On the Utility of the Lorentzian No-Boundary Wavefunction as a Precursor to Inflation

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Abstract

This thesis follows the path integral approach to quantize gravity. The Lorentzian No-Boundary Wavefunction (NBWF) is constructed with a straightforward incorporation of the potential of a single scalar field (to be regarded as the inflaton). The absolute-square of the NBWF is then analyzed for peaks with respect to the scalar field which can be interpreted as the predictions on the initial values of the field. The Starobinsky model of inflation is chosen for further analysis. It is demonstrated that this leads to a non-normalizable NBWF in the region of interest.

Possible provisions to work around the non-normalizability of the NBWF are suggested and supported by heuristic justifications. The breakdown of one of the aforementioned provisions is demonstrated. The other provision, which entails adding a 'correction' to the potential of Starobinsky inflation, allows to continue with the slow-roll analysis. The consequences on the Jordan frame action, of adding such a term in the Einstein frame action are studied carefully. Comparison of the slow-roll analysis results with the Planck results does not rule out this model, whereas the imposition of a quantum cosmological consistency condition suggests that the tree-level Lorentzian NBWF is not a useful precursor to Starobinsky inflation. Finally, a few words on the interpretation of the path integral in quantum cosmology are mentioned.

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Conventions

I work with units in which $c = 1 = \hbar$. In these units the Planck mass is $m_{\mathrm{P}} = 1/\sqrt{G}$, where G is the gravitational constant and I will use the reduced Planck mass, $M_{\mathrm{P}} = m_{\mathrm{P}}/\sqrt{8\pi}$. Greek and Latin alphabets respectively label spacetime and spatial indices. ∂_μ or $_{,\mu}$ denote partial differentiation with respect to x^μ, whereas ∇_μ or $_{;\mu}$ are used for covariant differentiation. Moreover, the following conventions are used:

- the metric signature is $(-+++)$
- The Christoffel symbols of the second kind are given by: $\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})$
- The Riemann curvature tensor, and the Ricci tensor are respectively given by: $R^{\lambda}{}_{\mu\sigma\nu} = \Gamma^{\lambda}{}_{\mu\nu,\sigma} - \Gamma^{\lambda}{}_{\mu\sigma,\nu} + \Gamma^{\lambda}{}_{\rho\sigma}\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\rho\nu}\Gamma^{\rho}{}_{\mu\sigma}, \quad R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$
- The Lagrangian of a scalar field is given by: $L = T - V = \int \mathrm{d}^3x(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi))$
- In sections where both Jordan and Einstein frames are relevant, the Einstein frame quantities are denoted by an overhead tilde.

The reduced Planck constant $\hbar$ is denoted explicitly wherever relevant. In absence of any ambiguity, the wavefunctional will be called the wavefunction. Commonly used abbreviations are General Relativity (GR), Einstein Field Equations (EFE), Quantum Gravity (QG), Quantum Cosmology (QC), and the No-Boundary Wavefunction (NBWF).

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1 Introduction

The general theory of relativity (GR) was put forth by Albert Einstein in 1915. At the time, it was an attempt to overcome the incompatibility of Newtonian gravity with the special theory of relativity. Not only was GR (by construction) successful in doing so, it completely revolutionized the idea of gravity as a force. GR attributed gravity to the intrinsic curvature of spacetime, and established how the presence of energy or matter influenced the said curvature (worked out in detail in Chapter 2). This new, revolutionary theory could correctly predict the anomalous (at the time) precession of the perihelion of Mercury's orbit, and also offered bold predictions like the propagation of light along curved paths (geodesics) near massive objects, gravitational radiation, and black holes (first hypothesized by Karl Schwarzschild). GR was put to test observationally by Arthur Eddington, and in 1919 the gravitational deflection of star-light around the sun was successfully observed. The work of Roger Penrose and Stephen Hawking in the 1960s established that black holes were robust predictions of general relativity. The direct detection of gravitational waves in 2015 by the LIGO mission, and the first image of a black hole by the EHT in 2019 were further crowning jewels in GR's legacy.

Besides the observational verifications mentioned above, GR has been immensely successful in the field of cosmology, and was pivotal in establishing modern cosmology. In the 1920s, the theoretical work by Alexander Friedmann on the basis of GR, and the observational discoveries of Edwin Hubble put cosmology on the map as a scientific field. The discoveries of cosmic microwave background (CMB) in 1964 and the accelerated expansion of the universe in 1998, and missions such as the Hubble Space Telescope, COBE, WMAP in the 1990s and the 2000s, not only strengthened the theoretical predictions, but also gave observational inputs to further theoretical developments. All this together led to a standard model of cosmology, which is covered in Chapter 2.

But, it is well documented by now that GR is incomplete. At a classical level, GR unavoidably leads to presence of singularities. These singularities are characterized by geodesic incompleteness, and result in the equations losing their predictive power in those regions of space-time. This is undesirable in any 'complete' theory. Apart from this, the other fundamental forces (electromagnetic, weak, strong) are not only quantized, but are also unified under the umbrella of the Standard Model of particle physics. On the other hand, gravity (as described by GR) has not yet been successfully quantized so that no subsequent unification with other forces has yet been possible. Approaches to quantize gravity can be broadly classified into two types: 1) Focus on quantizing gravity by starting from a classical theory, by applying one of the known quantization regimes (canonical, path integral), and 2) Focus on a quantized unification of all forces, and arrive at gravity as an emergent phenomenon. In the former approach, starting from GR as the classical theory and attempting to quantize canonically has lead to Quantum Geometrodynamics and Loop Quantum Gravity. Path integral quantization is also an important approach, especially in the context of cosmology. Attempts to quantize gravity under the latter approach mainly fall under the framework of string theory. See [1] for an overview on these approaches.

It is the hope that a full theory of quantum gravity (QG) will make sense of the classical singularities at small length/high energy scales. The scale at which it is unavoidable to take the quantum effects of gravity into account is the Planck scale, characterized by the

Planck mass m_{P}, or the Planck length l_{P}[1]. At the energy/length scales accessible to current observational instruments, this quantum theory should reduce to GR, as the astrophysical and large scale cosmological observations match the predictions of GR.

Although quantizing gravity is desirable, there exist certain problems in attempting to do so. The main problems are: 1) In quantizing other theories, the fields exist on a background spacetime. In GR, the field of interest is the spacetime metric itself, *ie.* the spacetime is dynamic, and not a background entity. 2) GR is non-renormalizable. The infinities that appear in its perturbative expansion cannot be accounted for by introducing a finite number of counter-terms. 3) In non-gravitational quantum theories, there exist transition amplitudes that evolve in time. In GR, as spacetime itself is dynamic, the existence and role of 'time' at a quantum level is not clear. This is innately related to the problem of choice of appropriate inner product.

Quantum field theory on curved spacetime is a well-established field. It is a good intermediate step between classical GR and QG, as it gives important clues regarding the nature of QG. Calculations in this field have given concrete predictions on black hole thermodynamics, and quantum fluctuations of the inflaton field. See [2] for a detailed introduction and applications of this field.

The standard model of cosmology is also incomplete in the sense that it is marred by several shortcomings, which are mentioned in Section 2.4. The paradigm of inflation provides elegant solutions to these problems (Chapter 3), and also provides mechanisms to generate the seeds of the inhomogeneities observed in the CMB, large scale structure, and the primordial gravitational waves that will hopefully be observed in the future[2]. According to [4], the Starobinsky inflation (see Section 3.4 for details) is observationally the most preferred model of inflation.

[1] $m_{\mathrm{P}} = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg}$, and $l_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}$.

[2] The LISA mission is a promising candidate to observe primordial gravitational waves [see 3].

Quantum cosmology (QC) is the application of a quantum theory of gravity to cosmological models. In this thesis, the path integral approach to quantize gravity (explained in detail in Section 4.2) is used. The wavefunctional of the universe is uniquely determined by specifying boundary conditions (see Section 4.3). In this thesis, the no-boundary proposal [5] put forth by Hartle and Hawking in 1983 is used to set the boundary conditions, giving rise to the no-boundary wavefunction (NBWF). To understand the relevance of the particular implementation of the no-boundary proposal used in this thesis, it is important to review various historical implementations of this proposal.

The idea of the original formulation of the no-boundary proposal was to sum over all geometries whose initial hypersurface (in context of cosmology, origin of the universe) is compact as well as regular. This facilitates the emergence of the universe out of 'zero size' as the initial hypersurface vanishes. It was thus aptly named the no-boundary proposal as there is no initial boundary. This idea could be straightforwardly implemented in the Euclidean path integral. The problems that face this implementation are: 1) The Euclidean path integral in QG/QC has certain shortcomings, which are presented in Section 4.2, 2) Quantum mechanically, a simultaneous impostion of compactness (size) and regularity (rate) is problematic, as size and rate would be conjugate variables in a quantum theory, and 3) In this approach, issues with stability of fluctuations are shown to exist in [6]; which are not seen in any observations of the early universe.

To rescue the no-boundary proposal from the drawbacks of the Euclidean path integral, in [7] the NBWF is constructed from a Lorentzian path integral, but still with the imposition of compactness of the initial hypersurface. Without requiring regularity there, the dominant (saddle-point) contribution of the path integral gave the correct NBWF. In [8] it was shown that this approach too leads to unstable fluctuations, the origin of which was determined to be the Dirichlet boundary condition (compactness) imposed on the initial hypersurface.

Apart from the general problems that plague the path integral approach to quantize gravity, the Lorentzian approach has no other conceptual shortcomings. As seen in QM and QFT, the Lorentzian approach naturally incorporates the notions of unitarity and causality. The technical difficulty in dealing with the Lorentzian path integral is its oscillatory integrand, which is resolved by applying the Picard-Lefschetz theory as explained in Appendix A. The problem of unstable fluctuations originating from imposing the Dirichlet boundary condition was resolved in [9] by instead imposing an appropriate imaginary Neumann boundary condition at the initial hypersurface which ensures regularity there. The correct NBWF again arises as a saddle-point contribution. According to this redefinition of the no-boundary proposal, the universe started out as purely Euclidean. The change to Lorentzian signature is facilitated by the presence of a scalar field, as shown in [10].

Thus, the construction of the NBWF in this thesis follows [9]. The aim of this thesis is to study whether the NBWF gives rise to predictions on the initial value of the scalar field (inflaton) which could be considered to be suitable for Starobinsky inflation.

The structure of this thesis is as follows: Chapters 2, 3, 4 build up the background knowledge on essential topics in general relativity, standard cosmology and its drawbacks, inflation and the Starobinsky model, quantizing gravity by the path integral approach and the associated problems, and the initial conditions of the universe's wavefunction. Chapter 5 sets up the NBWF, discusses the main problems, and motivates working around these problems. In Chapter 6, slow-roll analysis is attempted (and performed in detail where possible) based on the suggested workarounds to the problem of non-normalizability of the NBWF. Chapter 7 has a few words on the interpretation of the path integral in quantum cosmology, and finally the results of this work are summarized in Chapter 8. Appendices A and B contain proofs of certain properties and equations used throughout this thesis.

2 General Relativity and Standard Cosmology

This chapter provides an introduction to GR in the Lagrangian formalism. Knowledge of this topic is a prerequisite for the path integral approach to quantum gravity, which is employed in this thesis. Main aspects of standard cosmology and the theory's major shortcomings are also presented. Understanding the standard model and its drawbacks naturally paves the way to the inflationary paradigm, which offers an elegant resolution of these problems. Inflation is discussed in detail in Chapter 3.

The sections on GR, and Cosmology are respectively based on [11–13], and [13–15].

2.1 Einstein-Hilbert Action

$(\mathcal{M}, \mathbf{g})$ forms the dynamical spacetime in GR, where $\mathcal{M}$ is a four-dimensional pseudo-Riemannian manifold endowed with a metric $\mathbf{g}$. The vacuum field equations of GR are obtained by varying the action:

$$S_{\mathrm{EH}} = \frac{M_{\mathrm{P}}^2}{2} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g} R \tag{2.1}$$

Here, $g = \det \mathbf{g}$ and R is the Ricci curvature scalar. Contributions of matter fields and that of the cosmological constant can be added to the above action and their variation gives the EFE in its familiar form.

$$S_{tot} = \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}(R-2\Lambda) + \int \mathrm{d}^4x\,\sqrt{-g}\mathcal{L}_m \tag{2.2}$$

$\mathcal{L}_m$ is the Lagrangian density of all the matter fields involved.

To derive the EFE, consider first the action

$$S = \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}(R-2\Lambda)$$

It can be varied as follows:

$$\begin{aligned}\delta S &= \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\;\delta\big(\sqrt{-g}(R_{\mu\nu}g^{\mu\nu}-2\Lambda)\big)\\ &= \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\;\big(\delta(\sqrt{-g})(R-2\Lambda)+\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}+\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\big)\end{aligned} \tag{2.3}$$

Here,

$$\delta(\sqrt{-g}) = \frac{-1}{2\sqrt{-g}}\delta g \tag{2.4}$$

Consider a matrix M. Jacobi's formula for differentiating a determinant gives: $\mathrm{d}\det M = \mathrm{Tr}(\mathrm{adj}(M)\,\mathrm{d}M)$, where $\mathrm{adj}(M) = M^{-1}\det M$. Using this in Eq (2.4) for $M \equiv \mathbf{g}$ leads to:

$$\delta(\sqrt{-g}) = \frac{-1}{2\sqrt{-g}}gg^{\mu\nu}\delta g_{\mu\nu} \tag{2.5}$$

Also,

$$\begin{aligned} g_{\mu\nu}g^{\nu\lambda} &= \delta_\mu^\lambda\\ \therefore (\delta g_{\mu\nu})g^{\nu\lambda} &= -g_{\mu\nu}(\delta g^{\nu\lambda})\\ \therefore \delta g_{\mu\nu} &= -g_{\rho\nu}g_{\mu\sigma}\delta g^{\rho\sigma}\end{aligned} \tag{2.6}$$

Substituting Eq (2.6) in Eq (2.4) gives:

$$\begin{aligned}\delta(\sqrt{-g}) &= -\frac{-g}{2\sqrt{-g}}g_{\mu\nu}\delta g^{\mu\nu}\\ \therefore \delta(\sqrt{-g}) &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\end{aligned} \tag{2.7}$$

For $\delta R_{\mu\nu}$ consider the following:

$$
\begin{aligned}
R^{\lambda}{}_{\mu\rho\nu} &= \Gamma^{\lambda}{}_{\mu\nu,\rho} - \Gamma^{\lambda}{}_{\mu\rho,\nu} + \Gamma^{\lambda}{}_{\rho\sigma}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\mu\rho} \\
\therefore \delta R^{\lambda}{}_{\mu\rho\nu} &= \partial_{\rho}\delta\Gamma^{\lambda}{}_{\mu\nu} - \partial_{\nu}\delta\Gamma^{\lambda}{}_{\mu\rho} + (\delta\Gamma^{\lambda}{}_{\rho\sigma})\Gamma^{\sigma}{}_{\mu\nu} \\
&\quad + \Gamma^{\lambda}{}_{\rho\sigma}(\delta\Gamma^{\sigma}{}_{\mu\nu}) - (\delta\Gamma^{\lambda}{}_{\nu\sigma})\Gamma^{\sigma}{}_{\mu\rho} - \Gamma^{\lambda}{}_{\nu\sigma}(\delta\Gamma^{\sigma}{}_{\mu\rho})
\end{aligned}
$$

Since $\delta\Gamma^{\lambda}{}_{\mu\nu}$ is a difference between two connections, it is a tensor and its covariant derivative can be computed.

$$
\nabla_{\rho}\delta\Gamma^{\lambda}{}_{\mu\nu} = \partial_{\rho}\delta\Gamma^{\lambda}{}_{\mu\nu} + \Gamma^{\lambda}{}_{\rho\sigma}(\delta\Gamma^{\sigma}{}_{\mu\nu}) - \Gamma^{\sigma}{}_{\rho\mu}(\delta\Gamma^{\lambda}{}_{\sigma\nu}) - \Gamma^{\sigma}{}_{\rho\nu}(\delta\Gamma^{\lambda}{}_{\sigma\mu})
$$

Therefore the variation of the Riemann curvature tensor takes the form:

$$
\begin{aligned}
\delta R^{\lambda}{}_{\mu\rho\nu} &= \nabla_{\rho}\delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla_{\nu}\delta\Gamma^{\lambda}{}_{\mu\rho} \\
\because \delta R_{\mu\nu} &= \delta R^{\lambda}{}_{\mu\lambda\nu} \\
\therefore \delta R_{\mu\nu} &= \nabla_{\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla_{\nu}\delta\Gamma^{\lambda}{}_{\mu\lambda}
\end{aligned}
\tag{2.8}
$$

Using Eq (2.7) and Eq (2.8) in Eq (2.3) gives:

$$
\begin{aligned}
\delta S = \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\delta g^{\mu\nu}\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\Big) \\
+ \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}g^{\mu\nu}(\nabla_{\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla_{\nu}\delta\Gamma^{\lambda}{}_{\mu\lambda})
\end{aligned}
\tag{2.9}
$$

The second integral in the above expression can be rewritten as:

$$
\frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\nabla_{\lambda}A^{\lambda} \tag{2.10}
$$

where

$$
A^{\lambda} := (g^{\mu\nu}\delta\Gamma^{\lambda}{}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}{}_{\mu\nu})
$$

This is an integral of a total derivative, which can be expressed as a surface integral on $\partial\mathcal{M}$. Demanding that A^{λ} vanishes on $\partial\mathcal{M}$[1], this integral can be dropped and will not be considered again in this section.

[1]This is not rigorous as A^{λ} contains the metric as well as its derivatives. A detailed treatment of boundary terms is given in Section 2.2.

Substituting Eq (2.9) in the variation of Eq (2.2) and rewriting Eq (2.2) leads to:

$$\delta S_{tot} = \frac{M_{\mathrm{P}}^2}{2} \int \mathrm{d}^4x \sqrt{-g}\delta g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} - \frac{1}{M_{\mathrm{P}}^2} \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \right) \tag{2.11}$$

The Einstein tensor is given by

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

and the symmetric stress-energy tensor is

$$T_{\mu\nu} := \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$$

Setting $\delta S_{tot} = 0$ gives the EFE:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{M_{\mathrm{P}}^2} T_{\mu\nu} \tag{2.12}$$

2.2 Boundary terms

To have a consistent variation principle, boundary terms arising out of the variation need to be dealt with carefully. Varying S_{EH} leads to an integral of a total derivative, as seen in Eq (2.1) which can be expressed as an integral over a boundary. If A^μ in that integral is set to zero on $\partial\mathcal{M}$, this term would vanish. But A^μ is made up of the metric as well as its derivatives (contained in the Christoffel symbols). Setting them both to zero implies a simultaneous imposition of the Dirichlet and the Neumann boundary conditions, which renders the variation principle ill-defined. Thus a careful treatment of the boundary terms arising in the variation of S_{EH} is necessary and is presented in this section.

