: Rotation

$$|\psi\rangle = i0.43571 e^{i\Delta_{off}t_6}|e,1\rangle - 0.556 e^{i\Delta_{off}t_7}|g,0\rangle - i0.7071 e^{i\Delta_{off}t_7}|g,2\rangle \tag{2.29}$$

Step 5: Swap

$$c_{g_2}(t) = -i0.7071\cos\left(\frac{\Omega\sqrt{n}}{2}t_5\right)e^{i\Delta_{off}t_7} - 0.43571\sin\left(\frac{\Omega\sqrt{n}}{2}t_5\right)e^{i\Delta_{off}t_6} \tag{2.30}$$

set $c_{g_2}(t_5) = 0$

$$0.7071\cos\left(\frac{\Omega\sqrt{n}}{2}t_5\right)e^{i\Delta_{off}t_7} = i0.43571\sin\left(\frac{\Omega\sqrt{n}}{2}t_5\right)e^{i\Delta_{off}t_6} \tag{2.31}$$

$$-\frac{i0.7071}{0.43571}e^{i\Delta_{off}(t_7-t_6)} = \tan\left(\frac{\Omega\sqrt{n}}{2}t_5\right) \tag{2.32}$$

Set Ω to be real and positive:

$$-ie^{i\Delta_{off}(t_7-t_6)} = 1 \tag{2.33}$$

$$e^{i\Delta_{off}(t_7-t_6)} = i = e^{i\frac{\pi}{2}} \tag{2.34}$$

$$\Delta_{off}(t_7-t_6) = \frac{\pi}{2} \tag{2.35}$$

$$t_6 = t_7 - \frac{\pi}{2\Delta_{off}} \tag{2.36}$$

Solving for t_5:

$$\tan\left(\frac{\Omega\sqrt{n}}{2}t_5\right) = \frac{0.7071}{0.43571} \tag{2.37}$$

$$\Omega t_5 = 1.44 \tag{2.38}$$

More n=2 states. Note that c_{g_0} stays constant because there is no state for it to exchange excitations in.

$$c_{e_1}(t_5) = i0.43571\cos\left(\frac{\Omega\sqrt{2}}{2}t_5\right)e^{i\Delta_{off}t_6} + 0.7071\sin\left(\frac{\Omega\sqrt{n}}{2}t_5\right)e^{i\Delta_{off}t_7} \tag{2.39}$$

The final state is:

$$|\psi\rangle = (-0.7969 - 0.23393i)|e,1\rangle + (0.534368 + 0.156861i)|g,0\rangle \tag{2.40}$$

Step 4: Pulse

First, rotation.

$$e^{i\Delta_{off}\sigma^+\sigma^- t}|g\rangle = 1|g\rangle \tag{2.41}$$
$$e^{i\Delta_{off}\sigma^+\sigma^- t}|e\rangle = e^{i\Delta_{off}t}|e\rangle \tag{2.42}$$

The state becomes:

$$|\psi\rangle = (-0.796 - 0.2339i)\, e^{i\Delta_{off}t_4}|e,1\rangle - (0.53436 + 0.1568i)|g,0\rangle \tag{2.43}$$

Pulse:

$$|e,2\rangle \to i|g,2\rangle \tag{2.44}$$

$$\Omega_q t_4 = \pi \tag{2.45}$$

State:

$$|\psi\rangle = i(-0.7969 - 0.2339i)\, e^{i\Delta_{off}t_4}|g,1\rangle - i(0.5343 + 0.15686i)|e,0\rangle \tag{2.46}$$

Step 3: Rotation

$$|\psi\rangle = i(-0.7969 - 0.2339i)\, e^{i\Delta_{off}t_4}|g,1\rangle - i(0.5343 + 0.15686i)\, e^{i\Delta_{off}t_3}|e,0\rangle \tag{2.47}$$