The dependence of $A^\lambda = (g^{\mu\nu}\delta\Gamma^\lambda{}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^\nu{}_{\mu\nu})$ on the metric and its derivatives can be made more explicit. Metric compatibility of the covariant derivative gives:

$$\begin{aligned}
0 &= \nabla_\alpha g_{\mu\nu} \\
\therefore 0 &= \partial_\alpha g_{\mu\nu} - \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\alpha\nu} g_{\lambda\mu} \\
\therefore 0 &= \partial_\alpha \delta g_{\mu\nu} - \Gamma^\lambda{}_{\alpha\mu} \delta g_{\lambda\nu} - \Gamma^\lambda{}_{\alpha\nu} \delta g_{\lambda\mu} \\
&\quad - g_{\lambda\nu} \delta\Gamma^\lambda{}_{\alpha\mu} - g_{\lambda\mu} \delta\Gamma^\lambda{}_{\alpha\nu} \\
\therefore \nabla_\alpha \delta g_{\mu\nu} &= g_{\lambda\nu} \delta\Gamma^\lambda{}_{\alpha\mu} + g_{\lambda\mu} \delta\Gamma^\lambda{}_{\alpha\nu}
\end{aligned} \tag{2.13}$$

More algebraic manipulation leads to:

$$\begin{aligned}
\delta\Gamma^\lambda{}_{\mu\nu} &= \frac{1}{2} g^{\lambda\alpha} (\nabla_\nu \delta g_{\alpha\mu} + \nabla_\mu \delta g_{\nu\alpha} - \nabla_\alpha \delta g_{\mu\nu}) \\
\therefore g^{\mu\nu} \delta\Gamma^\lambda{}_{\mu\nu} &= \frac{1}{2} g^{\mu\nu} g^{\lambda\alpha} (\nabla_\nu \delta g_{\alpha\mu} + \nabla_\mu \delta g_{\nu\alpha} - \nabla_\alpha \delta g_{\mu\nu})
\end{aligned} \tag{2.14}$$

And further exploiting the dummy indices gives:

$$g^{\mu\lambda} \delta\Gamma^\nu{}_{\mu\nu} = \frac{1}{2} g^{\mu\nu} g^{\lambda\alpha} (\nabla_\nu \delta g_{\alpha\mu} + \nabla_\alpha \delta g_{\mu\nu} - \nabla_\mu \delta g_{\nu\alpha}) \tag{2.15}$$

Using Eq (2.14) and Eq (2.15) in the definition of A^λ leads to:

$$A^\lambda = g^{\mu\nu} g^{\lambda\alpha} (\nabla_\mu \delta g_{\nu\alpha} - \nabla_\alpha \delta g_{\mu\nu}) \tag{2.16}$$

Thus the expression (2.10) can be written as:

$$\begin{aligned}
I &:= \frac{M_{\mathrm{P}}^2}{2} \int_{\mathcal{M}} \mathrm{d}^4 x \, \sqrt{-g} \nabla_\lambda A^\lambda \\
\therefore I &= \frac{M_{\mathrm{P}}^2}{2} \int_{\partial\mathcal{M}} \mathrm{d}^3 y \, \sqrt{|h|} \epsilon n_\lambda A^\lambda \\
\therefore I &= \frac{M_{\mathrm{P}}^2}{2} \int_{\partial\mathcal{M}} \mathrm{d}^3 y \, \sqrt{|h|} \epsilon n_\lambda g^{\mu\nu} g^{\lambda\alpha} (\nabla_\mu \delta g_{\nu\alpha} - \nabla_\alpha \delta g_{\mu\nu})
\end{aligned} \tag{2.17}$$

Here, Gauss's theorem was applied in going from the first step to the second. $\{y^\alpha\}$ are the coordinates on $\partial\mathcal{M}$ and $h = \det \mathbf{h}$ is the determinant of the metric induced on $\partial\mathcal{M}$. n^μ is the unit normal to $\partial\mathcal{M}$ and ϵ is a parameter that depends on the nature of $\partial\mathcal{M}$.

$$h_{\mu\nu} = g_{\mu\nu} - \epsilon n_\mu n_\nu \tag{2.18}$$

$$n^\mu n_\mu = \epsilon = \begin{cases} -1 & \partial\mathcal{M} \text{ spacelike} \\ +1 & \partial\mathcal{M} \text{ timelike} \end{cases} \tag{2.19}$$

To proceed further, imposition of the Dirichlet boundary condition is chosen. The metric will be held fixed on $\partial\mathcal{M}$, that is:

$$\delta \mathbf{g}\big|_{\partial\mathcal{M}} = \delta \mathbf{h} = 0$$

This simplifies Eq (2.16) to:

$$A^{\lambda}\big|_{\partial\mathcal{M}} = g^{\mu\nu}g^{\lambda\alpha}(\partial_{\mu}\delta g_{\nu\alpha} - \partial_{\alpha}\delta g_{\mu\nu}) \equiv g^{\mu\nu}g^{\lambda\alpha}(\delta\partial_{\mu}g_{\nu\alpha} - \delta\partial_{\alpha}g_{\mu\nu}) \quad (2.20)$$

Using Eq (2.20) in Eq (2.17) results in:

$$I = \frac{M_{\mathrm{P}}^2}{2}\int_{\partial\mathcal{M}} \mathrm{d}^3y\, \sqrt{|h|}\epsilon n_{\lambda} g^{\mu\nu}g^{\lambda\alpha}(\delta\partial_{\mu}g_{\nu\alpha} - \delta\partial_{\alpha}g_{\mu\nu}) \quad (2.21)$$

The boundary term is non-vanishing even after imposing the Dirichlet condition. Thus, if a term defined on $\partial\mathcal{M}$ (so that it does not alter the bulk action) that exactly cancels out the contributions of Eq (2.21) is added to S_{EH}, it will result in a well-defined variation principle. This can be achieved with the following:

$$S_{\mathrm{GHY}} = M_{\mathrm{P}}^2 \int_{\partial\mathcal{M}} \mathrm{d}^3y\, \sqrt{|h|}K\epsilon \quad (2.22)$$

It is called the Gibbons-Hawking-York boundary term, which was first realized in [16] and made concrete in [17]. $K = g^{\mu\nu}K_{\mu\nu} = g^{\mu\nu}\nabla_{\mu}n_{\nu}$ is the trace of the extrinsic curvature. It can be shown (for example, see [18]) that $\delta S_{\mathrm{GHY}} = -I$. Therefore, the variation of $S = S_{\mathrm{EH}} + S_{\mathrm{GHY}}$ along with imposing the Dirichlet condition leads to the EFE without any boundary terms. S_{GHY} has further applications in quantum gravity and black hole thermodynamics.

2.3 ΛCDM model

The currently accepted standard model of cosmology is the Big Bang cosmology. The ΛCDM model is a particular parameterized form of big bang cosmology, which employs the least number of parameters to successfully predict and explain most phenomena observed in the universe. ΛCDM model is characterized by the total energy density of the universe, which includes contributions from dark energy, dark matter, baryonic matter, radiation (negligible today), and the curvature. Accurate predictions of accelerated expansion of the universe, Cosmic Microwave Background (CMB) Radiation, Large Scale Structure (LSS) of the universe, and baryogenesis are the highlights of this model.

The underlying assumption is that at the largest scales, the universe is homogeneous and isotropic. This is called the cosmological principle. Based on the cosmological principle, one arrives at the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a(t)^2\Big(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\Big) \tag{2.23}$$

Here, $a(t)$ is the scale factor which relatively parametrizes the expansion of the universe, and $k \in \{0, \pm 1\}$ is the curvature parameter. Its values 0, +1, and −1 correspond to spatially flat, closed, and open geometries. A useful quantity to talk about the expansion of the universe is the Hubble parameter:

$$H(t) = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \tag{2.24}$$

Eq (2.23) can also be written as:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a(t)^2\Big(\mathrm{d}\chi^2 + f_k(\chi)^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\Big) \tag{2.25}$$

$$\text{where } r = f_k(\chi) = \begin{cases} \sinh\chi & k=-1 \\ \chi & k=0 \\ \sin\chi & k=+1 \end{cases}$$

Solving EFE for the FLRW metric along with the symmetric stress-energy tensor of a perfect fluid with density ρ and pressure p

gives the Friedmann equations (overhead dots denote time derivative):

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{1}{3M_{\rm P}^2}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{2.26}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\rm P}^2}(\rho + 3p) + \frac{\Lambda}{3} \tag{2.27}$$

Demanding the covariant conservation of $T_{\mu\nu}$, or alternatively using Eqs (2.26) and (2.27) leads to:

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{2.28}$$

It is important to note that out of Eqs (2.26), (2.27), and (2.28) any two are independent. Using the expression for equation of state of a perfect fluid, $w = p/\rho$ in Eq (2.28) gives:

$$\rho \propto a^{-3(1+w)} \tag{2.29}$$

The density ρ is given by: $\rho = \rho_{dm} + \rho_m + \rho_\gamma$, where the contributors are cold dark matter (dm), baryonic matter (m), and radiation (photons and neutrinos) (γ). It is useful to define a critical density:

$$\rho_c \coloneqq 3M_{\rm P}^2 H^2 \tag{2.30}$$

Eq (2.26) can be rewritten as:

$$\begin{aligned} 3M_{\rm P}^2 H^2 &= \rho_{dm} + \rho_m + \rho_\gamma + M_{\rm P}^2\Lambda - \frac{3M_{\rm P}^2 k}{a^2} \\ \therefore 1 &= \Omega_{dm} + \Omega_m + \Omega_\gamma + \Omega_\Lambda + \Omega_k \end{aligned} \tag{2.31}$$

Ωs are the density parameters defined as a ratio between the density and the critical density. A convenient shorthand is: $1 = \Omega + \Omega_k$.

Quantities defined for 'today' ($t =$ age of the universe till today) are written with '0' as a subscript. $\rho_{c,0} = 3M_{\rm P}^2 H_0^2$, and $\Omega_{i,0} = \rho_{i,0}/\rho_{c,0}$, where according to Eq (2.29), $\rho_i = \rho_{i,0}a^{-3(1+w_i)}$ on setting $a_0 = 1$. Also worth noting is: $w_{dm} = 0 = w_m, w_\gamma = 1/3, w_\Lambda = -1$, and $w_k = -1/3$.

Using these relations in Eq (2.26) and dividing it by H_0^2 leads to:

$$\frac{H^2}{H_0^2} = \underbrace{(\Omega_{dm,0} + \Omega_{m,0})}_{\Omega_{\mathrm{M},0}} a^{-3} + \Omega_{\gamma,0} a^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0} a^{-2} \tag{2.32}$$

$$\therefore \dot{a} = a H_0 \sqrt{\Omega_{\mathrm{M},0} a^{-3} + \Omega_{\gamma,0} a^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0} a^{-2}} \tag{2.33}$$

An expression relating t_0 and H_0 can be obtained as follows:

$$\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}t}$$
$$\therefore \mathrm{d}t = \frac{\mathrm{d}a}{\dot{a}}$$

Using Eq (2.33) in the above expression and integrating along with the initial condition $a(t = 0) = 0$ gives:

$$t_0 = \frac{1}{H_0} \int_0^1 \frac{\mathrm{d}a}{a\sqrt{\Omega_{\mathrm{M},0} a^{-3} + \Omega_{\gamma,0} a^{-4} + \Omega_{\Lambda,0} + \Omega_{k,0} a^{-2}}} \tag{2.34}$$

As seen above, the age of the universe can be calculated in the ΛCDM model solely using observable parameters.

This concludes a brief introduction to standard cosmology based on ΛCDM model. The values of the various parameters that appear in Eq (2.34) are given in Table 2.1. These values are taken from [19, 20].

H_0 [km s^{-1} Mpc^{-1}]	$\Omega_{\mathrm{M},0}$	$\Omega_{\gamma,0}$	$\Omega_{\Lambda,0}$	$\Omega_{k,0}$
67.66 ± 0.42[2]	0.3111 ± 0.0056	$\sim 10^{-5}$	0.6889 ± 0.0056	0.0007 ± 0.0019

Table 2.1: Observed values of certain parameters in the ΛCDM model.

2.4 Drawbacks

As successful as the ΛCDM model has been in explaining most of the observable phenomena in the universe, it is still plagued with certain

[2]Data from missions studying the early universe and the late universe report different values of the Hubble parameter. This difference is statistically significant and it constitutes the problem of 'Hubble tension'. See [21] for a comprehensive review of the Hubble tension.

shortcomings. The universe appears to have evolved out of initial conditions which have been fine-tuned to very specific values for the universe to be observed as it is today. The ΛCDM model by itself does not offer any mechanism to enable the universe to evolve to its current state from generic initial conditions. The problems associated with such fine-tuning of initial conditions are the horizon problem and the flatness problem[3]. This section will present both these problems in detail.

2.4.1 Comoving particle horizon and the Hubble radius

It is often useful to write the FLRW metric in conformal time τ. Making the substitution $\mathrm{d}\tau = \mathrm{d}t / a(t)$ in Eq (2.25) gives:

$$\mathrm{d}s^2 = a(\tau)^2\big(-\mathrm{d}\tau^2 + \mathrm{d}\chi^2 + f_k(\chi)^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\big) \tag{2.35}$$

Eq (2.35) for radial, null geodesics reduces to:

$$0 = a(\tau)^2(-\mathrm{d}\tau^2 + \mathrm{d}\chi^2)$$

Using this one can define a quantity χ_p as:

$$\chi_p \coloneqq \tau - \tau_i = \int_{t_i}^{t} \frac{\mathrm{d}t}{a(t)} \tag{2.36}$$

This is called the particle horizon, which is the maximum comoving distance that light can travel between some initial time t_i and t. Thus events separated by distances $\chi > \chi_p$ could not have been in causal contact in the past (up to time t_i). Note that χ_p is the coordinate distance, and the physical distance is given by $d_p = a(t)\chi_p$. Also,

$$\frac{\mathrm{d}t}{a(t)} = \frac{1}{a}\frac{\mathrm{d}a}{aH} = \mathrm{d}\ln a(aH)^{-1}$$

$(aH)^{-1}$ is the comoving Hubble radius. Substituting the above in Eq (2.36) for $a(t_i = 0) = 0$ gives an expression for particle horizon in terms of the Hubble radius as follows:

$$\chi_p = \int_0^a \frac{1}{a'}\frac{\mathrm{d}a'}{a'H'} = \int_0^a \mathrm{d}\ln a'(a'H')^{-1} \tag{2.37}$$

[3]There also exists the monopole problem, discussing which is beyond the scope of this thesis.

2.4.2 The Horizon Problem

According to ΛCDM, in the early universe $a(t)$ is small and thus contributions of matter and radiation dominate ($w \geq 0$). Using Eq (2.29) in (a truncated version of) Eq (2.26), it can be shown that (for $w \geq 0$):

$$a(t) \propto t^{2/3(1+w)} \tag{2.38}$$

$$\therefore \frac{1}{\dot{a}} = (aH)^{-1} \propto t^{1-2/3(1+w)} \tag{2.39}$$

It is then evident that when $w \geq 0$, both the particle horizon and the Hubble radius increase monotonically. That is, the causally connected region around a point was smaller in the past, and gets bigger in the future.

Thus, there exist regions that are causally connected now but were not in causal contact at the time of CMB last scattering. Inhomogeneities, being gravitationally unstable would grow in time. So any inhomogeneities present over causally disconnected regions at CMB last scattering would be more enhanced at present. Despite that, the CMB observed today is nearly homogeneous. This implies that at CMB last scattering, homogeneity scales were larger than causality scales. Regions that are not causally connected being in equilibrium seems counter-intuitive. One can assume that all the disconnected regions had fine-tuned values of parameters, or classify this as a problem. In physics, it is preferred to proceed with the latter perspective.

2.4.3 The Flatness Problem

The nature of the geometry of spacetime is encoded in the curvature parameter k. As seen in Table 2.1, observations favour a spatially flat universe. $\Omega_0 \coloneqq \Omega_{\mathrm{M},0}+\Omega_{\gamma,0}+\Omega_{\Lambda,0} \sim \mathcal{O}(1)$ and that $\Omega_{k,0} \approx 0 \implies k = 0$ lies within the allowed uncertainties. Also, from earlier definitions:

$$\Omega_k = 1 - \Omega = \frac{-k}{(aH)^2} \tag{2.40}$$

As established in Eq (2.39), the Hubble radius $(aH)^{-1}$ grows with time. Using this in Eq (2.40) shows that $|\Omega - 1|$ keeps increasing, and thus should not be expected to be near zero today. But observations give $\Omega_0 \sim \mathcal{O}(1)$, which contradicts what is expected from standard cosmology. Again, one has a choice between accepting a fine-tuned value of k since the early universe, or to view this as a problem.

It pays to notice that the cause of both the horizon and the flatness problems is the monotonically increasing Hubble radius. This will be instrumental in setting up the premise for inflation.

3 Inflation

Standard cosmology and its shortcomings were discussed in Chapter 2. As illustrated earlier, the horizon and flatness problems both stem from a monotonically increasing Hubble radius in the early universe, as is expected in standard cosmology. It is appealing to think that these problems can be remedied by introducing a period of decreasing Hubble radius in the early universe. This is the central idea of cosmic inflation.

This chapter will cover how inflation solves the problems of standard cosmology, mechanisms that could potentially realize inflation, and a detailed discussion on Starobinsky's model of inflation, which is currently observationally the most favoured model.

References for this chapter include [15, 22, 23].

3.1 Introduction

To realize inflation, a period of decreasing Hubble radius needs to materialize in the early universe. From the expression for the Hubble radius $(aH)^{-1}$, it is evident that this can occur in case of exponential expansion of the universe, for when $a(t) \propto \exp(t)$, $H(t) = \text{constant}$. In this case, $(aH)^{-1}$ clearly decreases over time. Thus, inflation is considered to be a period of de Sitter-like expansion in the early universe. There are many models that offer mechanisms to drive such an exponential expansion, such as chaotic inflation, natural inflation, theories with non-minimal couplings and modified gravity, etc. See [24, and the references therein] for an encyclopaedic reference on models of inflation.