Step 2: Swap

$$c_{g_1}(t_2) = i(-0.7969 - 0.2339i)\, e^{i\Delta_{off}t_4}\cos\left(\frac{\Omega\sqrt{n}}{2}t_2\right) \tag{2.48}$$
$$- (0.53436 + 0.1568i)\, e^{i\Delta_{off}t_3}\sin\left(\frac{\Omega\sqrt{n}}{2}t_2\right) \tag{2.49}$$

$$c_{e_0}(t_2) = -i(0.53436 + 0.15686i)\, e^{i\Delta_{off}t_3}\cos\left(\frac{\Omega\sqrt{n}}{2}t_2\right) \tag{2.50}$$
$$- (-0.7969 - 0.2339i)\, e^{i\Delta_{off}t_4}\sin\left(\frac{\Omega\sqrt{n}}{2}t_2\right) \tag{2.51}$$

set $c_{g1}(t_2) = 0$

$$i\frac{(-0.7969-0.2339i)}{(0.53436+0.156861i)}e^{i\Delta_{off}t_4}\cos\left(\frac{\Omega\sqrt{n}}{2}t_2\right)=e^{i\Delta_{off}t_3}\sin\left(\frac{\Omega\sqrt{n}}{2}t_2\right) \tag{2.52}$$

$$(-1.49137i)\,e^{i\Delta_{off}(t_4-t_3)}=\tan\left(\frac{\Omega\sqrt{n}}{2}t_2\right) \tag{2.53}$$

Set Ω to be real and positive:

$$-ie^{i\Delta_{off}(t_4-t_3)}=1 \tag{2.54}$$

$$\Delta_{off}\left(t_4-t_3\right)=\frac{\pi}{2} \tag{2.55}$$

$$t_3=t_4-\frac{\pi}{2\Delta_{off}} \tag{2.56}$$

$$t_3=0.00368 \tag{2.57}$$

Solving for Ωt_2:

$$\tan\left(\frac{\Omega\sqrt{n}}{2}t_2\right)=1.491374 \tag{2.58}$$

$$\Omega t_2=1.9602 \tag{2.59}$$

The final state is:

$$|\psi\rangle=(-0.31944-0.2052i)\,|e,0\rangle \tag{2.60}$$

Step 1: Pulse

First, rotation.

$$e^{i\Delta_{off}\sigma^+\sigma^-t}\,|g\rangle=1\,|g\rangle \tag{2.61}$$
$$e^{i\Delta_{off}\sigma^+\sigma^-t}\,|e\rangle=e^{i\Delta_{off}t}\,|e\rangle \tag{2.62}$$

The state becomes:

$$|\psi\rangle=(-0.31944-0.205228i)\,e^{i\Delta_{off}t_1}\,|e,0\rangle \tag{2.63}$$

Pulse:

$$|e,0\rangle \to i\,|g,0\rangle \tag{2.64}$$

$$\Omega_q t_1 = \pi \tag{2.65}$$

$$t_1 = 0.00314 \tag{2.66}$$

State:

$$|\psi\rangle = i\,(-0.31944 - 0.20522i)\,e^{i\Delta_{off}t_1}\,|g,0\rangle \tag{2.67}$$

$$|\psi\rangle = (-0.28689 + 0.2487091j)\,|g,0\rangle \tag{2.68}$$

Times for each operation

Below we summarize the states of each of the processes and the times obtained.

time	time [ns]	accumulated time [ns]	state
t_8	15.1934	60.4676	$\vert g\rangle \otimes \left(\frac{1}{\sqrt{2}}\vert 1\rangle + \frac{i}{\sqrt{2}}\vert 3\rangle\right)$
t_7	3.14159	45.274	$0.43571\vert g,1\rangle$ $+i0.556\vert e,0\rangle$ $-0.7071\vert e,2\rangle$
t_6	3.68154	42.132	$\vert\psi\rangle = i0.43571\vert e,1\rangle$ $-(0.1568-0.5343i)\vert g,0\rangle$ $-(0.1991-0.6784i)\vert g,2\rangle$
t_5	12.0660	38.451	$\vert\psi\rangle =$ $(-0.4180-0.1227i)\vert e,1\rangle$ $-(0.1568-0.5343i)\vert g,0\rangle$ $-(0.1991-0.6784i)\vert g,2\rangle$
t_4	3.14159	26.384	$\vert\psi\rangle =$ $(-0.7969-0.23393i)\vert e,1\rangle +$ $(0.534368+0.156861i)\vert g,0\rangle$
t_3	3.68154	23.2433	$\vert\psi\rangle =$ $(-0.4489+0.6987i)\vert g,1\rangle +$ $(0.15686-0.5343i)\vert e,0\rangle$
t_2	16.4202	19.5618	$\vert\psi\rangle =$ $(-0.4489+0.6987i)\vert g,1\rangle +$ $(0.4685+0.30102i)\vert e,0\rangle$
t_1	3.14159	3.14159	$\vert\psi\rangle =$ $(-0.31944-0.2052i)\vert e,0\rangle$
t_{init}	0	0	$\vert\psi\rangle =$ $(-0.28689+0.2487091i)\vert g,0\rangle$