Conditions for inflation

As already discussed, a decreasing Hubble radius is a condition for inflation. Equivalently, it also implies an accelerated expansion, and the violation of the strong energy condition (SEC). The equivalence can be shown as follows:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} &= \frac{-\ddot{a}}{(aH)^2} \\ \therefore \frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0 &\iff \ddot{a} > 0 \\ \ddot{a} > 0 &\iff \rho + 3p < 0 \quad \text{(follows from Eq (2.27))} \end{aligned}$$

3.2 Resolution of Problems

This section shows how the problems faced by standard cosmology are solved in a straightforward and elegant manner by the introduction of an inflationary phase in the early universe. So, although it is tempting to extend the standard model to include inflation, it is important to note that the observations in support of inflation are not yet at desirably high enough confidence levels [see 4]. Observations have also not been able to single out a particular model of inflation, but the data favours some models more than others.

3.2.1 The Flatness Problem

For a non-flat universe, $k = \pm 1$. Eq (2.40) for $k = \pm 1$ gives:

$$|\Omega - 1| = ((aH)^{-1})^2 \tag{3.1}$$

As inflation causes $(aH)^{-1}$ to shrink, $|\Omega - 1|$ is driven to zero and thus, inflation makes $k = 0$ an attractor. This is in line with the observations and does not require k to specifically be zero.

3.2.2 The Horizon Problem

Instead of a monotonically increasing Hubble radius as in standard cosmology, if the Hubble radius decreases in the early universe, it implies that the Hubble radius (and hence the causality scale) was large enough in the very early universe to encompass the homogeneity scales (which are the various modes of density fluctuations) at that time, bringing them in causal contact. The Hubble radius keeps decreasing during inflation, causing the density fluctuations to be 'frozen out'. After the end of inflation, the Hubble radius starts increasing and these density fluctuations are observed to enter the horizon again. Inflation thus renders these density fluctuations to be causally connected in the very early universe, solving the horizon problem, or the need to have finely-tuned values of parameters. See Fig 3.1 for a pictorial and graphical depiction.

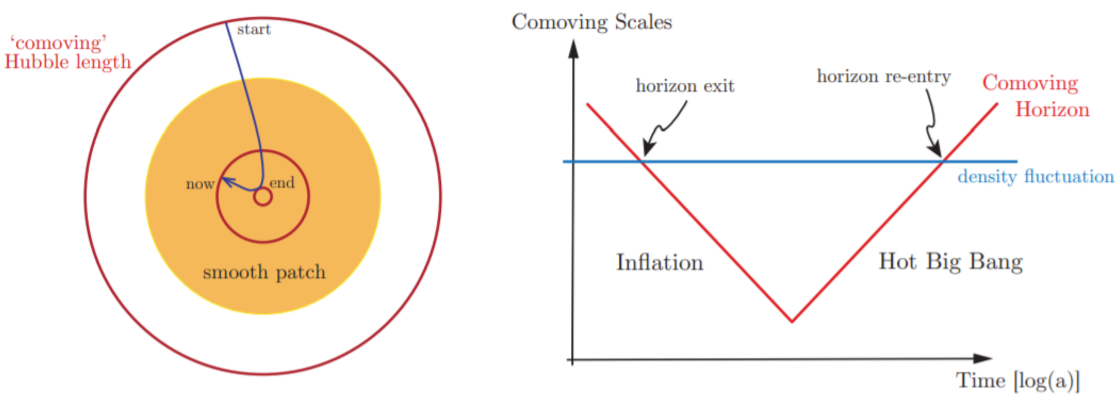

Figure 3.1: *Left:* Pictorial representation of 'bouncing' Hubble radius caused by start and end of inflation, and its effect on causal connectedness of homogeneity scales. *Right:* Graphical representation of the horizon exit and re-entry of density fluctuations caused by inflation. Image taken from [22].

3.3 Slow-roll Inflaton Dynamics

3.3.1 Inflaton dynamics

In the simplest case, dynamics of inflation can be explained by a single scalar field minimally coupled to gravity. This field ϕ is called the inflaton. The dynamics are governed by the actions:

$$S = \int \mathrm{d}^4x \sqrt{-g}\Big(\frac{M_\mathrm{P}^2}{2}R \underbrace{-\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - V(\phi)}_{\mathcal{L}_m}\Big) \tag{3.2}$$

Even this simple case can accommodate many inflation models for different $V(\phi)$. The symmetric energy-momentum tensor for Eq (3.2) is:

$$\begin{aligned} \because T_{\mu\nu} &:= \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \\ \therefore T_{\mu\nu} &= \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}_m \end{aligned} \tag{3.3}$$

Considering $g_{\mu\nu}$ to be the FLRW metric, assuming $\phi \equiv \phi(t)$ to be homogeneous, and comparing Eq (3.3) with $T_{\mu\nu} = (\rho_\phi + p_\phi)u_\mu u_\nu + g_{\mu\nu}p_\phi$ (energy momentum tensor of a perfect fluid) leads to:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.4}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{3.5}$$

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{3.6}$$

As $\ddot{a} > 0 \iff \rho + 3p < 0 \equiv w < -1/3$, accelerated expansion can be achieved from ϕ if $V(\phi)$ dominates $\dot{\phi}^2$ in Eq (3.6).

Varying Eq (3.2) wrt. ϕ and using Eq (3.4) in Eq (2.26) gives:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \tag{3.7}$$

$$H^2 = \frac{1}{3M_\mathrm{P}^2}\rho_\phi = \frac{1}{3M_\mathrm{P}^2}\Big(\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big) \tag{3.8}$$

3.3.2 Slow-roll dynamics

Using Eq (2.27) for ϕ, and introducing $\varepsilon := -\dot{H}/H^2$ gives:

$$\frac{\ddot{a}}{a} = H^2(1-\varepsilon) = -\frac{1}{6M_\mathrm{P}^2}(\rho_\phi + 3p_\phi) \tag{3.9}$$

Substituting Eq (3.8) in Eq (3.9), adding Eq (3.4) with Eq (3.5) and using the result with Eq (3.8) gives:

$$\varepsilon = \frac{3}{2}(w_\phi + 1) = \frac{1}{2M_\mathrm{P}^2}\frac{\dot{\phi}^2}{H^2} \tag{3.10}$$

ε is called the first slow-roll parameter.

In the de Sitter limit, that is $w_\phi = -1 \implies \varepsilon, \dot{\phi}^2 = 0$. But for inflation, merely a de Sitter-like expansion suffices. Thus, imposing the following is enough:

$$\dot{\phi}^2 \ll V(\phi) \implies \varepsilon \ll 1 \tag{3.11}$$

Eq (3.11) is called the first slow-roll condition. In the slow-roll regime, $H^2 \approx V/3M_\mathrm{P}^2 \approx$ constant.

To sustain the accelerated expansion for a sufficiently long time, $\ddot{\phi}$ needs to be small enough, specifically, $|\ddot{\phi}| \ll |3H\dot{\phi}|, |V_{,\phi}|$. This can be imposed by introducing a second slow-roll parameter η and demanding that $|\eta| \ll 1$ (second slow-roll condition).

$$\eta := -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{3.12}$$

Thus, slow-roll inflation is realized for ε, $|\eta| \ll 1$, and ends as $\varepsilon \sim 1$.

Potential slow-roll parameters

The slow-roll conditions presented above can be equivalently given as constraints on the shape of the inflaton potential by introducing the potential slow-roll paramters ϵ_V, η_V.

$$\epsilon_V = \frac{M_{\mathrm{P}}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \tag{3.13}$$

$$\eta_V = M_{\mathrm{P}}^2\left(\frac{V_{,\phi\phi}}{V}\right) \tag{3.14}$$

The end of inflation is then defined as $\varepsilon(\phi_{\mathrm{end}}) \approx \epsilon_V(\phi_{\mathrm{end}}) = 1$.

Number of e-folds

A useful quantity to define is the number of e-folds of expansion that have occurred during inflation. It is denoted by N, with $\mathrm{d}N \coloneqq \mathrm{d}\ln a$. Algebraic manipulation gives $\mathrm{d}\ln a = H\,\mathrm{d}t\; = (H/\dot{\phi})\,\mathrm{d}\phi$. During slow-roll, $\varepsilon \ll 1 \implies 3H^2 \approx V/M_{\mathrm{P}}^2$, and $|\eta| \ll 1 \implies \dot{\phi} \approx -V_{,\phi}/3H$.

$$\therefore \mathrm{d}N = \mathrm{d}\ln a = H\,\mathrm{d}t\; = \mathrm{d}\phi\,\frac{H}{\dot{\phi}} \approx -\frac{\mathrm{d}\phi}{M_{\mathrm{P}}^2}\frac{V}{V_{,\phi}}$$

$$N(\phi) = \int_a^{a_{\mathrm{end}}} \mathrm{d}\ln a = \ln\frac{a_{\mathrm{end}}}{a} \tag{3.15}$$

$$= \int_t^{t_{\mathrm{end}}} \mathrm{d}t\,H \tag{3.16}$$

$$= \int_\phi^{\phi_{\mathrm{end}}} \mathrm{d}\phi\,\frac{H}{\dot{\phi}} \tag{3.17}$$

$$\approx \int_{\phi_{\mathrm{end}}}^{\phi} \frac{\mathrm{d}\phi}{M_{\mathrm{P}}^2}\frac{V}{V_{,\phi}} \tag{3.18}$$

$N(\phi)$ is calculated at the moment of horizon crossing of k_*, an arbitrary pivot mode. Here k is the wave-number associated with a particular mode of density fluctuations. The moment of horizon crossing is $k_* = a(t_*)H(t_*)$[1]. The Planck mission uses $k_* = 0.002\,\mathrm{Mpc}^{-1}$ and $k_* = 0.05\,\mathrm{Mpc}^{-1}$ for its analysis of constraints on inflation [4].

[1] At horizon crossing, wavelength = Hubble radius, thus $\lambda \sim k^{-1} = (aH)^{-1}$. Expressions for sub- and superhorizon modes also follow accordingly.

Contact with observations

The density perturbations in the early universe generated by inflation can be decomposed into scalar, vector, and tensor components. It can be shown that the vector components decay rapidly, and are thus not of relevance. On the other hand, the scalar perturbations lay the seeds of anisotropies observed in the CMB, and the LSS of the universe. The tensor perturbations are said to give rise to primordial gravitational waves, which have not yet been observed. Hence, observations of the aforementioned artifacts lead to constraints on inflationary parameters.

Analysis of observations can be succinctly encoded in the parameters 'scalar spectral index' n_s, and 'tensor-to-scalar power ratio' r. See [15] for more details. These parameters can be expressed in terms of the potential slow-roll parameters as follows:

$$n_s = 1 + 2\eta_V - 6\epsilon_V \tag{3.19}$$

$$r = 16\epsilon_V \tag{3.20}$$

N_* can be interpreted as the number of e-fold expansion needed for the horizon crossing of a mode with wavenumber k_*. Constraints set by [4] on N_*, n_{s_*}, and r_* for $k_* = 0.002\,\mathrm{Mpc}^{-1}$ are:

- $N_* = 50 - 60$
- $n_{s_*} = 0.9649 \pm 0.0042$ at 68% CL
- $r_* < 0.10$ at 95% CL.
  Planck2018 + BICEP2/Keck Array: $r_* < 0.056$

3.4 Starobinsky Inflation

This section provides an introduction to Starobinsky's model of inflation (also called the R^2 inflation model), which according to [4] is

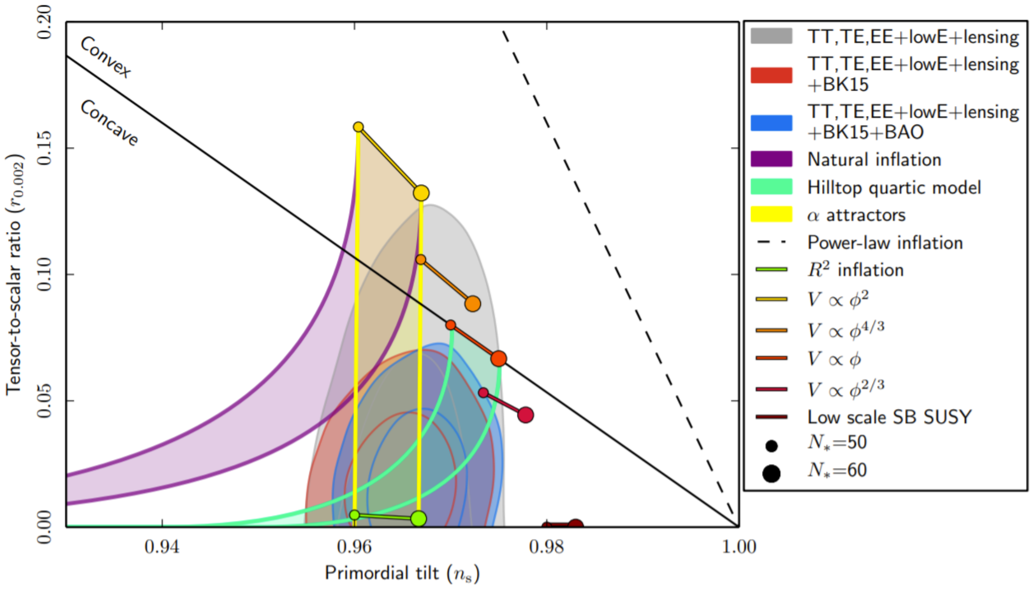

Figure 3.2: n_{s_*} and r_* for $N_* = 50 - 60$ at $k_* = 0.002\,\mathrm{Mpc}^{-1}$ for various models of inflation. Image taken from [4].

observationally the most preferred model of inflation as can be seen in Fig 3.2. Owing to this, the Starobinsky model of inflation will be used to perform further analysis in this thesis so that the results would bear some phenomenological significance. An important thing to mention in the context of Starobinsky's model is the issue of the equivalance and physicality of the Jordan and the Einstein frames. For the purpose of this thesis, the two frames will only be seen as frames related by a conformal transformation, without addressing any of the deeper issues. For an introduction to conformal transformations, Jordan and Einstein frames, see [25]. A modern approach to explore the frames' equivalence is presented in [26].

3.4.1 Introduction

In [27], it was shown that the addition of a quadratic curvature correction to S_{EH} results in a de Sitter-like solution. In Jordan frame, for a modified theory of gravity the action is written using a $f(R)$ term.

The Jordan frame action for Starobinsky's model is:

$$S_{\mathrm{J}} = \frac{M_{\mathrm{P}}^2}{2} \int \mathrm{d}^4x \, \sqrt{-g} \left(R + \frac{R^2}{6M^2} \right) \tag{3.21}$$

Here, M is a mass scale associated with the correction term.

3.4.2 Conformal transformation to the Einstein frame

The Einstein frame action consists of S_{EH} along with a scalar field minimally coupled to gravity. One can transform from Jordan frame to Einstein frame by the means of a suitable conformal transformation involving a scalar field. The scalar field may or may not be a new degree of freedom. Eq (3.21) can be written as Einstein gravity minimally coupled to a scalar field as in [28], and will also be proved here. The proof presented here is along the lines of [29].

The Einstein frame metric $\tilde{\mathbf{g}}$ is related to the Jordan frame metric $\mathbf{g}$ through:

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{3.22}$$

The Ricci scalar in the Jordan frame can be expressed in terms of Einstein frame quantities as:

$$R = \Omega^2 (\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu \omega \tilde{\nabla}_\nu \omega) \tag{3.23}$$

Where $\omega = \ln \Omega$. Eq (3.23) is proved in Appendix B. For $f(R) = R + R^2/6M^2$, Eq (3.21) can be rewritten as:

$$S = \int \mathrm{d}^4x \, \sqrt{-g} \left(\frac{M_{\mathrm{P}}^2}{2} FR - U \right), \text{ where } U \equiv (FR - f(R)) \frac{M_{\mathrm{P}}^2}{2}$$

for some auxilliary field $F \equiv \Omega^2$. Using Eq (3.23) $\sqrt{-g} = \Omega^{-4}\sqrt{-\tilde{g}}$ in the above equation gives:

$$S = \int \mathrm{d}^4x \, \sqrt{-\tilde{g}} \left(\frac{M_{\mathrm{P}}^2}{2} (\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu \omega \tilde{\nabla}_\nu \omega) - \Omega^{-4} U \right)$$

The $\tilde{\Box}\omega$ term is the integral of a total derivative, which can be written as an integral on the boundary and dropped by imposing $\tilde{\nabla}_\mu \omega = 0$ at the boundary. Further, $\phi/M_\mathrm{P} \coloneqq \sqrt{3/2}\ln F \implies \omega = \phi/\sqrt{6}M_\mathrm{P}$ and $V(\phi) \coloneqq \Omega^{-4}U$. Using these in the above equation leads to:

$$S_\mathrm{E} = \int \mathrm{d}^4x\, \sqrt{-\tilde{g}}\left(\frac{M_\mathrm{P}^2}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - V(\phi)\right) \tag{3.24}$$

Eq (3.24) is exactly the action of Einstein gravity minimally coupled to a scalar field.

The next task is to arrive at an expression for the Einstein frame potential $V(\phi)$. Since $F \equiv \Omega^2$ has so far been arbitrary, it has to be given a specific definition to proceed further.

$$\begin{aligned}
F &\coloneqq \frac{\partial}{\partial R}f(R) = \frac{\partial}{\partial R}\left(R + \frac{R^2}{6M^2}\right) = 1 + \frac{R}{3M^2} \\
\therefore R &= 3M^2(F-1) \\
\because V(\phi) &= \Omega^{-4}U = \frac{M_\mathrm{P}^2}{2}\left(\frac{FR - f(R)}{F^2}\right) \\
\therefore V(\phi) &= \frac{M_\mathrm{P}^2}{2}\left(\frac{F(F-1)3M^2 - (F-1)3M^2 - {}^{(F-1)^2 9M^4}/_{6M^2}}{F^2}\right) \\
\therefore V(\phi) &= \frac{3}{4}M_\mathrm{P}^2 M^2\left(\frac{F-1}{F}\right)^2 = \frac{3}{4}M_\mathrm{P}^2M^2\left(1 - \frac{1}{F}\right)^2
\end{aligned} \tag{3.25}$$

Also, $\phi/M_\mathrm{P} \coloneqq \sqrt{3/2}\ln F \implies F = \exp(\sqrt{2/3}\phi/M_\mathrm{P})$. Using this in Eq (3.25) results in:

$$V(\phi) = \frac{3}{4}M_\mathrm{P}^2M^2\left(1 - \exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_\mathrm{P}}\right)\right)^2 \tag{3.26}$$

The mass parameter M is replaced by $\alpha \coloneqq M^2/M_\mathrm{P}^2$. This gives:

$$V(\phi) = \frac{3}{4}\alpha M_\mathrm{P}^4\left(1 - \exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_\mathrm{P}}\right)\right)^2 \tag{3.27}$$

Constraints on the value of V leads to $\alpha \approx (1.13 \times 10^{-5})^2$ [30, 31].

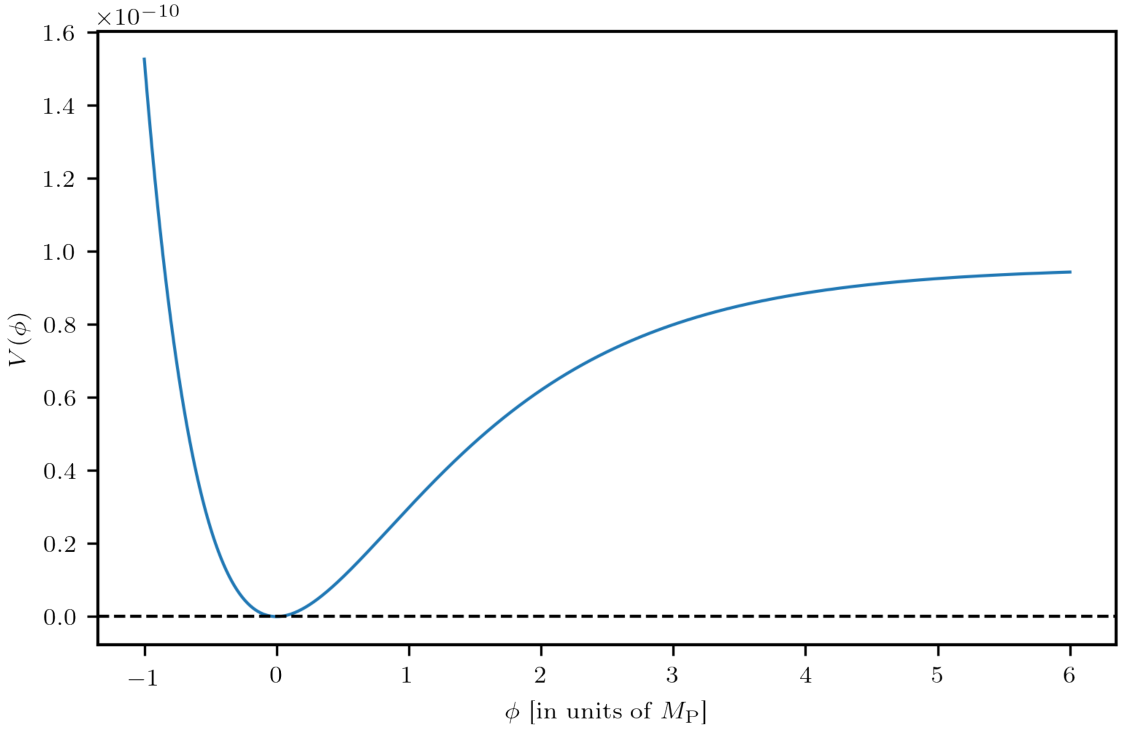

Figure 3.3: $V(\phi)$ vs. ϕ, the scale is set by $\alpha \approx (1.13 \times 10^{-5})^2$.

As can be seen in Fig 3.3, in the region $\phi/M_\mathrm{P} < 0$, $V(\phi)$ is very steep and thus slow-roll inflation cannot be sustained in this region. It will be labelled as the forbidden region for Starobinsky's model.

3.4.3 Slow-roll analysis

From Eqs (3.13), (3.14), and (3.18) for the potential in Eq (3.27) it is straightforward to compute the potential slow-roll parameters:

$$\epsilon_V(\phi) = \frac{4}{3}\left(\exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_\mathrm{P}}\right) - 1\right)^{-2} \tag{3.28}$$

$$\eta_V(\phi) = \frac{4}{3}\frac{\exp\left(-\sqrt{2/3}\,\phi/M_\mathrm{P}\right)\left(2\exp\left(-\sqrt{2/3}\,\phi/M_\mathrm{P}\right) - 1\right)}{\left(1 - \exp\left(-\sqrt{2/3}\,\phi/M_\mathrm{P}\right)\right)^2} \tag{3.29}$$

And the number of e-folds of expansion:

$$
\begin{aligned}
N_* &= \frac{1}{2M_\mathrm{P}}\sqrt{\frac{3}{2}}\int_{\phi_\mathrm{end}}^{\phi_*} \mathrm{d}\phi \left(\exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_\mathrm{P}}\right) - 1\right) \\
\therefore\ N_* &= \frac{3}{4}\left(\exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_\mathrm{P}}\right) - \sqrt{\frac{2}{3}}\frac{\phi}{M_\mathrm{P}}\right)\Bigg|_{\phi_\mathrm{end}}^{\phi_*}
\end{aligned}
\tag{3.30}
$$

ϕ_end can be obtained by requiring $\epsilon_V(\phi_\mathrm{end}) = 1$. This gives:

$$
\frac{\phi_\mathrm{end}}{M_\mathrm{P}} = \sqrt{\frac{3}{2}}\ln\left(1 + \frac{2}{\sqrt{3}}\right) \tag{3.31}
$$

Further, using Eq (3.31) in Eq (3.30), and requiring that $N_* = 50 - 60$, ϕ_* can be obtained numerically. The numerical computations for ϕ_* lead to:

$$
\frac{\phi_*}{M_\mathrm{P}} = \begin{cases} 5.2435 & N_* = 50 \\ 5.4532 & N_* = 60 \end{cases} \tag{3.32}
$$

These values of ϕ_* can be substituted in Eqs (3.28) and (3.29), which can then be used in Eqs (3.19) and (3.20) to arrive at values for n_{s_*} and r_* for the Starobinsky model. The result is encapsulated in Table 3.1. These values lie well within the constraints set by [4].

	$N_* = 50$	$N_* = 60$
n_{s_*}	0.9616	0.9678
r_*	0.0042	0.0030

Table 3.1: Values of the scalar spectral index and the tensor-to-scalar power ratio for Starobinsky's model of inflation.

4 Quantum Cosmology

As mentioned in the introduction, on the one hand there exist compelling arguments in favour of quantizing gravity, and on the other hand there are several difficulties in doing so. Owing to these difficulties, a complete theory of QG does not yet exist. Another major hurdle is the lack of direct experimental or observational input to steer the model-building, as energy scales close to m_{P} are not accessible to us either in terrestrial colliders or in cosmological observations[1]. The range of allowed initial values of inflaton and perturbations generated by the quantum fluctuations of inflaton are inferable from observations, and the predictions on the initial value and quantum corrections to the perturbations differ for different theories of QG. Therefore, QC used in the context of inflation is the best available guide to theorists.

The view adopted in this thesis will be that gravity should be attempted to be quantized without regard to unification of forces. Thus the canonical and path integral quantization regimes are the available choices. In both these approaches, the aim is to arrive at a wavefunction that encodes the dynamics of spacetime at a quantum level, and in the classical limit, would reduce to the classical theory that one started out with (usually GR). At the level of current mathematical advancement, it is only possible to arrive at a 'semiclassical' approximation of the wavefunction, which in the canonical approach corresponds to employing WKB and Born-Oppenheimer type approximation schemes, and in

[1]It is hypothesized that the universe's energy scale was $\sim m_{\mathrm{P}}$ soon after its 'creation' and this era is opaque to EM radiation as photons first decoupled only about 380,000 years later.

the path integral approach reduces to the saddle-point contribution. In going to the full theory, both approaches face certain problems which have not yet been resolved. The boundary conditions that are needed to determine the wavefunction uniquely are implemented straightforwardly using the path integral approach, and thus will be used here.

[1, 32] are the books that were used as references to write this chapter.

4.1 3+1 Formalism of GR

Before getting into the details of path integral quantization of QG, it is pertinent to be acquainted with the basics of $3+1$ formalism of GR, which will be presented here.

It is convenient in many applications of GR to foliate the global spacetime $(\mathcal{M}, \mathbf{g})$ into a set of three-dimensional spacelike hypersurfaces. The general covariance is preserved by admitting all possible foliations. This can be expressed as:

$$\mathcal{M} \cong \mathbb{R} \times \Sigma \tag{4.1}$$

Here, $\mathbb{R}$ denotes a global time function t, and Σ all the three-dimensional spacelike hypersurfaces, each of which is labelled as Σ_t. It is necessary that $(\mathcal{M}, \mathbf{g})$ is globally hyperbolic so that the aforementioned global time function can exist such that $t =$ constant corresponds to a Cauchy surface (on which initial data can be specified) Σ_t.

From Eqs (2.18) and (2.19), for a spacelike hypersurface:

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{4.2}$$

n^μ is the unit normal to Σ_t. From Eqs (4.2) and (2.19) it is clear that $h_{\mu\nu} n^\nu = 0$ and thus $h_{\mu\nu}$ is a three-dimensional entity and will be denoted as h_{ab}.

It is useful to introduce t^μ, a vector field denoting the flow of time which obeys the relation $t^\mu \nabla_\mu t = 1$. A lapse function N given by $N = -t^\mu n_\mu$ denotes the amount of time elapsed between two hypersurfaces. A shift vector N^μ gives the lateral shift in the coordinate of a point in going from one hypersurface to the other. Using these, t^μ can be written as (See Fig 4.1 for visualization):

$$t^\mu = N n^\mu + N^\mu \tag{4.3}$$

It can again be checked that N^μ is a three-dimensional object which will be denoted as N^a. With these, the spacetime metric can be separated into spatial and temporal components resulting in:

$$(g_{\mu\nu}) = \begin{pmatrix} N_i N^i - N^2 & N_a \\ N_a & h_{ab} \end{pmatrix} \tag{4.4}$$

This introduction to the $3+1$ formalism of GR is sufficient for the purpose of the thesis. More details can be found in [1, 11, 12].

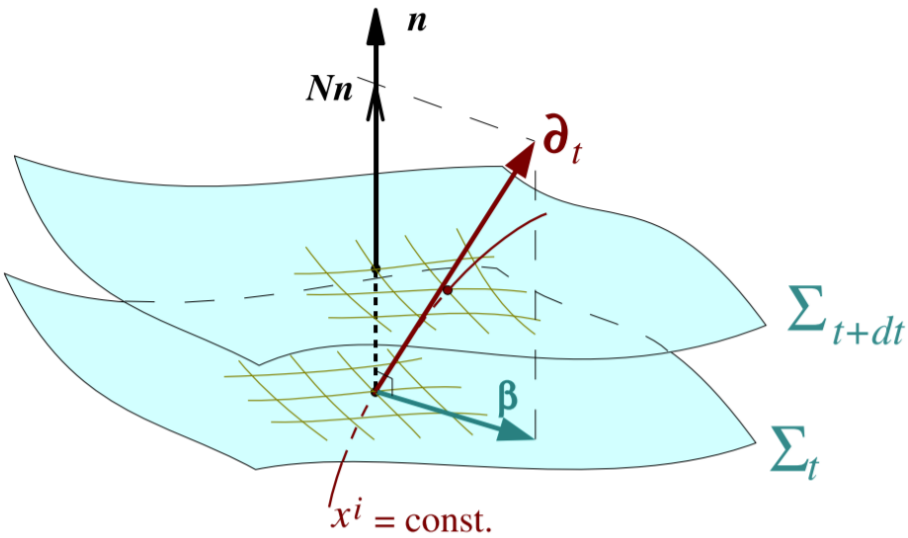

Figure 4.1: Coordinate x^i of a point on the hypersurface Σ_t corresponding to a point on the neighbouring hypersurface $\Sigma_{t+\mathrm{d}t}$. The lapse N and the shift vector $\boldsymbol{\beta}$ ($\mathbf{N}$ in the present terminology) encode how the coordinates change between neighbouring hypersurfaces. Image taken from [33].

4.2 Path Integral Formalism

The path integral formalism is a covariant approach to quantize gravity. Like in quantum mechanics (QM) and quantum field theory (QFT), the path integral is written as a sum over all possible histories[2] (states of existence) of the gravitational field, *ie.* the spacetime metric as seen in Eq (4.5). Unlike in QM and QFT, where the path integral is understood as a propagator between an initial and a final state, in QG its meaning is not yet so well understood. The absence of a background entity, especially the lack of external time in QG entails a rethinking of the path integral's meaning. A few words on its interpretation in the case of quantum cosmology can be found in Chapter 7.

Problems

It is important to note that $\mathcal{D}g_{\mu\nu}(x)$ in Eq (4.5) is only a formal notation. In QM the path integral measure is well defined. In QFT and in QG, a rigorous definition of the integral measure is lacking. The operator ordering problem which is evident in the canonical approach to QG is reflected here also in the ambiguity in defining the integral measure. As in other approaches to QG, the problem of time persists in this formalism. Perturbative expansion also needs to be treated with utmost care since gravity as described by GR is non-renormalizable. Moreover, it is not clear whether an additional sum over topologies is to be included, or whether it is even possible to do so.

4.2.1 Lorentzian path integral

Despite the aforementioned problems, it is of some value to extend the path integral formalism as known from QM and QFT to gravity. In the case of several simple models, the QG wavefunction constructed canonically (under the framework of Quantum Geometrodynamics) matches

[2] To ensure that gauge equivalent paths are not overcounted, gauge fixing and ghost terms need to be introduced as per the Faddeev-Popov prescription. For details, see [1, Chapter 2].

the one constructed by the path integral. Results obtained using this approach could possess important clues or present heuristic motivations for future attempts to arrive at a full theory.

In analogy to QM and QFT, the Lorentzian path integral for QG and QC can be formally written as:

$$\Psi = \int \mathcal{D}g_{\mu\nu}(x)\, e^{iS[g_{\mu\nu}(x),\, \phi(x)]} \tag{4.5}$$

The wavefunction is expressed as an integral over the sum of all possible four-dimensional pseudo-Riemannian metrics (modulo the group of diffeomorphisms of $\mathcal{M}$). The contribution of each metric configuration included in the sum is weighted by the factor e^{iS}, where the action is the Einstein-Hilbert action plus matter contributions (for simplicity, only one scalar field ϕ is considered here).

Besides the problems associated with path integrals in QG/QC, the Lorentzian path integral presents no other conceptual problems. From QM and QFT, it also known that the notions of unitarity and causality are preserved in the Lorentzian version. The only problem faced here is technical in nature. It is the integrand which is oscillatory, and thus not absolutely convergent. A framework to deal with this is presented in Appendix A.

4.2.2 Euclidean path integral

Based on the techniques available to deal with path integrals in QFT, and owing to the complex nature of Eq (4.5) it is tempting to perform a Wick rotation. A transformation such as $t \to \tau = it$ in Eq (4.5) gives[3]:

$$\Psi_{\mathrm{E}} = \int \mathcal{D}g_{\mathrm{E}_{\mu\nu}}(x)\, e^{-S_{\mathrm{E}}[g_{\mathrm{E}_{\mu\nu}}(x),\, \phi(x)]} \tag{4.6}$$

The Euclidean regime is entered upon performing Wick rotation. But the situation here is unlike in QFT in Minkowski spacetime. In case

[3]Here, the subscript E refers to 'Euclidean', and not 'Einstein frame'.

of a background Minkowski spacetime, Wick rotation transforms the symmetry group from the Lorentz group to the group of rotations in four dimensions. The dynamic spacetime present in GR does not possess any such symmetries. Moreover, the effect of a Wick rotation is unclear in case of spacetimes with $\mathrm{d}t\,\mathrm{d}x$ terms in their line elements, as signature won't simply change to $(++++)$ in such cases. The Euclidean path integral is also unbounded from below. This is illustrated by the conformal factor in the following way:

$$g_{\mathrm{E}_{\mu\nu}} \to \tilde{g}_{\mathrm{E}_{\mu\nu}} = \Omega^2 g_{\mathrm{E}_{\mu\nu}}$$

Using Eq (3.23) and dropping the total derivative term gives:

$$\implies S_{\mathrm{E}}[\tilde{g}_{\mathrm{E}}] = -\frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-\tilde{g}_{\mathrm{E}}}\Big(\frac{\tilde{R} - 6(\tilde{\nabla}\ln\Omega)^2}{\Omega^2} - 2\frac{\Lambda}{\Omega^4}\Big)$$

Thus, in cases where the conformal factor is highly varying, the presence of its gradient can allow the action to be arbitrarily high in magnitude with a negative sign. Therefore, the action is unbounded from below.

4.3 Boundary Conditions of the Universe

As in any other (quantum) theory, boundary conditions are essential to uniquely determine the wavefunction. But unlike in other cases, where the boundary conditions can be fixed externally; the boundary conditions of the universe are not available for manipulation. Thus in QC, boundary conditions can only be proposed and the implications of these proposals are then studied to determine their viability. Highlights of two important (well-studied) proposals, the no-boundary and the tunneling proposals for boundary conditions in QC are presented here.

4.3.1 The no-boundary proposal

The no-boundary proposal from Hartle and Hawking in 1983 [5] describes the boundary conditions needed for the universe to emerge out of a zero size. In its original implementation, the Euclidean path integral along with a Dirichlet boundary condition on the initial hypersurface was used. The different implementations of this proposal [7, 9] that have since been introduced also lead to results similar to the original work. In case of a closed universe ($k = 1$), the probability distribution for the initial values of the scalar field (inflaton) ϕ is given by[4]:

$$\mathcal{P}_{\rm NB} \propto \exp\left(\frac{M_{\rm P}^4}{\hbar V(\phi)}\right) \tag{4.7}$$

4.3.2 The tunneling proposal

The tunneling proposal was first put forth by A. Vilenkin in 1982 [34] and was subsequently made more rigorous in [35, 36]. The idea stemmed from the equation of a in FLRW universe for exponential expansion: $a(t) = H^{-1}\cosh(Ht)$. For $t < 0$ this gives a contracting universe, reaching a minimum size $a = H^{-1}$ at $t = 0$, and expanding for $t > 0$. Vilenkin imagined that, like in QM, the universe could be said to tunnel directly into 'existence' in a state corresponding to $t \geq 0$. This could be implemented by imposing that at the boundaries, the resultant wavefunction Ψ should only be outgoing. Unlike in QM, in QG/QC due to the lack of an external time it is not possible to identify or distinguish incoming and outgoing modes. But at a semiclassical level, a consistent definition of outgoing modes is possible. Taking into account the effect on the probability current of the restriction on Ψ, and that the density be positive definite leads to the following closed universe probability distribution for the initial values of the inflaton:

$$\mathcal{P}_{\rm NB} \propto \exp\left(-\frac{M_{\rm P}^4}{\hbar V(\phi)}\right) \tag{4.8}$$

Notice that Eqs (4.7) and (4.8) differ only in sign. This is of importance in QC, as starting with either proposal leads to significantly different predictions for inflationary cosmology.