Table 2.1: Qubit states and times obtained for an example arbitrary final state

Chapter 3

Displace with caution

In order to determine whether the experimentally-formed final states matched with theory, Wigner tomography was used to evaluate the experimental results. The probability amplitudes were determined by using a "photon number readout" on the resonator. It was unclear from the paper how this was done. The probability amplitudes for experiment and theory are shown for the state $|\psi_a\rangle = 1 + |3\rangle$ and $|\psi_b\rangle = 1 + i\,|3\rangle$ in Fig. (3.1 Alex) and Fig. (3.1 Bhavin), respectively. In Fig. (3.1 Chaoran), the simulated Wigner tomography representation is shown, displaying the phase rotation in the state $1 + i\,|3\rangle$.

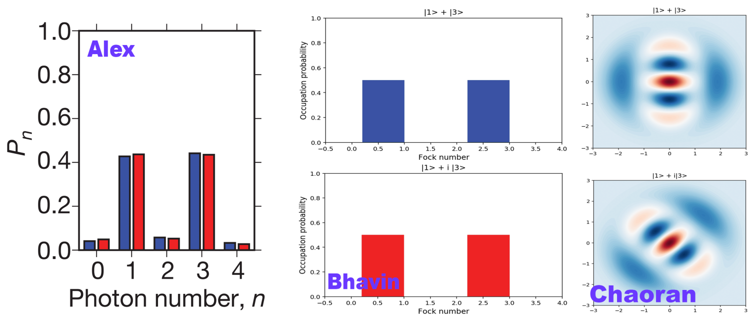

Figure 3.1: Photon number distributions for $|\psi_a\rangle$ (blue) and $|\psi_b\rangle$ (red). Both states are equal superpositions of $|1\rangle$ and $|3\rangle$ but the phase information that distinguishes the two states is lost. Alex is showing experiment, while Bhavin is showing the simulations. Chaoran is showing the phase representation using Wigner simulations.

Since Fig. (3.1 Alex) and Fig. (3.1 Bhavin) does not show the phase distribution, Hofheinz and his gang of experimentalists used Wigner tomography to determine the phase space distributions. In Fig. (3.2 Hsuan-Tung), we show their experimental results, and in Fig. (3.2 Philip), we show our simulations. Negative quasi- probabilities are clearly measured. Fig. (3.3) shows our simulations matching with their simulation and experiment when the phase is constant. A hand-wavy explanation for how the Wigner functions can be understood is as follows:

1. Displace the resonator state $|\psi\rangle$ by $D(-\alpha)$ using a microwave pulse.

2. Measure the photon number probabilities P_n.
3. Evaluate the parity: $\langle \Pi \rangle = \sum_n (-1)^n P_n$
4. Evaluate the Wigner function: $W(\alpha) = \frac{2}{\pi} \mathrm{Tr}\left(D(-\alpha)\rho D(\alpha)\Pi\right)$

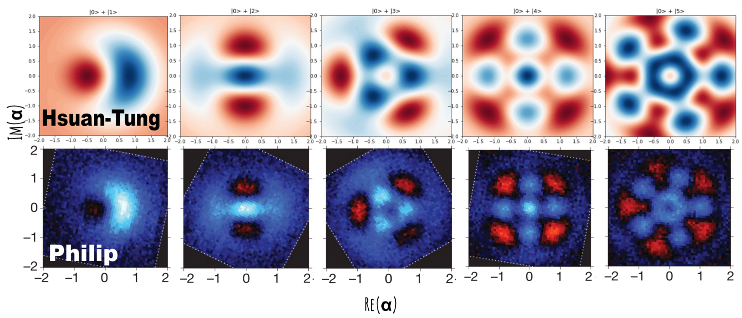

Figure 3.2: Wigner tomography of superpositions of resonator Fock states $|0\rangle + |N\rangle$. Hsuan-Tung is displaying the theoretical form of the Wigner function $W(\alpha)$ as a function of the complex resonator amplitude α in photon number units, for states $N = 1$ to 5. Philip shows the measured Wigner functions, with the color scale bar on the far right. The experimental Wigner functions have been rotated to match theory, compensating for a phase delay between the qubit and resonator microwave lines; the measured area is bounded by a dotted white line.