[4] This can be verified by substituting $k = 1$ in Eq (5.26).

5 Lorentzian Path Integral for the No-Boundary Proposal

In this chapter, detailed calculations for constructing the Lorentzian NBWF are presented, and the main problem associated with it is highlighted. The no-boundary proposal is implemented through the imposition of appropriate boundary conditions on the equation of motion of the scale factor, and the inclusion of corresponding boundary terms with the action. Further, the non-normalizable nature of the NBWF is illustrated but on the basis of prior works, the existence of provisions to work around the this problem is motivated.

5.1 Setting up the Wavefunction

The construction of the NBWF presented here closely follows [9].
To set up the quantum cosmological wavefunction, the following form of the FLRW metric is used ($N \equiv$ the lapse function):

$$\mathrm{d}s^2 = -N(t)^2\,\mathrm{d}t^2 + a(t)^2\Big(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\Big) \tag{5.1}$$

For the task at hand, it is convenient to rescale the lapse function: $N \to N/a$. The metric can then be written as:

$$\mathrm{d}s^2 = -\frac{N(t)^2}{a(t)^2}\,\mathrm{d}t^2 + a(t)^2\Big(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\Big) \tag{5.2}$$

For Eq (5.2), it is straightforward to calculate that the Ricci scalar is:

$$R = \frac{6}{N^2 a^2}(a^3\ddot{a} + 2a^2\dot{a}^2 + kN^2) \tag{5.3}$$

The overhead dot represents differentiation with respect to t.
For convenience, a new variable q is introduced, and is defined as:

$$\begin{aligned} q &\coloneqq a^2 \\ \therefore \dot{q} = 2a\dot{a} &\implies \frac{\dot{q}^2}{4} = a^2\dot{a}^2 \\ \therefore \ddot{a} = 2(a\ddot{a} + \dot{a}^2) &\implies \frac{q\ddot{q}}{2} = a^3\ddot{a} + a^2\dot{a}^2 \end{aligned} \tag{5.4}$$

Using Eqs (5.4) in Eq (5.3) gives:

$$R = \frac{3}{2qN^2}(2q\ddot{q} + \dot{q}^2 + 4kN^2) \tag{5.5}$$

The aim is to finally study the relation between the NBWF, and inflation. It is thus feasible to use $\Lambda \equiv 3H^2$ which holds during exponential expansion. The gravitational action in presence of a boundary term S_{B}, for the metric in Eq (5.2) can therefore be written as:

$$S = \frac{M_{\mathrm{P}}^2}{2}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{-g}(R - 6H^2) + S_{\mathrm{B}} \tag{5.6}$$

$$\therefore \frac{S}{V_3} = M_{\mathrm{P}}^2 \int_0^1 \mathrm{d}t\,\left(\frac{3}{2N}q\ddot{q} + \frac{3}{4N}\dot{q}^2 + 3N(1 - H^2 q)\right) + S_{\mathrm{B}}' \tag{5.7}$$

Here V_3 is the volume of a 3−dimensional hypersurface. $t = 0$ and $t = 1$ can be thought of as merely labels for the initial and final hypersurfaces Σ_0 and Σ_1, and $S_{\mathrm{B}}' \coloneqq S_{\mathrm{B}}/V_3$. Varying Eq (5.5) leads to:

$$\frac{\delta S}{V_3} = M_{\mathrm{P}}^2 \int_0^1 \mathrm{d}t\,\left(\frac{3}{2N}\ddot{q} - 3NH^2\right) + M_{\mathrm{P}}^2\left(\frac{3}{2N}q\delta\dot{q}\right)\Big|_{t=0}^{t=1} + \delta S_{\mathrm{B}}' \tag{5.8}$$

Integration by parts was used in varying the term with the second derivative. Eq (5.8) gives the equation of motion of q on setting $\delta S = 0$.

$$\ddot{q} = 2N^2 H^2 \tag{5.9}$$

In Eq (5.8), setting $S_\mathrm{B} = 0$ leads to the Neumann boundary condition. Recall from Section 2.2 that the Dirichlet condition for gravity is given by Eq (2.22). For a spacelike boundary hypersurface, $\epsilon = -1$, and the trace of the second fundamental form for Eq (5.2) is:

$$\begin{aligned} K &= h^{ij}K_{ij} = h^{ij}\frac{\sqrt{q}}{2N}\frac{\partial}{\partial t}h_{ij} \\ \therefore K &= \frac{3}{2N}\frac{\dot{q}}{\sqrt{q}} \end{aligned} \tag{5.10}$$

As in Section 2.2, $\mathbf{h}$ is the metric induced on the boundary hypersurface.

$$\because S_\mathrm{B} = -M_\mathrm{P}^2 \int_{\partial\mathcal{M}} \mathrm{d}^3y \sqrt{|h|}K = -V_3 M_\mathrm{P}^2 \Big(\frac{3}{2N}q\dot{q}\Big)\Big|_{t=0}^{t=1} \tag{5.11}$$

$$\therefore \delta S'_\mathrm{B} = -M_\mathrm{P}^2 \Big(\frac{3}{2N}q\delta\dot{q} + \frac{3}{2N}\dot{q}\delta q\Big)\Big|_{t=0}^{t=1} \tag{5.12}$$

It is easy to verify that substituting Eq (5.12) in Eq (5.8) does lead to the Dirichlet boundary condition.

For the metric in Eq (5.2), the general expression for a Lorentzian path integral, Eq (4.5) reduces to:

$$\Psi = \int_{\Sigma_0}^{\Sigma_1} \mathrm{d}N\, \delta q\, e^{iS/\hbar} \tag{5.13}$$

The RHS of Eq (5.13) is an oscillatory integral and hence is not absolutely convergent. It can be converted to an absolutely convergent integral by deforming the lapse contour to pass through the critical point(s) of the integrand. This is an application of the Picard-Lefschetz theory. A detailed explanation of this process is given in Appendix A.

The no-boundary proposal is realized here by the imposition of appropriate boundary conditions in solving the equation of motion of q, which in effect corresponds to choosing the right boundary terms for the action. It is found that for the Lorentzian path integral, imposing a Neumann boundary condition on the initial hypersurface, and a Dirichlet boundary condition on the final hypersurface results in the correct implementation of the no-boundary proposal. The boundary conditions subjected to the equation of motion of q, Eq (5.9) are[1]:

$$\frac{\dot{q}}{2N}\bigg|_{t=0} = +i \tag{5.14}$$

$$q\big|_{t=1} = q_1 \tag{5.15}$$

Solving Eq (5.9) subject to Eqs (5.14) and (5.15) gives:

$$q = N^2H^2t^2 + 2iNt + q_1 - N^2H^2 - 2iN \tag{5.16}$$

As discussed earlier, Neumann condition corresponds to $S'_\mathrm{B} = 0$, and for the Dirichlet condition, Eq (5.11) gives S'_B. Therefore,

$$\begin{aligned} S'_\mathrm{B}\big|_{t=0} &= 0 \\ S'_\mathrm{B}\big|_{t=0} &= -M_\mathrm{P}^2\Big(\frac{3}{2N}q\dot{q}\Big)\bigg|_{t=1} \end{aligned} \tag{5.17}$$

Using Eqs (5.16) and (5.17) in Eq (5.7) leads to:

$$\frac{S(N)}{V_3} = M_\mathrm{P}^2(H^4N^3 + 3iH^2N^2 - 3H^2q_1N - 3iq_1 - 3(1-k)N) \tag{5.18}$$

The action is manifestly complex, which is an artefact of the imaginary Neumann boundary condition imposed on the initial hypersurface, which is needed to ensure a regular initial geometry.

For convergence, the lapse contour needs to be deformed to make it pass through the critical points of the integrand in Eq (5.13). Since the integrand is an exponential function, it suffices to find the critical points of S. Setting $S_{,N} = 0$ gives the critical points, which are[2]:

$$N_\pm = -\frac{i}{H^2} \pm \frac{\sqrt{H^2q_1 - k}}{H^2} \tag{5.19}$$

Substituting Eq (5.19) in Eq (5.18) results in:

$$\frac{S(N_+)}{V_3} = M_\mathrm{P}^2\Big(-\frac{2}{H^2}(H^2q_1 - k)^{3/2} - \frac{i}{H^2}(5-3k)\Big) \tag{5.20}$$

$$\frac{S(N_-)}{V_3} = M_\mathrm{P}^2\Big(\frac{2}{H^2}(H^2q_1 - k)^{3/2} - \frac{i}{H^2}(3k-1)\Big) \tag{5.21}$$

[1]The choice $\frac{\dot{q}}{2N}\big|_{t=0} = -i$ would lead to the tunneling wavefunction.
[2]$\mathfrak{Re}[N_\pm]$ with opposite signs denote sums over expanding and contracting universes.

Since the contour of integration has been deformed to pass through $N_\pm$, at a semiclassical level it is enough to consider only contributions of the saddle-points to the entire path integral. This corresponds to the 'tree-level' approximation. Thus at the tree-level, Eq (5.13) reduces to:

$$\Psi \approx e^{iS(N_+)/\hbar} + e^{iS(N_-)/\hbar} \tag{5.22}$$

Using Eqs (5.20) and (5.21) in Eq (5.22) gives:

$$\begin{aligned}\Psi = &\exp\left(\frac{V_3 M_\mathrm{P}^2}{\hbar H^2}(3k-1)\right)\exp\left(\frac{2V_3 M_\mathrm{P}^2 i}{\hbar H^2}(H^2 q_1 - k)^{3/2}\right)\\ &+\exp\left(\frac{V_3 M_\mathrm{P}^2}{\hbar H^2}(5-3k)\right)\exp\left(-\frac{2V_3 M_\mathrm{P}^2 i}{\hbar H^2}(H^2 q_1 - k)^{3/2}\right)\end{aligned} \tag{5.23}$$

Incorporating the inflaton

The Einstein frame action for Starobinsky inflation, Eq (3.24) in the slow-roll regime (negligible kinetic contribution) can be written as:

$$S = \frac{M_\mathrm{P}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\left(R - 2\frac{V(\phi)}{M_\mathrm{P}^2}\right) \tag{5.24}$$

Instead of a cosmological constant, the slowly varying potential of the inflaton can be considered to be a cosmological constant-like term. Substituting $\Lambda_\mathrm{eff} := V(\phi)/M_\mathrm{P}^2$ and $H_\mathrm{eff}^2 := \Lambda_\mathrm{eff}/3$ in Eq (5.24) makes it similar to Eq (5.6) but with $H \to H_\mathrm{eff}$. Thus the result in Eq (5.23) can also be used with $H^2 \to H_\mathrm{eff}^2 = V(\phi)/3M_\mathrm{P}^2$. This leads to:

$$\begin{aligned}\Psi = &\exp\left(\frac{3V_3 M_\mathrm{P}^4}{\hbar V(\phi)}(3k-1)\right)\exp\left(\frac{6V_3 M_\mathrm{P}^4 i}{\hbar V(\phi)}\left(\frac{q_1 V(\phi)}{3M_\mathrm{P}^2} - k\right)^{3/2}\right)\\ &+\exp\left(\frac{3V_3 M_\mathrm{P}^4}{\hbar V(\phi)}(5-3k)\right)\exp\left(-\frac{6V_3 M_\mathrm{P}^4 i}{\hbar V(\phi)}\left(\frac{q_1 V(\phi)}{3M_\mathrm{P}^2} - k\right)^{3/2}\right)\end{aligned} \tag{5.25}$$

Here, $V(\phi)$ is given by Eq (3.27).

Eq (5.25) is the tree-level Lorentzian NBWF. It is straightforward to verify that for $k = 1$ it is real, in line with the result of [5].

5.2 Problem with Predictability

In a quantum theory, peaks in the absolute square of the wavefunction are interpreted as predictions of the theory. As Ψ is constructed by integrating out all degrees of freedom but the inflaton, peaks in $|\Psi|^2$ should correspond to predictions on the initial value of ϕ. Computing the absolute square of Ψ in Eq (5.25) gives:

$$\begin{aligned}|\Psi|^2 = &\exp\left(\frac{2A}{V(\phi)}(3k-1)\right) + \exp\left(\frac{2A}{V(\phi)}(5-3k)\right) \\ &+ 2\exp\left(\frac{4A}{V(\phi)}\right)\left(2\cos^2\left(\frac{B}{V(\phi)}(CV(\phi)-k)^{3/2}\right)-1\right)\end{aligned} \tag{5.26}$$

Where,

$$A \equiv \frac{3V_3M_\mathrm{P}^4}{\hbar},\ B \equiv \frac{6V_3M_\mathrm{P}^4}{\hbar},\ C \equiv \frac{q_1}{3M_\mathrm{P}^2}$$

On inspecting Eq (5.26), one can deduce that the maxima of $|\Psi|^2$ occurs at the minima of $V(\phi)$. On the other hand, from Eq (3.27) it is clear that the minima of $V(\phi)$ occurs at $\phi_{\mathrm{min}} = 0$, and is $V(\phi_{\mathrm{min}}) = 0$ (see Fig 3.3). This causes $|\Psi|^2$ to blow up at ϕ_{min} (see Fig 5.1).

Thus, using the tree-level Lorentzian NBWF in conjunction with the (Einstein frame) potential of Starobinsky's model of inflation renders the NBWF non-renormalizable at ϕ_{min}. This results in loss of predictability in a region of interest. It is not possible to perform any further analysis unless this problem is dealt with.

5.3 Motivations for Workarounds

The NBWF is explored at a 1-loop level in [37]. At a 1-loop level, their calculations lead to a normalized probability density for the NBWF. In [30] based on a phase-space analysis of initial values of the inflaton, it was found that for plateau-like models of inflation (*eg.* Starobinsky's

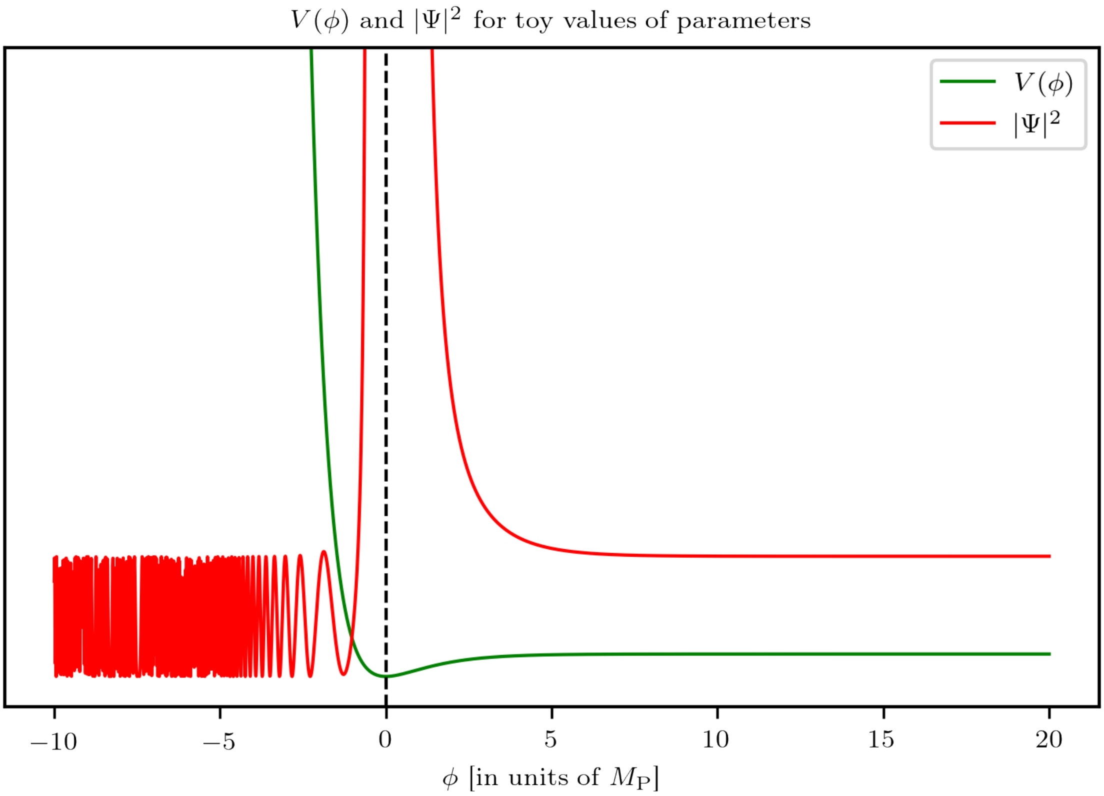

Figure 5.1: A juxtaposition of $V(\phi)$ and $|\Psi|^2$. To make essential features of both simultaneously visible, toy values of parameters are used.

model), inflation can also begin and be sustained for required number of e-folds for initial values of the field around $\phi = 0$.

With regards to the present analysis, the aforementioned works are suggestive in the following way: 1) The problem of non-normalizability appears to be an artefact of working at the tree-level, and 2) There is an opportunity for interesting physics to occur in the region around $\phi_{\mathrm{min}} = 0$. These act as heuristic justifications for the provisions that will be laid down in Chapter 6 to work around the non-normalizability.

While a 1-loop analysis was beyond the scope of this master thesis, attempts have been made to 'hide away' the non-normalizability so that further analysis could be carried out at the tree-level without worrying about the failure of predictability.

6 Slow-Roll Analysis

In this chapter, two provisions are suggested which, at first sight, appear to successfully work around the problem of non-normalizability of the tree-level Lorentzian NBWF. The two provisions are labelled 'δ as a regulator', and 'δ as a correction'. A provision that is successful in hiding away the non-normalizability should then offer a reasonable pathway to perform slow-roll analysis. Based on this requirement, it will be shown that 'δ as a regulator' falls short, while 'δ as a correction' naturally leads to the ensuing slow-roll analysis.

While a consistent slow-roll analysis can be performed in the case of 'δ as a correction', it is then shown, by the imposition of a quantum cosmological consistency condition, that even though measures are implemented against the non-normalizability, the tree-level Lorentzian NBWF simply does not peak at a field value that could be considered as a good prediction for the initial value of the inflaton.