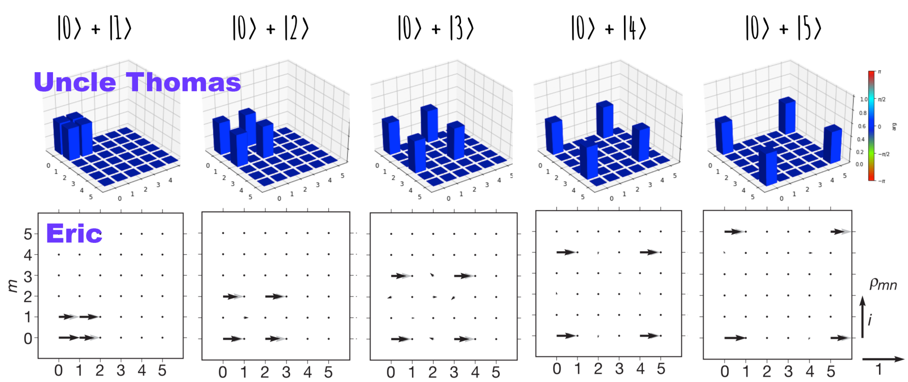

Figure 3.3: Resonator density matrix the ρ, projected onto the number states $\rho_{nm} = \langle m| \rho |n\rangle$. Uncle Thomas is showing our simulated model. Eric is showing the calculated (grey) and measured (black) values from Hofheinz. The magnitude and phase of ρ_{nm} is represented by the length and direction of an arrow in the complex plane.

We also did simulations for states with arbitrary phases, and compared it with Hofheinz's simulations and experiments. In Fig. (3.4) and Fig. (3.5) we show our simulations matching with Hofheinz's experiments for the Wigner distributions and phases, respectively, in phases $e^{ik\pi/8}$ for five values of k.

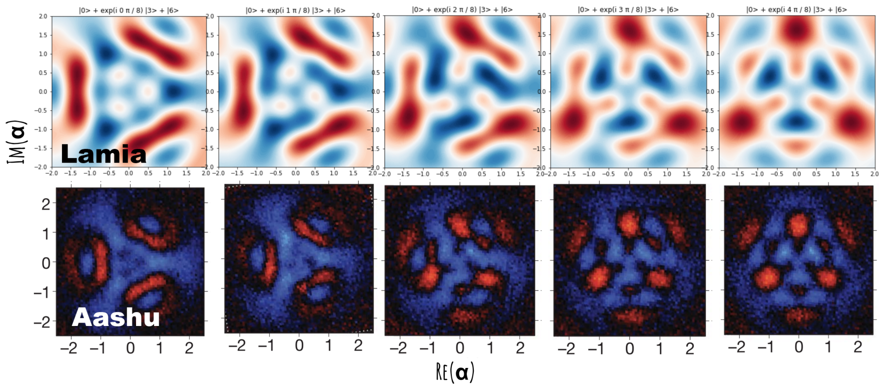

Figure 3.4: Wigner tomography of the states $|0\rangle + e^{ik\pi/8}|3\rangle + |6\rangle$ for five values of phase $k = 0$ to 4. Lamia is showing her own simulations while Aashu is showing Hofheinz's experiments.

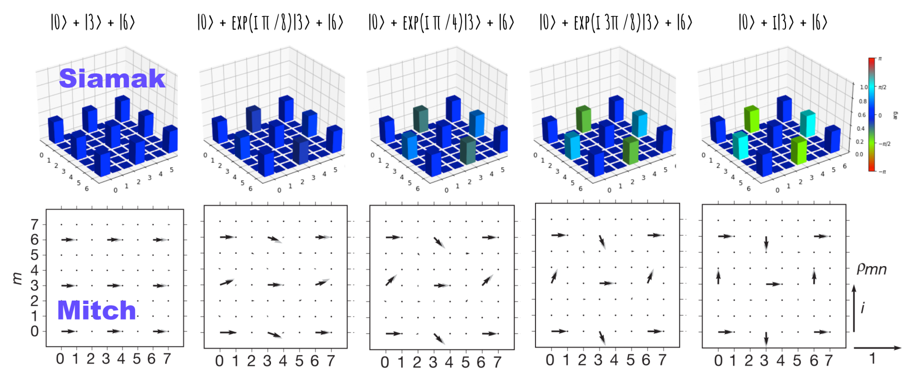

Figure 3.5: Resonator density matrix the ρ, projected onto the number states $\rho_{nm} = \langle m|\,\rho\,|n\rangle$. Siamak is showing our simulated model. Mitch is showing the calculated (grey) and measured (black) values from Hofheinz. The magnitude and phase of ρ_{nm} is represented by the length and direction of an arrow in the complex plane.

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Appendix

Qubit-Cavity dynamics

The qubit-cavity system dynamics for the two-level qubit can be derived as follows:
Hamiltonian:

$$\hat{H} = \hbar\omega_0\hat{N} + \frac{\Omega}{2}\hbar\left(\sigma^+a + a^\dagger\sigma^-\right) \tag{3.1}$$

Schrödinger Equation:

$$i\hbar\frac{d}{dt}\left|\tilde{\psi}(t)\right\rangle = \hat{H}\left|\tilde{\psi}(t)\right\rangle \tag{3.2}$$

Assume the following solution for the two-level system:

$$|\psi(t)\rangle = c_{e,n-1}(t)\,|e, n-1\rangle + c_{g,n}(t)\,|g, n\rangle \tag{3.3}$$

Sub in these coefficients

$$i\dot{c}_{e,n-1}(t)\,|e, n-1\rangle + i\dot{c}_{g,n}(t)\,|g, n\rangle = \begin{cases} & \omega_0 n c_{e,n-1}(t)\,|e, n-1\rangle + \omega_0 n c_{g,n}(t)\,|g, n\rangle \\ + & \frac{\Omega}{2}c_{e,n-1}(t)n\,|g, n\rangle + c_{g,n}(t)\sqrt{n}\,|e, n-1\rangle \end{cases} \tag{3.4}$$

Separate equations for like terms:

$$|e, n-1\rangle \qquad \dot{c}_{e,n-1}(t) = -i\omega_0 n c_{e,n-1}(t) - ig\sqrt{n}c_{g,n}(t) \tag{3.5}$$

$$|g, n\rangle \qquad \dot{c}_{g,n}(t) = -i\omega_0 n c_{g,n}(t) - ig\sqrt{n}c_{e,n-1}(t) \tag{3.6}$$

Enter the rotating frame:

$$\tilde{c} = ce^{i\omega_0 nt} \tag{3.7}$$

$$\dot{\tilde{c}}_{e,n-1}e^{-i\omega_0 n} - i\omega_0 n\tilde{c}_{e,n-1}e^{-i\omega_0 n} = -i\omega_0 n\tilde{c}_{e,n-1}(t)e^{-i\omega_0 n} - i\frac{\Omega}{2}\sqrt{n}\tilde{c}_{g,n}(t)e^{-i\omega_0 n} \tag{3.8}$$

We get these equations:

$$\dot{\tilde{c}}_{e,n-1} = -i\frac{\Omega}{2}\sqrt{n}\tilde{c}_{g,n}(t) \tag{3.9}$$

$$\dot{\tilde{c}}_{g,n} = -i\frac{\Omega}{2}\sqrt{n}\tilde{c}_{e,n-1}(t) \tag{3.10}$$

From eq. (3.10):

$$\dot{\tilde{c}}_{e,n-1} = \frac{i}{\frac{\Omega}{2}\sqrt{n}}\ddot{\tilde{c}}_{g,n} \tag{3.11}$$

Sub into eq. (3.9)

$$\frac{i}{\frac{\Omega}{2}\sqrt{n}}\ddot{\tilde{c}}_{g,n}(t) = -i\frac{\Omega}{2}\sqrt{n}\tilde{c}_{g,n}(t) \tag{3.12}$$