6.1 δ as a Regulator

6.1.1 Proposal

To keep the tree-level NBWF from blowing up, a new parameter δ is introduced and added to Eq (3.27) in the following manner:

$$\mathcal{V}(\phi) := V(\phi) + \delta \tag{6.1}$$

The role of δ is to act as a regulator such that even at $V(\phi_{\text{min}}) = 0$, $|\Psi|^2$ does not go to infinity. The presence of δ is desirable only at this stage of checking for predictions, and it should not have any impact on phenomenology. Thus, the results of the slow-roll analysis should be computed in the limit $\delta \to 0$.

6.1.2 Viability

For the computation of slow-roll results in the limit $\delta \to 0$ to be well-defined, the continuity of all the relevant functions must be checked in the same limit. Moreover, these functions should also be well-defined in the region of interest, *ie.* around $\phi_{\text{min}} = 0$. Thus, continuity should be checked in the limit $(\phi, \delta) \to (0, 0)$.

To make calculations easier, it is convenient to introduce α':

$$\alpha' = \frac{4}{3}\frac{\delta}{\alpha M_{\text{P}}^4}$$

Using this in Eq (6.1) results in:

$$\mathcal{V}(\phi) = \frac{3}{4}\alpha M_{\text{P}}^4\left(\left(1 - \exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_{\text{P}}}\right)\right)^2 + \alpha'\right) \tag{6.2}$$

Since $\alpha' \propto \delta$, $\alpha' \to 0 \equiv \delta \to 0$. Computing the first potential slow-roll parameter for Eq (6.2) gives:

$$\begin{aligned}\epsilon_{\mathcal{V}}(\phi) &:= \frac{M_{\text{P}}^2}{2}\left(\frac{\mathcal{V}_{,\phi}}{\mathcal{V}}\right)^2 \\ &= \frac{4}{3}\left(\exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_{\text{P}}}\right) - 1 + \alpha'\frac{\exp\left(2\sqrt{2/3}\,\phi/M_{\text{P}}\right)}{\exp\left(\sqrt{2/3}\,\phi/M_{\text{P}}\right) - 1}\right)^{-2}\end{aligned}$$

To check for the continuity of $\epsilon_{\mathcal{V}}(\phi)$ in the limit $(\phi, \alpha') \to (0, 0)$, consider the third term's limit along two different paths: 1) the limit along $(\phi, \alpha') \to (\text{const.}, 0)$ is 0, and 2) the limit along $(\phi, \alpha') \to (0, \text{const.})$ is not defined. As the limit along two different paths does not match, $\epsilon_{\mathcal{V}}(\phi)$ is not continuous as $(\phi, \alpha') \to (0, 0)$. Therefore, it is not feasible to perform slow-roll analysis under this provision

6.2 δ as a Correction

6.2.1 Proposal

The redefined potential $\mathcal{V}(\phi)$ can still be used to rescue $|\Psi|^2$ from non-normalizability if the condition $\alpha' \to 0$ is not imposed. In such a case, the aforementioned problem in defining the slow-roll parameters will not arise, and thus slow-roll analysis is feasible by construction.

As per this provision, α' will need to have a fixed, non-zero value. Hence, it can be thought of as a 'correction' to the potential that is brought upon by the imposition of the no-boundary proposal. The effect on $|\Psi|^2$, of implementing this provision can be seen in Fig 6.1.

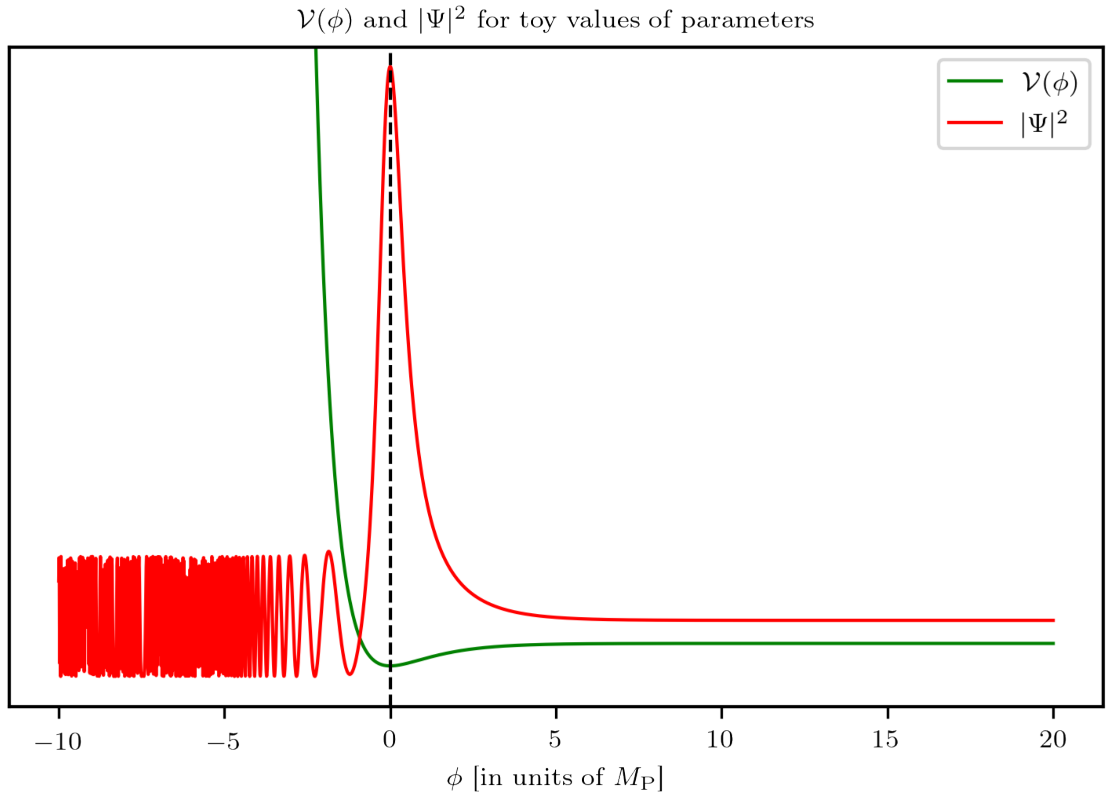

Figure 6.1: A juxtaposition of $\mathcal{V}(\phi)$ and $|\Psi|^2$ for a fixed, non-zero value of α'. To make essential features of both simultaneously visible, toy values of parameters are used.

6.2.2 Viability

As explained, by construction, this provision leads to slow-roll analysis. But, as a permanent modification is being made to the Einstein frame potential of the Starobinsky model, it is pertinent to check the consequences of this change on the Jordan frame action. It should be verified that the dynamics are not altered drastically.

The modified Einstein frame action for $\mathcal{V}(\phi)$ given by Eq (6.2) is:

$$\mathcal{S}_{\mathrm{E}} = \int \mathrm{d}^4x\,\sqrt{-\tilde{g}}\left(\frac{M_{\mathrm{P}}^2}{2}\tilde{R} - \frac{1}{2}\tilde{\nabla}^{\mu}\phi\tilde{\nabla}_{\mu}\phi - \mathcal{V}(\phi)\right) \tag{6.3}$$

This leads to the following modified Jordan frame action:

$$\mathcal{S}_{\mathrm{J}} = \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\left(\left(R + \frac{R^2}{6\alpha M_{\mathrm{P}}^2}\right)(1-\alpha') - \frac{3\alpha M_{\mathrm{P}}^2}{2}\alpha'\right) \tag{6.4}$$

This result is proved in Appendix B.

The equation of motion of S_{J} as given in Eq (3.21) is:

$$\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{1}{3\alpha M_{\mathrm{P}}^2}\left(RR_{\mu\nu} - \frac{1}{4}R^2 g_{\mu\nu} + g_{\mu\nu}\Box R - R_{;\mu\nu}\right) = 0 \tag{6.5}$$

Whereas, the equation of motion of $\mathcal{S}_{\mathrm{J}}$ as given in Eq (6.4) is:

$$\begin{aligned}&\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{3\alpha M_{\mathrm{P}}^2}{4}\frac{\alpha'}{(1-\alpha')}g_{\mu\nu}\right)\\ &+ \frac{1}{3\alpha M_{\mathrm{P}}^2}\left(RR_{\mu\nu} - \frac{1}{4}R^2 g_{\mu\nu} + g_{\mu\nu}\Box R - R_{;\mu\nu}\right) = 0\end{aligned} \tag{6.6}$$

Eqs (6.5) and (6.6) are also derived in Appendix B.

Thus, the equations of motion of S_{J} and $\mathcal{S}_{\mathrm{J}}$ differ only by the presence of a cosmological constant-like term in the latter. Further analysis (which will be presented later) leads to $\alpha' = 1/(3+2\sqrt{3})$ (see Eq (6.14)), which ensures that this additional term is positive. As worked out in [27], S_{J} leads to a de Sitter-like solution, and as $\mathcal{S}_{\mathrm{J}}$ adds only a positive cosmological constant term, the nature of the dynamics is not altered. The expansion continues to be exponential, but at a modified rate.

6.2.3 Slow-roll results

It is first necessary to arrive at the expressions of the slow-roll parameters for the corrected potential of the Starobinsky model. Direct computation leads to the following results for $\epsilon_{\mathcal{V}}(\phi)$, $\eta_{\mathcal{V}}(\phi)$, and N_*:

$$\begin{aligned}\epsilon_{\mathcal{V}}(\phi) &= \frac{M_{\mathrm{P}}^2}{2}\left(\frac{\mathcal{V}_{,\phi}}{\mathcal{V}}\right)^2 \\ &= \frac{4}{3}\left(\exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_{\mathrm{P}}}\right) - 1 + \alpha' \frac{\exp\left(2\sqrt{2/3}\,\phi/M_{\mathrm{P}}\right)}{\exp\left(\sqrt{2/3}\,\phi/M_{\mathrm{P}}\right) - 1}\right)^{-2} \end{aligned} \tag{6.7}$$

$$\begin{aligned}\eta_{\mathcal{V}}(\phi) &= M_{\mathrm{P}}^2\left(\frac{\mathcal{V}_{,\phi\phi}}{\mathcal{V}}\right) \\ &= \frac{4}{3}\frac{\exp\left(-\sqrt{2/3}\,\phi/M_{\mathrm{P}}\right)\left(2\exp\left(-\sqrt{2/3}\,\phi/M_{\mathrm{P}}\right) - 1\right)}{\left(1 - \exp\left(-\sqrt{2/3}\,\phi/M_{\mathrm{P}}\right)\right)^2 + \alpha'} \end{aligned} \tag{6.8}$$

$$\begin{aligned}N_* &= \int_{\phi_{\mathrm{end}}}^{\phi_*} \frac{\mathrm{d}\phi}{M_{\mathrm{P}}^2}\frac{\mathcal{V}}{\mathcal{V}_{,\phi}} \\ &= \frac{3}{4}\left(\exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_{\mathrm{P}}}\right) - \sqrt{\frac{2}{3}}\frac{\phi}{M_{\mathrm{P}}}\right. \\ &\qquad \left.\left. + \alpha'\left(\exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_{\mathrm{P}}}\right) + \ln\left(\exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_{\mathrm{P}}}\right) - 1\right)\right)\right)\right|_{\phi_{\mathrm{end}}}^{\phi_*} \end{aligned} \tag{6.9}$$

It can be verified that as $\alpha' \to 0$, *ie.* as $\mathcal{V}(\phi) \to V(\phi)$, Eqs (6.7), (6.8), and (6.9) reduce to Eqs (3.28), (3.29), and (3.30) respectively, as is required for consistency.

Constraints on n_s from Eq (3.19), r from Eq (3.20), and N_* can be used to either arrive at a value for α', or at least establish constraints on its possible values. As a first step towards determining the value of α', the following conditions are imposed: 1) $\epsilon_{\mathcal{V}}(\phi_{\mathrm{end}}) = 1$, and 2) for $\alpha' = 0$, $\epsilon_{\mathcal{V}}(\phi_{\mathrm{end}}) = \epsilon_V(\phi_{\mathrm{end}})$. This gives:

$$\exp\left(\sqrt{\frac{2}{3}}\frac{\phi_{\mathrm{end}}}{M_{\mathrm{P}}}\right) = \frac{1 + \sqrt{3} + \sqrt{1 - (3 + 2\sqrt{3})\alpha'}}{\sqrt{3}(1 + \alpha')} \tag{6.10}$$

The LHS of Eq (6.10) is an exponential of a real-valued field. Thus, the RHS has to be a positive real number. This constrains α' as follows:

$$-1 < \alpha' \leq \frac{1}{3 + 2\sqrt{3}} \approx 0.1547 \tag{6.11}$$

Further, numerical computation was used to compute values of n_{s_*} and r_*, for different values of α' throughout its allowed range in (6.11), for both $N_* = 50$, and $N_* = 60$. The results have been tabulated for a few representative values of α' in Table 6.1.

	$N_* = 50$		$N_* = 60$	
α'	n_{s*}	r_*	n_{s*}	r_*
-0.99	0.9626	0.0039	0.9686	0.0028
-0.75	0.9624	0.0039	0.9685	0.0028
-0.50	0.9622	0.0040	0.9683	0.0028
-0.25	0.9619	0.0041	0.9681	0.0029
0	0.9616	0.0042	0.9678	0.0030
$1/(3+2\sqrt{3})$	0.9611	0.0043	0.9675	0.0030

Table 6.1: Variation in n_{s_*} and r_* with respect to α'.

It is observed that for values of α' throughout its allowed range (6.11), the parameters n_{s_*} and r_* take on values that lie within the constraints set by [4]. Thus, apart from giving a preliminary constraint on α', slow-roll analysis fails to either further tighten the constraints on α', or rule out this theory.

6.2.4 Quantum cosmological consistency condition

Results of the slow-roll analysis necessitate further investigation into the theory. One way ahead is to impose the quantum cosmological consistency condition first introduced in [38], in the context of exploring the tunneling wavefunction's predictions on inflation. The quantum cosmological consistency condition acts as a link between quantum cosmology and inflation phenomenology. It is expressed as:

$$E_{\mathrm{QC}} \approx E_{\mathrm{model}} \tag{6.12}$$

Where $E_{\mathrm{QC}} := \mathcal{V}(\phi_{\min})^{1/4}$ is the quantum cosmological prediction for inflationary energy, and $E_{\mathrm{model}} := \mathcal{V}(\phi_*)^{1/4}$ is the prediction for inflationary energy from the model. Provided that the foundational assumptions are valid, this relation can also hold exactly, but only the saddle-point contribution of the Lorentzian path integral is considered here, which gives a semiclassical approximation of E_{QC}.

Using the definitions of E_{QC} and E_{model}, Eq (6.12) for the tree-level Lorentzian NBWF reads:

$$\mathcal{V}(\phi_{\min} = 0) \approx \mathcal{V}(\phi_*) \tag{6.13}$$

For this relation to be satisfied, ϕ_* would have to be, within the constraints of the theory, as close as possible to $\phi_{\min} = 0$. Variation in ϕ_* with respect to α' should be analyzed and the value of α' for which ϕ_* is the closest to 0 will be singled out. At this value of α', validity of the quantum cosmological consistency condition will be checked.

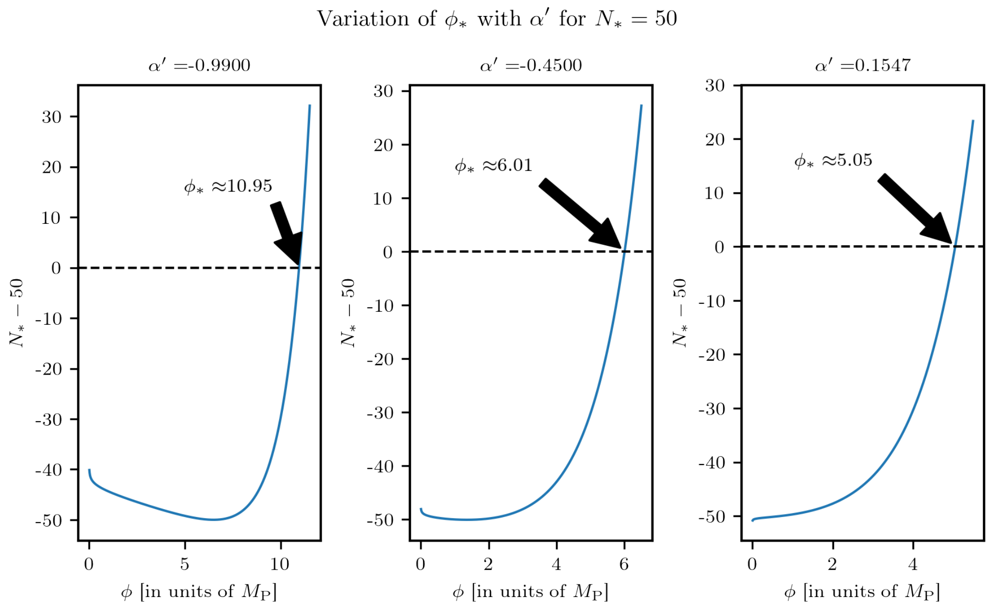

Figure 6.2: Decreasing trend seen in ϕ_* for increasing α' in the case of $N_* = 50$

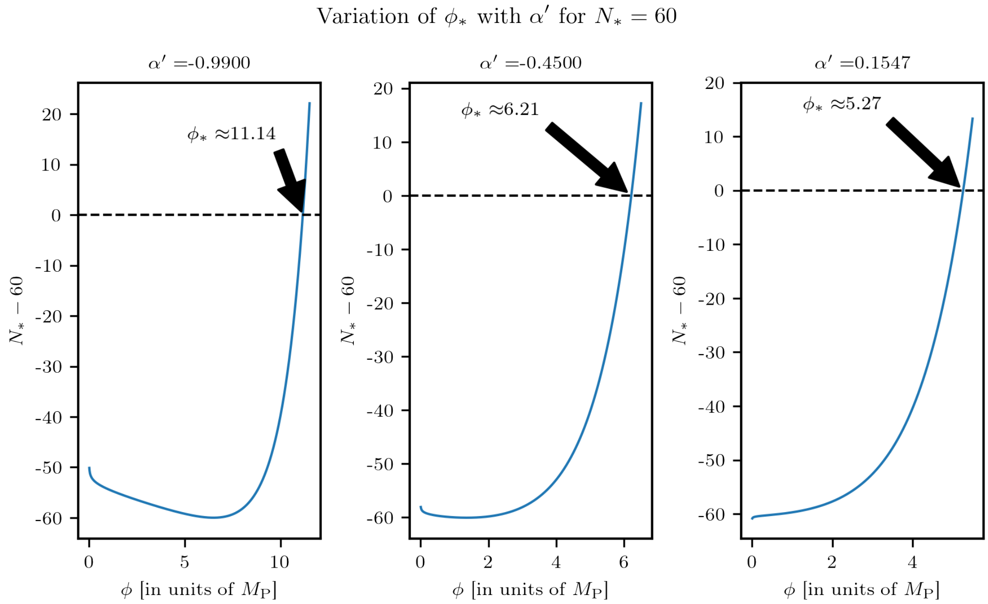

Figure 6.3: Decreasing trend seen in ϕ_* for increasing α' in the case of $N_* = 60$

The result of computational analysis as seen in Figs 6.2 and 6.3 shows a decreasing ϕ_* for increasing values of α' for both $N_* = 50$, and $N_* = 60$. To further verify this trend, in Fig 6.4, ϕ_* is plotted against α' in a reduced domain consisting only of its non-negative (higher) values from (6.11), again for both values of N_*.