Simplify

$$\ddot{\tilde{c}}_{g,n}(t) + \left(\frac{\Omega}{2}\right)^2 n\tilde{c}_{g,n}(t) = 0 \tag{3.13}$$

$$\tilde{c}_{g,n} = c_{g_n}(0)\cos\left(\frac{\Omega}{2}\sqrt{n}t\right) + ic_{e_{n-1}}(0)\sin\left(\frac{\Omega}{2}\sqrt{n}t\right) \tag{3.14}$$

$$c_{g_n}(t) = c_{g_n}(0)\cos\left(\frac{\Omega\sqrt{n}}{2}t\right) + ic_{e_{n-1}}(0)\sin\left(\frac{\Omega\sqrt{n}}{2}t\right) \tag{3.15}$$

$$c_{e_{n-1}}(t) = c_{e_{n-1}}(0)\cos\left(\frac{\Omega\sqrt{n}}{2}t\right) + ic_{g_n}(0)\sin\left(\frac{\Omega\sqrt{n}}{2}t\right) \tag{3.16}$$

Off-resonance pulse and rotation

$$H_{off} = \Delta_{off}\sigma^+\sigma^- + \left(\frac{\Omega_q}{2}\sigma^+e^{-i\Delta_{off}t} + \frac{\Omega_q^*}{2}\sigma^-e^{i\Delta_{off}t}\right)$$
$$i\frac{\partial\left|\psi(t)\right\rangle}{\partial t} = H_{off}(t)\left|\psi\right\rangle$$

Reframing the picture:

$$\left|\tilde{\psi}(t)\right\rangle = e^{i\Delta_{off}\sigma^+\sigma^-t}\left|\psi(t)\right\rangle$$
$$\left|\psi(t)\right\rangle = e^{-i\Delta_{off}\sigma^+\sigma^-}\left|\tilde{\psi}(t)\right\rangle$$

$$\Delta_{off}\sigma^+\sigma^- e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|\tilde{\psi}(t)\right\rangle + ie^{-i\Delta_{off}\sigma^+\sigma^- t}\frac{\partial}{\partial t}\left|\tilde{\psi}(t)\right\rangle$$
$$= H_{off}e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|\tilde{\psi}(t)\right\rangle$$

$$\begin{aligned}
i\frac{\partial}{\partial t}\left|\tilde{\psi}(t)\right\rangle &= e^{i\Delta_{off}\sigma^+\sigma^- t}\left(H_{off} - \Delta_{off}\sigma^+\sigma^-\right)e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|\tilde{\psi}(t)\right\rangle \\
&= e^{i\Delta_{off}\sigma^+\sigma^- t}\left(\frac{\Omega_q}{2}\sigma^+ e^{-i\Delta_{off}t} + \frac{\Omega_q^*}{2}\sigma^- e^{i\Delta_{off}t}\right)e^{-i\Delta_{off}\sigma^+\sigma^- t}\left|\tilde{\psi}(t)\right\rangle \\
&= \left[\frac{\Omega_q}{2}e^{i\Delta_{off}\sigma^+\sigma^- t}\sigma^+ e^{-i\Delta_{off}\sigma^+\sigma^- t}\cdot e^{-i\Delta_{off}t}\right. \\
&\quad \left. + \frac{\Omega_q^*}{2}e^{i\Delta_{off}\sigma^+\sigma^- t}\sigma^- e^{-i\Delta_{off}\sigma^+\sigma^- t}\cdot e^{i\Delta_{off}t}\right]\left|\tilde{\psi}(t)\right\rangle
\end{aligned} \tag{3.17}$$

The middle of eq. (3.17) can be simplified.
Acting on the excited state:

$$e^{-i\Delta_{off}\sigma^+\sigma^- t}|e\rangle = e^{-i\Delta_{off}t}|e\rangle$$
$$\sigma^+|e\rangle = 0$$

Acting on the ground state:

$$e^{-i\Delta_{off}\sigma^+\sigma^- t}|g\rangle = |g\rangle$$
$$\sigma^+|g\rangle = |e\rangle$$
$$e^{i\Delta_{off}\sigma^+\sigma^- t}|e\rangle = e^{i\Delta_{off}t}|e\rangle$$

Which is the same for $\sigma^+ e^{i\Delta_{off}t}$. A similar argument can be made for using $\sigma^- e^{-i\Delta_{off}t}$:
Acting on the ground state:

$$e^{-i\Delta_{off}\sigma^+\sigma^- t}|g\rangle = |g\rangle$$
$$\sigma^-|g\rangle = 0$$