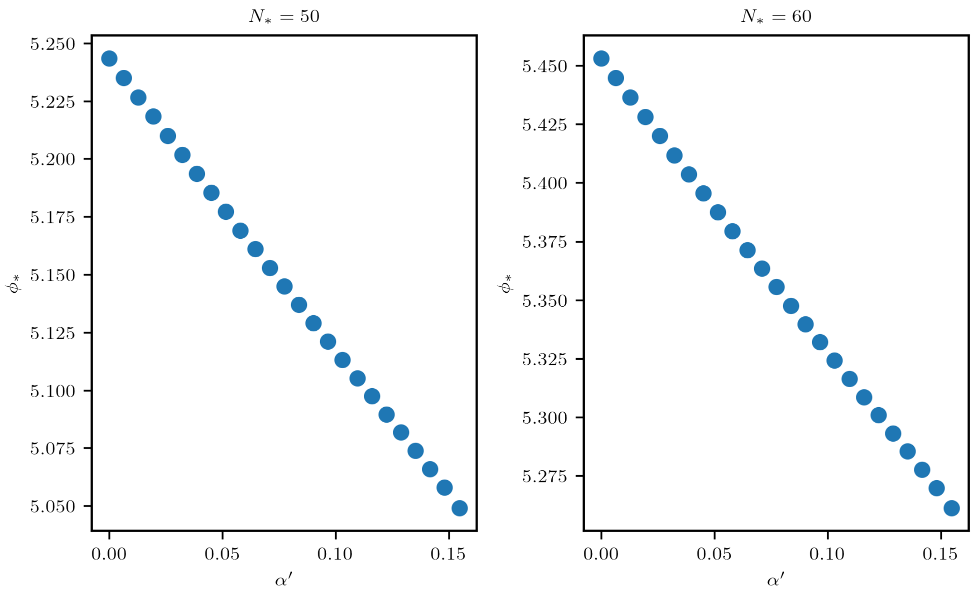

Figure 6.4: Monotonically decreasing ϕ_* for $0 \leq \alpha' \leq 1/(3 + 2\sqrt{3})$

As ϕ_* remains positive for all values of α', the lowest allowed value of ϕ_* would be closest to $\phi_{\text{min}} = 0$. This occurs for the highest allowed value of α', which is:

$$\alpha' = \frac{1}{3 + 2\sqrt{3}} \tag{6.14}$$

For this value of α',

$$\frac{\phi_*}{M_{\text{P}}} = \begin{cases} 5.0491 & N_* = 50 \\ 5.2612 & N_* = 60 \end{cases} \tag{6.15}$$

On comparing with Eq (3.32), it is clear that introducing the correction term to the Starobinsky potential does not significantly alter the value of ϕ_* (and consequently all the quantities that depend on it).

For $\alpha' = 1/(3+2\sqrt{3})$ and the resultant values of ϕ_* for each value of N_*, E_{QC} and E_{model} are computed and checked for closeness. The result is in Table 6.2.

	$N_* = 50$	$N_* = 60$
$E_{\text{model}}/E_{\text{QC}}$	4.4720	4.6694

Table 6.2: The ratio of E_{QC} to E_{model} computed at $\alpha' = 1/(3 + 2\sqrt{3})$.

It is thus clear that the quantum cosmological consistency condition is not satisfied closely enough (if expressed in terms of potentials instead of energies, the ratio in Table 6.2 would be ~ 450). Therefore, even with corrections to rescue from non-normalizability, the tree-level Lorentzian NBWF does not peak at a field value that could be considered to be a good prediction for (Starobinsky) inflation.

7 Intepretation of Path Integrals in Quantum Cosmology

The interpretation of path integrals in QM and QFT is straightforward. The presence of an external time allows for path integrals to be thought of as probability amplitudes for a given state to transition to another state over time. It is thus a propagator, or a time-dependent Green's function. Whereas in QG/QC the same interpretation cannot be given to the path integral, due to the absence of external time. It is thus important to rethink the path integral's role in present context.

From a first glance at the QC path integral as seen in Eq (4.5), it is clear that the integration is over all possible metric configurations. Integrating out the spacetime metric results in a timeless wavefunction. It was shown in [39] that starting from the Hamiltonian of the theory enables a detailed exploration of the nature of this path integral. The perspectives presented here are in agreement with the analysis performed in [39], the crux of which is demonstrated in the following.

Consider a superspace[1] spanned by the coordinates $\{q^a\}$[2] (gravitational + matter degrees of freedom) with a superspace metric denoted by G_{ab}. The Hamilton is then given by:

$$H = \frac{1}{2}G^{ab}(q)p_a p_b + V(q) \tag{7.1}$$

Where p_a is the conjugate momentum. To proceed with the path in-

[1]Superspace is the configuration space in a theory of gravity, introduced by John Wheeler in [11].

[2]Latin alphabets are used here to denote the superspace indices.

tegral quantization, the prescription laid out in [40] gives ($\hbar = 1$):

$$\Psi = \int \mathcal{D}N\mathcal{D}p_a\mathcal{D}q^a \exp\Big(i\int \mathrm{d}t\,(p_a\dot{q}^a - NH)\Big) \tag{7.2}$$

N is introduced here as a Lagrange multiplier. In cosmological context, it is straightforward to recognize it as the lapse function. Thus it clear that the N integral gives the total time elapsed, say from 0 to T. Therefore, Eq (7.2) leads to:

$$\begin{aligned}\Psi &= \int \mathrm{d}T\,\mathcal{D}p_a\mathcal{D}q^a \exp\Big(i\int_0^T \mathrm{d}t\,(p_a\dot{q}^a - H)\Big) \\ &= \int \mathrm{d}T\,\langle q'', T; q', 0\rangle \end{aligned} \tag{7.3}$$

$\langle q'', T; q', 0\rangle$ is the QM propagator that would satisfy the Schrödinger equation. It is thus clear that Ψ is not the propagator as one would expect from the QM path integral.

The meaning of Ψ can be elucidated as follows:

$$\Psi \equiv G(q', q'') = 2\pi\tilde{G}(q;, q''; E)|_{E=0} \tag{7.4}$$

$\tilde{G}$ is the energy-dependent Green's function which is given as:

$$\tilde{G}(q', q''; E) = \frac{1}{2\pi}\int \mathrm{d}T\, e^{iET}\langle q'', T; q', 0\rangle \tag{7.5}$$

By identifying Ψ as the energy-dependent Green's function (computed at $E = 0$), it is clear that the composition law for path integrals in QM does not apply and so Ψ cannot be thought of as a propagator between an initial and a final state.

The canonical quantization method offers further motivation to reinforce this interpretation of Ψ. The Wheeler-DeWitt equation of the canonical quantization approach[3], $H\Psi = 0$ can be rewritten as an eigenvalue equation $H\Psi = E\Psi$ for $E = 0$.

[3] Follows from the Hamiltonian constraint for an appropriately constructed Hamiltonian. See [1, Chapter 5] for details.

For completeness, it is important to also discuss the interpretation of QM itself in the context of QC. In QC, the 'wavefunction of the universe' is the central object of interest. The presence of this entity naturally leads to the many-worlds interpretation (MWI) of QM. This was proposed by Hugh Everett (so it is also called the Everettian interpretation) in 1957 [41] to explicitly be able to apply QM for the entire universe. MWI does away with the popular Copenhagen interpretation's requirement for inherently classical entities (*eg.* the observer) apart from the quantum realm. Disregarding the Copenhagen interpretation also frees the theory from the non-local concept of collapse of the wavefunction. MWI is a decoherence-based interpretation of QM. Decoherence is the emergence of classicality through the inevitable interactions of a quantum system with its environment. This concept was first introduced by H. Dieter Zeh in 1970 [42][4]. The emergence of classical properties in QC using decoherence is discussed in [44]. In MWI, the semiclassical wavefunction that is used for analysis in this thesis is considered as a branch of the 'universal wavefunction'.

[4]An introduction to this topic along with various examples can be found in [43].

8 Summary, Conclusions, Outlook

The main aim of this work was to analyze the NBWF to check if its predictions on the initial value of the inflaton match those suggested by the specific model of inflation under consideration. In other words, to study whether the boundary condition on the initial hypersurface as set by the no-boundary proposal (see Eq (5.14)) leads to a wavefunction that could that could be considered 'useful' in giving rise to inflation.

Having decided upon the path integral approach to quantize gravity, a further choice needed to be made between the Euclidean and the Lorentzian path integrals. While the Euclidean version has unresolved conceptual problems, the problem Lorentzian path integral is technical in nature. A framework to deal with this problem of non-convergence of the integral due to the oscillatory nature of the integrand was provided, in the form of an application of the Picard-Lefschetz theory. On appropriately deforming the contour of the lapse integral, the overall integral converges absolutely. Further, as previous implementations of the no-boundary proposal [5, 7] were shown to have problems with stability of fluctuations [6, 8], a redefined implementation of the no-boundary proposal [9] was preferred. In this redefinition, instead of imposing a Dirichlet boundary condition on the initial hypersurface, an imaginary Neumann boundary condition was imposed there. Using these prescriptions, the path integral corresponding to the no-boundary proposal was set up, and the NBWF was calculated in the saddle-point approximation. On incorporating the potential of a single scalar field, the inflaton, the NBWF took the form seen in Eq (5.25).

Starobinsky's model of inflation was chosen for model-specific analysis owing to it being favoured by CMB observations. The NBWF would peak (has maxima) at the minima of the inflaton potential. The potential of the Starobinsky model had a minima $V(\phi_{\text{min}} = 0) = 0$. As a result, the NBWF is non-normalizable in a region of interest. Previous works [30, 37] motivated working around this problem at the tree-level as a one-loop analysis was beyond the scope of this thesis. Two workarounds were suggested. While one of the suggested workaround did not allow for slow-roll analysis, a detailed slow-roll analysis was possible with the other workaround.

Introducing a fixed, non-zero δ in the Starobinsky potential kept the NBWF from blowing up at $\phi_{\text{min}} = 0$. Results of the slow-roll analysis gave an allowed range of values for δ, but no further tightening of constraints or falsification of this theory was possible based on the slow-roll results. This necessitated further investigation which was done by imposing the quantum cosmological consistency condition, seen in Eq (6.12). The quantum cosmological consistency condition establishes a relation between quantum cosmology and inflation phenomenology.

It was found that the quantum cosmological consistency condition is not satisfied, thus the conclusion of this work is that despite taking measures to work around the non-normalizability of the NBWF, the NBWF simply does not peak at a field value that can be termed as good prediction on the initial value of the inflaton. This result is in line with previous works such as [37] which showed that at the tree-level the NBWF is non-normalizable, and [38, 45] which showed that it was the tunneling proposal that led to a wavefunction (both at tree- and one-loop level) which gave a favourable prediction for the initial value of inflaton.

As a consequence of this result, the validity of either the no-boundary proposal or inflation (manifested by Starobinsky model) is called into question. Inflation not only resolves elegantly the problems of standard cosmology, but also provides a mechanism for the generation of the perturbations that are observed in CMB, LSS, and hopefully the primordial gravitational waves. Observational confidence in inflation is also growing based on the results of analysis of data from the Planck mission [4]. Thus, the view adopted in this work is that the result disfavours the no-boundary proposal, and not inflation.

It is important to mention that certain assumptions go into the validity of the quantum cosmological consistency condition. The major ones are: 1) The semiclassical quantum cosmological description of the early universe is accurate, and 2) Performing the inflation-era computations by considering a quantum field on a curved, non-dynamic background spacetime is justified. As these are the best tools available at the moment, they are considered axiomatic. In the future, research could be performed to carefully analyze the validity of the quantum cosmological consistency condition by scrutinizing the aforementioned assumptions. Another possible line of research that could be pursued is to attempt to take into account the 'beyond tree-level' contributions in arriving at the wavefunction for the redefined no-boundary proposal. This could be done via direct computation, or by making use of renormalization group techniques. Additionally, instead of using Eq (5.14) in constructing the wavefunction, as mentioned in the first footnote on page 40, the boundary condition for the tunneling proposal could be used as well, leading to the tunneling wavefunction. The resultant wavefunction could again be tested for utility in predicting inflation.

A Convergence of Lorentzian Path Integrals

This appendix briefly outlines the framework under which a (highly) oscillatory integrand of the Lorentzian path integral is rendered absolutely convergent, via an application of the Picard-Lefschetz theory. The Picard-Lefschetz theory was first used in physical context in [46], whereas its use in Lorentzian quantum cosmology was first given in [47]. The outline sketched in this appendix is based on [47], and the same can be referred for a more detailed proof of convergence.

Consider an oscillatory integral of the type:

$$I = \int_D \mathrm{d}x\, e^{iS[x]/s} \tag{A.1}$$

Where s is a 'small' real parameter, and S is a real-valued function of x over a real domain D. In quantum mechanical applications, $s \equiv \hbar$, and $S[x]$ is the action. In present context, Eq (5.13) reduces to Eq (A.1) after solving the q integral (which was done exactly) for $x \equiv N$. To apply the Picard-Lefschetz theory, $S[x]$ needs to be interpreted as a holomorphic function of $x \in \mathbb{C}$. Thus, by applying the Cauchy's integral theorem, the real domain D can be deformed into a contour onto the complex plane, say $\mathcal{C}$; as long as the endpoints stay the same. The goal is to deform the original contour into a contour of steepest descent, passing through the critical points of the integrand. As the integrand is an exponential, the critical points are simply given by $\partial_x S = 0$.

Let $\mathcal{I} := iS/s$, and $h := \mathfrak{Re}[\mathcal{I}]$, $H := \mathfrak{Im}[\mathcal{I}]$. Thus, h controls the magnitude of the integrand, whereas H influences its oscillatory behaviour. The steepest descent contour through a critical point is defined as the path along which h decreases as rapidly as possible. Here, the steepest descent contours are called the Lefschetz thimbles.

Let u^1 and u^2 be coordinates that span $\mathbb{C}$ and g_{ij} be (components of) the corresponding metric on $\mathbb{C}$. Freedom to choose the coordinates implies a freedom to choose the metric, which can be exploited to fit the needs of the task at hand. For some parameter λ along the flow, downward flow originating from a critical point is defined as:

$$\frac{\mathrm{d}u^i}{\mathrm{d}\lambda} = -g^{ij}\frac{\partial h}{\partial u^j} \tag{A.2}$$

Using Eq (A.2) it can be shown that h decreases monotonically along the downward flow from a critical point in the following way:

$$\frac{\mathrm{d}h}{\mathrm{d}\lambda} = \frac{\partial h}{\partial u^j}\frac{\mathrm{d}u^j}{\mathrm{d}\lambda} = -g^{ij}\frac{\partial h}{\partial u^j}\frac{\partial h}{\partial u^i} = -\Sigma_i\left(\frac{\partial h}{\partial u^i}\right)^2 < 0 \tag{A.3}$$

For $x \in \mathbb{C}$, the metric is $\mathrm{d}s^2 = |\mathrm{d}x|^2$. Using the coordinates $(u^1, u^2) \equiv (u, \bar{u}) := \big((\mathfrak{Re}[x] + i\mathfrak{Im}[x]), (\mathfrak{Re}[x] - i\mathfrak{Im}[x])\big)$ leads to a metric whose components are:

$$g_{uu} = g_{\bar{u}\bar{u}} = 0, \; g_{u\bar{u}} = g_{\bar{u}u} = \frac{1}{2}$$

In these coordinates, $h = (\mathcal{I} + \bar{\mathcal{I}})/2$. Using this, the relation in (A.3) is confirmed as follows:

$$\frac{\mathrm{d}h}{\mathrm{d}\lambda} = -\Sigma_i\left(\frac{\partial h}{\partial u^i}\right)^2 = -\frac{1}{4}\left[\left(\frac{\partial \mathcal{I}}{\partial u}\right)^2 + \left(\frac{\partial \bar{\mathcal{I}}}{\partial \bar{u}}\right)^2\right] < 0 \tag{A.4}$$

Further, it can be checked that:

$$\frac{\mathrm{d}u}{\mathrm{d}\lambda} = -\frac{\partial \bar{\mathcal{I}}}{\partial \bar{u}}, \; \frac{\mathrm{d}\bar{u}}{\mathrm{d}\lambda} = -\frac{\partial \mathcal{I}}{\partial u} \tag{A.5}$$

Using Eq (A.5) with $H = (\mathcal{I} - \bar{\mathcal{I}})/2i$ gives:

$$\frac{\mathrm{d}H}{\mathrm{d}\lambda} = 0 \tag{A.6}$$

Therefore, from Eqs (A.4) and (A.6), it is established that along a downward flow from a critical point, the real part of the exponent of the integrand in Eq (A.1) decreases monotonically, whereas the imaginary part remains constant. Thus, $\exp(iS[x]/s)$ which was purely oscillatory while integrating along the original (real) contour, does not oscillate anymore when evaluated along a contour of downward flow from a critical point. Instead, it decreases monotonically. Therefore, using this prescription it is possible to arrive at the steepest descent contour. Integrating along this contour renders the oscillatory Lorentzian path integral in Eq (A.1) absolutely convergent.

There are more subtleties involved in the complete proof of convergence, as given in [47]. But the outline sketched here is sufficient to show that the Lorentzian path integral evaluated along the steepest descent contour through the critical points does indeed converge. As long as the integral is well-defined throughout the contour (which is guaranteed by the part of the proof not presented here), the exact shape of the contour is not of interest in this application as only the saddle-point contribution is taken into account, for which it suffices that the contour passes through the critical points.

B Conformal Transformations to the Jordan Frame

This appendix contains a compilation of proofs of equations or properties that were used in the main chapters of the thesis, regarding the conformal transformations from the Einstein frame to the Jordan frame.

B.1 Jordan frame Ricci scalar

Eq (3.23) which was used in going from the Jordan frame action of the Starobinsky model to the Einstein frame will be proved in this section (along the lines of a proof presented in [12] for the Einstein frame Ricci scalar). A new notation for the (anti-)symmetrization of indices is introduced here. Parentheses (round brackets) around indices imply a symmetrization over those indices, whereas square brackets imply an antisymmetrization. For example:

$$A_{(\mu}B_{\nu)} \coloneqq \frac{1}{2}(A_\mu B_\nu + A_\nu B_\mu)$$
$$A_{[\mu}B_{\nu]} \coloneqq \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu)$$

For any two derivatives ∇^1 and ∇^2, there exists a relation (for an arbitrary 4-vector $\mathbf{v}$):

$$\nabla^2{}_\mu v_\nu - \nabla^1{}_\mu v_\nu = -C^\sigma{}_{\mu\nu} v_\sigma \tag{B.1}$$

It is clear that the tensor $\mathbf{C}$ is symmetric in its lower indices.

Let $\nabla^1 \equiv \tilde{\nabla}$ and $\nabla^2 \equiv \nabla$ be the covariant derivatives associated with the metrics $\tilde{\mathbf{g}}$ and $\mathbf{g}$ respectively, where $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. For this choice of metrics,

$$C^{\sigma}{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\tilde{\nabla}_\mu g_{\nu\rho} + \tilde{\nabla}_\nu g_{\mu\rho} - \tilde{\nabla}_\rho g_{\mu\nu}) \tag{B.2}$$

Consider,

$$\begin{aligned} \tilde{\nabla}_\rho g_{\mu\nu} &= \tilde{\nabla}_\rho \Omega^{-2} \tilde{g}_{\mu\nu} \\ &= g_{\mu\nu} \Omega^2 \tilde{\nabla}_\rho \Omega^{-2} \\ \therefore \tilde{\nabla}_\rho g_{\mu\nu} &= -2 g_{\mu\nu} \tilde{\nabla}_\rho \underbrace{\ln \Omega}_{=:\omega} \end{aligned} \tag{B.3}$$

Using Eq (B.3) in Eq (B.2) gives:

$$C^{\sigma}{}_{\mu\nu} = -(\delta^\sigma_\nu \tilde{\nabla}_\mu \omega + \delta^\sigma_\mu \tilde{\nabla}_\nu \omega - \tilde{g}_{\mu\nu} \tilde{\nabla}^\sigma \omega) \tag{B.4}$$

Eq (B.1) implies $\nabla_\mu v_\nu = \tilde{\nabla}_\mu v_\nu - C^{\sigma}{}_{\mu\nu} v_\sigma$. Thus, for any $(0,2)$ rank tensor,

$$\nabla_\mu F_{\rho\sigma} = \tilde{\nabla}_\mu F_{\rho\sigma} - C^{\lambda}{}_{\mu\rho} F_{\lambda\sigma} - C^{\lambda}{}_{\mu\sigma} F_{\lambda\rho}$$

Using this,

$$(\nabla_\mu \nabla_\lambda - \nabla_\lambda \nabla_\mu) v_\nu = (\tilde{\nabla}_\mu \tilde{\nabla}_\lambda - \tilde{\nabla}_\lambda \tilde{\nabla}_\mu) v_\nu - 2\tilde{\nabla}_{[\mu} C^{\sigma}{}_{\lambda]\nu} v_\sigma + 2 C^{\rho}{}_{\nu[\mu} C^{\sigma}{}_{\lambda]\rho} v_\sigma \tag{B.5}$$

Also, $R_{\mu\lambda\nu}{}^{\sigma} v_\sigma = (\nabla_\mu \nabla_\lambda - \nabla_\lambda \nabla_\mu) v_\nu$. Using this in Eq (B.5) leads to:

$$R_{\mu\lambda\nu}{}^{\sigma} = \tilde{R}_{\mu\lambda\nu}{}^{\sigma} - 2\tilde{\nabla}_{[\mu} C^{\sigma}{}_{\lambda]\nu} + 2 C^{\rho}{}_{\nu[\mu} C^{\sigma}{}_{\lambda]\rho} \tag{B.6}$$

Where $\mathbf{v}$ is dropped since it is arbitrary.

Substituting Eq (B.4) in Eq (B.6), and further algebraic manipulation results in (the shorthand $(\tilde{\nabla}\omega)^2 \equiv \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \omega \tilde{\nabla}_\nu \omega$ is used):

$$\begin{aligned} R_{\mu\lambda\nu}{}^{\sigma} = \tilde{R}_{\mu\lambda\nu}{}^{\sigma} &- 2\delta^\sigma_{[\mu} \tilde{\nabla}_{\lambda]} \tilde{\nabla}_\nu \omega + 2\tilde{g}_{\nu[\mu} \tilde{\nabla}_{\lambda]} \tilde{\nabla}^\sigma \omega \\ &+ 2\tilde{g}_{\nu[\mu} \tilde{\nabla}_{\lambda]} \omega \tilde{\nabla}^\sigma \omega + 2\delta^\sigma_{[\lambda} \tilde{\nabla}_{\mu]} \omega \tilde{\nabla}_\nu \omega - 2\delta^\sigma_{[\lambda} \tilde{g}_{\mu]\nu} (\tilde{\nabla}\omega)^2 \end{aligned} \tag{B.7}$$

Exploiting the symmetries of the Riemann curvature tensor gives:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = -R^{\lambda}{}_{\mu\nu\lambda} = -R_{\lambda\mu\nu}{}^{\lambda} = R_{\mu\lambda\nu}{}^{\lambda}$$

It is then straightforward to work out that:

$$\begin{aligned} R_{\mu\nu} &= \tilde{R}_{\mu\nu} - 2\delta^{\lambda}_{[\mu}\tilde{\nabla}_{\lambda]}\tilde{\nabla}_{\nu}\omega + 2\tilde{g}_{\nu[\mu}\tilde{\nabla}_{\lambda]}\tilde{\nabla}^{\lambda}\omega \\ &\quad + 2\tilde{g}_{\nu[\mu}\tilde{\nabla}_{\lambda]}\omega\tilde{\nabla}^{\lambda}\omega + 2\delta^{\lambda}_{[\lambda}\tilde{\nabla}_{\mu]}\omega\tilde{\nabla}_{\nu}\omega - 2\delta^{\lambda}_{[\lambda}\tilde{g}_{\mu]\nu}(\tilde{\nabla}\omega)^2 \qquad \text{(B.8)} \\ \therefore R_{\mu\nu} &= \tilde{R}_{\mu\nu} + \tilde{g}_{\mu\nu}\tilde{\Box}\omega + (n-2)(\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\omega + \tilde{\nabla}_{\mu}\omega\tilde{\nabla}_{\nu}\omega - \tilde{g}_{\mu\nu}(\tilde{\nabla}\omega)^2) \\ &\qquad \text{(B.9)} \end{aligned}$$

Further contracting Eq (B.9) with $g^{\mu\nu} = \Omega^2\tilde{g}^{\mu\nu}$ leads to:

$$R = \Omega^2(\tilde{R} + 2(n-1)\tilde{\Box}\omega - (n-1)(n-2)(\tilde{\nabla}\omega)^2) \qquad \text{(B.10)}$$

Eq (B.10) reduces to Eq (3.23) for $n = 4$. ■

B.2 Jordan frame action for modified Einstein frame potential

It was shown in Section 3.4 that one can transform S_{J} in Eq (3.21) to S_{E} in Eq (3.24) by the means of a conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ for a suitably chosen value of Ω. Later, in Section 6.2, $\mathcal{S}_{\mathrm{E}}$ was encountered in Eq (6.3), which is a modified version of S_{E}. In this section, it will be proved that the Jordan frame equivalent of $\mathcal{S}_{\mathrm{E}}$ is given by $\mathcal{S}_{\mathrm{J}}$, seen in Eq (6.4), while keeping the choice of Ω same as in the original case. This ensures that the scalar field appearing in the Einstein frame is the same in both instances.

The aim is to arrive at a Jordan frame action of the form:

$$\mathcal{S}_{\mathrm{J}} = \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,\varphi(R) \qquad \text{(B.11)}$$

That is, the goal is to determine the expression for $\varphi(R)$ which corresponds to $\mathcal{S}_{\mathrm{E}}$ in Eq (6.3). Recall from Section 3.4,

$$f(R) = R + \frac{R^2}{6M^2}$$
$$e^{2\omega} = \Omega^2 = F = f(R)_{,R}$$
$$\phi = \sqrt{6}M_{\rm P}\,\omega$$
$$V(\phi) = \frac{U}{\Omega^4} = \frac{M_{\rm P}^2}{2}\Big(\frac{FR - f(R)}{F^2}\Big)$$

It is useful to define $\mathcal{U} \coloneqq \Omega^4\mathcal{V}(\phi)$. Using these in Eq (6.3) gives:

$$\mathcal{S}_{\rm E} = \int \mathrm{d}^4x\,\sqrt{-\tilde{g}}\bigg(\frac{M_{\rm P}^2}{2}\tilde{R} - \frac{M_{\rm P}^2}{2}(6(\tilde{\nabla}\omega)^2) - \frac{\mathcal{U}}{\Omega^4}\bigg) \tag{B.12}$$

Here, the shorthand $(\tilde{\nabla}\omega)^2 \equiv \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\omega\tilde{\nabla}_\nu\omega$ is used. In analogy to U, let $\mathcal{U} = M_{\rm P}^2(FR - \varphi(R))/2$. Then Eq (B.12) becomes:

$$\mathcal{S}_{\rm E} = \frac{M_{\rm P}^2}{2}\int \mathrm{d}^4x\,\sqrt{-\tilde{g}}\bigg(\tilde{R} - 6(\tilde{\nabla}\omega)^2 - \frac{FR - \varphi(R)}{\Omega^4}\bigg) \tag{B.13}$$

In Section 3.4, ω was subject to $\tilde{\nabla}_\mu\omega = 0$ at the boundary, so that the integral of the total derivative could be dropped. Owing to this boundary condition, the same integral of the total derivative can be reintroduced as it does not contribute. This leads to:

$$\mathcal{S}_{\rm E} = \frac{M_{\rm P}^2}{2}\int \mathrm{d}^4x\,\sqrt{-\tilde{g}}\bigg(\tilde{R} + 6\tilde{\Box}\omega - 6(\tilde{\nabla}\omega)^2 - \frac{\Omega^2R - \varphi(R)}{\Omega^4}\bigg) \tag{B.14}$$

Using $\sqrt{-\tilde{g}} = \Omega^4\sqrt{-g}$ and Eq (B.10) for $n = 4$ in Eq (B.14) gives:

$$\begin{aligned}\mathcal{S}_{\rm E} &= \frac{M_{\rm P}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}(\Omega^2R - \Omega^2R + \varphi(R))\\ &= \frac{M_{\rm P}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,\varphi(R)\end{aligned} \tag{B.15}$$

Eq (B.15) is the same as Eq (B.11).

To determine $\varphi(R)$, consider the following:

$$
\begin{aligned}
\mathcal{V}(\phi) &= \frac{\mathcal{U}}{\Omega^4} \\
\therefore \mathcal{V}(\phi) &= \frac{M_{\mathrm{P}}^2}{2}\frac{(FR-\varphi(R))}{F^2} \\
\therefore \frac{2}{M_{\mathrm{P}}^2}\mathcal{V}(\phi) &= \frac{FR-\varphi(R)}{F^2}
\end{aligned} \tag{B.16}
$$

Using Eq (3.27) for $V(\phi)$ and Eq (6.2) for $\mathcal{V}(\phi)$ in Eq (B.16) results in:

$$
\begin{aligned}
\frac{FR-\varphi(R)}{F^2} &= V(\phi) + \frac{3\alpha M_{\mathrm{P}}^2}{2}\alpha' \\
\therefore \frac{FR-\varphi(R)}{F^2} &= \frac{FR-f(R)}{F^2} + \frac{3\alpha M_{\mathrm{P}}^2}{2}\alpha' \\
\therefore \varphi(R) &= f(R) - F^2\frac{3\alpha M_{\mathrm{P}}^2}{2}\alpha'
\end{aligned} \tag{B.17}
$$

$$
\because F = f(R)_{,R} \implies F^2 = \left(1 + \frac{R}{3\alpha M_{\mathrm{P}}^2}\right)^2 \tag{B.18}
$$

Substituting Eq (B.18) in Eq (B.17) and performing alegrbraic manipulations leads to:

$$
\varphi(R) = \left(R + \frac{R^2}{6\alpha M_{\mathrm{P}}^2}\right)(1-\alpha') - \frac{3\alpha M_{\mathrm{P}}^2}{2}\alpha' \tag{B.19}
$$

This is non-trivial and well-defined for $\alpha' \neq -1$, which is true from (6.11). Substituting Eq (B.19) in Eq (B.11) gives:

$$
\mathcal{S}_{\mathrm{J}} = \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\left(\left(R + \frac{R^2}{6\alpha M_{\mathrm{P}}^2}\right)(1-\alpha') - \frac{3\alpha M_{\mathrm{P}}^2}{2}\alpha'\right) \quad \blacksquare
$$

B.3 EOMs of Jordan frame actions

The equations of motions of the Jordan frame actions seen in Eqs (3.21) and (6.4), were given in Eqs (6.5) and (6.6) respectively. In this section, the EOMs will be derived by explicitly varying the action. For the sake of brevity, upon performing the variation, only the bulk terms will be considered, and those terms which can be expressed as boundary terms will be ignored. Consider a Jordan frame action:

$$S = \frac{M_{\rm P}^2}{2}\int {\rm d}^4x\,\sqrt{-g}(R + mR^2 - 2\Lambda) \tag{B.20}$$

$$S = \frac{M_{\rm P}^2}{2}\int {\rm d}^4x\,\underbrace{\sqrt{-g}}_{a}\underbrace{R}_{b}\underbrace{(1 + mR_{\mu\nu}g^{\mu\nu})}_{c} - 2\sqrt{-g}\Lambda$$

$$\therefore\,\delta S = \frac{M_{\rm P}^2}{2}\int {\rm d}^4x\,\big((\delta a)b + a(\delta b)\big)c + ab(\delta c) - 2\Lambda\delta a \tag{B.21}$$

In the above, m is some constant of dimensions $[m] = [M^{-2}]$. Eq (2.7) gives the required expression for δa, and:

$$\delta b = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}$$
$$\delta c = m(R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu})$$

Substituting these in Eq (B.21) and dropping the boundary term gives:

$$\delta S = \frac{M_{\rm P}^2}{2}\int {\rm d}^4x\,\sqrt{-g}\bigg(\delta g^{\mu\nu}\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + 2m\Big(RR_{\mu\nu} - \frac{1}{4}R^2 g_{\mu\nu}\Big)\Big) + 2mRg^{\mu\nu}\delta R_{\mu\nu}\bigg) \tag{B.22}$$

For the last term, in Section 2.1 $g^{\mu\nu}\delta R_{\mu\nu} \equiv \nabla_\lambda A^\lambda$ was established, where A^λ is given by Eq (2.16). Using Eq (2.6) in Eq (2.16) leads to:

$$A^\lambda = \nabla^\lambda g_{\mu\nu}\delta g^{\mu\nu} - \nabla_\rho \delta g^{\rho\lambda}$$

$$\therefore\,2mRg^{\mu\nu}\delta R_{\mu\nu} = 2mR\big((\nabla_\lambda\nabla^\lambda g_{\mu\nu}\delta g^{\mu\nu}) - (\nabla_\mu\nabla_\nu\delta g^{\mu\nu})\big) \tag{B.23}$$

Since the term in Eq (B.23) appears under an integral, it can be integrated by parts. Doing so twice, and again dropping the boundary terms results in:

$$2mRg^{\mu\nu}\delta R_{\mu\nu} = 2m(g_{\mu\nu}\Box R - R_{;\mu\nu})\delta g^{\mu\nu}$$

Substituting this in Eq (B.22) gives:

$$\delta S = \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,\delta g^{\mu\nu}\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} \\ + 2m\Big(RR_{\mu\nu} - \frac{1}{4}R^2 g_{\mu\nu} + g_{\mu\nu}\Box R - R_{;\mu\nu}\Big)\Big) \tag{B.24}$$

Setting $\delta S = 0$ then gives the EOM of Eq (B.20):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + 2m\Big(RR_{\mu\nu} - \frac{1}{4}R^2 g_{\mu\nu} + g_{\mu\nu}\Box R - R_{;\mu\nu}\Big) = 0 \tag{B.25}$$

For $\Lambda = 0$ and $m = 1/6\alpha M_{\mathrm{P}}^2$, Eq (B.25) gives Eq (6.5). ■

Now consider another Jordan frame action given by:

$$\mathcal{S} = \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\big((R + mR^2)\gamma - 2\Lambda\big) \tag{B.26}$$

For some constant $\gamma \neq 0$. Then Eq (B.26) can be written as:

$$\mathcal{S} = \frac{M_{\mathrm{P}}^2\gamma}{2}\int \mathrm{d}^4x\,\sqrt{-g}\Big(R + mR^2 - 2\frac{\Lambda}{\gamma}\Big) \tag{B.27}$$

Varying Eq (B.27) leads to:

$$\delta\mathcal{S} = \frac{M_{\mathrm{P}}^2\gamma}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,\delta g^{\mu\nu}\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{\Lambda}{\gamma}g_{\mu\nu} \\ + 2m\Big(RR_{\mu\nu} - \frac{1}{4}R^2 g_{\mu\nu} + g_{\mu\nu}\Box R - R_{;\mu\nu}\Big)\Big) \tag{B.28}$$

Again, setting $\delta\mathcal{S} = 0$ gives the EOM of Eq (B.27)[1]:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{\Lambda}{\gamma}g_{\mu\nu} + 2m\Big(RR_{\mu\nu} - \frac{1}{4}R^2 g_{\mu\nu} + g_{\mu\nu}\Box R - R_{;\mu\nu}\Big) = 0 \tag{B.29}$$

Then, for $\Lambda = 3\alpha M_{\mathrm{P}}^2\alpha'/4$, $m = 1/6\alpha M_{\mathrm{P}}^2$, and $\gamma = 1 - \alpha'$, Eq (B.29) gives Eq (6.6). ■

[1]Presence of such a γ alters the mass scale from M_{P}^2 to γM_{P}^2, but its effect is not seen in the resultant EOM in absence of matter fields.

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