Acting on the excited state:

$$e^{-i\Delta_{off}\sigma^+\sigma^- t}|e\rangle = |e\rangle$$
$$\sigma^-|e\rangle = |g\rangle$$
$$e^{i\Delta_{off}\sigma^+\sigma^- t}|g\rangle = e^{i\Delta_{off}t}|g\rangle$$

Therefore, eq. (3.17) can be written as:

$$i\frac{\partial}{\partial t}\left|\tilde{\psi}(t)\right\rangle = \left(\frac{\Omega_q}{2}\sigma^+ + \frac{\Omega_q^*}{2}\sigma^-\right)\left|\tilde{\psi}(t)\right\rangle$$

Solving this expression:

$$\left|\tilde{\psi}(t)\right\rangle = \exp\left\{-i\left(\frac{\Omega_q}{2}\sigma^+ + \frac{\Omega_q^*}{2}\sigma^-\right)t\right\}\left|\tilde{\psi}(0)\right\rangle$$

Taking the picture out of the rotating frame:

$$|\psi(t)\rangle = e^{-i\Delta_{off}\sigma^+\sigma^- t}\exp\left\{-i\left(\frac{\Omega_q}{2}\sigma^+ + \frac{\Omega_q^*}{2}\sigma^-\right)t\right\}|\psi(0)\rangle$$

When time is reversed, note that the Schrödinger Eq. becomes:

$$i\frac{\partial\,|\psi(-t)\rangle}{\partial(-t)} = H(-t)\,|\psi\rangle \tag{3.18}$$

$$-i\frac{\partial\,|\psi(t)\rangle}{\partial(t)} = H(t)\,|\psi\rangle \tag{3.19}$$

not sure if this is right

And the operators switch their order.
Also, using the σ^+ and σ^- operators shown further below, we can obtain:

$$e^{i\Delta_{off}\sigma^+\sigma^- t}\,|g\rangle = 1\,|g\rangle \tag{3.20}$$

$$e^{i\Delta_{off}\sigma^+\sigma^- t}\,|e\rangle = e^{i\Delta_{off}t}\,|e\rangle \tag{3.21}$$

Pulse Derivation

Here, we show an example derivation for how the π-pulse is in fact a rotation over π radians.

Our state before the pulse is:

$$|\psi(t_8)\rangle = 0.43571\,|g,1\rangle + i0.556e^{i\Delta_{off}t_7}\,|e,0\rangle - 0.7071e^{i\Delta_{off}t_7}\,|e,2\rangle \tag{3.22}$$

The Hamiltonian evolves in time as such:

$$
\begin{aligned}
e^{iHt} &= e^{i\left(\frac{\Omega_q}{2}\sigma^+ + \frac{\Omega_q^*}{2}\sigma^-\right)t} && (3.23)\\
&= e^{\frac{i}{2}\left((\mathrm{Re}\Omega_q + i\mathrm{Im}\Omega_q)\sigma^+ + (\mathrm{Re}\Omega_q - i\mathrm{Im}\Omega_q)\sigma^-\right)} && (3.24)\\
&= e^{\frac{i}{2}\left(\mathrm{Re}\Omega_q\left(\sigma^+ + \sigma^-\right) + i\mathrm{Im}\Omega_q\left(\sigma^+ - \sigma^-\right)\right)} && (3.25)\\
&= e^{\frac{i}{2}(\mathrm{Re}\Omega_q \sigma^x - \mathrm{Im}\Omega_q \sigma^y)} && (3.26)\\
&= e^{i\frac{|\Omega_q|}{2}\left[\frac{\mathrm{Re}\Omega_q}{|\Omega_q|}\sigma^x - \frac{\mathrm{Im}\Omega_q}{|\Omega_q|}\sigma^y\right]t} && (3.27)\\
&= e^{i\frac{|\Omega_q|}{2}\hat{n}\cdot\vec{\sigma}t} && (3.28)
\end{aligned}
$$

where:

$$\hat{n} = \frac{\mathrm{Re}\Omega_q}{|\Omega_q|}\hat{x} - \frac{\mathrm{Im}\Omega_q}{|\Omega_q|}\hat{y} \tag{3.29}$$

Expanding eq. (3.28):

$$\mathbb{I}\cos\left(\frac{|\Omega_q|}{2}t\right) + i\left[\frac{\mathrm{Re}\Omega_q}{|\Omega_q|}\sigma^x - \frac{\mathrm{Im}\Omega_q}{|\Omega_q|}\sigma^y\right]\sin\left(\frac{|\Omega_q|}{2}t\right) \tag{3.30}$$

Before applying eq. (3.30), let's simplify it a little: we can control the drive on the qubit, so we can control the phase as well, so let's set that equal to zero. That means that this is true: $\frac{\mathrm{Im}\Omega_q}{|\Omega_q|} = 0$ and $\frac{\mathrm{Re}\Omega_q}{|\Omega_q|} = 1$.

Eq. (3.30) becomes:

$$\mathbb{I}\cos\left(\frac{|\Omega_q|}{2}t\right) + i\sigma^x\sin\left(\frac{|\Omega_q|}{2}t\right) \tag{3.31}$$

Applying eq. (3.31) to the state (**??**), term by term:

$$
\begin{aligned}
|\psi(t_7)\rangle = &\cos\left(\frac{|\Omega_q|}{2}t_7\right)0.43571\,|g,1\rangle \\
&+ i\sin\left(\frac{|\Omega_q|}{2}t_7\right)0.43571\,|e,1\rangle \\
\\
&+ \cos\left(\frac{|\Omega_q|}{2}t_7\right)i0.556e^{i\Delta_{off}t_7}\,|e,0\rangle \\
&- \sin\left(\frac{|\Omega_q|}{2}t_7\right)0.556e^{i\Delta_{off}t_7}\,|g,0\rangle \\
\\
&- \cos\left(\frac{|\Omega_q|}{2}t_7\right)0.7071e^{i\Delta_{off}t_7}\,|e,2\rangle \\
&+ i\sin\left(\frac{|\Omega_q|}{2}t_7\right)0.7071e^{i\Delta_{off}t_7}\,|g,2\rangle
\end{aligned}
\tag{3.32}
$$

The time t_2 must be set so that the $|e,2\rangle$ term is zero so that the excitation goes into the cavity. This time for the "pi" pulse is always the same, since Ω_q can always be manually set as described.

$$
\frac{|\Omega_q|}{2}t_7 = \frac{\pi}{2} \tag{3.33}
$$

$$
\rightarrow t_7 = \frac{\pi}{|\Omega_q|} \tag{3.34}
$$

Therefore, eq. (3.32) becomes:

$$
|\psi\rangle = i0.43571\,|e,1\rangle - 0.556e^{i\Delta_{off}t_7}\,|g,0\rangle - i0.7071e^{i\Delta_{off}t_7}\,|g,2\rangle \tag{3.35}
$$

The time t_7 is left in for convenience.

Spin Matrices

$$\sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.36}$$

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.37}$$

$$\sigma^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{3.38}$$

$$\sigma^+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{3.39}$$

$$\sigma^- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.40}$$

$$\sigma^x = \sigma^+ + \sigma^- \tag{3.41}$$

$$\sigma^y = i\left(\sigma^+ - \sigma^-\right) \tag{3.42}$$

$$\sigma^+\sigma^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.43}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.44}$$

$$= \frac{1}{2}\left(-\sigma^z + 1\right) \tag{3.45}$$

$\sigma^x \left|g\right\rangle$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left|e\right\rangle \tag{3.46}$$

$\sigma^y \left|g\right\rangle$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -i\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i\left|e\right\rangle \tag{3.47}$$

$\sigma^z \left|g\right\rangle$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\left|g\right\rangle \tag{3.48}$$

$\sigma^+ \left|g\right\rangle$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \tag{3.49}$$

$\sigma^- |g\rangle$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |e\rangle \tag{3.50}$$

$\sigma^x |e\rangle$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |g\rangle \tag{3.51}$$

$\sigma^y |e\rangle$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = i \begin{bmatrix} 0 \\ 1 \end{bmatrix} = i\,|g\rangle \tag{3.52}$$

$\sigma^z |e\rangle$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |e\rangle \tag{3.53}$$

$\sigma^+ |e\rangle$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |g\rangle \tag{3.54}$$

$\sigma^- |e\rangle$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \tag{3.55}$$